Ergodic Properties of Plane Billiards with Symmetric Potentials

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Abstract. We study chaotic behaviour of the motion of a particle moving like in a billiard table outside some disks where a symmetric potential acts. Quadratic forms introduced in (Markarian, 1988) to study non-vanishing Lyapunov exponents are used.

1. Introduction

A. The purpose of this paper is to prove some ergodic properties (see C. below) of dynamical systems defined by the motion of a point mass in a bounded connected region Q of the plane. The particle moves like in a billiard table outside some disk contained in Q, and under the action of symmetric potentials inside them.

B. This problem was discussed (on the 2-torus) in Sinai (1963). In Kubo (1976) and Kubo and Murata (1981) it is proved that some conditions on the rotation function (see Sect. 2 of this paper) are sufficient for the ergodicity (and the Bernoulli property) of the dynamical system on the torus; Kubo mentions some repelling potentials that verify these conditions.

During the International Conference and Workshop on Dynamical Systems, I.M.P.A., Rio de Janeiro, August 1989, Victor Donnay announced some results (obtained by himself and Carlangelo Liverani) on the same subject, using invariant cone techniques; and Yacob Sinai suggested me to study related problems using quadratic forms, a method I used to deduce conditions on the boundaries for chaotic billiards.

C. Let μ be a probability measure on a compact 2-manifold M, K a subset such that $\mu(K) = 0$, $H = M \backslash K$ and $T : H \to H$ the restriction to H of a $\mathbb{C}^r$-diffeomorphism, $r \geq 1$, defined on an open subset of M that preserved the measure μ. We will assume that

$$\log^+ \|(T^{\pm 1})'_x\| \in L^1(H, \mu) \quad (\log^+ s = \max\{\log s, 0\}),$$

a condition that permits to apply the ergodic multiplicative theorem of Oseledets.

$\Sigma(T)$ will denote the Pesin region, that is the set of regular points that have only non-zero Lyapunov exponents.

If $\mu(\Sigma(T)) = 1$ we will say that the map T (or the dynamical system it defines is non-uniform hyperbolic or has chaotic behaviour. In Markarian (1990) a characterization of non-uniform hyperbolicity of smooth maps with singularities in terms of Lyapunov quadratic forms is given.

$B : TM \to \mathbb{R}$ is a quadratic form on M if $B_x : T_x M \to \mathbb{R}$ is a quadratic form in the usual sense. If $f : M \to M$ is a diffeomorphism we denote by $f^\# B$ (pullback of B by f) the quadratic form $(f^\# B)_x u = B_{f(x)}(f'_x u)$. B is non-degenerate on a subset $N \subset M$ if B_x is non-degenerate for every $x \in N$, that is, the matrix associate to B_x with respect to any base and the inner product has non-zero eigenvalues. B is positive in N if $B_x u > 0$ for every $x \in N$ and every non-zero $u \in T_x M$.

Theorem. *Let $B : TM \to R$ be a quadratic form such that B_x depends measurably on x, is non-degenerate in H and $P_x = (T^\# B - B)_x$ is positive for every $x \in H$. Then $\mu(\Sigma(T)) = 1$.*

A slightly different version of the theorem (B_x depends continuously on x on a set of measure one) was proved in Markarian (1988).

D. It is well known that the existence of non-zero Lyapunov exponents for T allows to construct locally invariant submanifolds. If all Lyapunov exponents are different from zero in a set Λ, a non-uniform hyperbolic decomposition of $T_x M$ is obtained at each point $x \in \Lambda$ and if Λ has positive measure there is a countable number of invariant sets Λ_i of positive measure ($\Sigma \mu(\Lambda_i) = \mu(\Lambda)$) such that $T|_{\Lambda_i}$ is ergodic. See, for instance Pesin and Sinai (1981) for a survey of such results. The ergodicity of T follows from the uniqueness of the ergodic components Λ_i. And this can be done studying the behaviour of T on K, the set of its singularities.

So, applying the theorem in order to have non-vanishing Lyapunov exponents on sets of measure one we will construct quadratic forms. Conditions on the elements that determine the dynamical system will appear as a consequence of the hypothesis on the quadratic forms.

E. To study the dynamical system that is the object of this paper we will work on the space M_1 of the pairs $(q, v) = x$, where q is a point of the set $\mathscr{B}$ union of the regular components of the boundary of Q and the circumferences determined by the disks where the potentials are defined, and v is the velocity vector of the trajectory leaving q. Tx will be the pair (if it exists) defined by the position and the velocity of the particle at the next "leaving position" on $\mathscr{B}$.

Then we study the action of T' on the tangent space to M_1. This is done on a special parametrization of TM_1 (inspired on the study of geodesic flows) that allows to write the quadratic forms in a very simple way.

Conditions on the rotation function and on the geometry of the problem will appear after establishing the positivity of

$$P = T^\# B - B.$$

This is studied in general in Sect. 3 and in a particular example ($U(r) = U$ (const)) in Sect. 4, where some conjectures proposed in Baldwin (1988) are proved.

We do not study in general the potentials admissible with those rotation functions. However we indicate the relationship between rotation functions (and their derivatives) and $U(r)$ in some particular important cases.

In order to have ergodic properties of the flow determined by the motion, some conditions that must be imposed on the potential are evident: there cannot be sets of positive measure of trajectories that finish or are contained in the disks.

2. Central Potentials in a Circle

Let Q be a bounded, connected region of the plane whose boundary consists of a finite set of curves ∂Q_i, $\mathbb{C}^{r+1}$, $r \geq 2$ with curvature ($|k_i|$) bounded. The regular components of the boundary $\partial \tilde{Q}_i = \partial Q_i \setminus \bigcup_{i \neq j} \partial Q_j$ have positive (focusing components), negative (dispersing) or zero curvature (neutral).

Let $\mathscr{D}$ be a disk completely contained in Q, with boundary $\mathscr{C}$ such that $d(\mathscr{C}, \partial Q_i)$ > 0 for every i, d is the euclidean distance in $\mathbb{R}^2$. In $\mathscr{D}$ is defined a central potential $U(r)$.

We consider the motion of a point unit mass that is reflected elastically in $\partial \tilde{Q} = \bigcup \partial \tilde{Q}_i$ which moves freely with velocity of norm one outside the disk, and under the action of the potential inside the disk.

If $n(q)$ is the unit inward normal (relative to $Q \setminus \mathscr{D}$) in $q \in \partial \tilde{Q} \bigcup \mathscr{C} = \mathscr{B}$, then the parametrization $q(s)$ and the curvature $K(s)$, where s is the arc length on $\mathscr{B}$, are defined by the equations

$$q''(s) = K\vec{n} = Kiq'(s),$$

where i is the $\pi/2$-angle counter clockwise rotation operator, $K = -\dfrac{1}{R}$ on $\mathscr{C}$.

In order to determine the movement inside the circle we consider its equations with energy $E = \frac{1}{2}$, and angular momentum M. Its position in polar coordinates $(r(t), \varphi(t))$ is determined by

$$M = r^2 \dot{\varphi}, \qquad \tfrac{1}{2} = \tfrac{1}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2) + U(r),$$

with initial conditions $(r(0), \varphi(0)) = (R, \varphi_2) = \bar{q}_2$ (Fig. 1). See, for example, Landau and Lifchitz (1981, Sect. 14).
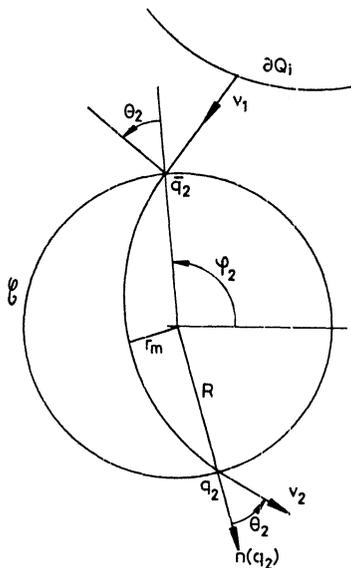

Fig. 1

If $M \neq 0$ there is a closest distance of approach to the center $r_m = r(T)$ and we have

$$\dot{r} = \sqrt{1 - 2U(r) - M^2/r^2}, \qquad r \in [r_m, R]$$

and, finally

$$\frac{d\varphi}{dr} = \frac{M}{r\sqrt{(1 - 2U(r))r^2 - M^2}}.$$

If θ_2 is the angle of "reflection" – not properly the angle of incidence (see Fig. 1) – the angular momentum evaluated at $t = 0$ and

$$t = T \quad \text{is} \quad M = |(\bar{q}_2 - 0)) \wedge \vec{v}_1| = R \sin \theta_2 = \sqrt{r_m^2(1 - 2U(r_m))},$$

supposing $\theta_2 \in [0, \pi/2)$.

The symmetry of the potential implies that a particle that enters at the point $\bar{q}_2$ with the indicated angle θ_2 will leave the disk at $q_2 = (R, \varphi_2 + g(\theta_2))$. The angle between $n(q_2)$ and the velocity v_2 of the particle is θ_2.

In the following section we will take coordinates (s, θ) on the phase space. In this case s is measured clockwise on the circumference; so if we take the same origin ($\varphi = 0$, for example) it results $s = -R\varphi$.

The function $g \colon \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ is called the *rotation function*. It depends on the potentials $U(r)$ in the following way:

$$g(\theta_2) = 2 \int\limits_{r_m}^{R} \frac{M\,dr}{r\sqrt{(1 - 2U(r))r^2 - M^2}} \quad \text{if} \quad \theta_2 \in (0, \pi/2).$$

Since a particle that enters in the disk with angle $\theta_2 \in (0, \pi/2)$ will rotate counter clockwise the same amount that an orbit entering with angle $-\theta_2$ will rotate clockwise, for $-\dfrac{\pi}{2} < \theta < 0$ we define $g(\theta) = -g(-\theta)$. Thus, in general, g is discontinuous at $\theta = 0$ and we will ignore the trajectories that enter the disk with angle $\theta_2 = 0$; they determine a set of measure zero on our phase space. If $\lim\limits_{\theta_2 \to 0^+} g(\theta_2) = n\pi$, for any $n \in \mathbb{Z}$ we are able to define a continuous function:

$$g(\theta_2) = 2n\pi - g(-\theta_2) \quad \text{if} \quad -\frac{\pi}{2} < \theta_2 \leq 0$$

We suppose that $G(\theta_2) = g'(\theta_2)$ exists for every $\theta_2 \neq 0$.

Since it can be useful for applications, we calculate explicitly $G(\theta_2)$ in the case $U(r)$ continuous for $r \in (0, R]$, $U(R) = 0$, $U'(r)$ continuous for $r \in (0, R)$. Some additional conditions must be imposed in order to have convergent integrals.

We study first the relation between θ_2 and r_m. Studying the velocity of the particle at q_2, $t = 2T$, we obtain

$$\cos \theta_2 = \frac{\dot{r}(2T)}{\sqrt{R^2\dot{\varphi}^2(2T) + \dot{r}^2(2T)}}.$$

From the energy equation it results $1 = \dot{r}^2(2T) + R^2\dot{\varphi}^2(2T)$. So $\dot{r}^2(2T) = 1 - M^2/R^2$ and $\cos \theta = \dfrac{\sqrt{R^2 - M^2}}{R}$.

Supposing $\theta_2 > 0$, we have M, $\dot{\varphi} > 0$ and $\theta_2 = \text{Arcos} \dfrac{\sqrt{R^2 - M^2}}{R}$ with $M = r_m(1 - 2U(r_m))^{1/2}$. So

$$\frac{d\theta_2}{dr_m} = \frac{r_m(1 - 2U(r_m) - r_m U'(r_m))}{M\sqrt{R^2 - M^2}}.$$

In order to calculate $\dfrac{dg}{dr_m}$ is better to make the substitution $r = e^s r_m$. Then

$$g(r_m) = 2 \int\limits_0^{\log R/r_m} \frac{(1 - 2U(r_m))^{1/2} ds}{[e^{2s}[1 - 2U(e^s r_m)]e^{2s} - 1 + 2U(r_m)]^{1/2}},$$

and

$$\frac{dg}{dr_m} = \frac{-2(1 - 2U(r_m))^{1/2}}{\sqrt{R^2 - M^2}} + \frac{2r_m}{M} \int\limits_0^{\log R/r_m} I(s) ds$$

with

$$I(s) = \frac{e^{3s}(1 - 2U(r_m))U'(e^s r_m) - e^{2s}(1 - 2U(e^s r_m))U'(r_m)}{[e^{2s}[1 - 2U(e^s r_m)] - 1 + 2U(r_m)]^{3/2}}.$$

And finally

$$G(\theta_2) = \frac{-2(1 - 2U(r_m))}{1 - 2U(r_m) - r_m U'(r_m)} + \frac{2\sqrt{R^2 - M^2}}{1 - 2U(r_m) - r_m U'(r_m)} \int\limits_0^{\log R/r_m} I(s) ds.$$

3. Dynamics and Chaotic Behaviour

As usually, let be

$$M_1 = \{(q, v) : q \in \mathscr{B}, \|v\| = 1, \langle v, n(q) \rangle > 0\}.$$

Given $x_1 = (q_1, v_1) \in M_1$, Tx_1 (if defined) is defined moving forward in the direction v_1, a distance (time) t_1 till the intersection either with $\partial \tilde{Q}$ in q_2 or with $\mathscr{C}$ in $\bar{q}_2$. In the first case let v_2 be the reflected vector: $v_2 = v_1 - 2\langle n(q_2), v_1 \rangle n(q_2)$; in the second one, let q_2 be as in the previous section, and $v_2 = e^{i\theta_2} n(q_2)$, where $e^{i\theta_2}$ is the rotation operator of angle $\theta_2(e^{i\pi/2} = i)$. In any case $Tx_1 = x_2 = (q_2, v_2)$. We do not define it on S, determined by the $x_i \in M$, such that $q_2 \in \partial Q \backslash \partial \tilde{Q}$, or v_1 is tangent to $\mathscr{C}$ or $\theta_2 = 0$ if $\bar{q}_2 \in \mathscr{C}$. Let $K = T^k S$, $k \in \mathbb{Z}$. If as usual $d\mu = d\theta\, ds\, \cos\theta$, normalized and $H = M_1 \backslash K$ it results that $T : H \to H$ is measurable, $\mathbb{C}^r$ diffeomorphism and $\mu(K) = 0$.

Small modifications must be done in the proof of Lemma 1, Chap. 6 of Cornfeld et al. (1982) in order to prove that T is μ-measure preserving. The proof that T verifies the general conditions on the map f of Sect. 1 and that it is a discontinuous dynamical system (or a smooth map with singularities) are given in details in Part V (Plane billiards as smooth dynamical systems, by J.M. Strelcyn) of Katok and Strelcyn (1986).

A wave front is given by the curve $(q(s), v(s))$ in M_1 and so an element in the tangent space at $x = (q(0), v(0)) \in M_1$ is $(q'(0), v'(0)) = (q', v')$. We restrict now the

analysis to the case $q_1 \in \partial \tilde{Q}$, $q_2 \in \mathscr{C}$. The other two cases ($q_1, q_2 \in \partial \tilde{Q}$, $q_1 \in \mathscr{C}$, $q_2 \in \partial \tilde{Q}$) are studied as the billiard map.

Let be γ_1, γ_2 the parametrization of $\partial \tilde{Q}$ and $\mathscr{C}$, respectively, with parameters the arc lengths; s_1, s_2 the arc lengths of γ_1, γ_2 in neighbourhoods of q_1, $\bar{q}_2$; $\theta_j(s_j)$ the angles of $n(q_j(s_j))$ with $v_j(s_j)$; and $K_j(s_j)$ the curvatures in $q_j(s_j)$, $j = 1, 2$. In this analysis $K_2 = -\dfrac{1}{R}$.

If q' and v' are projected on iv, the positive $\pi/2$-rotated of v, we obtain a natural parametrization of the tangent space. Let be $V_j = \langle v'_j, iv_j \rangle$, $\alpha_j = \langle q'_j, iv_j \rangle$, $j = 1, 2$. We have that

$$q_1(s_1) = \gamma_1(s_1), \qquad s_2 = s_2(s_1),$$

$$v_1(s_1) = i\gamma'_1(s_1)e^{i\theta_1(s_1)} = -i\gamma'_2(s_2)e^{-\theta_2(s_2)}, \tag{1}$$

$$\bar{q}_2(s_2) = \gamma_1(s_1) + t_1(s_1)v_1(s_1), \tag{2}$$

$$q_2(s_2) = \gamma_2(s_2 - Rg(\theta_2(s_2))) = 0 + e^{ig(\theta_2(s_2))}(\gamma_2(s_2) - 0), \tag{3}$$

$$v_2(s_2) = i\gamma'_2(s_2 - Rg(\theta_2(s_2))e^{i\theta_2(s_2)}. \tag{4}$$

If we denote $\dfrac{ds_2}{ds_1} = \sigma'$, $\dfrac{d\theta_2}{ds_2} = \theta'_2$, $\dfrac{d\theta_1}{ds_1} = \theta'_1$ and take derivatives with respect to s_1 in the previous formulas we obtain from (1),

$$i\gamma''_1(s_1)e^{i\theta_1(s_1)} - \gamma'_1(s_1)\theta'_1(s_1)e^{i\theta_1(s_1)} = \sigma'[-i\gamma''_2(s_2) - \gamma'_2(s_2)\theta'_2(s_2)]e^{-i\theta_2(s_2)},$$

and so

$$K_1 + \theta'_1 = \sigma'[K_2 - \theta'_2]; \tag{5}$$

and from (2), taking products by $-iv_1 = -\gamma'_2 e^{-i\theta_2}$,

$$\langle \gamma'_2 \sigma', -\gamma'_2 e^{-i\theta_2} \rangle = \langle \gamma'_1 + t'_1 v_1 + t_1 v'_1, \gamma'_1 e^{i\theta_1} \rangle,$$
$$-\sigma' \cos\theta_2 = \cos\theta_1 - t_1(K_1 + \theta'_1),$$

and so

$$\cos\theta_1 = \sigma'[t_1(K_2 - \theta'_2) - \cos\theta_2]. \tag{6}$$

Finally,

$$\alpha_2 = \langle q'_2, iv_2 \rangle = \langle \gamma'_2(s_2 - Rg(\theta_2))(1 - Rg'(\theta_2)\theta'_2)\sigma', i\gamma'_2(s_2 - Rg(\theta_2))e^{i\theta_2} \rangle$$
$$= -\cos\theta_2(1 - Rg'(\theta_2)\theta'_1)\sigma',$$
$$V_2 = \langle v'_2, iv_2 \rangle = \langle i\gamma''_2(s_2 - Rg(\theta_2))(1 - Rg'(\theta_2)\theta'_2)\sigma' e^{i\theta_2}$$
$$\qquad - \gamma'_2(s_2 - Rg(\theta_2))\theta'_2\sigma' e^{i\theta_2}, -\gamma'_2(s_2 - Rg(\theta_2))e^{i\theta_2} \rangle$$
$$= \sigma'[K_2(1 - Rg'(\theta_2)\theta'_2) + \theta'_2],$$
$$V_1 = \langle v'_1, iv_1 \rangle = K_1 + \theta'_1 = \sigma'(K_2 - \theta'_2),$$
$$\alpha_1 = \langle q'_1, iv_1 \rangle = -\cos\theta_1 = \sigma'[\cos\theta_2 - t_1(K_2 - \theta'_2)].$$

The general expression of a quadratic form B_x, $x = (q, v)$ is $B_x(q', v') = a\alpha^2 + 2b\alpha V + cV^2$, where a, b, c are functions of s, θ (using the natural parametrization of M_1). In order to verify condition (ii) of Theorem 1, a, b and c must be measurable

functions of x and $ac - b^2 \neq 0$. We take $b = 1$ in order to have control of non-degeneracy of B and to simplify the calculus. We will use θ instead of θ_2, K instead of K_2 and G instead of $g'(\theta_2)$. Then

$$
\begin{aligned}
P_{x_1}&(q', v') \\
&= (T^\# B - B)_{x_1}(q', v') = a_2\alpha_2^2 - 2\alpha_2 V_2 + c_2 V_2^2 - a_i\alpha_1^2 - 2\alpha_1 V_1 - c_1 V_1^2 \\
&= (a_2 - a_1)\alpha_2^2 + (c_2 - c_1)V_2^2 + a_1(\alpha_2^2 - \alpha_1^2) + 2(\alpha_2 V_2 - \alpha_1 V_1) + c_1(V_2^2 - V_1^2) \\
&= \sigma'^2\{\theta'^2[(a_2 - a_1)R^2 G^2 \cos^2\theta + (c_2 - c_1)(1 - KRG)^2 + a_1(R^2 G^2 \cos^2\theta - t_1^2) \\
&\quad - 2KR^2 G^2 \cos\theta + 2RG\cos\theta + 2t_1 + c_1(K^2 R^2 G^2 - 2KRG)] \\
&\quad + 2\theta'[-(a_2 - a_1)RG\cos^2\theta + (c_2 - c_1)(K - K^2 RG) \\
&\quad + a_1(Kt_1^2 - RG\cos^2\theta - t_1\cos\theta) + 2KRG\cos\theta - 2Kt_1 + c_1(2K - K^2 RG)] \\
&\quad + (a_2 - a_1)\cos^2\theta + (c_2 - c_1)K^2 + a_1(2Kt_1\cos\theta - K^2 t_1^2) \\
&\quad - 4K\cos\theta + 2K^2 t_1\}. \tag{7}
\end{aligned}
$$

If $a_i = 0$, $P > 0$ if and only if

$$
\begin{aligned}
&[(c_2 - c_1)(1 - KRG)^2 - 2KR^2 G^2 \cos\theta + 2RG\cos\theta \\
&\quad + 2t_1 + c_1(K^2 R^2 G^2 - 2KRG)]\theta'^2 + 2\theta'[(c_2 - c_1)(K - K^2 RG) \\
&\quad + 2KRG\cos\theta - 2Kt_1 + c_1(2K - K^2 RG)] \\
&\quad + (c_2 - c_1)K^2 - 4K\cos\theta + 2K^2 t_1 > 0, \tag{8}
\end{aligned}
$$

and this is true if and only if

$$
(c_2 - c_1)K^2 - 4K\cos\theta + 2K^2 t_1 > 0,
$$

and,

$$
\begin{aligned}
&[c_2(K - K^2 RG) + c_1 K + 2KRG\cos\theta - 2Kt_1]^2 \\
&\quad - [c_2(1 - KRG)^2 - c_1 - 2KR^2 G^2 \cos\theta + 2RG\cos\theta + 2t_1] \\
&\quad \times [(c_2 - c_1)K^2 - 4K\cos\theta + 2K^2 t_1] < 0.
\end{aligned}
$$

If $c_i = 0$, these inequalities become ($K = -1/R$),

$$
\frac{2\cos\theta}{R} + \frac{t_1}{R^2} > 0.
$$

$$
\left(-2G\cos\theta + 2\frac{t_1}{R}\right)^2 - (2RG^2\cos\theta + 2RG\cos\theta + 2t_1)\left(\frac{4\cos\theta}{R} + \frac{2t_1}{R^2}\right) < 0.
$$

The first one is always true and the second one is

$$
\cos\theta\left\{-G(G+2)\cos\theta - \frac{t_1}{R}(G+1)(G+2)\right\} < 0.
$$

As $\cos\theta > 0$ it must be $(G+2)[RG\cos\theta + t_1(G+1)] > 0$ which is true either if $G < -2$ or $G > \dfrac{-t_1}{t_1 + R\cos\theta} > -1$.

If $G = -2$, very simple considerations allow to deduce that each trajectory goes out the disk as it does not exist; that is $v_1 = v_2$. If the case Q were a square, we have a dynamical system that has vanishing Lyapunov exponents and is not ergodic.

Indeed, if $G = -2$, the expression that must be studied to know the sign of P is $(a_i = c_i = 0)$,

$$(4R\cos\theta + 2t_1)\theta_1^2 + 2\left(4\cos\theta + \frac{2t_1}{R}\right) + \frac{4}{R}\cos\theta + \frac{2t_1}{R}$$

$$= \left(\frac{4}{R}\cos\theta + \frac{2}{R^2}t_1\right)(R\theta' + 1)^2 > 0 \quad \text{if} \quad \theta' \neq -\frac{1}{R} = K.$$

From (5), $\theta' = \theta_2' = K_2$ implies $\theta_1' = -K_1 = 0$. Now consider the wave front that has entered the disk with $\theta_2' = K_2$: at the moment it leaves the circle it has $\bar\theta_1 = -K_2$ because the circulation on the circumference of inward and outward trajectories are opposite. Then, as (5) is valid for billiards, at the next bounce against the rectilinear wall, we have $\theta_2' = 0$. This implies that $P = 0$ along these trajectories and we cannot apply Theorem 1.

We study now what happens if $-2 < G < \dfrac{-t_1}{t_1 + R\cos\theta}$. By looking at expression (8) we deduced that the positivity of P is not difficult to study when $c_i = c$ (const). In this case P is positive if and only if

$$\frac{(G+2)}{R^2}\{4R\cos\theta[-RG\cos\theta - t_1(G+1)] + c(G+2)(c - 2t_1)\} < 0. \qquad (9)$$

As we are supposing $-RG\cos\theta - t_1(G+1) > 0$, the expression between the parenthesis is smaller than

$$-4R[RG + t_1(G+1)] + c(G+2)(c - 2t_1).$$

We suppose now, for example, that $t_1 \geq c$; then a sufficient condition for verifying (9) is

$$-4R[RG - c(G+1)] - c^2(G+2) < 0;$$

and, if $G + 2 \geq d > 0$, $f(c) = dc^2 + 4Rc(d-1) + 4R^2(d-2) > 0$.

But $f(c) = d(c + 2R)\left[c - \dfrac{2R(2-d)}{d}\right]$ which is positive, for positive c, if

$$c > \frac{2R(2-d)}{d} = c_1. \qquad (10)$$

This is not a good condition since $c_1 \to +\infty$ if $d \to 0$ but the method used indicates how to work in specific problems.

As another example, observe that if $t_1 > \dfrac{c}{2}$, a sufficient condition for verifying (9) is

$$-RG - \frac{c}{2}(G+1) < 0;$$

and if $G > -1$, this is verified if

$$c > \frac{-2RG}{G+1}.$$

If $q_2 \in \partial\tilde Q_i$, the positivity of P must be studied as in the billiard map. This was done in our paper (Markarian 1990), formulas (6), (7), and (7'), where it is proved

that sufficient and necessary condition for P being positive are

$$c_2 - c_1 - a_1 t_1^2 + 2t_1 > 0,$$

$$(c_1 - 2t_1 + a_1 t_1^2)\left(4c_2 - 2E + (a_2 - a_1)\frac{E^2}{4}\right)$$

$$+ c_2 E\left(2 - 2a_1 t_1 - (a_2 - a_1)\frac{E}{4}\right)$$

$$+ \frac{a_1^2 t_1^2 E^2}{4} < 0, \quad \text{with} \quad E = \frac{2\cos\theta}{K_2}; \quad \text{if} \quad K_2 \neq 0, \tag{12}$$

$$(a_2 - a_1)(c_1 - c_2 - 2t_1) + a_1 a_2 t_1^2 < 0, \quad \text{if} \quad K = 0. \tag{12'}$$

We have used two quadratic forms:

$$a_i = 0, \quad c_i = 0$$
$$a_i = 0, \quad c_i = c\,(\text{const}).$$

In both cases formula (10) is verified, (12') is transformed in an equality and $P = 0$ for a value of θ'.

But if almost all trajectories enter in the disk – and thus $P > 0$ – there is no problem with this "bad" wave front. This is the case when Q is the unit square centered in the center of $\mathscr{C}$ with $R < 1/2$.

If $K_2 < 0$, (12) is verified and we can use any type of C^3 dispersing curve as ∂Q_i, obtaining non-uniform hyperbolic systems.

We can also combine neutral and dispersing components as parts of $\mathscr{B}\backslash\mathscr{C}$ if we take care of not generating sets of positive measure of trajectories that bounce only at the neutral ones.

4. Constant Potentials

In this section we study some of the results conjectured in Baldwin, 1988) for constant potentials $U(r) = U\,(\text{const})$, $0 \leq r < R$ in the torus. See Sect. 5 of his paper and remember that only chaotic behaviour is observed by numerical methods; not ergodicity. As he works with total energy $E = 1$, his U corresponds to our $2U$.

We calculate first the value of $G(\theta_2) = g'(\theta_2)$.

Inside the disk the trajectory is rectilinear. For $\theta_2 \in [0, \pi/2)$, we have (Fig. 2, observe how we measure $\bar{\theta}_2$) $r_m = R\sin\bar{\theta}_2$ and the angular momentum is

$$M = R\sin\theta_2 = R\sin\bar{\theta}_2\sqrt{1 - 2U}.$$

So

$$g(\theta_2) = 2\int_{r_m}^{R} \frac{M\,dr}{r\sqrt{(1 - 2U)r^2 - M^2}}$$

and, if

$$z^2 = \frac{1 - 2U}{M^2}r^2 - 1,$$

$$g(\theta_2) = 2\int_{r_M}^{R_M} \frac{dz}{z^2 + 1} = 2\,\text{Arctg}\sqrt{\frac{1 - 2U}{\sin^2\theta_2} - 1} = 2\,\text{Arcos}\frac{\sin\theta_2}{\sqrt{1 - 2U}}$$
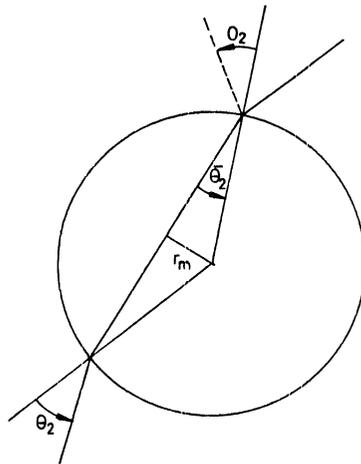
Fig. 2

(we take the determination of Arcos z such that $\pi \geq \text{Arcos } z \geq 0$). In order to make g continuous at $\theta_2 = 0$, we take, for $\theta_2 \in (-\pi/2, 0)$, $g(\theta_2) = 2\pi - g(-\theta_2)$.

Observe that if $U > 0$, for certain angles ($\sin \theta_2 > \sqrt{1 - 2U}$) the expression of $g(\theta_2)$ does not make sense. In these cases the particle does not enter in the disk: it is reflected as in a billiard, and the graph of g is as in Fig. 3
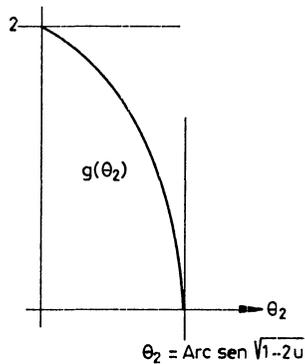

Fig. 3

If $g(\theta_2)$ is defined,

$$G(\theta_2) = \frac{-2\cos \theta_2}{\sqrt{\cos^2 \theta_2 - 2U}},$$

which graph is in Fig. 4a for $U < 0$, and in Fig. 4b for $0 < U < 1/2$.

Observe that from the formula for V_1, α_1, that are valid for billiards, as we calculate P, neutral components can be ignored if t_1 indicates time between two successive touchs with $\mathscr{C}$. Indeed, if $K_1 = 0$ and the subindex 0 is used for all the elements in

the previous (to q_1) bounce, then

$$K_0 + \theta_0' = \frac{ds_1}{ds_0}(K_1 - \theta_1') = -\frac{ds_1}{ds_0}\theta_1' = -\frac{ds_2}{ds_0}(K_2 - \theta_2');$$

$$-\cos\theta_0 = \frac{ds_1}{ds_0}[\cos\theta_1 - t_0(K_1 - \theta_1')]$$

$$= \frac{ds_2}{ds_0}[-\cos\theta_2 + (t_0 + t_1)(K_2 - \theta_2')].$$

So, the study of chaotic behaviour of a particle moving on a torus, freely outside a disk and submitted to the action of a potential inside, it can be considered as in a square Q. If the square has unit sides, and the disk has radius R, both centered in $(0,0)$, then $t_1 \geq 2(\frac{1}{2} - R) = 1 - 2R$.

If $U(r) = U$, we must distinguish two cases:
a) $0 < U < 1/2$. In this case we use the quadratic form with $a_1 = c_1 = 0$. If the particle enters in the disk, $G < -2$ (see Fig. 4b) and $P > 0$. If it does not enter, it is reflected as in a billiard, and from our previous work (Markarian 1988, Sect. 2.A), it is deduced again that $P > 0$. So in this case the dynamical system has chaotic behaviour.
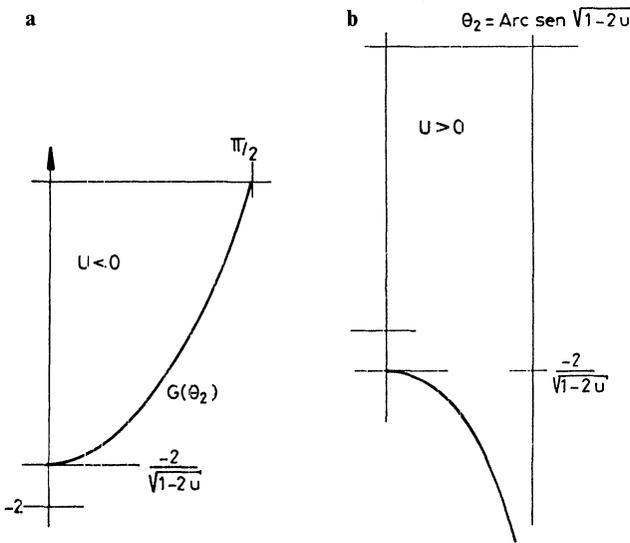

Fig. 4

b) $U < 0$. We use the quadratic form with $a_i = 0$, $c_1 = c$ (constant).

As $G \geq \dfrac{-2}{\sqrt{1 - 2U}}$, we take in (10) $d = \dfrac{-2 + 2\sqrt{1 - 2U}}{\sqrt{1 - 2U}}$, and so $c_1 = \dfrac{2R}{-1 + \sqrt{1 - 2U}}$. As $t_1 \geq 1-2R$, we can take $c = 1-2R$ if $1-2R > \dfrac{2R}{-1 + \sqrt{1 - 2U}}$.

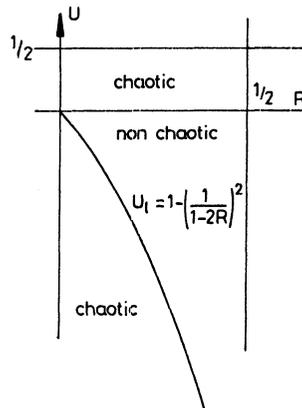This implies $2U < 1 - \left(\dfrac{1}{1 - 2R}\right)^2 = U_l$.

Fig. 5

Baldwin proved – using KAM theory – that the system neither is ergodic nor has Pesin region of measure one if $2U > U_l$. He conjectured (5.4) that the system is ergodic for $2U < u_l$; we have proved that it has chaotic behaviour (see Fig. 5).

In this case the quadratic form $B(q', v') = 2\alpha V + (1 - 2R)V^2$ is the best one since it allows to deduce chaotic behaviour in the maximal set $2U < U_l$.

As I was revising this paper I knew that A. Knauf had proved the results in Sect. 4 using invariant cones techniques: Physica D **36**, 259–262 (1989).

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