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Cyclic Homology of Differential Operators, the Virasoro Algebra and a q-Analogue

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Abstract. We show how methods from cyclic homology give easily an explicit 2-cocycle φ on the Lie algebra of differential operators of the circle such that φ restricts to the cocycle defining the Virasoro algebra. The same methods yield also a q-analogue of φ as well as an infinite family of linearly independent cocycles arising when the complex parameter q is a root of unity. We use an algebra of q-difference operators and q-analogues of Koszul and de Rham complexes to construct these "quantum" cocycles.

The Virasoro algebra Vir is the universal central extension of the Lie algebra $\mathrm{Der}(\mathbf{C}[x,x^{-1}])$ of derivations of the algebra $\mathbf{C}[x,x^{-1}]$ of complex Laurent polynomials. This extension

$$0\to\mathbf{C}\to\mathrm{Vir}\to\mathrm{Der}(\mathbf{C}[x,x^{-1}])\to 0$$

has a one-dimensional centre and is defined by the following 2-cocycle α on $\mathrm{Der}(\mathbf{C}[x,x^{-1}])$:

$$\alpha\left(P\frac{d}{dx}, Q\frac{d}{dx}\right)=\frac{1}{12}\mathrm{res}\begin{vmatrix}P' & Q'\\ P'' & Q''\end{vmatrix}=\frac{1}{6}\mathrm{res}(QP''')$$

with $P,Q\in\mathbf{C}[x,x^{-1}]$. Here P' denotes the derived polynomial of P and res is the residue map. Set $L_n=x^{n+1}d/dx$; then the cocycle α takes the familiar form

$$\alpha(L_m,L_n)=\frac{m^3-m}{6}\delta_{m+n,0}\,,$$

where $\delta_{i,j}$ is the Kronecker symbol.

We now embed $\mathrm{Der}(\mathbf{C}[x,x^{-1}])$ in the associative algebra $\mathscr{D}=\mathrm{Diff}(\mathbf{C}[x,x^{-1}])$ of all algebraic differential operators on $\mathbf{C}[x,x^{-1}]$. The set $\{x^i(d/dx)^j\}_{i\in\mathbf{Z},\,j\in\mathbf{N}}$ is a basis of the complex vector space $\mathscr{D}$.

In [5] Kac and Peterson proved that the Virasoro algebra is a Lie subalgebra of a central extension of $\mathscr{D}$ considered as a Lie algebra (see also [8] for a generalization and [4] for related results). More precisely,

Theorem 1. *Let φ be the antisymmetric bilinear form on $\mathscr{D}$ defined by*

$$\varphi\left(x^i\left(\frac{d}{dx}\right)^j, x^k\left(\frac{d}{dx}\right)^l\right) = \begin{cases} (-1)^l j!\, l! \left(\sum_{p=j}^{j-i-1} \binom{p}{j}\binom{k-p-1}{l}\right)\delta_{i+k,\, j+l} & \text{if} \quad i<0<k \\ -\varphi\left(x^k\left(\frac{d}{dx}\right)^l, x^i\left(\frac{d}{dx}\right)^j\right) & \text{if} \quad k<0<i \\ 0 & \text{otherwise.} \end{cases}$$

Then φ is a 2-cocycle for the Lie algebra $\mathscr{D}$. The restriction of φ to the Lie subalgebra $\mathrm{Der}(\mathbf{C}[x, x^{-1}])$ *is the Virasoro cocycle α.*

We shall first give another proof of Theorem 1 based on the following elementary observation from cyclic homology theory: any cyclic 1-cocycle (or equivalently antisymmetric Hochschild 1-cocycle) ψ on an associative algebra A, i.e. an antisymmetric bilinear form ψ on A such that for all a_0, a_1, a_2 in A we have

$$\psi(a_0 a_1, a_2) - \psi(a_0, a_1 a_2) + \psi(a_2 a_0, a_1) = 0,$$

is a 2-cocycle for the Lie algebra A with Lie bracket given by the commutators. Now cyclic cohomology is easier to compute than Lie algebra cohomology. As a matter of fact, the Hochschild and cyclic cohomology of differential operators was determined by Kassel-Mitschi (see [2]), Wodzicki [7], and Brylinski-Getzler [3]. In particular, the cyclic cohomology group $HC^1(\mathscr{D})$ of $\mathscr{D}$ turns out to be one-dimensional. We compute a generator φ which is the desired Lie 2-cocycle. Our main construction is the commutative diagram in Sect. 2. It involves five quasi-isomorphic chain complexes and relates the standard Hochschild complex of $\mathscr{D}$ to the de Rham complex of $\mathbf{C}[x, x^{-1}]$.

In the second part of the paper we observe that the above-mentioned diagram can easily be quantized, thus giving a non-commutative generalization of the constructions of the first part. This is done by considering a q-analogue of the algebra of differential operators, namely the algebra $\mathscr{D}_q$ of q-difference operators generated by x, x^{-1}, ∂_q and the relation

$$\partial_q x - q x \partial_q = 1$$

which is the q-analogue of the classical Heisenberg relation. From the quantized diagram we get a Hochschild 1-cocycle φ_q on the algebra $\mathscr{D}_q$. It is a one-parameter deformation of the cocycle of Theorem 1. Moreover, when q is a root of unity $\neq 1$, we obtain an *infinite family* of cocycles whose cohomology classes in the Hochschild group $H^1(\mathscr{D}, \mathscr{D}^*)$ are linearly independent. Such a phenomenon is reminiscent of what happens for de Rham cohomology in positive characteristic. The Virasoro generators L_n deform to elements $L_n(q)$ whose linear span is no longer closed under the commutator operation – which is not surprising in "non-commutative geometry" –; however, they generate $\mathscr{D}_q$ as an associative algebra. This suggests the algebra of q-difference operators as a q-analogue of the Virasoro algebra.

Let us sketch the contents of the paper.

In Sect. 1 we give a Koszul resolution for $\mathscr{D}$ which we compare with its standard Hochschild resolution. This enables us to construct in Sect. 2 five quasi-isomorphic chain complexes whose homology is the Hochschild homology of $\mathscr{D}$. Composing the homology isomorphisms connecting these complexes yields the cocycle φ (Sect. 3). We introduce the algebra $\mathscr{D}_q$ in Sect. 4 and build up the homological machinery necessary to deal with it in Sect. 5. In Sect. 6 we give explicit formulas for φ_q and the infinite family of "exotic" cocycles arising in the root of unity case.

1. Comparison of Resolutions for $\mathscr{D}$

Any associative algebra $\mathscr{D}$ has a canonical resolution by free $\mathscr{D}$-bimodules, namely the Hochschild resolution C'_*, b' where $C'_n = \mathscr{D} \otimes \mathscr{D}^{\otimes n} \otimes \mathscr{D}$ and

$$b'(D_0 \otimes D_1 \otimes \dots \otimes D_{n+1}) = \sum_{i=0}^{n} (-1)^i D_0 \otimes \dots \otimes D_i D_{i+1} \otimes \dots \otimes D_{n+1},$$

all tensor products being taken over the field of complex numbers. The Hochschild resolution is too big to allow the computation of the Hochschild groups of $\mathscr{D}$. In this section we construct a length-two resolution K_*, β' for $\mathscr{D}$. We also build a chain map

$$j' : C'_*, b' \to K_*, \beta'$$

over the identity.

We need the following notations. First, let $\partial = d/dx$ denote the usual derivation on the Laurent polynomials. Let V be a two-dimensional vector space with basis $\{dx, d\partial\}$. We denote by $\mathscr{D}^o$ the algebra $\mathscr{D}$ with opposite multiplication. We now introduce the chain complex K_*, β'. As a graded vector space it is defined by

$$K_* = \mathscr{D} \otimes \mathscr{D}^o \otimes \Lambda^* V .$$

The differential β' is the $\mathscr{D} \otimes \mathscr{D}^o$-linear degree -1 map given by

$$\beta'(1 \otimes 1 dx \wedge d\partial) = (1 \otimes x - x \otimes 1) d\partial - (1 \otimes \partial - \partial \otimes 1) dx,$$

$$\beta'(1 \otimes 1 dx) = 1 \otimes x - x \otimes 1,$$

$$\beta'(1 \otimes 1 d\partial) = 1 \otimes \partial - \partial \otimes 1 .$$

Before we state the main result of this section, let us adopt the following convention: if a, b are commuting elements in an associative algebra and if $i > 0$, we define

$$\frac{a^i - b^i}{a - b} = a^{i-1} + a^{i-2} b + \dots + a b^{i-2} + b^{i-1} .$$

Proposition 1. *The complex K_*, β' is a free $\mathscr{D} \otimes \mathscr{D}^o$-resolution of $\mathscr{D}$. There exists a $\mathscr{D} \otimes \mathscr{D}^o$-linear chain map*

$$j' : C'_*, b' \to K_*, \beta'$$

such that j'_0 is the identity of $\mathscr{D} \otimes \mathscr{D}^o$ and $j'_1(1 \otimes x^i \partial^j \otimes 1)$ is equal to

$$-(1 - \delta_{i,0})(1 \otimes \partial^j) \frac{1 \otimes x^i - x^i \otimes 1}{1 \otimes x - x \otimes 1} dx - (1 - \delta_{j,0})(x^i \otimes 1) \frac{1 \otimes \partial^j - \partial^j \otimes 1}{1 \otimes \partial - \partial \otimes 1} d\partial$$

if $i, j \geqq 0$ and to

$$(1 \otimes \partial^j)(x^i \otimes x^i) \frac{1 \otimes x^{-i} - x^{-i} \otimes 1}{1 \otimes x - x \otimes 1} dx - (1 - \delta_{j,0})(x^i \otimes 1) \frac{1 \otimes \partial^j - \partial^j \otimes 1}{1 \otimes \partial - \partial \otimes 1} d\partial$$

if $i < 0 \leqq j$.

Proof. Let us start with the following lemma.

Lemma 1. *The set* $\{1\otimes x - x\otimes 1, 1\otimes\partial - \partial\otimes 1\}$ *is a regular sequence of commuting elements in* $\mathscr{D}\otimes\mathscr{D}^o$.

Proof. We compute

$$[1\otimes x - x\otimes 1, 1\otimes\partial - \partial] = 1\otimes[x,\partial] + [x,\partial]\otimes 1 = 0$$

since $[x,\partial] = -1$ in $\mathscr{D}$ and $[x,\partial] = +1$ in $\mathscr{D}^o$. Let us prove these elements form a regular sequence.

The algebra $\mathscr{D}\otimes\mathscr{D}^o$ has no zero divisors. The quotient $(\mathscr{D}\otimes\mathscr{D}^o)/(1\otimes\partial - \partial\otimes 1)$ is isomorphic to the algebra generated by x, x^{-1}, x', x'^{-1}, ∂, ∂' and the relations

$$[x,x'] = [\partial,\partial'] = 0 \quad \text{and} \quad [\partial,x] = -[\partial',x'] = 1,$$

which is an iterated Ore extension and therefore has no zero divisors. This proves the lemma.

It is easy to check that the complex K_*, β' is the Koszul resolution attached to this regular sequence (see [1, Sect. 9]). It remains to check that

$$\beta' j'_1 = j'_0 b' = b',$$

which is done by a straightforward computation. We shall do it in the case $i<0<j$. Then in $\mathscr{D}\otimes\mathscr{D}^o$ we have

$$\begin{aligned}
\beta' j'_1(1\otimes x^i\partial^j\otimes 1) &= (1\otimes\partial^j)(x^i\otimes x^i)\frac{1\otimes x^{-i} - x^{-i}\otimes 1}{1\otimes x - x\otimes 1}(1\otimes x - x\otimes 1) \\
&\quad -(x^i\otimes 1)\frac{1\otimes\partial^j - \partial^j\otimes 1}{1\otimes\partial - \partial\otimes 1}(1\otimes\partial - \partial\otimes 1) \\
&= (1\otimes\partial^j)(x^i\otimes 1 - 1\otimes x^i) - (x^i\otimes 1)(1\otimes\partial^j - \partial^j\otimes 1) \\
&= x^i\partial^j\otimes 1 - 1\otimes x^i\partial^j = b'(1\otimes x^i\partial^j\otimes 1).
\end{aligned}$$

2. A Diagram with Five Chain Complexes

Consider the diagram

$$\begin{array}{ccccccccc}
\ldots\xrightarrow{b} & \mathscr{D}^{\otimes 4} & \xrightarrow{b} & \mathscr{D}^{\otimes 3} & \xrightarrow{b} & \mathscr{D}\otimes\mathscr{D} & \xrightarrow{b} & \mathscr{D} & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & \\
\ldots\longrightarrow & 0 & \longrightarrow & \mathscr{D}\otimes\Lambda^2 V & \xrightarrow{\beta} & \mathscr{D}\otimes V & \xrightarrow{\beta} & \mathscr{D} & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} & \\
\ldots\longrightarrow & 0 & \longrightarrow & \Omega_S^0 & \xrightarrow{d} & \Omega_S^1 & \xrightarrow{d} & \Omega_S^2 & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow & \\
\ldots\longrightarrow & 0 & \longrightarrow & \Omega_A^0 & \xrightarrow{d} & \Omega_A^1 & \longrightarrow & 0 & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow & \\
\ldots\longrightarrow & 0 & \longrightarrow & H_{DR}^0(A) = \mathbf{C}\cdot 1 & \xrightarrow{0} & H_{DR}^1(A) = \mathbf{C}\dfrac{dx}{x} & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

which we describe now.

1. The upper two rows are obtained from the $\mathscr{D}\otimes\mathscr{D}^o$-linear chain map

$$j': C'_*, b' \to \mathscr{D}\otimes\mathscr{D}^o\otimes\Lambda^* V, \beta'$$

by applying the functor $\mathscr{D}\otimes_{\mathscr{D}\otimes\mathscr{D}^o}-$. Then the upper row is the standard Hochschild complex of the algebra $\mathscr{D}$.

2. We denote $A=\mathbf{C}[x,x^{-1}]$ and Ω_A^*, d the corresponding de Rham complex with cohomology groups $H_{DR}^*(A)$. The map $\Omega_A^0 \to H_{DR}^0(A)$ is given by $P \mapsto P(0)$ and the map $\Omega_A^1 \to H_{DR}^1(A)$ by

$$P dx \mapsto \operatorname{res}(P)\frac{dx}{x}.$$

3. The algebra S is the graded algebra associated to the filtration on $\mathscr{D}$ by the order of differential operators. It can be seen as the algebra of polynomial functions on the cotangent bundle over $\mathbf{C}\backslash\{0\}$. As an algebra

$$S \cong A[\xi] \cong \mathbf{C}[x,x^{-1},\xi].$$

The vertical map $\pi_*: \Omega_S^* \to \Omega_A^*$ is induced by the null-section, i.e. by the algebra map sending ξ to 0.

4. The maps σ_* are defined by

$$\begin{aligned}\sigma_0(x^i\partial^j) &= -x^i\xi^j dx\wedge d\xi,\\ \sigma_1(x^i\partial^j dx) &= -x^i\xi^j dx,\\ \sigma_1(x^i\partial^j d\partial) &= -x^i\xi^j d\xi,\\ \sigma_0(x^i\partial^j dx\wedge d\partial) &= x^i\xi^j.\end{aligned}$$

They are obtained by composing a generalized symbol map $\mathscr{D}\otimes\Lambda^* V \to \Omega_S^*$ with the duality isomorphism induced by the symplectic 2-form $dx\wedge d\xi$.

Proposition 2. *The above diagram is commutative; its vertical maps are homology isomorphisms and for any differential operator D in $\mathscr{D}$, $j_1(D\otimes x^i\partial^j)$ is equal to*

$$-(1-\delta_{i,0})\left(\sum_{p=0}^{i-1} x^{i-p-1}\partial^j D x^p\right)dx-(1-\delta_{j,0})\left(\sum_{p=0}^{j-1}\partial^{j-p-1}Dx^i\partial^p\right)d\partial$$

if $i,j\geqq 0$ and to

$$\left(\sum_{p=-1}^{i} x^{i-p-1}\partial^j D x^p\right)dx-(1-\delta_{j,0})\left(\sum_{p=0}^{j-1}\partial^{j-p-1}Dx^i\partial^p\right)d\partial$$

if $i<0\leqq j$.

As a corollary, we recover the Hochschild groups of $\mathscr{D}$, namely

$$H_i(\mathscr{D},\mathscr{D})=\operatorname{Tor}_i^{\mathscr{D}\otimes\mathscr{D}^o}(\mathscr{D},\mathscr{D})\cong\begin{cases}\mathbf{C} & \text{if } i=1,2\\ 0 & \text{otherwise}.\end{cases}$$

Proof of Proposition 2. Since j is obtained from a chain map over the identity between resolutions, j is a chain map and a homology isomorphism. Let us compute j_1 when $i<0\leqq j$. We leave the other cases to the reader. We have

$$\begin{aligned} j_1(D\otimes x^i\partial^j) &= D\otimes_{\mathscr{D}\otimes\mathscr{D}^o} j'_1(1\otimes x^i\partial^j\otimes 1) \\ &= \left(\sum_{p=-1}^{i} D(1\otimes\partial^j)(x^p\otimes x^{i-p-1})\right)dx \\ &\quad -(1-\delta_{j,0})\left(\sum_{p=0}^{j-1} D(x^i\otimes 1)(\partial^p\otimes\partial^{j-p-1})\right)d\partial \\ &= \left(\sum_{p=-1}^{i} x^{i-p-1}\partial^j D x^p\right)dx-(1-\delta_{j,0})\left(\sum_{p=0}^{j-1}\partial^{j-p-1}Dx^i\partial^p\right)d\partial\,. \end{aligned}$$

The map $\beta=\mathrm{id}_{\mathscr{D}}\otimes_{\mathscr{D}\otimes\mathscr{D}^o}\beta'$ is given by

$$\beta(Ddx\wedge d\partial)=[x,D]d\partial-[\partial,D]dx\,,$$

$$\beta(Ddx)=[x,D]\,,$$

$$\beta(Dd\partial)=[\partial,D]\,.$$

With these formulas it is easy to check that σ is a chain map. It is clearly an isomorphism, hence a homology isomorphism.

Finally, π is a homology isomorphism because of Poincaré's lemma.

3. Proof of Theorem 1

We now prove Theorem 1. We define $\varphi:\mathscr{D}\otimes\mathscr{D}\to\mathbf{C}$ as

$$\varphi=\mathrm{res}\circ\pi_1\circ\sigma_1\circ j_1\,.$$

Since the diagram in Sect. 2 is commutative, we have $\varphi\circ b=0$, which means that for any triple (D_0,D_1,D_2) of differential operators,

$$\varphi(D_0D_1\otimes D_2)-\varphi(D_0\otimes D_1D_2)+\varphi(D_2D_0\otimes D_1)=0\,.$$

In other words, φ is a Hochschild 1-cocycle. By Proposition 2, φ is an homology isomorphism. Therefore, its cohomology class generates $H^1(\mathscr{D},\mathscr{D}^*)\cong\mathbf{C}$.

We now compute φ. We need the following well-known formula.

Lemma 2. *Let $P\in\mathbf{C}[x,x^{-1}]$. Then in D*

$$\partial^n P=\sum_{r=0}^{n}\binom{n}{r}P^{(r)}\partial^{n-r}\,,$$

where $P^{(r)}$ is the r-th derivative of P.

Lemma 3. *Let $i,k\in\mathbf{Z}$ and $j,l\in\mathbf{N}$. Then $\varphi(x^i\partial^j\otimes x^k\partial^l)$ is equal to*

$$(-1)^l j!\,l!\left(\sum_{p=j}^{j-i-1}\binom{p}{j}\binom{k-p-1}{l}\right)\delta_{i+k,\,j+l}$$

if $i<0<k$, to

$$-(-1)^j j!\,l!\left(\sum_{p=k}^{-1}\binom{j-p-1}{j}\binom{l-k+p}{l}\right)\delta_{i+k,\,j+l}$$

if $k<0<i$ and is zero otherwise.

Proof. Firstly, by definition of $\mathrm{res}\circ\pi_1\circ\sigma_1$ we have

$$(\mathrm{res}\circ\pi_1\circ\sigma_1)(x^i\partial^j dx) = -\delta_{i,-1}\delta_{j,0}$$

and

$$(\mathrm{res}\circ\pi_1\circ\sigma_1)(x^i\partial^j d\partial) = 0.$$

In order to compute $\varphi(x^i\partial^j\otimes x^k\partial^l)$ we have to express $x^{k-p-1}\partial^l x^i\partial^j x^p$ in the basis $\{x^i\partial^j\}$ of $\mathcal{D}$. Now by Lemma 2,

$$\begin{aligned} x^{k-p-1}\partial^l x^i\partial^j x^p &= \sum_{r=0}^{j}\sum_{s=0}^{l}\binom{j}{r}\binom{l}{s}x^{k-p-1}(x^i(x^p)^{(r)})^{(s)}\partial^{l+j-r-s} \\ &= \sum_{r=0}^{j}\sum_{s=0}^{l}\binom{j}{r}\binom{l}{s}p(p-1)\dots(p-r+1)(i+p-r)(i+p-r-1)\dots \\ &\quad \dots(i+p-r-s+1)x^{i+k-r-s-1}\partial^{l+j-r-s}. \end{aligned}$$

We have to look for all monomials whose degree in ∂ is zero. These are the terms with $r=j$ and $s=l$. Hence

$$\begin{aligned} &(\mathrm{res}\circ\pi_1\circ\sigma_1)(x^{k-p-1}\partial^l x^i\partial^j x^p dx) \\ &\quad = -(p(p-1)\dots(p-j+1)(i-j+p)(i-j+p-1)\dots(i-j+p-l+1))\delta_{i+k,\,j+l}. \end{aligned}$$

Now there are three cases:

(a) If $k=0$, then $j_1(x^i\partial^j\otimes\partial^l)=0$ and therefore $\varphi(x^i\partial^j\otimes\partial^l)=0$.
(b) Let $k-1\geqq p\geqq 0$. Then

$$Z_p = (\mathrm{res}\circ\pi_1\circ\sigma_1)(x^{k-p-1}\partial^l x^i\partial^j x^p dx) = 0$$

if $p<j-1$. If $p\leqq j$, then

$$i+p-j-l+1 = p-k+1 \leqq 0 \quad\text{and}\quad i+p-j \geqq i-1.$$

Therefore, if $i>0$, $Z_p=0$. If $i=0$, then $k=j+l$ and $i+p-j\geqq -1$. Then $Z_p=0$ if $p-j\geqq 0$. It remains to consider the case $p=j-1$ for which $Z_p=0$ again.

The conclusion is that for $k>0$, $\varphi(x^i\partial^j\otimes x^k\partial^l)=0$ unless $i<0$ in which case it has the desired form.

(c) Let $k\leqq p\leqq -1$. Then necessarily $i=j+l-k>0$ and

$$\varphi(x^i\partial^j\otimes x^k\partial^l) = -(-1)^j j!\,l!\left(\sum_{p=k}^{-1}\binom{j-p-1}{j}\binom{l-k+p}{l}\right)\delta_{i+k,\,j+l}.$$

Lemma 4. *The Hochschild cocycle φ is antisymmetric and hence defines a 2-cocycle for the Lie algebra underlying $\mathcal{D}$.*

Proof. It is enough to consider the case $i<0<k$. Then by the previous lemma

$$\varphi(x^k\partial^l\otimes x^i\partial^j) = -(-1)^l j!\,l!\left(\sum_{q=i}^{-1}\binom{l-q-1}{l}\binom{j-i+q}{j}\right)\delta_{i+k,\,j+l}.$$

Setting $p=j-i+q=k-l+q$, we get

$$\begin{aligned} \varphi(x^k\partial^l\otimes x^i\partial^j) &= -(-1)^l j!\,l!\left(\sum_{p=j}^{j-i-1}\binom{k-p-1}{l}\binom{p}{j}\right)\delta_{i+k,\,j+l} \\ &= -\varphi(x^i\partial^j\otimes x^k\partial^l), \end{aligned}$$

which proves the antisymmetry of φ.

We now complete the proof of Theorem 1 by showing that φ restricts to the Virasoro cocycle α.

Lemma 5. $\varphi(x^i\partial\otimes x^k\partial)=\alpha(x^i\partial\otimes x^k\partial)$.

Proof. Let $i<0<k$. Then

$$\begin{aligned}\varphi(x^i\partial\otimes x^k\partial)&=-\left(\sum_{p=0}^{k-1}p(k-p-1)\right)\delta_{i+k,2}\\&=\left((1-k)\left(\sum_{p=0}^{k-1}p\right)+\left(\sum_{p=0}^{k-1}p^2\right)\right)\delta_{i+k,2}\\&=\left(-\frac{k(k-1)^2}{2}+\frac{k(k-1)(2k-1)}{6}\right)\delta_{i+k,2}\\&=-\frac{k(k-1)(k-2)}{6}\delta_{i+k,2}=\alpha(x^i\partial\otimes x^k\partial).\end{aligned}$$

The other cases follow by antisymmetry.

4. The Algebra of q-Difference Operators

Let q be a complex number $\neq 0,1$. The q-analogue of the algebra $\mathscr{D}$ is the algebra $\mathscr{D}_q$ of q-difference operators on $\mathbf{C}[x,x^{-1}]$. By definition $\mathscr{D}_q$ is the algebra of all linear endomorphisms of $\mathbf{C}[x,x^{-1}]$ generated by multiplications by Laurent polynomials and by Jackson's q-differentiation operator ∂_q defined for any polynomial P by

$$\partial_q(P)=\frac{P(qx)-P(x)}{qx-x}.$$

As a complex associative algebra $\mathscr{D}_q$ is generated by x, x^{-1}, and ∂_q and the relation

$$\partial_q x-qx\partial_q=1$$

which is the q-analogue of the Heisenberg relation for differential operators. The family $\{x^i\partial_q^j\}_{i\in\mathbf{Z},\,j\in\mathbf{N}}$ is a basis of $\mathscr{D}_q$. It is convenient to introduce the algebra automorphism τ_q of $\mathbf{C}[x,x^{-1}]$ defined by

$$\tau_q(x)=qx.$$

Since $\tau_q=1+(q-1)x\partial_q$, the automorphism τ_q belongs to $\mathscr{D}_q$. We have the additional relations

$$\partial_q x-x\partial_q=\tau_q \quad\text{and}\quad \tau_q x=qx\tau_q.$$

The q-differentiation operator is not a derivation, but a τ_q-derivation; namely for all P,Q in $\mathbf{C}[x,x^{-1}]$ we have

$$\partial_q(PQ)=\tau_q(P)\partial_q(Q)+\partial_q(P)Q.$$

It is easy to check that $\{x^i\partial_q\}_{i\in\mathbf{Z}}$ is a basis of the vector space $\mathrm{Der}_q(\mathbf{C}[x,x^{-1}])$ of all τ_q-derivations of $\mathbf{C}[x,x^{-1}]$.

For integers $n\in\mathbf{Z}$ and $r>0$, set

$$(n)_q=1+q+\ldots+q^{n-1},$$

$$(r!)_q=(1)_q(2)_q\ldots(r)_q,$$

and

$$\binom{n}{r}_q = \frac{(n)_q(n-1)_q \ldots (n-r+1)_q}{(r!)_q}.$$

It is well-known that $\binom{n}{r}_q$ is a polynomial in the variable q. Therefore, $\binom{n}{r}_q$ is well-defined for all complex numbers q. We have the following identities

$$(-n)_q = -q^{-n}(n)_q, \quad \binom{n}{r}_q = (-1)^r q^{r(2n-r+1)/2} \binom{r-1-n}{r}_q,$$

and

$$\binom{n}{r}_q = \binom{n-1}{r}_q + q^{n-r}\binom{n-1}{r-1}_q$$

with the convention $\binom{n}{0}_q = 1$. Notice also that if q is a root of unity of order d, then $(n)_q = 0$ for all multiples n of d. Using the above identities, we have the following q-analogue of Lemma 2.

Lemma 6. *Let $P \in \mathbb{C}[x, x^{-1}]$. Then in $\mathscr{D}_q$,*

$$\partial_q^n P = \sum_{r=0}^{n} \binom{n}{r}_q (\tau_q^{n-r}\partial_q^r)(P)\partial_q^{n-r}.$$

5. Homology of $\mathscr{D}_q$

Under the hypotheses of Sect. 4 we define a complex $K_*(q), \beta'_q$ which is a deformation of the Koszul complex K_*, β' of Sect. 1. Let V_q be a two-dimensional vector space with basis $\{dx, d\partial_q\}$. As a graded space

$$K_*(q) = \mathscr{D}_q \otimes \mathscr{D}_q^o \otimes \Lambda^* V_q.$$

The differential β'_q is the $\mathscr{D}_q \otimes \mathscr{D}_q^o$-linear map given by

$$\begin{aligned} \beta'_q(1 \otimes 1 dx \wedge d\partial_q) &= (1 \otimes x - qx \otimes 1)d\partial_q - (q \otimes \partial_q - \partial_q \otimes 1)dx, \\ \beta'_q(1 \otimes 1 dx) &= (1 \otimes x - x \otimes 1), \\ \beta'_q(1 \otimes 1 d\partial_q) &= (1 \otimes \partial_q - \partial_q \otimes 1). \end{aligned}$$

We have $\beta_q'^2 = 0$ because of the q-Heisenberg relation $\partial_q x - qx\partial_q = 1$.

Proposition 3. *The complex $K_*(q), \beta'_q$ is a free $\mathscr{D}_q \otimes \mathscr{D}_q^o$-resolution of $\mathscr{D}_q$.*

Proof. Filter $\mathscr{D}_q$ by the powers of ∂_q. The associated graded algebra $S_q = \mathrm{gr}(\mathscr{D}_q)$ is the algebra generated by x, x^{-1}, ∂_q and the relation

$$\partial_q x = qx\partial_q.$$

The filtration on $\mathscr{D}_q$ induces a filtration on the chain complex $K(q), \beta'_q$. In the resulting spectral sequence we have

$$E^0 = S_q \otimes S_q^o \otimes \Lambda^* V_q,$$

the differential d^0 being given by the same formulas as β'_q. Now the acyclicity of E^0, d^0 is proved in [6]. The lemma follows by a standard spectral sequence argument.

Corollary 1. *The Hochschild homology groups of $\mathscr{D}_q$ are the homology groups of the complex*

$$0 \longrightarrow \mathscr{D}_q \otimes \Lambda^2 V_q \xrightarrow{\beta_q} \mathscr{D}_q \otimes V_q \xrightarrow{\beta_q} \mathscr{D}_q \longrightarrow 0$$

defined for any $D \in \mathscr{D}_q$ by

$$\beta_q(Ddx \wedge d\partial_q) = (xD - qDx)d\partial_q - (q\partial_q D - D\partial_q)dx,$$

$$\beta_q(Ddx) = [x, D],$$

$$\beta_q(Dd\partial_q) = [\partial_q, D].$$

Moreover, there is a homology isomorphism $j_*(q)$ from the standard Hochschild complex of $\mathscr{D}_q$ to the complex $\mathscr{D}_q \otimes \Lambda^* V_q$, β_q such that $j_0(q)$ is the identity on $\mathscr{D}_q$ and for any $D \in \mathscr{D}_q$, $j_1(q)(D \otimes x^i \partial_q^j)$ is equal to

$$-(1-\delta_{i,0})\left(\sum_{p=0}^{i-1} x^{i-p-1}\partial_q^j D x^p\right)dx - (1-\delta_{j,0})\left(\sum_{p=0}^{j-1} \partial_q^{j-p-1} D x^i \partial_q^p\right)d\partial_q$$

if $i, j \geqq 0$ and to

$$\left(\sum_{p=-1}^{i} x^{i-p-1}\partial_q^j D x^p\right)dx - (1-\delta_{j,0})\left(\sum_{p=0}^{j-1} \partial_q^{j-p-1} D x^i \partial_q^p\right)d\partial_q$$

if $i < 0 \leqq j$.

Proof. The first assertion is a straightforward consequence of the lemma. By the comparison theorem of resolutions there is a homology isomorphism $j'_*(q)$ from the standard Hochschild resolution to the resolution $K_*(q)$, β'_q such that $j'_0(q)$ is the identity. Since β'_q has the same form as β' on $\mathscr{D}_q \otimes V_q$, we may take $j'_1(q) = j'_1$. By tensoring with $\mathscr{D}_q$ over $\mathscr{D}_q \otimes \mathscr{D}_q^o$, we get $j_1(q)$ which is the same as j_1 in Sect. 2.

We proceed now as in Sect. 2 and compare the complex $\mathscr{D}_q \otimes \Lambda^* V_q$, β_q with a q-analogue of the de Rham complex of $S = \mathbf{C}[x, x^{-1}, \xi]$. Let us define a degree $+1$ differential on Ω_S^* by

$$\delta_q = \sigma_* \beta_q \sigma_*^{-1},$$

where $\sigma_p : \mathscr{D}_q \otimes \Lambda^p V_q \to \Omega_S^{2-p}$ is the linear isomorphism given by

$$\sigma_0(x^i \partial_q^j) = -x^i \xi^j dx \wedge d\xi,$$

$$\sigma_1(x^i \partial_q^j dx) = -x^i \xi^j dx,$$

$$\sigma_1(x^i \partial_q^j d\partial_q) = -x^i \xi^j d\xi,$$

$$\sigma_2(x^i \partial_q^j dx \wedge d\partial_q) = x^i \xi^j.$$

A straightforward computation yields

Lemma 7. *We have*

$$\delta_q(x^i \xi^j) = (q(i)_q x^{i-1}\xi^j + (q^{i+1}-1)x^i \xi^{j+1})dx + (q(j)_q x^i \xi^{j-1} + (q^{j+1}-1)x^{i+1}\xi^j)d\xi,$$

$$\delta_q(x^i \xi^j dx) = -((j)_q x^i \xi^{j-1} + (q-1)x^{i+1}\xi^j)dx \wedge d\xi,$$

$$\delta_q(x^i \xi^j d\xi) = ((i)_q x^{i-1}\xi^j + (q-1)x^i \xi^{j+1})dx \wedge d\xi.$$

We shall not compute the Hochschild groups of $\mathscr{D}_q$, i.e. the cohomology groups of the twisted de Rham complex Ω_S^*, δ_q. Instead we map the latter onto the q-de Rham complex Ω_A^*, d_q, where

$$d_q(P) = q\partial_q(P)dx$$

for $P \in A = \mathbf{C}[x, x^{-1}]$. The cohomology of Ω_A^*, d_q is easy to compute.

Proposition 4. (a) *If $q \neq 0$ is not a root of unit,*

$$H^i(\Omega_A^*, d_q) \cong H^i(\Omega_A^*, d) \cong \begin{cases} \mathbf{C} \cdot 1 & \text{if } i=0 \\ \mathbf{C} \cdot \dfrac{dx}{x} & \text{if } i=1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If q is of order $d > 1$, we have*

$$H^i(\Omega_A^*, d_q) \cong \begin{cases} \mathbf{C} \cdot 1 \oplus \bigoplus_{N \in \mathbf{Z} \setminus \{0\}} \mathbf{C}x^{Nd} & \text{if } i=0 \\ \mathbf{C} \cdot \dfrac{dx}{x} \oplus \bigoplus_{N \in \mathbf{Z} \setminus \{0\}} \mathbf{C}x^{Nd} \dfrac{dx}{x} & \text{if } i=1 \\ 0 & \text{otherwise.} \end{cases}$$

We denote by res_q the projection of Ω_A^1 onto $\mathbf{C}\dfrac{dx}{x}$ and $\mathrm{res}_q^{(Nd)}$ the projection of Ω_A^1 onto the summand $\mathbf{C}x^{Nd}\dfrac{dx}{x}$. The generalized residue maps $\mathrm{res}_q^{(Nd)}$ $(N \neq 0)$ vanish unless q is a root of unity $\neq 1$.

Consider the projection $\pi_* : \Omega_S^* \to \Omega_A^*$ defined in Sect. 2.

Lemma 8. *The projection π_* is a chain map from Ω_S^*, δ_q onto Ω_A^*, d_q and induces a surjection from $H^1(\Omega_S^*, \delta_q) = H_1(\mathscr{D}_q, \mathscr{D}_q)$ onto $H^1(\Omega_A^*, d_q)$.*

Proof. The first assertion follows from a simple computation. As for the second one, it is easy to lift the generators of $H^1(\Omega_A^*, d_q)$ into 1-cocycles for Ω_S^*, δ_q.

To sum up we have the following commutative diagram which is the q-analogue of the diagram in Sect. 2.

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{b} & \mathscr{D}_q^{\otimes 4} & \xrightarrow{b} & \mathscr{D}_q^{\otimes 3} & \xrightarrow{b} & \mathscr{D}_q \otimes \mathscr{D}_q & \xrightarrow{b} & \mathscr{D}_q & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle j_2(q)} & & \downarrow{\scriptstyle j_1(q)} & & \downarrow{\scriptstyle j_0(q)} & \\
\cdots \longrightarrow & 0 & \longrightarrow & \mathscr{D}_q \otimes \Lambda^2 V_q & \xrightarrow{\beta_q} & \mathscr{D}_q \otimes V_q & \xrightarrow{\beta_q} & \mathscr{D}_q & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} & \\
\cdots \longrightarrow & 0 & \longrightarrow & \Omega_S^0 & \xrightarrow{\delta_q} & \Omega_S^1 & \xrightarrow{\delta_q} & \Omega_S^2 & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & \Omega_A^0 & \xrightarrow{d_q} & \Omega_A^1 & \longrightarrow & 0 & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{res}_q} & & \downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & H^0(\Omega_A^*, d_q) & \xrightarrow{0} & H^1(\Omega_A^*, d_q) & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

6. A q-Analogue of the Virasoro Cocycle

As in Sect. 3 we define a Hochschild 1-cocycle φ_q on $\mathscr{D}_q$ by

$$\varphi_q = \mathrm{res}_q \circ \pi_1 \circ \sigma_1 \circ j_1(q).$$

Since $\pi_1 \circ \sigma_1 \circ j_1(q)$ induces surjections on homology, the cohomology class of φ_q in $H^1(\mathscr{D}_1, \mathscr{D}_q^*)$ is not zero. If moreover q is a root of unity of order $d > 1$, the 1-cocycles

$$\varphi_q^{(Nd)} = \mathrm{res}_q^{(Nd)} \circ \pi_1 \circ \sigma_1 \circ j_1(q)$$

represent an infinite family of linearly independent cohomology classes.

We give now explicit formulas for these quantum cocycles.

Theorem 2. (a) *We have*

$$\varphi_q(x^i\partial_q^j, x^k\partial_q^l)$$
$$= \begin{cases} (-1)^l (j!)_q (l!)_q \left(\sum_{p=j}^{j-i-1} q^{l(l-2k+2p+1)/2} \binom{p}{j}_q \binom{k-p-1}{l}_q \right) \delta_{i+k,\, j+l} & \\ \quad = (j!)_q (l!)_q \left(\sum_{p=j}^{j-i-1} \binom{p}{j}_q \binom{i-j+p}{l}_q \right) \delta_{i+k,\, j+l} & \text{if } i<0<k \\ \quad -\varphi_q(x^k\partial_q^l, x^i\partial_q^j) & \text{if } k<0<i \\ 0 & \text{otherwise}. \end{cases}$$

(b) *If q is a root of unity of order $d>1$ and N is an integer $\neq 0$, then $\varphi_q^{(Nd)}(x^i\partial_q^j, x^k\partial_q^l)$ is equal to*

$$\left(\sum_{p=j}^{\min(k-1,d-1)} [(p)_q (p-1)_q \dots (p-j+1)_q] \right.$$
$$\left. \times [(i-j+p)_q (i-j+p-1)_q \dots (i-j+p-l+1)_q] \right) \delta_{i+k,\, j+l+Nd}$$

if $k \geqq 1$, to 0 *if $k=0$ and to*

$$\left(\sum_{\max(k,\, j-d)}^{-1} [(p)_q (p-1)_q \dots (p-j+1)_q] \right.$$
$$\left. \times [(i-j+p)_q (i-j+p-1)_q \dots (i-j+p-l+1)_q] \right) \delta_{i+k,\, j+l+Nd}$$

if $k<0$.

Proof. As in Sect. 3 we have to compute

$$(\mathrm{res}_q^{(Nd)} \circ \pi_1 \circ \sigma_1)(x^{k-p-1}\partial_q^l x^i \partial_q^j x^p dx).$$

Firstly, we have

$$(\mathrm{res}_q^{(Nd)} \circ \pi_1 \circ \sigma_1)(x^i\partial_q^j dx) = -\delta_{i,\, Nd-1}\delta_{j,0}$$

Now by Lemma 6 of Sect. 4,

$$x^{k-p-1}\partial_q^l x^i \partial_q^j x^p = \sum_{r=0}^{j}\sum_{s=0}^{l}\binom{j}{r}_q\binom{l}{s}_q (p)_q(p-1)_q\dots(p-r+1)_q$$
$$\times(i+p-r)_q(i+p-r-1)_q\dots(i+p-r-s+1)_q$$
$$q^{(j-r)(p-r)+(l-s)(i+p-r-s)}x^{i+k-r-s-1}\partial_q^{l+j-r-s}.$$

We have to look for all monomials whose degree in ∂_q is zero. These are the terms with $r=j$ and $s=l$. Hence

$$(\mathrm{res}_q^{(Nd)}\circ\pi_1\circ\sigma_1)(x^{k-p-1}\partial_q^l x^i\partial_q^j x^p dx)$$
$$=-((p)_q(p-1)_q\dots(p-j+1)_q(i-j+p)_q$$
$$\times(i-j+p-1)_q\dots(i-j+p-l+1)_q)\delta_{i+k,j+l+Nd}.$$

Composing with $j_1(q)$ yields Part (b) of the theorem. Using the same arguments as in Sect. 3, we deduce Part (a).

We conclude this paper by evaluating the cocycles $\varphi_q^{(Nd)}$ on the q-analogues

$$L_n(q)=x^{n+1}\partial_q$$

of the generators of the Virasoro algebra. These elements form a basis of the vector space $\mathrm{Der}_q(\mathbf{C}[x,x^{-1}])$ of the τ_q-derivations of $\mathbf{C}[x,x^{-1}]$. In the associative algebra $\mathscr{D}_q$ we have

$$L_m(q)L_n(q)=q^{n+1}x^{n+m+2}\partial_q^2+(n+1)_q x^{n+m+1}\partial_q,$$

which shows that $\mathrm{Der}_q(\mathbf{C}[x,x^{-1}])$ is not closed under ordinary commutators. Nevertheless, we have the following q-commutator relations

$$L_m(q)L_n(q)-q^{n-m}L_n(q)L_m(q)=(n-m)_qL_{n+m}(q).$$

The reader may check the following formulas.

Proposition 5. *For all pairs* (m,n) *of integers, we have*

$$\varphi_q^{(Nd)}(L_m(q),L_n(q))$$
$$=\begin{cases}\dfrac{q^{-n}(n+1)_{q^2}-(q^{-n}+1)(n+1)_q+(n+1)}{(q-1)^2}\delta_{m+n,Nd} & \text{if } n\geqq 0\\ -\dfrac{q^{n+2}(-n-1)_{q^2}-q(q^n+1)(-n-1)_q+(-n-1)}{(q-1)^2}\delta_{m+n,Nd} & \text{if } n\leqq -2\\ 0 & \text{otherwise}.\end{cases}$$

One verifies that the above fractions tend to the "classical" $-(n^3-n)/6$ when q tends to 1. The special case

$$\varphi_q^{(2Nd)}(L_{Nd}(q),L_{Nd}(q))=\frac{Nd}{(q-1)^2}\neq 0$$

shows that the exotic cocycles $\varphi_q^{(2Nd)}$ are not antisymmetric when $N\neq 0$.

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