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On a Symmetry of Turbulence

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Abstract. This paper presents results on symmetries of the spectrum of singularities for random cascades found in the statistical theory of turbulence. It is shown that empirical dimension curves possess a natural symmetry whose presence restricts the class of allowable probability distributions of the cascade generator in a simple manner. In particular, necessary and sufficient conditions on the probability distribution of the generator are obtained for symmetry of the singularity spectrum within a large class of cascade models.

1. Introduction and Main Results

Recent analysis of the Anselmet, Gagne, Hopfinger, Antonia (1984) turbulence data by Meneveau and Sreenivasan (1987a) provides a comparison of empirical curves with those predicted by various cascade models, including the lognormal cascade, the binomial cascades, and the beta model. Based on comparisons of "singularity spectra" one finds models with remarkably good fit (e.g. either random or deterministic binomial cascades), as well as examples of departure (e.g. lognormal cascade). The statistical problem of precise dimension estimation is well-known to be non-trivial; see Cutler (1991a, b) for a comprehensive treatment of the statistical problems. In place of fitting the probability distribution of the generator, the focus here is on a qualitative property of the dimension spectrum which will be shown to restrict the class of generators in a natural way.

A precise description of the random cascade goes as follows. For simplicity let us consider partitions of the unit interval J into b subintervals of equal lengths $\frac{1}{b}$; the empirical curves are for one-dimensional cross sections. Let $W(\sigma_1, \ldots, \sigma_n)$, for $\sigma_i \in \{0, 1, \ldots, b-1\}$, be independent and identically distributed (iid) mean one positive random variables. These are referred to as the *cascaded variables* or *generators* of the cascade. Define a sequence of random measures μ_n on J by

$$d\mu_n(x) = p_n(x)\,dx\,, \tag{1}$$

where the density $p_n(x)$ is the step function defined by

$$p_n(x) = W(\sigma_1)W(\sigma_1, \sigma_2)\ldots W(\sigma_1, \ldots, \sigma_n),\ x \in J_n(\sigma_1, \ldots, \sigma_n). \tag{2}$$

One may apply the martingale convergence theorem to show that with probability one the sequence $\{\mu_n\}$ will converge to a random measure μ_∞ which defines the random cascade; see Kahane and Peyrière (1976).

Let W denote a generic random variable carrying the common distribution of the $W(\sigma_1, \ldots, \sigma_n)$'s. The *completely random multinomial cascade* is a special case of the general *random cascade* obtained by taking $W = bp_i$ with equal probabilities $\frac{1}{b}$, for $i = 0, \ldots, b-1$; "completely random" is added here for emphasis on equal probabilities. The *completely random binomial cascade* is obtained by taking $b = 2$. The *lognormal cascade* refers to the case in which W has a lognormal distribution.

The binomial case $b = 2$, $p_1 = p$ was considered in the analysis of one-dimensional sections of turbulence data by Meneveau and Sreenivasan (1987a). The value $p = 0.7$ was seen to provide a "good fit" of the singularity spectrum, which for this model is defined as a dimension function

$$f(\alpha) = \dim F(\alpha) \tag{3}$$

of the set

$$F(\alpha) = \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{\log_b \mu_\infty(J_n(x))}{-n} = \alpha \right\}, \tag{4}$$

where $J_n(x) = J(\sigma_1, \ldots, \sigma_n)$, for $x = \sum_{i=0}^{\infty} \sigma_i b^{-i}$ in base b. This example is well-known and an implicit formula for the singularity spectrum as the Legendre transform

$$f(\alpha) = \min_q \{\alpha q + \tau(q)\} \tag{5}$$

of the Renyi exponent,

$$\tau(q) = \lim_{n\to\infty} \frac{\log_b \sum' \mu_\infty^q(J(\sigma_1, \ldots, \sigma_n))}{n} \tag{6}$$

is known rigorously by combinatorics and large deviation computations; see Collet, Lebowitz, Porzio (1987), Feder (1988), Falconer (1990), Brown, Minchon, Peyrière (1990). The prime in the sum in (6) indicates a sum over those intervals with positive mass.

The existence and computation of the singularity spectrum and the Renyi scaling exponent for the random cascade was recently obtained by Holley and Waymire (1992) under conditions on the W's which bound them away from 0 and below the branching number b. While the results are applicable to the case of the random multinomial cascades, they do not cover the lognormal cascade, for example. In any case they are sufficient to show that the singularity spectrum and the Renyi function for the completely random multinomial cascades and the deterministic multinomial cascade coincide as described in (5), (6) above. Moreover, it follows from Holley and Waymire (1992) that under strong boundedness conditions on the W's one has

$$\tau(q) \equiv \chi_b(q) \tag{7}$$

and

$$f(\alpha) = \min_q \{\alpha q + \chi_b(q)\}, \tag{8}$$

where

$$\chi_b(q) = \log_b EW^q - (q-1)\,. \tag{9}$$

To state precise conditions on the cascaded distribution (generator) under which this holds let us first define

$$\varrho(q) = \frac{E\{W^q \log_b W\}}{EW^q}\,. \tag{10}$$

Then (7) and (8) hold whenever

i. $0 < a \le W < b$ with probability one for some a.

ii. $\dfrac{EW^{2q}}{(EW^q)^2} < b$

iii. $\chi_b(\alpha) > 0$ and $E\left(\dfrac{W}{\|W\|_\infty}\right)^\beta > b^{-1}$, where $\varrho(\beta) = 1-\alpha$.

We assume the strong boundedness throughout.

An inspection of the graph in Meneveau and Sreenivasan (Fig. 2, p. 1425, 1987a) for the "fit" of the binomial cascade with $b = 2$, $p = 0.7$ strongly suggests a natural symmetry in the turbulence data. It follows from the results of Holley and Waymire (1992) that both the completely random multinomial cascades and the deterministic multinomial cascades have the same singularity spectrum. Moreover, it follows that for the (completely random and/or deterministic) binomial cascade with parameters $b = 2$, $p \in (0,1)$ one has an implicit formula for the singularity spectrum $f(\alpha)$ in terms of the Legendre transform relations $\left(\dfrac{df}{d\alpha} = q, \dfrac{d\tau}{dq} = -\alpha\right)$

$$f(\alpha) = \min_q\{\alpha q + \tau(q)\} = \alpha q(\alpha) + \tau(q(\alpha))\,, \tag{11}$$

where

$$\tau(q) = \chi_2(q) = \log_2\{p^q + (1-p)^q\}\,, \tag{12}$$

and

$$\alpha \equiv \alpha(q) = -\frac{d\tau}{dq} = -\frac{p^q \log_2 p + (1-p)^q \log_2(1-p)}{p^q + (1-p)^q}\,. \tag{13}$$

We leave it to the reader to directly establish the *symmetry* (i.e. unimodal with symmetry about the mode) for arbitrary p as a warm-up to the general theorem below; also see the paragraph following Remark 2.

Remark 1. Although it has not been rigorously proven, conventional wisdom and formal calculations for the lognormal cascade suggest that $f(\alpha)$ may be a quadratic function and a Legendre transform pair with $\tau(q) = \chi_b(q)$ (e.g. see Meneveau and Sreenivasan, p. 64, 1987b, or Mandelbrot 1989); and therefore symmetric about its mode. This would also be anticipated by the results of Holley and Waymire (1992) if the conditions on the bounds required there can be relaxed. However, surprisingly, this is quite possibly *not* the case in view of preliminary computations by Durrett and Gravner (personal communication).

In order to obtain a general symmetry condition we shall consider classes of random cascades for which one has the Legendre transform relation

$$f(\alpha) = \min_q\{q\alpha + \chi_b(q)\}\,. \tag{14}$$

As noted above, this is known to include the random multinomial cascades, as a consequence of Holley and Waymire (1992), as well as other strongly bounded random cascades. One may check directly from (14) that $f(\alpha)$ is concave since for any $0 < r < 1$, α_1, α_2,

$$\begin{aligned} f(r\alpha_1 + (1-r)\alpha_2) &= \min_q\{q(r\alpha_1 + (1-r)\alpha_2) + \chi_b(q)\} \\ &= \min_q\{r[q\alpha_1 + \chi_b(q)] + (1-r)[q\alpha_2 + \chi_b(q)]\} \\ &\geq rf(\alpha_1) + (1-r)f(\alpha_2) . \end{aligned} \tag{15}$$

Theorem 1. *Consider strongly bounded random cascades for which* (14) *holds. Then the singularity spectrum* $f(\alpha)$ *is symmetric about its mode if and only if*

$$\frac{EW^q}{EW^{-q}} = e^{cq}, \tag{16}$$

where $c = -\log E \log W^{-1}$.

Proof. From (14), (15) one has

$$\alpha = \alpha(q) = -\frac{d\chi_b(q)}{dq} = 1 - \varrho(q), \tag{17}$$

where (cf. (10))

$$\varrho(q) = \frac{d}{dq}\log_b EW^q = \frac{E\{W^q \log_b W\}}{EW^q}. \tag{18}$$

Now observe from this that $\alpha(q) + \alpha(-q) = 2\alpha(0)$ if and only if

$$\varrho(q) + \varrho(-q) = 2\varrho(0). \tag{19}$$

Taking base-b logarithms in (16) and differentiation with respect to q shows that (16) implies (19). Similarly, one may integrate (19) to get (16). The constant follows by taking $q = 0$. Next observe from (14) that one can express $f(\alpha(q)) = f(\alpha(-q))$ as

$$\alpha(q)q + \chi_b(q) = -\alpha(-q)q + \chi_b(-q). \tag{20}$$

Equivalently, therefore, one has for (14)

$$\chi_b(q) - \chi_b(-q) = -(\alpha(q) + \alpha(-q))q. \tag{21}$$

Now, as already noted, the symmetry condition (16) is equivalent to (19) which, in turn by (17), is equivalent to

$$\alpha(q) + \alpha(-q) = 2\alpha(0). \tag{22}$$

Thus, from this and the definition of $\chi_b(q)$ one gets after taking logarithms that (21) is equivalent to (16). ■

Remark 2. The curve $\varrho(q)$ plays a basic role in the analysis in Holley and Waymire (1992). Properties and a graph of $\varrho(q)$ are given in Lemma 3.3, Fig. 3.1 therein which remain valid in the unbounded cases as well.

While the symmetry condition as stated is easily computed, e.g. $\frac{EW^q}{EW^{-q}} = [4p(1-p)]^q$ in the completely random binomial case, the following corollary makes its meaning transparent.

Corollary 1. *Under condition* (14), *one has a symmetric singularity spectrum for strongly bounded random cascades if an only if there is an* $a > 0$ *such that*

$$W \overset{\text{dist}}{=} aW^{-1}.$$

Proof. If one assumes the condition of the corollary then W^q and $a^q W^{-q}$ have the same distribution. Taking expected values, the symmetry condition of the theorem follows immediately from this with $c = \log a$. Conversely, the symmetry condition of the theorem implies that $\log W$ and $\log(aW^{-1})$ have the same moment generating function and therefore the same probability distribution. The symmetry condition of the corollary therefore follows with $a = e^c$. ■

Remark 3. In view of Corollary 1 the symmetry condition is that up to a rescaling W and W^{-1} have the same distribution. This makes the completely random binomial example transparent for all values of p merely by rescaling W^{-1} to W!

Another way to view this is to consider probability distributions on the *multiplicative group* $\Gamma = \mathscr{R}_+$ of positive real numbers with identity $e = 1$. Recall that symmetric probability distributions on the additive group $G = \mathscr{R}$ about $b \in G$ can be viewed as follows. A real-valued random variable X has a symmetric distribution about (the additive identity) 0 iff X and (its additive inverse) $-X$ have the same distribution. Secondly, X has a symmetric distribution about $a \in R$ iff (its additive translate) $X - a$ is symmetric about 0. In particular, such a symmetry has a natural abstraction to a property of a probability distribution on an arbitrary group. With this we observe that our symmetry condition expressed in Corollary 1 is simply the condition that a probability distribution on the (multiplicative) group Γ be "symmetric about $a \in \Gamma$." Thus one determines the basic symmetry in the otherwise seemingly asymmetric cases of random binomial, lognormal, etc. generators.

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