Commun. Math. Phys. 150, 519–535 (1992)

Communications in Mathematical Physics

Chern–Simons Solitons, Toda Theories and the Chiral Model*

Gerald Dunne†

Department of Mathematics and Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, U.S.A.

Received March 15, 1992; in revised form May 2, 1992

Abstract. The two-dimensional self-dual Chern–Simons equations are equivalent to the conditions for static, zero-energy solutions of the $(2 + 1)$-dimensional gauged nonlinear Schrödinger equation with Chern–Simons matter-gauge dynamics. In this paper we classify all finite charge $SU(N)$ solutions by first transforming the self-dual Chern–Simons equations into the two-dimensional chiral model (or harmonic map) equations, and then using the Uhlenbeck–Wood classification of harmonic maps into the unitary groups. This construction also leads to a new relationship between the $SU(N)$ Toda and $SU(N)$ chiral model solutions.

1. Introduction

The study of the nonlinear Schrödinger equation in $2 + 1$-dimensional space-time is partly motivated by the well-known successes of the $1 + 1$-dimensional nonlinear Schrödinger equation. Here we consider a *gauged* nonlinear Schrödinger equation in which we have not only the nonlinear potential term for the matter fields, but also we have a coupling of the matter fields to gauge fields. Furthermore, this matter-gauge dynamics is chosen to be of the Chern–Simons form rather than the conventional Yang–Mills form. Such a choice is motivated by the fact that the resulting Schrödinger equation is related to a non-relativistic field theory for the many-body anyon system.

The theory with an Abelian gauge field was analyzed by Jackiw and Pi [11] who found static, zero energy solutions which arise from a two-dimensional notion of self-duality. The static, self-dual matter density satisfies the Liouville equation which is known to be integrable and, indeed, solvable in the sense that the general (real) solution may be expressed in terms of an arbitrary holomorphic function [14]. The gauged nonlinear Schrödinger equation with a *non-Abelian* Chern–Simons matter-gauge dynamics has also been considered [8, 4, 5], and once

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069, and NSF grant #87-08447

† Present address: Physics Department, University of Connecticut, Storrs CT 06269, USA

again static, zero energy solutions (referred to as self-dual Chern–Simons solitons) have been found to arise from an analogous, but much richer, two-dimensional self-duality condition. These two-dimensional self-duality equations are formally integrable and as special cases they reduce to the classical and affine Toda equations, both well-known integrable nonlinear systems of partial differential equations [21, 16, 7, 15, 1]. For the classical Toda equations one can exhibit explicit exact solutions with finite charge [8, 4, 5] using the results of Kostant [12] and Leznov and Saveliev [13] concerning the integration of the Toda equations.

In this paper I classify *all* finite charge solutions to the self-dual Chern–Simons equations. This classification is achieved by first showing that the self-duality equations are equivalent to the classical equations of motion of the (Euclidean) two-dimensional chiral model (also known as the harmonic map equations). One can then make use of some deep classification theorems due to K. Uhlenbeck [22] (see also subsequent work by J. C. Wood [26]) which classify all chiral model solutions (for $U(N)$ and $SU(N)$) with finite classical chiral model action. The chiral model action is in fact proportional to the net gauge invariant *charge* Q in the matter-Chern–Simons system, and so the classification of all finite charge solutions is complete.

Another harmonic map result, due to G. Valli [23], shows that the charge Q is actually *quantized*, in integral multiples of $2\pi\kappa$ (where κ is the Chern–Simons coupling strength) – a fact already observed for the special case of the classical Toda solutions found in Ref. [4].

The explicit description of $SU(N)$ chiral model solutions rapidly becomes algebraically complicated, even for $N \geqq 3$. Wood [26] has given an explicit parametrization involving sequences of holomorphic maps into Grassmannians and an algorithm involving only algebraic and integral transform operations. The $SU(3)$ and $SU(4)$ cases have been studied in great detail in Refs. [18, 26]. The harmonic maps into individual Grassmannians (also known as the CP^{N-1} model) have been known for some time [2, 20] – for a review see Ref. [28]. In this paper I present the explicit "uniton" decomposition of a class of solutions to the $SU(N)$ chiral model equations for *any* N. The matrices are expressed in terms of $(N-1)$ arbitrary holomorphic functions, and this class of solutions has the remarkable property that when the matter density is diagonalized it satisfies the classical $SU(N)$ Toda equations. At first sight, such a direct correspondence between the Toda equations and the chiral model equations seems very surprising, but within the context of the self-duality equations the correspondence arises quite naturally.

The outline of this paper is as follows. In Sect. 2, I describe the derivation of the self-dual Chern–Simons equations as the conditions (in fact, necessary and sufficient) for the static, zero energy solutions to the $2+1$-dimensional gauged nonlinear Schrödinger equation with Chern–Simons coupling. In Sect. 3, I show how the self-duality equations reduce to the integrable Toda equations in special cases. The equivalence between the self-dual Chern–Simons equations and the chiral model (or harmonic map) equations is demonstrated in Sect. 4, and in Sect. 5 I show how to classify all solutions using the results of Uhlenbeck and Wood. Finally, in Sect. 6 I present a special class of explicit harmonic maps which correspond (via a unitary transformation) to the known classical $SU(N)$ Toda solutions of the self-dual Chern–Simons equations. A brief conclusion is devoted to comments and suggestions for further investigation.

2. The Self-Dual Chern–Simons Equations

The gauged non-linear Schrödinger equation reads

$$iD_0\Psi = -\frac{1}{2}\vec{D}^2\Psi + \frac{1}{\kappa}[\Psi^\dagger, \Psi]\Psi\,, \tag{1}$$

where the gauge covariant derivative is $D_\mu \equiv \partial_\mu + [A_\mu, \,]$, and both the gauge potential A_μ and the matter field matrix Ψ are Lie algebra valued: $A_\mu = A_\mu^a T^a$, $\Psi = \Psi^a T^a$. The Lie algebra generators satisfy:

$$[T^a, T^b] = f^{abc}T^c\,, \quad T^{a\dagger} = -T^a\,, \quad \mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}\,. \tag{2}$$

Our results are presented for the Lie algebra of $SU(N)$, but the formulation generalizes naturally for any compact Lie algebra. In $2+1$ dimensions we choose to couple the matter and gauge fields via the Chern–Simons equation

$$F_{\mu\nu} = \frac{1}{\kappa}\varepsilon_{\mu\nu\rho}J^\rho\,, \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the gauge curvature, κ is a coupling constant and J^ρ is the covariantly conserved ($D_\mu J^\mu = 0$) matter current

$$\begin{aligned} J^0 &= [\Psi, \Psi^\dagger]\,, \\ J^i &= \frac{1}{2}([\Psi^\dagger, D_i\Psi] - [(D_i\Psi)^\dagger, \Psi])\,. \end{aligned} \tag{4}$$

We can also define the scalar current V^μ,

$$\begin{aligned} V^0 &= \mathrm{tr}(\Psi^\dagger\Psi) \\ V^i &= \frac{1}{2}\mathrm{tr}(\Psi^\dagger D_i\Psi - (D_i\Psi)^\dagger\Psi)\,, \end{aligned} \tag{5}$$

which is ordinarily conserved ($\partial_\mu V^\mu = 0$).

Note that the Schrödinger equation (1) and the Chern–Simons equations (3) are invariant under the gauge transformation

$$\begin{aligned} &\Psi \to g^{-1}\Psi g\,, \\ &A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g\,, \end{aligned} \tag{6}$$

where $g \in SU(N)$. Furthermore, the Schrödinger equation (1) may be expressed as the Heisenberg equation of motion

$$i\partial_0\Psi = \frac{\delta H}{\delta\Psi^\dagger}\,, \tag{7}$$

where the Hamiltonian H is

$$H = -\int d^2x\,\mathrm{tr}((D_+\Psi^\dagger)D_-\Psi) = \frac{1}{2}\int d^2x (D_+\Psi^\dagger)^a (D_-\Psi)^a\,. \tag{8}$$

Here $D_\pm \equiv D_1 \pm iD_2$ and the gauge fields $A_\pm$ appearing in $D_\pm$ are determined by (3)–(4). Note that the simple, manifestly positive, form (8) for the Hamiltonian relies on the fact that we have chosen the nonlinear coupling coefficient in (1) to be $1/\kappa$, the *same* as the Chern–Simons coupling strength in (3).

We begin by seeking solutions which satisfy the *self-dual Ansätz*

$$D_-\Psi = 0\,. \tag{9}$$

From (8) we see that such solutions minimize the energy and thus, by (7), must correspond to *static* solutions. This fact, together with the self-dual *Ansätz* (9), leads to the following concise form of the Chern–Simons equation (3):

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = \frac{2}{\kappa}[\Psi^\dagger, \Psi] \,. \tag{10}$$

Here $A_\pm = A_1 \pm iA_2$, $\partial_\pm = \partial_1 \pm i\partial_2$, and $A_\pm = -(A_\pm)^\dagger$. Equations (9) and (10) are referred to collectively as the *self-dual Chern–Simons equations*. From the above discussion we see that solutions of these equations yield *static, minimum* (in fact, zero) energy solutions to the gauged nonlinear Schrödinger equation (1) with Chern–Simons coupling (3). In fact, owing to a remarkable dynamical $SO(2, 1)$ symmetry of (1) and (3), it is possible to show that *all* static solutions of (1) and (3) must be self-dual [4].

To conclude this summary we recall that the self-dual Chern–Simons equations (9) and (10) have arisen previously in another context [9, 3] – they are the dimensional reduction (from Euclidean four dimensions to Euclidean two dimensions) of the four-dimensional self-dual Yang–Mills equations [8, 4, 5].

3. Algebraic Reduction to Classical and Affine Toda Equations

Before classifying the general solutions to the self-dual Chern–Simons equations we show that certain simplifying algebraic *Ansätze* for the fields reduce (9) and (10) to familiar integrable nonlinear equations. First, choose

$$A_i = \sum_\alpha A_i^\alpha H_\alpha \,, \tag{11a}$$

$$\Psi = \sum_\alpha \psi^\alpha E_\alpha \,, \tag{11b}$$

where the sums are over all positive, simple roots α of the algebra (for $SU(N)$ we may take $\alpha = 1 \ldots N-1$), and H_α, E_β are the Cartan subalgebra and step operator generators (respectively) in the Chevalley basis [10]. In this Lie algebra basis, the commutation relations have an especially concise form (for α, β simple roots):

$$[H_\alpha, H_\beta] = 0, \quad [H_\alpha, E_{\pm\beta}] = \pm K_{\alpha\beta} E_{\pm\beta} \,,$$
$$[E_\alpha, E_{-\beta}] = \delta_{\alpha\beta} H_\alpha \,. \tag{12}$$

Here $K_{\alpha\beta}$ is the (classical) Cartan matrix for the Lie algebra. For $SU(N)$, K is the $(N-1)\times(N-1)$ symmetric tridiagonal matrix

$$K_{\alpha\beta} = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}, \tag{13}$$

which is familiar from the discrete approximation to the second derivative in numerical analysis.

With this choice (11) for the matter and gauge fields, the self-dual Chern–Simons equations combine to yield the system of equations

$$\nabla^2 \ln \rho_\alpha = -\frac{2}{\kappa} K_{\alpha\beta} \rho_\beta \quad (\alpha = 1 \ldots N-1)\,, \tag{14}$$

where $\rho_\alpha \equiv |\psi^\alpha|^2$. This system is known as the two-dimensional classical Toda equations. For $SU(2)$, (14) becomes the Liouville equation

$$\nabla^2 \ln \rho = -\frac{4}{\kappa} \rho\,, \tag{15}$$

which Liouville showed to be integrable and indeed "solvable" [14] – in the sense that the general (real) solution could be expressed in terms of a single arbitrary holomorphic function $f = f(x^-)$:

$$\rho = \frac{\kappa}{2} \nabla^2 \ln(1 + f(x^-)\bar{f}(x^+))\,. \tag{16}$$

Kostant [12], and Leznov and Saveliev [13] showed that the two-dimensional Toda equations (14) are integrable (with K the Cartan matrix of any simple Lie algebra), and that the solutions are intimately related to the representation theory of the corresponding Lie algebra (see also Ref. [16]). The general (real) solutions for ρ_α may be expressed in terms of r arbitrary holomorphic functions, where r is the rank of the algebra. Indeed, explicit formulas may be given expressing the ρ_α as a matrix element of certain path-ordered exponentials in the α^{th} fundamental representation [13]. In Ref. [4] the $SU(N)$ Toda solutions were expressed in an equivalent but simpler form, more reminiscent of Liouville's solution (16) for the $SU(2)$ case:

$$\rho_\alpha \equiv \frac{\kappa}{2} \nabla^2 \ln \det(M_\alpha^\dagger M_\alpha) \quad (\alpha = 1 \ldots N-1)\,, \tag{17}$$

where M_α is the $N \times \alpha$ *rectangular* matrix

$$M_\alpha = (u \;\; \partial_- u \;\; \partial_-{}^2 u \ldots \partial_-^{\alpha-1} u)\,, \tag{18}$$

with u being an N-component column vector

$$u = \begin{pmatrix} 1 \\ f_1(x^-) \\ f_2(x^-) \\ \vdots \\ f_{N-1}(x^-) \end{pmatrix}, \tag{19}$$

involving $(N-1)$ arbitrary holomorphic functions $f_\alpha(x^-)$. For $N = 2$, this trivially reduces to Liouville's solution (16). We shall discuss the general $SU(N)$ solution (17)–(19) in more detail in Sect. 6 in relation to the chiral model.

By extending the algebraic *Ansätz* (11b) for the matter fields (while retaining (11a) for the gauge fields) to

$$\Psi = \sum_{\substack{\text{positive} \\ \alpha = \text{simple} \\ \text{roots}}} \psi^{\alpha} E_{\alpha} + \psi^{M} E_{-M} , \tag{20}$$

where E_{-M} is the step operator corresponding to minus the maximal root, the self-dual Chern–Simons equations (9) and (10) combine into the *affine* Toda equations

$$\nabla^2 \ln \rho_a = -\frac{2}{\kappa} \tilde{K}_{ab} \rho_b , \tag{21}$$

where $\tilde{K}_{ab}$ is the $(r+1) \times (r+1)$ *affine* Cartan matrix. For $SU(N)$, $\tilde{K}_{ab}$ is the $N \times N$ symmetric matrix

$$\tilde{K}_{ab} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} . \tag{22}$$

The *affine* Toda equations (23) are integrable, but it is not possible to give convergent expressions for the general solutions in terms of a certain number of arbitrary functions.

4. Equivalence between Self-Dual Chern–Simons Equations and Chiral Model Equations

In Ref. [4] it was shown that it is possible to make a gauge transformation g^{-1} (as in Eq. (6)) which combines the self-dual Chern–Simons equations into a single matrix equation

$$\partial_{-} \chi = [\chi^{\dagger}, \chi] , \tag{23}$$

where

$$\chi = \sqrt{\frac{2}{\kappa}} g \Psi g^{-1} . \tag{24}$$

The existence of such a gauge transformation g^{-1} follows from the following zero-curvature relation. To see this, define

$$\mathscr{A}_{+} \equiv A_{+} - \sqrt{\frac{2}{\kappa}} \Psi , \tag{25a}$$

$$\mathscr{A}_{-} \equiv A_{-} + \sqrt{\frac{2}{\kappa}} \Psi^{\dagger} . \tag{25b}$$

Then the self-dual Chern–Simons equations imply that

$$\partial_- \mathscr{A}_+ - \partial_+ \mathscr{A}_- + [\mathscr{A}_-, \mathscr{A}_+] = 0 , \tag{26}$$

which means that we can (locally) write $\mathscr{A}_\pm$ as

$$\mathscr{A}_\pm = g^{-1} \partial_\pm g , \tag{27}$$

for some $g \in SU(N)$. Then, defining χ as in (24), we see that (25) and (27) imply that

$$D_- \Psi = \sqrt{\frac{\kappa}{2}} g^{-1} (\partial_- \chi - [\chi^\dagger, \chi]) g , \tag{28}$$

and

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] - \frac{2}{\kappa} [\Psi^\dagger, \Psi] = g^{-1} (\partial_- \chi + \partial_+ \chi^\dagger - 2[\chi^\dagger, \chi]) g . \tag{29}$$

This shows that the self-dual Chern–Simons equations (9) and (10) are equivalent to the single equation (23).

Equation (23) may now be rewritten as the *chiral model* equation:

$$\partial_+ (h^{-1} \partial_- h) + \partial_- (h^{-1} \partial_+ h) = 0 , \tag{30}$$

where $h \in SU(N)$ is related to χ as:

$$h^{-1} \partial_+ h = 2\chi , \tag{31a}$$

$$h^{-1} \partial_- h = -2\chi^\dagger . \tag{31b}$$

Note that if we define $J_+ = 2\chi$ and $J_- = -2\chi^\dagger$, then (23) (and its conjugate) become

$$\partial_+ J_- + \partial_- J_+ = 0 , \tag{32a}$$

$$\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0 , \tag{32b}$$

which express the fact that J has zero divergence and zero curl (in the non-Abelian sense). For this reason, the chiral model equations (30) are also known as the harmonic map equations. Equation (32b) shows that J is a pure gauge, which justifies (31a, b) – then the chiral model equation (30) is simply the zero divergence equation (32a) expressed in terms of h.

We conclude this section by stressing that given *any* solution h of the chiral model equation (30), the matrices χ and $\chi^\dagger$ defined in (31a) automatically solve (23). We thereby obtain a solution

$$\Psi^{(0)} = \sqrt{\frac{\kappa}{2}} \chi , \tag{33a}$$

$$A_+^{(0)} = \chi , \tag{33b}$$

$$A_-^{(0)} = -\chi^\dagger , \tag{33c}$$

of the self-dual Chern–Simons equations.

In order to compare these solutions with the Toda solutions discussed in the previous section we note that, with the algebraic *Ansätze* (11) and (20), the (Hermitian) non-Abelian charge density $\rho = [\Psi, \Psi^\dagger]$ is *diagonal* (it is also traceless

and so it may be decomposed in terms of the Cartan subalgebra elements). In contrast, the solutions obtained from the chiral model have charge density $\rho^{(0)} = \kappa/2[\chi, \chi^\dagger]$ which need not be diagonal. However, $\rho^{(0)}$ *is* Hermitian and so it *can* be diagonalized by a unitary matrix g. This diagonalizing matrix is precisely the gauge transformation relating Ψ and χ in (24). It is an algebraically non-trivial task to construct explicitly this gauge transformation and obtain the diagonal form of $\rho^{(0)}$ – however, we shall present such an explicit diagonalization in Sect. 6, relating certain chiral model solutions to the classical $SU(N)$ Toda solutions (17)–(19).

5. Classification of Solutions

The main point of exhibiting the equivalence of the self-dual Chern–Simons equations (9)–(10) to the chiral model equation (30) is that *all* solutions to the latter have been classified (subject to a finiteness condition which has direct physical relevance in the Chern–Simons language). Recall that this amounts to classifying *all* zero-energy static solutions to the gauged nonlinear Schrödinger equation (1) with Chern–Simons coupling (3).

In the two-dimensional Euclidean chiral model the "classical action" or "energy functional" is

$$\mathscr{E}(h) = -\frac{1}{2}\int d^2x \operatorname{tr}(h^{-1}\partial_- h h^{-1}\partial_+ h)\,, \tag{34}$$

which is manifestly positive. The classification (to be described in detail below) of solutions to the chiral model equation (30) is achieved subject to the finiteness condition

$$\mathscr{E}(h) < \infty\ . \tag{35}$$

Such a finiteness condition is appropriate in the 2 + 1-dimensional non-relativistic matter-Chern–Simons system ((1) and (3)) because

$$\mathscr{E}(h) = 2\int d^2x \operatorname{tr}(\chi\chi^\dagger) = \frac{4}{\kappa}\int d^2x \operatorname{tr}(\Psi\Psi^\dagger) = \frac{4}{\kappa}\int d^2x\, V^0 \equiv \frac{4}{\kappa}Q\,, \tag{36}$$

where Q is the net gauge invariant charge, and we have used the relations (31), (24) and (5). Thus, the "finite energy" condition (35) of the chiral model is precisely the "finite charge" condition of the Chern–Simons system. As well as being physically significant, the finiteness condition (35) is mathematically *crucial* because the classification results of Uhlenbeck [22], Wood [26] and Valli [23] are actually formulated for chiral model solutions on S^2, rather than on $\mathbf{R}^2$. However, when the "charge" (or "action") in (36) is finite, h extends to the conformal compactification $\mathbf{R}^2 \cup \{\infty\} = S^2$ of $\mathbf{R}^2$, and so the classification of finite charge solutions on $\mathbf{R}^2$ is equivalent to the classification of solutions on S^2 [25, 18]. This fortuitous correspondence permits us to take over directly the following results from the mathematical literature regarding the classification of solutions to the chiral model equations.

Theorem (K. Uhlenbeck [22]; see also J. C. Wood [26]). *Every finite action solution h of the $SU(N)$ chiral model equation* (30) *may be uniquely factorized as a product*

$$h = \pm h_0 \prod_{i=1}^{m} (2p_i - 1)\,, \tag{37}$$

where:

a) $h_0 \in SU(N)$ *is constant*;
b) *each* p_i *is a Hermitian projector* ($p_i^\dagger = p_i$ *and* $p_i^2 = p_i$);
c) *defining* $h_j = h_0 \prod_{i=1}^{j} (2p_i - 1)$, *the following linear relations must hold*:

$$(1 - p_i)\left(\partial_+ + \frac{1}{2} h_{i-1}^{-1} \partial_+ h_{i-1}\right) p_i = 0 ,$$

$$(1 - p_i) h_{i-1}^{-1} \partial_- h_{i-1} p_i = 0 ;$$

d) $m \leqq N - 1$.

Each factor $(2p_i - 1)$ is referred to as a "uniton" factor [22].

The $\pm$ sign in (37) has been inserted to allow for the fact that Uhlenbeck and Wood considered the case of $U(N)$ rather than $SU(N)$. However, since $\det(2p_i - 1) = (-1)^{N-d_i}$ if p_i is an $N \times N$ Hermitian projector onto a d_i-dimensional subspace, we see that we can obtain a solution $h \in SU(N)$ simply by choosing $h_0 \in SU(N)$ and the appropriate overall $\pm$ sign.

Note that Uhlenbeck's theorem tells us that for $SU(2)$ *all* finite action solutions of the chiral model have the form

$$h = -h_0(2p - 1) , \tag{38}$$

where p is a holomorphic projector

$$(1 - p)\partial_+ p = 0 . \tag{39}$$

(These, then, are essentially the CP^1 model solutions of Din and Zakrzewski [2] – see also the work of Sasaki [20]. Here we spell out these solutions in order to clarify the relationship with the self-dual Chern–Simons solutions, and to set the stage for the general $SU(N)$ results presented in Sect. 6.)

Since $p^2 = p$, condition (39) is equivalent to $\partial_+ pp = 0$. All such holomorphic projectors may be written as the projection matrix

$$p = M(M^\dagger M)^{-1} M^\dagger , \tag{40}$$

where $M = M(x^-)$ is any *rectangular* matrix depending only on the x^- variable (so $M^\dagger = M^\dagger(x^+)$ is a function only of x^+). It is straightforward to verify that such a projector satisfies (39). In order to obtain an h in the defining representation of $SU(2)$ we choose

$$M = \begin{pmatrix} 1 \\ f(x^-) \end{pmatrix} \tag{41}$$

so that

$$p = \frac{1}{1 + f\bar{f}} \begin{pmatrix} 1 & \bar{f} \\ f & f\bar{f} \end{pmatrix} . \tag{42}$$

(Note that in general one should consider projection onto the space spanned by $M = \begin{pmatrix} f_1(x^-) \\ f_2(x^-) \end{pmatrix}$, but since only the *direction* is important for p, this reduces to

either (41) or to $\begin{pmatrix} f(x^-) \\ 1 \end{pmatrix}$. The formulas in the latter case are analogous, and we shall in fact see that in the final result (48) there is no distinction – a *single* function $f(x^-)$ suffices to determine the diagonal form of $[\chi, \chi^\dagger]$.)

The corresponding solution χ of the self-dual Chern–Simons equation is given by (see (31))

$$\chi = \frac{1}{2}h^{-1}\partial_+ h = \frac{1}{2}(2p-1)\cdot 2\partial_+ p = 2p\partial_+ p - \partial_+ p = \partial_+ p$$

$$= \frac{f\partial_+ \bar{f}}{(1+f\bar{f})^2}\begin{pmatrix} -1 & \frac{1}{\bar{f}} \\ -f & 1 \end{pmatrix}. \tag{43}$$

The corresponding matter density is

$$[\chi, \chi^\dagger] = \frac{\partial_+ \bar{f}\partial_- f}{(1+f\bar{f})^3}\begin{pmatrix} 1-f\bar{f} & 2\bar{f} \\ 2f & -1+f\bar{f} \end{pmatrix}. \tag{44}$$

This may be diagonalized using the unitary matrix

$$g = \frac{1}{\sqrt{1+f\bar{f}}}\begin{pmatrix} 1 & -\bar{f} \\ f & 1 \end{pmatrix}, \tag{45}$$

which also diagonalizes p:

$$g^{-1}pg = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{46}$$

$$g^{-1}\chi g = \frac{\partial_+ \bar{f}}{(1+f\bar{f})}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{47}$$

$$g^{-1}[\chi, \chi^\dagger]g = \frac{\partial_+ \bar{f}\partial_- f}{(1+f\bar{f})^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \partial_+\partial_- \ln(1+f\bar{f})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \partial_+\partial_- \ln\det(M^\dagger M)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{48}$$

But this is precisely the classical $SU(2)$ Toda solution (16). Thus we see that the $SU(2)$ element g in (45) is the gauge transformation which converts between the classical Toda solution obtained using the *Ansätz* (11) (compare with (47)) and Uhlenbeck's chiral model solution (38) with p as in (42). Interestingly, Uhlenbeck's result tells us that this classical Toda solution is the *only* one with finite charge for $SU(2)$ (note that the charge is gauge invariant). In particular, as was argued in Ref. [4], there is no finite charge solution for the $SU(2)$ *affine* Toda equation arising from the algebraic *Ansätz* in (20) (this is consistent with the index theory analysis of

Ref. [27]). In the next section we will construct the analogous gauge transformation relating a particular solution of the $SU(N)$ chiral model equations in Uhlenbeck's factorized form (37), with the *explicit* $SU(N)$ Toda solutions (17)–(19).

It becomes significantly more involved to describe systematically all possible uniton factorizations of solutions to the $SU(N)$ chiral model equations for $N \geqq 3$. Wood [26] has given an explicit construction and parametrization of *all* $SU(N)$ solutions in terms of sequences of Grassmannian factors. These Grassmannian factors are the fundamental building blocks, and are modelled on the original CP^{N-1} solutions of Din and Zakrzewski [2, 28] and Sasaki [20]. Wood's parameterization involves an algorithm which uses only algebraic, derivative and integral transform operations. The $N = 3$ and $N = 4$ [18, 26] cases have been studied in great detail. In the next section we present the uniton factorization of a general class of harmonic maps for *any* N.

To conclude this section we quote a result due to Valli:

Theorem (G. Valli [23]). *Let h be a solution of the chiral model equation* (30). *Then the action*

$$-\frac{1}{2}\int d^2x \operatorname{tr}(h^{-1}\partial_- h h^{-1}\partial_+ h)$$

is quantized in integral multiples of 8π.

Given the relation (36) between the chiral model action and the Chern–Simons model charge Q, we obtain, as a corollary of Valli's theorem, the result that the charge is quantized in integral multiples of $2\pi\kappa$. This fact had been independently noted (see Appendix of Ref. [4]) for the classical $SU(N)$ Toda solutions.

6. Toda Solutions and the Chiral Model

In this section we generalize the single uniton $SU(2)$ solution discussed in (38)–(48) to a multi-uniton $SU(N)$ solution. This solution to the chiral model equation (30) has the remarkable property that it is gauge equivalent to the $SU(N)$ Toda solutions in (17)–(19). Let us first state the result, and then prove its validity.

Main Result. *The following matrix*

$$h = (-1)^{\frac{1}{2}N(N+1)} \prod_{\alpha=1}^{N-1} (2p_\alpha - 1)\,, \tag{49}$$

where p_α is the Hermitian holomorphic projector $p_\alpha = M_\alpha (M_\alpha^\dagger M_\alpha)^{-1} M_\alpha^\dagger$ for the matrix M_α in (18), *is a solution of the chiral model equation $\partial_+(h^{-1}\partial_- h) + \partial_-(h^{-1}\partial_+ h) = 0$. Furthermore, with χ and $\chi^\dagger$ related to h as in* (31), (i.e. $\chi \equiv \frac{1}{2}h^{-1}\partial_+ h$, $\chi^\dagger \equiv -\frac{1}{2}h^{-1}\partial_- h$) *there exists a unitary transformation g which diagonalizes the charge density matrix $[\chi, \chi^\dagger]$ so that*

$$g^{-1}[\chi, \chi^\dagger] g = \sum_{\alpha=1}^{N-1} \{\partial_+\partial_- \ln\det(M_\alpha^\dagger M_\alpha)\} H_\alpha\,, \tag{50}$$

where H_α are the CSA *generators of $SU(N)$ in the Chevalley basis for the defining representation. Recalling* (11) *and* (17) *we recognize this diagonalized form* (50) *as the $SU(N)$ Toda solution to the self-dual Chern–Simons equations.*

Proof. The matrix h in (49) is clearly unitary, and the $(-1)^{\frac{1}{2}N(N+1)}$ factor ensures that $h \in SU(N)$. The matrix g appearing in (50) (the $SU(N)$ matrix which provides the gauge transformation between the chiral model and Toda solutions) is simply the unitary matrix

$$g = (e_1 e_2 \ldots e_N) \tag{51}$$

which simultaneously diagonalized all the p_α. Thus the column vectors $\{e_\alpha\}$ are just the orthonormal basis elements constructed by the Gramm–Schmidt process beginning with the column vectors $u, \partial_- u, \partial_-^2 u, \ldots, \partial_-^{N-1} u$ (which are assumed to be linearly independent). Since the vectors $e_1, \ldots, e_\alpha$ span the same space as the vectors $u, \partial_- u, \ldots, \partial_-^{\alpha-1} u$ it is clear that

$$p_\alpha = \tilde{M}_\alpha (\tilde{M}_\alpha^\dagger \tilde{M}_\alpha)^{-1} \tilde{M}_\alpha^\dagger , \tag{52}$$

where $\tilde{M}_\alpha = (e_1 e_2 \ldots e_\alpha)$. And since the e_α's are orthonormal ($e_\alpha^\dagger e_\beta = \delta_{\alpha\beta}$) we find a simple expression for p_α:

$$p_\alpha = \sum_{\beta=1}^{\alpha} e_\beta e_\beta^\dagger . \tag{53}$$

Note however that the column vectors e_α depend on *both* x^- and x^+ (unlike the column vectors $\partial_-^\alpha u$ which only depend on x^-) – this dependence enters through the Gramm–Schmidt procedure:

$$e_\alpha = \frac{(1 - p_{\alpha-1})\partial_-^{\alpha-1} u}{\sqrt{\partial_+^{\alpha-1} u^\dagger (1 - p_{\alpha-1}) \partial_-^{\alpha-1} u}} , \quad (\alpha = 1 \ldots N) , \tag{54}$$

where $p_0 \equiv 0$. The unitary matrix g in (51) diagonalizes *each* p_α projection matrix. In fact, due to the orthonormality of the columns it is easy to see that

$$g^{-1} p_\alpha g = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & 0 & & & & & 0 \end{pmatrix} , \tag{55}$$

where the first α entries on the diagonal are 1, all other entries being 0. (This is hardly surprising, as p_α is a projector onto the α-dimensional subspace spanned by $e_1, \ldots e_\alpha$.)

To verify that h in (49) does indeed solve the chiral model equation (30) we first show that

$$h^{-1} \partial_+ h = 2 \sum_{\alpha=1}^{N-1} \partial_+ p_\alpha . \tag{56}$$

Then, since h is unitary and each p_α is Hermitian, we deduce that

$$h^{-1}\partial_- h = -2\sum_{\alpha=1}^{N-1}\partial_- p_\alpha\,, \tag{57}$$

from which the chiral model relation $\partial_+(h^{-1}\partial_- h) + \partial_-(h^{-1}\partial_+ h) = 0$ follows immediately.

To prove (56) we first note that each p_α is a holomorphic projector: i.e. $p_\alpha\partial_+ p_\alpha = \partial_+ p_\alpha$ for each $\alpha = 1 \ldots N-1$. Therefore, from (49) we have

$$\begin{aligned} h^{-1}\partial_+ h &= \sum_{\beta=1}^{N-1}\{(2p_{N-1}-1)\ldots(2p_{\beta+1}-1)(2p_\beta-1)2\partial_+ p_\beta(2p_{\beta+1}-1)\ldots \\ &\qquad (2p_{N-1}-1)\} \\ &= \sum_{\beta=1}^{N-1}\{(2p_{N-1}-1)\ldots(2p_{\beta+1}-1)2\partial_+ p_\beta(2p_{\beta+1}-1)\ldots(2p_{N-1}-1)\}\,. \end{aligned} \tag{58}$$

The result (56) follows now if we can prove that

$$[\partial_+ p_\alpha, p_\beta] = 0 \quad \forall \alpha < \beta\,. \tag{59}$$

From (55) we have that $\partial_+(g^{-1}p_\alpha g) = 0$, or equivalently:

$$g^{-1}(\partial_+ p_\alpha)g = [g^{-1}\partial_+ g, g^{-1}p_\alpha g]\,, \tag{60}$$

and we note the particularly simple form of $g^{-1}p_\alpha g$ as in (55). Now $g^{-1}\partial_+ g$ is an $N\times N$ matrix whose $(\alpha\beta)^{\text{th}}$ entry is $e_\alpha^\dagger\partial_+ e_\beta$. From the Gramm–Schmidt procedure (54) it is clear that $\partial_+ e_\beta$ is a linear combination of $e_1 \ldots e_\beta$, and so, by the orthonormality of the basis,

$$e_\alpha^\dagger\partial_+ e_\beta = 0 \quad \forall \alpha > \beta\,. \tag{61}$$

Similarly, since $\partial_+(e_\alpha^\dagger e_\beta) = \partial_+(\delta_{\alpha\beta}) = 0$,

$$\begin{aligned} e_\alpha^\dagger\partial_+ e_\beta &= -\partial_+ e_\alpha^\dagger e_\beta \\ &= -(e_\beta^\dagger\partial_- e_\alpha)^\dagger \\ &= 0 \quad \forall \beta > \alpha + 1\,, \end{aligned} \tag{62}$$

where in the last step we used the fact that $\partial_- e_\alpha$ is a linear combination of $e_1, \ldots, e_{\alpha+1}$. Thus, the matrix $g^{-1}\partial_+ g$ has the following simple form, with non-zero entries only on and immediately above the diagonal:

$$g^{-1}\partial_+ g = \begin{pmatrix} e_1^\dagger\partial_+ e_1 & e_1^\dagger\partial_+ e_2 & & 0 \\ \ddots & \ddots & & \\ & e_\alpha^\dagger\partial_+ e_\alpha & e_\alpha^\dagger\partial_+ e_{\alpha+1} & \\ & \ddots & \ddots & \\ 0 & & & e_{N-1}^\dagger\partial_+ e_N \\ & & & e_N^\dagger\partial_+ e_N \end{pmatrix}. \tag{63}$$

Equations (60) and (55) then imply $g^{-1}(\partial_+ p_\alpha)g$ has only *one* non-zero entry, in the $(\alpha(\alpha+1))^{\text{th}}$ place:

$$g^{-1}(\partial_+ p_\alpha)g = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & e_\alpha^\dagger \partial_+ e_{\alpha+1} & & \\ & & 0 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} . \tag{64}$$

It follows that $[g^{-1}(\partial_+ p_\alpha)g, g^{-1}p_\beta g] = 0$ for $\alpha < \beta$, which is just the statement (59).

This completes the proof that the matrix h in (49) does indeed satisfy the chiral model equation. We now proceed to prove the diagonalization formula (50). Observe that

$$\begin{aligned} g^{-1}\chi g &= \frac{1}{2} g^{-1}(h^{-1}\partial_+ h)g \\ &= \sum_{\alpha=1}^{N-1} g^{-1}(\partial_+ p_\alpha)g \\ &= \sum_{\alpha=1}^{N-1} (e_\alpha^\dagger \partial_+ e_{\alpha+1}) E_\alpha , \end{aligned} \tag{65}$$

where E_α are the $SU(N)$ positive simple root step operators in the defining representation: $(E_\alpha)_{ab} = \delta_{\alpha a}\delta_{\alpha+1,b}$. It should be noted that this is of the same algebraic *Ansätz* form as the $SU(N)$ *classical* Toda *Ansätz* for the matter field Ψ in (11b) for the Chern–Simons model. To make the comparison complete we note that

$$e_\alpha^\dagger \partial_+ e_{\alpha+1} = \sqrt{\partial_+ \partial_- \ln \det(M_\alpha^\dagger M_\alpha)} . \tag{66}$$

This may be verified directly from the Gramm–Schmidt projections (54) or by noting that

$$\begin{aligned} \partial_+ \partial_- \ln \det(M_\alpha^\dagger M_\alpha) &= \partial_+ \partial_- \operatorname{tr} \ln(M_\alpha^\dagger M_\alpha) \\ &= \partial_+ \operatorname{tr}((M_\alpha^\dagger M_\alpha)^{-1} M_\alpha^\dagger \partial_- M_\alpha) \\ &= \operatorname{tr}((1-p_\alpha)\partial_- M_\alpha (M_\alpha^\dagger M_\alpha)^{-1} \partial_+ M_\alpha^\dagger (1-p_\alpha)) \\ &= \operatorname{tr}(\partial_- p_\alpha \partial_+ p_\alpha) \\ &= |e_\alpha^\dagger \partial_+ e_{\alpha+1}|^2 , \end{aligned} \tag{67}$$

where in the last step we have used (64) (and its conjugate).

It now follows immediately that the diagonalized charge density is

$$g^{-1}[\chi, \chi^\dagger]g = \sum_{\alpha=1}^{N-1} \{\partial_+ \partial_- \ln \det(M_\alpha^\dagger M_\alpha)\} H_\alpha , \tag{68}$$

as claimed in (50). This should be compared with the $SU(N)$ Toda solution (17) (see (48) for the explicit $SU(2)$ case). Finally, the net Abelian charge is

$$\begin{aligned} Q &= \frac{\kappa}{2}\int d^2x \,\mathrm{tr}(\chi\chi^\dagger) \\ &= \frac{\kappa}{2}\sum_{\alpha=1}^{N-1}\int d^2x\,\partial_+\partial_-\ln\det(M_\alpha^\dagger M_\alpha) \\ &= \sum_{\alpha=1}^{N-1}\int d^2x\,\rho_\alpha \end{aligned} \tag{69}$$

with the ρ_α being the non-Abelian charge densities of the Toda solution (17).

7. Concluding Comments

In summary, we have shown that the static, self-dual zero-energy solutions of the 2 + 1-dimensional gauged nonlinear Schrödinger equation (1), with Chern–Simons coupling (3), may be classified in terms of the Uhlenbeck–Wood classification of solutions to the chiral model (or harmonic map) equations (30). The gauge invariant charge is quantized in integral multiples of $2\pi\kappa$. We have also found the explicit uniton factorization of a general class of harmonic maps into $SU(N)$, this class being distinguished by the fact that the corresponding matter density matrix, when diagonalized, satisfies the classical $SU(N)$ Toda equations.

For $SU(2)$, the Toda-type solutions exhaust *all* finite charge solutions, while for the $SU(N)$ $N \geqq 3$ systems this is *not* the case – there are harmonic map solutions which are not of Toda-type. This may be seen already for $SU(3)$ using the general solutions in Refs. [26, 18]; however, the non-Toda solutions for $SU(3)$ are somewhat awkward to present explicitly. The simplest non-Toda harmonic map solution arises in the $SU(4)$ model when we choose a one-uniton solution $h = 2p - 1$ with p being a holomorphic projector onto a two-dimensional holomorphic subspace. (Note that for $SU(3)$ we can only project onto a *one-dimensional* subspace, a two-dimensional subspace simply being the orthogonal complement of some one-dimensional subspace – thus $2p - 1$ just changes sign.) Thus p may be written as $p = M(M^\dagger M)^{-1}M^\dagger$, where M is a 4×2 holomorphic matrix whose column space may be specified by six arbitrary holomorphic functions. Then the (gauge invariant!) charge density V^0 is given by $V^0 = \partial_+\partial_-\ln\det(M^\dagger M)$ also depending on six functions. However, the $SU(4)$ Toda solution has V^0 as in (69) depending on only three arbitrary holomorphic functions (recall M_α is given by (18–(19)). This Toda solution is therefore less general.

The relationship between the chiral model and Toda equations is especially intriguing when viewed in light of the importance of the two-dimensional self-duality equations to the theory of integrable partial differential equations in two dimensions [24]. Ward [25] has also shown that the harmonic map equations may be understood in the setting of algebraic geometry, adapting results from "twistor" constructions of monopoles and Yang–Mills instantons. He had concentrated on the $SU(2)$ case – it would be interesting to see if similar constructions for $SU(N)$ of the Toda-type harmonic maps presented here leads to a deeper understanding of the Toda and/or chiral model equations.

Yet another mysterious correspondence arises from the work of Forgács et al. [6] and O'Raifeartaigh et al. [17], who have shown that the Toda systems arise as certain special cases not of the chiral model but of the Wess–Zumino–Novikov–Witten model. It is unclear what is the connection between their results and those presented here. The work of Piette et al. [19], which relates solutions of the chiral model with and without the Wess–Zumino term, may hold the key to this connection.

Finally, it would be interesting to consider the generalization of these results to Lie algebras other than $SU(N)$. For the compact simple algebras, the special Toda solutions to the self-dual Chern–Simons equations are known [12, 13]. Can one find the transformation g^{-1} which transforms these solutions to chiral model solutions, thereby obtaining explicit harmonic maps into the corresponding Lie group?

Acknowledgements. I thank Professor R. Jackiw for discussions and special thanks to Professor R. Ward for helpful correspondence.

References

1. Bilal, A., Gervais, J.-L.: Extended $c = \infty$ Conformal Systems from Classical Toda Field Theories. Nucl. Phys. **B314**, 646–686 (1989)
2. Din, A., Zakrzewski, W.: Properties of General Classical CP^{N-1} Solutions. Phys. Lett. **95B**, 419–422 (1980); General Classical Solutions in the CP^{N-1} Model. Nucl. Phys. **B174**, 397–406 (1980); Interpretation and Further Properties of General Classical CP^{N-1} Solutions. Nucl. Phys. **B182**, 151–157 (1981)
3. Donaldson, S.: Twisted Harmonic Maps and the Self-Duality Equations. Proc. Lond. Math. Soc. **55**, 127–131 (1987)
4. Dunne, G., Jackiw, R., Pi, S.-Y., Trugenberger, C.: Self-Dual Chern–Simons Solitons and Two-Dimensional Nonlinear Equations. Phys. Rev. **D43**, 1332–1345 (1991)
5. Dunne, G.: Self-Duality and Chern–Simons Theories. In: Proceedings of the XX$^{\text{th}}$ International Conference on Differential Geometric Methods in Theoretical Physics. Catto, S., Rocha, A. (eds). New York: Baruch College (1991)
6. Forgács, P., Wipf, A., Balog, J., Fehér, F., O'Raifeartaigh, L.: Liouville and Toda Theories as Conformally Reduced WZNW Theories. Phys. Lett. **B227**, 214–220 (1989); Toda Theory and W Algebra from a Gauged WZNW Point of View. Ann. Phys. (NY) **203**, 76–136 (1990)
7. Ganoulis, N., Goddard, P., Olive, D.: Self-Dual Monopoles and Toda Molecules. Nucl. Phys. **B205**, 601–636 (1982)
8. Grossman, B.: Hierarchy of Soliton Solutions to the Gauged Nonlinear Schrödinger Equation on the Plane. Phys. Rev. Lett. **65**, 3230–3232 (1990)
9. Hitchin, N.: The Self-Duality Equations on a Riemann Surface. Proc. Lond. Math. Soc. **55**, 59–126 (1987)
10. See e.g. Humphreys, J.: Introduction to Lie Algebras and Representation Theory. Berlin, Heidelberg, New York: Springer-Verlag 1990
11. Jackiw, R., Pi, S.-Y.: Soliton Solutions to the Gauged Non-Linear Schrödinger Equation on the Plane. Phys. Rev. Lett. **64**, 2969–2972 (1990); Classical and Quantum Non-Relativistic Chern–Simons Theory. Phys. Rev. **D42**, 3500–3513 (1990)
12. Kostant, B.: The Solution to a Generalized Toda Lattice and Representation Theory. Adv. Math. **34**, 195–338 (1979)
13. Leznov, A., Saveliev, M.: Representation of Zero Curvature for the System of Non-linear Partial Differential Equations $x_{\alpha,z\bar{z}} = \exp(kx)_\alpha$ and its Integrability. Lett. Math. Phys. **3**, 389–494 (1979); Representation Theory and Integration of Nonlinear Spherically Symmetric Equations of Gauge Theories. Commun. Math. Phys. **74**, 111–118 (1980)

14. Liouville, J.: Sur l'équation aux différences partielles $\frac{d^2}{dudv} log\ \lambda \pm \frac{\lambda}{2a^2} = 0$. J. Math. Pures, Appl. **18**, 71–72 (1853)
15. Mansfield, P.: Solution of Toda Systems. Nucl. Phys. **B208**, 277–300 (1982)
16. Mikhailov, A., Olshanetsky, M., Perelomov, A.: Two-Dimensional Generalized Toda Lattice. Commun. Math. Phys. **79**, 473–488 (1981)
17. O'Raifeartaigh, L., Ruelle, P., Tsutsui, I., Wipf, A.: W-Algebras for Generalized Toda Theories. Commun. Math. Phys. **143**, 333–354 (1992)
18. Piette, B., Zakrzewski, W.: General Solutions of the $U(3)$ and $U(4)$ Chiral Sigma Models in Two Dimensions. Nucl. Phys. **B300**, 207–222 (1988); Some Classes of General Solutions of the $U(N)$ Chiral σ Models in Two Dimensions. J. Math. Phys. **30**, 2233–2237 (1989)
19. Piette, B., Zait, R., Zakrzewski, W.: Solutions of the $U(N)$ Sigma Models with the Wess–Zumino Term. Zeit. für Phys. **C39**, 359–364 (1988)
20. Sasaki, R.: General Classical Solutions of the Complex Grassmannian and CP^{N-1} Sigma Models. Phys. Lett. **130B**, 69–72 (1983)
21. Toda, M.: Studies of a Nonlinear Lattice. Phys. Rep. **8**, 1–125 (1975)
22. Uhlenbeck, K.: Harmonic Maps into Lie Groups (Classical Solutions of the Chiral Model). J. Diff. Geom. **30**, 1–50 (1989)
23. Valli, G.: On the Energy Spectrum of Harmonic Two-Spheres in Unitary Groups. Topology **27**, 129–136 (1988)
24. Ward, R.S.: Integrable and Solvable Systems and Relations Among Them. Phil. Trans. Roy. Soc. Lond. **A315**, 451–457 (1985)
25. Ward, R.S.: Classical Solutions of the Chiral Model, Unitons and Holomorphic Vector Bundles. Commun. Math. Phys. **128**, 319–332 (1990)
26. Wood, J.C.: Explicit Construction and Parametrization of Harmonic Two-Spheres in the Unitary Group. Proc. Lond. Math. Soc. **58**, 608–624 (1989)
27. Yoon, Y.: Zero Modes of the Non-Relativistic Self-Dual Chern–Simons Vortices on the Toda Backgrounds. Ann. Phys. (NY) **211**, 316–333 (1991)
28. Zakrzewski, W.: Low Dimensional Sigma Models. London: Adam Hilger 1989

Communicated by K. Gawedzki