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The Ground State Energy of Schrödinger Operators

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Abstract. We study $e(\lambda) = \inf \operatorname{spec}(-\Delta + \lambda V)$ and examine when $e(\lambda) < 0$ for all $\lambda \neq 0$. We prove that $-c\lambda^2 \leqq e(\lambda) \leqq -d\lambda^2$ for suitable V and all small $|\lambda|$.

1. Introduction

In this paper we want to look at the "ground state energy," $e(\lambda) = \inf \operatorname{spec}(-\Delta + \lambda V)$, of a Schrödinger operator $-\Delta + \lambda V$ for V's which do *not* decay at infinity – think of periodic or almost periodic problems. In particular, we want to see when $e(\lambda)$ is *strictly* negative for all $\lambda \neq 0$. There is a large literature on this problem and the weaker $e \leqq 0$ result, most of it in one dimension. These examples typically have only essential spectrum so $e(\lambda) < 0$ is equivalent to solutions of $-u'' + \lambda V u = 0$ having an infinite number of zeros. The one-dimensional results often are phrased in these terms ("$-d^2/dx^2 + \lambda V$ is oscillatory").

The earliest results we are aware of are those of Wintner [19], who studied $\frac{-d^2}{dx^2} + \lambda V$ with $V(x+1) = V(x)$. He showed that

$$\lambda \int_0^1 V(x)dx - C\lambda^2 \int_0^1 V^2(x)dx \leqq e(\lambda) \leqq \lambda \int_0^1 V(x)dx \tag{1.1}$$

holds with $C = 1$. Kato [8] then improved this to $C = 1/16$. The question about the optimal C has been raised in [6, 8, 12, 19]. In Sect. 6 we will show that $C = (2\pi)^{-2}$ is best possible, the first inequality in (1.1) being strict for $\lambda \neq 0$.

In Sect. 5 we will recover Kato's result.

A series of authors (Moore [11], Blumenson [1], Ungar [18] and Staněk [17]) proved in the one-dimensional periodic case that $e(\lambda) < 0$ for all $\lambda \neq 0$ if $\int_0^1 V(x)dx = 0$ (note the strict inequality). By a Bloch wave analysis and eigenvalue perturbation theory [13], this result is easy, not only in one dimension but also for ν-dimensional periodic potentials (Eastham [4, 5] only proves $e(\lambda) \leqq 0$) if V is periodic with $\int_{\text{unit cell}} V(x)dx = 0$.

For the almost periodic one-dimensional case, various authors (Markus–Moore [9], Scharf [15] (who only showed $e(\lambda) \leqq 0$), Coppel [2], Halvorsen–Mingarelli [7], Dzurnak–Mingarelli [3] – see the review in [10]) proved $e(\lambda) < 0$ under some circumstances.

Our original goal was to understand how to prove similar results for multidimensional almost periodic models. In fact, we found such general methods that we feel the problem is rather transparent. Our results do not assume that V is almost periodic but only that it persists at infinity in some sense. For short range, V, the results are very different. For example, if V has compact support then $e(\lambda) = 0$ for λ small if $\nu \geqq 3$. So our hypotheses will have to be such that they exclude the short range case.

Our main results in this paper are three theorems, all in $\mathbb{R}^\nu$. The first theorem uses an abelian average

$$\mathrm{Av}_\varepsilon(V) = \int e^{-\varepsilon\langle x\rangle} V(x)dx \Big/ \int e^{-\varepsilon\langle x\rangle} dx ,$$

where, as usual, $\langle x\rangle = (|x|^2 + 1)^{1/2}$.

Theorem 1. *Let V obey:*

(i) *V is $C^1(\mathbb{R}^\nu)$ with $\|V\|_\infty$ and $\|\nabla V\|_\infty$ finite.*
(ii) $\mathrm{Av}_\varepsilon(V) \to 0$ *as* $\varepsilon \downarrow 0$.
(iii) $\mathrm{Av}_\varepsilon(V^2) \to a > 0$ *as* $\varepsilon \downarrow 0$.

Define $e(\lambda) = \inf \operatorname{spec}(-\Delta + \lambda V)$, $\lambda \in \mathbb{R}\setminus\{0\}$. *Then,* $e(\lambda) < 0$ *for all* $\lambda \in \mathbb{R}\setminus\{0\}$.
Indeed for some $b > 0$ and all $0 < |\lambda| \leqq 1$,

$$e(\lambda) \leqq -b\lambda^2 .$$

Remarks. 1. It suffices that $\mathrm{Av}_{\varepsilon_n}(V) \to 0$, $\mathrm{Av}_{\varepsilon_n}(V^2) \to a > 0$ for some sequence $\varepsilon_n \downarrow 0$.
2. Notice how we avoid the short range case where $\mathrm{Av}_\varepsilon(V^2) \to 0$.
3. Our proof shows one can take

$$b = a^2/[\|\nabla V\|_\infty^2 + 8\|V\|_\infty^3]$$

and in fact $\|\nabla V\|_\infty^2$ can be replaced by $\lim_{\varepsilon\downarrow 0} \mathrm{Av}_\varepsilon((\vec{\nabla} V)^2)$ if it exists.

Given Theorem 1, it is natural to ask about a lower bound on $e(\lambda)$ quadratic in λ. The hypotheses of Theorem 1 do not suffice for this, for consider $V(x) = \tanh(x)$ in one dimension. It is easy to see that the hypotheses of Theorem 1 hold but $e(\lambda) = -|\lambda|$. The key to eliminating linear terms is to deal with averages centered at arbitrary points, not just at the origin. Let $C_\alpha(a)$ with $a \in \mathbb{R}^\nu$ and $\alpha \in [0, \infty)$ be the hypercube in $\mathbb{R}^\nu$ of side α centered at a. Define

$$A_\pm(l) \equiv \sup_{\substack{\inf \\ C_\alpha(a);\ \text{all } a;\ \text{all } \alpha > l}} \left\{\alpha^{-\nu} \int_{C_\alpha(a)} V(x)dx\right\}$$

and

$$V_\pm = \lim_{l\to\infty} A_\pm(l)$$

Theorem 2. *Let V be bounded on $\mathbb{R}^\nu$. Then*

$$\lim_{\lambda\downarrow 0} \frac{e(\lambda)}{\lambda} = V_-, \quad \lim_{\lambda\uparrow 0} \frac{e(\lambda)}{\lambda} = V_+ .$$

Given this result, it is natural to expect that a quadratic lower bound on the energy will depend not only on $A_\pm(l) \to 0$ but on how fast it goes to zero. Intuitively, if the ground state is spread uniformly over a distance l, the kinetic energy is $O(l^{-2})$. If $A_\pm(l) = O(l^{-\alpha})$, then

$$\min_l (l^{-2} - \lambda l^{-\alpha})$$

is $-O(\lambda^{2/2-\alpha})$. This suggests that the correct condition for a quadratic lower bound is $A_\pm(l) = O(l^{-1})$. This is a natural condition in that in the periodic case, if the average of V over a fundamental cell is zero, then $A_\pm$ is indeed $O(l^{-1})$. We will need a somewhat stronger condition:

Theorem 3. *Suppose V is bounded with $V = \mathrm{div}(\vec{W})$ where $\vec{W}$ is $C^1(\mathbb{R}^\nu)$ function and $\|\vec{W}\|_\infty < \infty$. Then*

$$e(\lambda) \geqq -\lambda^2 \|\vec{W}\|_\infty^2 .$$

Notice that by Gauss' theorem, if $V = \mathrm{div}(\vec{W})$, then $\int_{C_\alpha(a)} V(x)dx$ is a surface term, so $\|\vec{W}\|_\infty < \infty$ implies that $A_\pm(l) = O(l^{-1})$. In one dimension, we can show that $A_\pm(l) = O(l^{-1})$ is equivalent to $\|W\|_\infty < \infty$.

2. Upper Bound: Proof of Theorem 1

Let $\varphi_\varepsilon(x)$ be the trial vector

$$\varphi_\varepsilon(x) = e^{-\varepsilon\langle x\rangle/2} / \| e^{-\varepsilon\langle x\rangle/2} \|_2 .$$

Then a simple calculation shows that

$$\| D^\alpha \varphi_\varepsilon \|_2 = O(\varepsilon^{|\alpha|})$$

for any multi-index α. In particular (with $H = -\Delta + \lambda V$ on $H^2(\mathbb{R}^\nu)$),

$$\langle \varphi_\varepsilon, H\varphi_\varepsilon \rangle = O(\varepsilon^2) + \lambda \mathrm{Av}_\varepsilon(V) , \tag{2.1}$$

$$\langle \varphi_\varepsilon, H^2\varphi_\varepsilon \rangle = O(\varepsilon^2) + \lambda^2 \mathrm{Av}_\varepsilon(V^2) , \tag{2.2}$$

if V is bounded. Finally,

$$\langle \varphi_\varepsilon, H^3\varphi_\varepsilon \rangle = \langle \nabla H\varphi_\varepsilon, \nabla H\varphi_\varepsilon \rangle + \lambda \langle H\varphi_\varepsilon, VH\varphi_\varepsilon \rangle .$$

If V is bounded

$$\langle H\varphi_\varepsilon, VH\varphi_\varepsilon \rangle = O(\varepsilon^2) + \lambda^2 \mathrm{Av}_\varepsilon(V^3) .$$

If ∇V is bounded

$$\langle \nabla H\varphi_\varepsilon, \nabla H\varphi_\varepsilon \rangle = O(\varepsilon^2) + \lambda^2 \mathrm{Av}_\varepsilon(\nabla(V)^2) .$$

Thus

$$\langle \varphi_\varepsilon, H^3\varphi_\varepsilon \rangle = O(\varepsilon^2) + \mathrm{Av}_\varepsilon(\lambda^3 V^3 + \lambda^2 (\nabla V)^2) . \tag{2.3}$$

Proof of Theorem 1. Suppose $e(\lambda) \geqq 0$. Then $H \geqq 0$ and so by the Schwarz inequality,

$$\langle \varphi_\varepsilon, H^2 \varphi_\varepsilon \rangle^2 \leqq \langle \varphi_\varepsilon, H\varphi_\varepsilon \rangle \langle \varphi_\varepsilon, H^3 \varphi_\varepsilon \rangle . \tag{2.4}$$

Using (2.1–3) and taking ε to zero, we see that

$$\lambda^4 [\lim \mathrm{Av}_\varepsilon(V^2)]^2 \leqq \lambda[\lim \mathrm{Av}_\varepsilon(V)]\, [\overline{\lim}\, \mathrm{Av}_\varepsilon(\lambda^3 V^3 + \lambda^2 (\nabla V)^2)]$$

which is impossible, since by hypothesis the left side is $\lambda^4 a^2 > 0$ and the right side zero.

To get a bound on $e(\lambda)$, use the fact that $(H - e(\lambda)) \geqq 0$, so (2.4) holds if H is replaced by $H - e(\lambda)$. Since $\langle \varphi_\varepsilon, H\varphi_\varepsilon \rangle \to 0$ as $\varepsilon \to 0$, in that limit $\langle \varphi, (H-e)^2 \varphi \rangle = \langle \varphi, H^2 \varphi \rangle + e^2 \geqq \langle \varphi, H^2 \varphi \rangle$. Thus

$$\lambda^4 a^2 \leqq |e(\lambda)| [\lambda^2 \| \nabla V \|_\infty^2 + |\lambda|^3 \| (V - e(\lambda)) \|_\infty^3] .$$

Since we consider only $|\lambda| \leqq 1$ and $|e(\lambda)| \leqq \| V \|_\infty$, we see that

$$\lambda^2 a^2 \leqq |e(\lambda)| [\| \nabla V \|_\infty^2 + 8 \| V \|_\infty^3] . \qquad \square$$

3. Lower Bound: Proof of Theorem 3

By hypothesis $V = \vec{\nabla} W$. Thus, if $u \in C_0^\infty$:

$$\begin{aligned} \langle u, Vu \rangle &= \int (\vec{\nabla} \cdot W) |u|^2 dx \\ &= -\int W \vec{\nabla} |u|^2 dx \end{aligned}$$

so

$$|\langle u, Vu \rangle| \leqq 2 \| W \|_\infty \| \nabla u \|_2 \| u \|_2 .$$

If $\| u \|_2 = 1$, we have that

$$\begin{aligned} \langle u, (-\Delta + \lambda V) u \rangle &\geqq \| \nabla u \|_2^2 - 2\lambda \| W \|_\infty \| \nabla u \|_2 \\ &\geqq -\lambda^2 \| W \|_\infty^2 \end{aligned}$$

by completing the square. Since C_0^∞ is a core for H, we have that

$$e(\lambda) \geqq -\lambda^2 \| W \|_\infty^2 .$$

4. Calculation of $D^\pm e/D\lambda$ at $\lambda = 0$ (Proof of Theorem 2)

We will only prove $\lim_{\lambda \downarrow 0} \lambda^{-1} e(\lambda) = V_-$ since the other limit then follows by replacing V by $-V$. Given α and a, define $\psi_{\alpha,a}$ to be the function which is

$$\psi_{\alpha,a}(x) = \alpha^{-1/2} \min(\alpha^{1/2}, \mathrm{dist}(x, \mathbb{R}^\nu \setminus C_\alpha(a)))$$

so $\psi_{\alpha,a}$ vanishes outside $C_\alpha(a)$, is 1 on $C_\alpha(a)$ with a collar of size $\alpha^{1/2}$ removed from the set and "linear" on the collar. Since V is bounded

$$\langle \psi_{\alpha,a}, V\psi_{\alpha,a} \rangle / \langle \psi_{\alpha,a}, \psi_{\alpha,a} \rangle = \alpha^{-\nu} \int_{C_\alpha(a)} V(x) dx + O(\alpha^{-1/2}) .$$

Moreover, it is easy to see that

$$\langle \nabla\psi_{\alpha,a}, \nabla\psi_{\alpha,a}\rangle / \langle \psi_{\alpha,a}, \psi_{\alpha,a}\rangle = O(\alpha^{-3/2}) .$$

It follows from the variational principle that

$$e(\lambda) \leqq \lambda V_- \quad \text{for all } \lambda \geqq 0 .$$

To get the lower bound, we introduce a map τ_α from $L^2(\mathbb{R}^\nu)$ to $L^2(\mathbb{R}^\nu)$ by

$$(\tau_\alpha\psi)(x) = \alpha^{-\nu} \int_{C_\alpha(a)} \psi(y)dy \quad \text{if } x \in C_\alpha(a),\ a \in \alpha\mathbb{Z}^\nu .$$

We claim that

$$\| \varphi - \tau_\alpha(\varphi) \|_2^2 \leqq c\alpha^2 \| \nabla\varphi \|_2^2 \tag{4.1}$$

for $\varphi \in Q(-\Delta)$ and will prove this below. Since τ_α is positivity preserving, self-adjoint and preserves the $L^1(\mathbb{R}^\nu)$ norm

$$\| \tau_\alpha(\varphi) \|_2^2 \leqq \| \varphi \|_2^2 .$$

(This also follows from the Schwarz inequality.)

Without loss of generality, we can suppose $e(\lambda) < 0$ for $\lambda \geqq 0$, e.g., replace V by $V - \| V \|_\infty - 1$. In that case for $\lambda \leqq 1/2$, find unit vectors ψ_λ so that

$$\langle \psi_\lambda, (-\Delta + \lambda V)\psi_\lambda \rangle \leqq e(\lambda)(1-\lambda) . \tag{4.2}$$

Since $e(\lambda) < 0$, we have

$$\langle \psi_\lambda, -\Delta\psi_\lambda \rangle \leqq \lambda \| V \|_\infty . \tag{4.3}$$

Write

$$\langle \psi_\lambda, (-\Delta + \lambda V)\psi_\lambda \rangle = a_1 + a_2 + a_3 + a_4$$

with (α to be fixed later)

$$a_1 = \langle (\psi_\lambda - \tau_\alpha(\psi_\lambda)), \lambda V(\psi_\lambda - \tau_\alpha(\psi_\lambda)) \rangle ,$$

$$a_2 = 2\mathrm{Re}\langle \tau_\alpha(\psi_\lambda), \lambda V(\psi_\lambda - \tau_\alpha(\psi_\lambda)) \rangle ,$$

$$a_3 = \langle \tau_\alpha(\psi_\lambda), \lambda V \tau_\alpha(\psi_\lambda) \rangle ,$$

$$a_4 = \langle \psi_\lambda, -\Delta\psi_\lambda \rangle ,$$

$a_4 \geqq 0$ and by (4.1) and (4.3)

$$|a_1| + |a_2| \leqq c\lambda^2\alpha^2 + c\lambda^{3/2}\alpha .$$

Clearly, $a_3 \geqq \lambda A_-(\alpha)$. Thus, by (4.2),

$$(1-\lambda)\lambda^{-1}e(\lambda) \geqq A_-(\alpha) - c\lambda\alpha^2 - c\lambda^{1/2}\alpha .$$

Take $\alpha = \ln \lambda^{-1}$ and find that

$$\underline{\lim}\, \lambda^{-1} e(\lambda) \geqq V_-$$

completing the proof of Theorem 2 modulo the lemma below. □

Lemma 4.1.

$$\| \varphi - \tau_\alpha(\varphi) \|_2^2 \leqq c\alpha^2 \| \nabla\varphi \|_2^2 .$$

Proof. If we prove this for φ supported in a single closed cube, C, where φ need not vanish on ∂C (Neumann form), we obtain it for all φ by summing all cubes. By scaling we can take $\alpha = 1$. So in a unit cube we need to prove that with P the projection onto the functioin 1:

$$\| (1 - P)\varphi \|_2^2 \leqq c \| \nabla\varphi \|_2^2 \tag{4.4}$$

with $\| \nabla\varphi \|_2^2$ the Neumann form. Since $-\Delta_N$ has 1 as eigenfunction with eigenvalue 0 and first eigenvalue π^2, (4.4) holds with $c = \pi^{-2}$. □

Remark. The lower bound proof can be pushed to get a power lower bound on $e(\lambda)$ if, say, $A_-(\alpha) = O(\alpha^{-1})$ but not a $-\lambda^2$ bound.

5. Examples

Let us first remark that using a special case of Theorem 3, Kato [8] obtained $C = 1/16$ in (1.1).

Next consider the almost periodic case. By an elementary calculation $\mathrm{Av}_\varepsilon(V) \to \mu(V)$, the Bohr mean of V. Here is a typical result that follows directly from our theorems:

Theorem. *Let* $V(x) = \sum_n c_n e^{2\pi i \vec{\alpha}_n \cdot \vec{x}}$ *on* $\mathbb{R}^\nu$, *where* $\vec{\alpha}_n$ *are arbitrary vectors and:*

(i) c_n *are not all zero,*
(ii) $\sum_n |c_n|(|\alpha_n| + |\alpha_n|^{-1}) < \infty$.

Then $-a\lambda^2 \leqq e(\lambda) \leqq -b\lambda^2$ *for some* $a, b > 0$ *and* $|\lambda|$ *small enough.*

Proof. Let

$$\vec{W}(x) = \sum_n c_n \frac{\vec{\alpha}_n}{|\alpha_n|^2} (2\pi i)^{-1} e^{2\pi i \vec{\alpha}_n \cdot \vec{x}}$$

which converges uniformly by (ii) so W is uniformly bounded. Clearly $\vec{\nabla} \cdot \vec{W} = V$ and by hypothesis

$$\| \nabla V \|_\infty \leqq 2\pi \sum_n |c_n| |\vec{\alpha}_n| < \infty ,$$

so Theorems 1 and 3 apply. □

As a final example of Theorem 3, consider the Hydrogen atom Hamiltonian

$$H = -\Delta + \lambda V, \quad V = -|\vec{x}|^{-1} .$$

Then $V(x) = \vec{\nabla} \cdot \vec{W}$ where $\vec{W}(x) = -\frac{1}{2}\vec{x}/|\vec{x}|$ and $\| \vec{W} \|_\infty^2 = \frac{1}{4}$. Since V is not bounded and W is not C^1, the theorem as stated does not apply but the proof does! The net result is

$$e(\lambda) \geqq -\lambda^2/4$$

which is exact! It is exact, of course, because the ground state obeys $\vec{W}\cdot\vec{\nabla}\varphi = c\varphi$ for the ground states (so the Schwarz inequality is an equality).

6. Two applications of the Riccati Equation

In this section we shall find the optimal constant in the Wintner–Kato estimate (1.1). Moreover, we shall give an alternate proof of Theorem 1.

Theorem. *Consider* $0 \neq V \in L^2(\mathbb{R}/\mathbb{Z})$ *with* $\langle V\rangle := \int_0^1 V(x)dx = 0$. *Then*

$$e(V) > -C\|V\|_2^2 \tag{6.1}$$

for all such V *and* $C = (2\pi)^{-2}$ *but for no smaller* C.

Proof. $H = -\dfrac{d^2}{dx^2} + V$ has a ground state $u > 0$ in $H_2^2(\mathbb{R}/\mathbb{Z})$. Its logarithmic derivative $m = u'/u$ satisfies

$$\langle m\rangle = 0 \tag{6.2}$$

and the Riccati equation

$$m' + m^2 + e(V) = V\,. \tag{6.3}$$

Taking averages before and after squaring gives

$$-e(V) = \langle m^2\rangle$$

and

$$\begin{aligned}\langle V^2\rangle &= \langle (m' + m^2 - \langle m^2\rangle)^2\rangle \\ &= \langle (m')^2\rangle + \langle (m^2 - \langle m^2\rangle)^2\rangle \\ &\equiv A + B\,,\end{aligned} \tag{6.4}$$

since the cross term is a perfect derivative and thus averages to zero. Here $B > 0$. Indeed, $m^2 = \langle m^2\rangle$ and m continuous imply that m is a constant and thus zero by (6.2). But then $V = 0$ by (6.3). Also by (6.2), $A = \|m'\|_2^2 \geqq (2\pi)^2\|m\|_2^2$. So we get $\|V\|_2^2 > -(2\pi)^2 e(V)$. Taking $m(x) = \lambda\cos(2\pi x)$ the inequality for A is saturated, $-e(V) = \lambda^2/2$ and $B = O(\lambda^4)$ as $\lambda \to 0$. Thus

$$\lim_{\lambda\to 0}\frac{\|V\|_2^2}{-e(V)} = (2\pi)^2\,. \qquad \square$$

Actually, the weak coupling limit of the Mathieu equation is the only regime where (6.1) with $C = (2\pi)^{-2}$ is (asymptotically) saturated, for we have:

Proposition. *Let* V *be as in the theorem. Then*

$$\frac{e(V)}{\|V\|_2^2} \to -(2\pi)^{-2}$$

if and only if $\|V\|_2 \to 0$ *and* $\|QV\|_2/\|V\|_2 \to 0$. *Here* $Q\colon L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})$ *is the orthogonal projection onto the Fourier modes with* $|k| \geqq 2$.

Lemma. *Let* $\mathscr{D} = \{m \in H_1^2(\mathbb{R}/\mathbb{Z}) | \langle m \rangle = 0,\ \|m\|_2 = 1\}$ *and set*

$$\Phi(m) = (\|m'\|_2^2 - (2\pi)^2) + (\|m\|_4^4 - 1) \equiv \Phi_1(m) + \Phi_2(m)$$

for $m \in \mathscr{D}$. *Then*

$$\inf_{m \in \mathscr{D}} \Phi(m) > 0 .$$

Remarks. 1. $\Phi(m)$ is well-defined since $H_1^2 \subset L^4$ (with bounded embedding).
2. $\inf_{m \in \mathscr{D}} \Phi_i(m) = 0$ for $i = 1, 2$.

Proof. Let $\{m\}$, $m \in \mathscr{D}$ be a minimizing sequence. Then $2\pi \cdot \|m\|_2 \leqq \|m'\|_2$ is bounded. Thus, upon passing to a subsequence, $m \xrightarrow{w} m_*$ in H_1^2 by the Banach–Alaoglu theorem (see e.g. [14]). Since the embedding $H_1^2 \subset L^2$ is compact, $m \to m_*$ in L^2. Thus, $\|m_*\|_2 = 1$, $\langle m_* \rangle = 0$, i.e. $m_* \in \mathscr{D}$. Moreover, $\|m_*'\|_2 \leqq \underline{\lim}\, \|m'\|_2$, $\|m_*\|_4 \leqq \underline{\lim} \|m\|_4$, showing that $\inf_{m \in \mathscr{D}} \Phi(m) = \Phi(m_*)$. Now $\Phi(m_*) = 0$ is impossible since it implies $Qm_* = 0$ and then $\|m_*\|_4^4 - 1 > 0$. □

Proof of the Proposition. Note that $-e(V) = \|m\|_2^2 > 0$ for $V \neq 0$. Thus

$$\begin{aligned} \frac{\|V\|_2^2}{-e(V)} - (2\pi)^2 &= \left(\frac{\|m'\|_2^2}{\|m\|_2^2} - (2\pi)^2 \right) + \left(\frac{\|m\|_4^4}{\|m\|_2^2} - \|m\|_2^2 \right) \\ &= \Phi_1(\tilde{m}) + \|m\|_2^2 \Phi_2(\tilde{m}) \\ &\geqq \frac{\|m\|_2^2}{1 + \|m\|_2^2} \Phi(\tilde{m}), \end{aligned} \tag{6.5}$$

where $\tilde{m} = m/\|m\|_2$. Then $\|V\|_2^2/(-e(V)) \to (2\pi)^2$ implies $\|m'\|_2/\|m\|_2 \to 2\pi$ by (6.5) and $\|m\|_2 \to 0$ because of the lemma. Therefore,

$$\frac{\|m^2\|_2}{\|m\|_2} = \frac{\|m\|_4^2}{\|m\|_2} \leqq \text{const} \frac{\|m'\|_2^2}{\|m\|_2} \to 0$$

and

$$\frac{\|V\|_2}{\|m\|_2} \to 2\pi$$

because of (6.3). In particular, $\|V\|_2 \to 0$. Moreover, $\|(1-Q)m'\|_2 = 2\pi \|(1-Q)m\|_2$ and $\|Qm'\|_2 \geqq 4\pi \|Qm\|_2$ imply

$$\begin{aligned} \|m'\|_2^2 &= (2\pi)^2 (\|m\|_2^2 - \|Qm\|_2^2) + \|Qm'\|_2^2 \\ &\geqq (2\pi)^2 \|m\|_2^2 + \left(1 - \frac{1}{4}\right) \|Qm'\|_2^2 . \end{aligned}$$

Thus, $\|QV\|_2/\|m\|_2 \leqq (\|Qm'\|_2 + \|m^2\|_2)/\|m\|_2 \to 0$, showing $\|QV\|_2/\|V\|_2 \to 0$.

Conversely, let $\|V\|_2 \to 0$, $\|QV\|_2/\|V\|_2 \to 0$. By (6.4), we have $2\pi\|m\|_2 \leqq \|m'\|_2 \leqq \|V\|_2$ and $\|m^2\|_2 = \|m\|_4^2 \leqq \text{const.} \|m'\|_2^2 \leqq \text{const} \|V\|_2^2$. Using (6.3) we estimate

$$1 \geqq \frac{\|m'\|_2}{\|V\|_2} \geqq 1 - 2\frac{\|m^2\|_2}{\|V\|_2} \geqq 1 - \text{const} \|V\|_2 ,$$

$$\frac{\|Qm'\|}{\|V\|_2} \leqq \frac{\|QV\|_2}{\|V\|_2} + \frac{\|m^2\|_2}{\|V\|_2} \to 0 .$$

Then

$$\|m\|_2^2 \geqq \|(1-Q)m\|_2^2 = (2\pi)^{-2}\|(1-Q)m'\|_2^2 = (2\pi)^{-2}(\|m'\|_2^2 - \|Qm'\|_2^2)$$

implies $\|m\|_2/\|V\|_2 \to (2\pi)^{-1}$. Now $\|V\|_2^2/(-e(V)) \to (2\pi)^{-2}$ follows from (6.5).□

Based on the Riccati equation, we can give the

Alternate Proof of Theorem 1. By the Allegretto–Piepenbrink theorem

$$-\Delta u = (e-V)u$$

has a non-negative distributional solution u. It is continuous and satisfies the Harnack inequality

$$c^{-1} < \frac{u(x)}{u(y)} < c \qquad (|x|, |y| \leqq 1)\,.$$

Here c denotes a generic constant depending only on a bound on $\|V\|_\infty$ or, as below, on $\|\nabla V\|_\infty$ as well. See e.g. [16] for the above. Then

$$u(x) = \int_{|y|<1} G_D(x,y)\Delta u(y)dy + \int_{|y|=1} P(x,y)u(y)d\sigma(y) \qquad (|x|<1)\,, \tag{6.6}$$

where G_D and P are the Dirichlet–Green's function respectively the Poisson kernel for the unit ball. Indeed, both sides share the same boundary values at $|x|=1$ and the equal upon applying Δ. As a result $\nabla u(x)$ is continuous and

$$|\nabla u(x)| \leqq c \sup_{|y|\leqq 1} |u(y)| \qquad (|x| < \tfrac{1}{2})\,.$$

Letting $m = \nabla u/u$, we then find

$$|m(x)| \leqq c \qquad (x \in \mathbb{R}^\nu)$$

by translating the origin and using Harnack's inequality. Again, m satisfies the Riccati equation

$$m^2 + \nabla\cdot m = V - e$$

showing $|\nabla\cdot m(x)| \leqq c$ for $x \in \mathbb{R}^\nu$.

Let us set $\langle f\rangle = \lim_{\varepsilon\downarrow 0} \mathrm{Av}_\varepsilon(f)$ for functions f on $\mathbb{R}^\nu$, provided the limit exists. Then

$$\mathrm{Av}_\varepsilon(\nabla\cdot m) = \frac{\int e^{-\varepsilon(x)}\,\nabla\cdot m(x)dx}{\int e^{-\varepsilon\langle x\rangle}dx} = O(\varepsilon)$$

after integration by parts, and hence

$$e = -\langle m^2\rangle\,,$$

$$\langle V^2\rangle \leqq \langle (V-e)^2\rangle = \langle (m^2+\nabla\cdot m)^2\rangle \leqq 2\,\overline{\lim_{\varepsilon\downarrow 0}}\,(\mathrm{Av}_\varepsilon(m^4) + \mathrm{Av}_\varepsilon(\nabla\cdot m)^2)\,.$$

Here $\mathrm{Av}_\varepsilon(m^4) \leqq c\langle m^2\rangle$ because m is bounded, and

$$\begin{aligned}\mathrm{Av}_\varepsilon(\nabla\cdot m)^2 &= \mathrm{Av}_\varepsilon((\nabla\cdot m)(V-e-m^2))\\ &\leqq \|\nabla V\|_\infty \,\mathrm{Av}_\varepsilon(m^2)^{1/2} + c\mathrm{Av}_\varepsilon(m^2) + O(\varepsilon)\end{aligned}$$

by integrating by parts, using the Schwarz inequality and the boundedness of m, $\nabla \cdot m$. Summing up:

$$\langle V^2 \rangle \leqq 2 \| \nabla V \|_\infty \langle m^2 \rangle^{1/2} + c\langle m^2 \rangle$$

which shows that $\langle V^2 \rangle > 0$ implies $-e = \langle m^2 \rangle > 0$. Also, replacing V by $\lambda V(|\lambda| < 1)$ does not affect c and implies $e(\lambda) \leqq -c\lambda^2$ with $c > 0$. □

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References

1. Blumenson, L.E.: On the eigenvalues of Hill's equation. Commun. Pure Appl. Math. **16**, 261–266 (1963)
2. Coppel, W.A.: "Disconjugacy." Lecture Notes in Mathematics, Vol. **220**. Berlin, Heidelberg, New York: Springer 1971
3. Dzurnak, A., Mingarelli, A.B.: Sturm–Liouville equations with Besicovitch almost-periodicity. Proc. AMS **106**, 647–653 (1989)
4. Eastham, M.S.P.: On the limit points of the spectrum. J. Lond. Math. Soc. **43**, 253–260 (1968)
5. Eastham, M.S.P.: The first stability interval of the periodic Schrödinger equation. J. Lond. Math. Soc. (2) **4**, 587–592 (1972)
6. Eastham, M.S.P.: "Results and Problems in the Spectral Theory of Periodic Differential Equations." Lecture Notes in Mathematics, Vol. **448**. Berlin, Heidelberg, New York: Springer 1975
7. Halvorsen, S.G., Mingarelli, A.B.: On the oscillation of almost-periodic Sturm–Liouville operators with an arbitrary coupling constant. Proc. AMS **97**, 269–272 (1986)
8. Kato, T.: Note on the least eigenvalue of the Hill equation. Quart. Appl. Math. **10**, 292–294 (1952)
9. Markus, L., Moore, R.A.: Oscillation and disconjugacy for linear differential equations with almost periodic coefficients. Acta Math. **96**, 99–123 (1956)
10. Mingarelli, A.B., Halvorsen, S.G.: Non-Oscillation Domains of Differential Equations with Two Parameters, Lecture Notes in Mathematics, Vol. **1338**. Berlin, Heidelberg, New York: Springer 1988
11. Moore, R.A.: The least eigenvalue of Hill's equation. J. Anal. Math. **5**, 183–196 (1956/57)
12. Putnam, C.R.: On the least eigenvalue of Hill's equation. Quart. Appl. Math. **9**, 310–314 (1951)
13. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. IV. Analysis of Operators. London: Academic Press 1978
14. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. I. Functional Analysis. London: Academic Press 1980
15. Scharf, G.: Fastperiodische Potentiale. Helv. Phys. Acta **38**, 573–605 (1965)
16. Simon, B.: Schrödinger semigroups. Bull. AMS **7**, 447–526 (1982)
17. Staněk, S.: A note on the oscillation of solutions of the differential equation $y'' = \lambda q(t) y$ with a periodic coefficient. Czech. Math. J. **29**, 318–323 (1979)
18. Ungar, P.: Stable Hill equations. Commun. Pure Appl. Math. **14**, 707–710 (1961)
19. Wintner, A.: On the non-existence of conjugate points. Am. J. Math. **73**, 368–380 (1951)

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