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Spectral Properties of a Class of Operators Associated with Conformal Maps in Two Dimensions

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Abstract. If f is a rational map of the Riemann sphere, define the transfer operator $\mathscr{L}$ by

$$\mathscr{L}\Phi(z)=\sum_{Z:fZ=z} g(Z)\Phi(Z).$$

Let also $\mathscr{B}$ be the Banach space of functions for which the second derivatives are measures. If $g\in\mathscr{B}$ and g satisfies a simple integrability condition (implying that g vanishes at critical points and multiple poles of f) then $\mathscr{L}$ is a bounded linear operator on $\mathscr{B}$. The essential spectral radius of $\mathscr{L}$ can be estimated and, under suitable conditions, proved to be strictly less than the spectral radius. Similar estimates for more general operators $\mathscr{L}$ are also obtained.

1. Assumptions and Generalities

Let X be a bounded open subset of $\mathbb{C}$. If the second derivatives in the sense of distributions of $\varphi:\mathbb{C}\mapsto\mathbb{C}$ are bounded measures, we write (using a "functional" notation for measures):

$$\operatorname{Var}\varphi=\int_{\mathbb{C}} dx\,dy\,|\partial^2\varphi|,$$

where $|\partial^2\varphi|$ denotes the norm of the 2×2 matrix of second derivatives (i.e. the norm bilinear of forms on $\mathbb{C}^2$, where $\mathbb{C}^2$ has the usual Hilbert norm). In particular

$$\operatorname{Var}\varphi\geqq\frac{1}{4}\int_{\mathbb{C}} dx\,dy\left[\left|\frac{\partial^2\varphi}{\partial x^2}\right|+2\left|\frac{\partial^2\varphi}{\partial x\partial y}\right|+\left|\frac{\partial^2\varphi}{\partial y^2}\right|\right].$$

We define

$$\mathscr{B}=\{\varphi:\operatorname{Var}\varphi<\infty \text{ and } \varphi \text{ vanishes on } \mathbb{C}\backslash X\}.$$

Then $\mathscr{B}$ is a Banach space with respect to the norm Var (which has properties similar to the total variation for functions on $\mathbb{R}$, particularly that it behaves well under conformal changes of coordinates, as we shall see). We may write (using again a functional notation)

$$\begin{aligned}\left|\frac{\partial}{\partial x}\varphi(x+iy)\right| &\leqq \frac{1}{2}\int_{-\infty}^{y} dt\left|\frac{\partial^2}{\partial x\partial t}\varphi(x+it)\right| + \frac{1}{2}\int_{y}^{\infty} dt|\cdots| \\ &\leqq \frac{1}{2}\int_{-\infty}^{\infty} dt\left|\frac{\partial^2}{\partial x\partial t}\varphi(x+it)\right| \\ &= \frac{1}{2}\mu(x),\end{aligned}$$

where μ is a positive measure on $\mathbb{R}$, independent of y and with total mass

$$\|\mu\| \leqq \int_{\mathbb{C}} dx dy\left|\frac{\partial^2\varphi}{\partial x\partial y}\right| \leqq \operatorname{Var}\varphi.$$

Similarly,

$$\left|\frac{\partial}{\partial y}\varphi(x+iy)\right| \leqq \frac{1}{2}\nu(y)$$

with

$$\|\nu\| \leqq \operatorname{Var}\varphi.$$

Note that μ is nonatomic, otherwise $\frac{\partial^2\varphi}{\partial x^2}$ would contain δ' singularities and could not be a measure. Similarly ν is nonatomic. As a consequence, $\varphi(x+iy)$ is continuous in x, uniformly in y, and continuous in y, uniformly in x. Therefore $\varphi: x+iy \mapsto \varphi(x+iy)$ is continuous[1] or more precisely φ is, in the sense of distributions, equal to a continuous function. We have

$$\|\varphi\|_0 = \sup_z |\varphi(z)| \leqq \tfrac{1}{4}\|\mu\| \leqq \tfrac{1}{4}\operatorname{Var}\varphi.$$

We also have

$$\begin{aligned}\int dx dy|\partial\varphi|^2 &= \int \mathrm{d}x dy\left(\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2\right) \\ &= -\int dx dy\,\varphi\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) \\ &\leqq \tfrac{1}{4}\operatorname{Var}\varphi\cdot 2\operatorname{Var}\varphi \\ &\leqq \tfrac{1}{2}(\operatorname{Var}\varphi)^2.\end{aligned}$$

For each ω in some countable index set Ω, the following are supposed to be given:

(i) an open set $V_\omega \subset X$,

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[1] This is a special case of the theorem of Fuglede–Schwartz [3]. I am indebted to F. Trèves for this reference

(ii) a map $\psi_\omega : V_\omega \mapsto X$ such that $V_\omega \mapsto \psi_\omega V_\omega$ is a conformal homeomorphism,

(iii) a continuous function φ_ω vanishing on $\mathbb{C} \backslash V_\omega$ such that φ_ω is the uniform limit of functions $\varphi_{\omega i}$ with support in V_ω satisfying $|\varphi_{\omega i}| \leqq |\varphi_\omega|$.[2]

Furthermore, we assume that

$$K = \sum_\omega \sup_i \operatorname{Var} \varphi_{\omega i} < \infty,$$

$$L = \sum_\omega \left[\frac{1}{2} \int dx dy \left| \varphi_\omega \cdot \frac{\psi''_\omega}{\psi'_\omega} \right|^2 \right]^{1/2} < \infty.$$

(ψ_ω is locally holomorphic or antiholomorphic, ψ'_ω and ψ''_ω denote its first and second derivatives with respect to z or $\bar{z}$; note that replacing φ_ω by $\varphi_\omega \circ \psi_\omega^{-1}$ and ψ_ω by ψ_ω^{-1} we do not change L; note also that $\varphi_\omega \in \mathscr{B}$.)

We introduce now $\mathscr{M}$ and $\mathscr{M}'$ by

$$\mathscr{M} \Phi(z) = \sum_\omega \varphi_\omega(z) \Phi(\psi_\omega z),$$

$$\mathscr{M}' \Phi(z) = \sum_\omega \varphi_\omega(\psi_\omega^{-1} z) \Phi(\psi_\omega^{-1} z).$$

Note that $\mathscr{M}$ and $\mathscr{M}'$ are formally adjoint to each other.

1.1. Proposition. *$\mathscr{M}$ is a bounded operator on $\mathscr{B}$.*

We can approximate φ_ω by $\varphi_{\omega i}$ with compact support in V_ω such that $\operatorname{Var} \varphi_{\omega i}$ is bounded uniformly in i, and

$$\int dx dy \left| \varphi_{\omega i} \frac{\psi''_\omega}{\psi'_\omega} \right|^2 \to \int dx dy \left| \varphi_\omega \frac{\psi''_\omega}{\psi'_\omega} \right|^2.$$

We shall use this to estimate $\|\mathscr{M}\|$ after we have established some general inequalities.

Suppose that φ is continuous with compact support in $\mathbb{C}$, ψ a conformal homeomorphism on a neighborhood of $\operatorname{supp} \varphi$, and $\operatorname{Var} \Phi < \infty$. We have

$$\int dx dy |\varphi| \cdot |\partial^2 (\Phi \circ \psi)| \leqq \int dx dy |\varphi| \cdot [|(\partial^2 \Phi) \circ \psi| \cdot |\psi'|^2 + |(\partial \Phi) \circ \psi| \cdot |\psi''|],$$

where ψ', ψ'' are the first and second derivatives of ψ with respect to z or $\bar{z}$. Therefore

$$\begin{aligned} &\int dx dy |\varphi| \cdot |\partial^2 (\Phi \circ \psi)| \\ &\leqq \int dx dy |\varphi \circ \psi^{-1}| \cdot |\partial^2 \Phi| + \left[\int dx dy |\partial \Phi|^2\right]^{1/2} \left[\int dx dy \left| \varphi \frac{\psi''}{\psi'} \right|^2 \right]^{1/2} \\ &\leqq \operatorname{Var} \Phi \left[\|\varphi\|_0 + \left(\frac{1}{2} \int dx dy \left| \varphi \frac{\psi''}{\psi'} \right|^2 \right)^{1/2} \right]. \end{aligned}$$

[2] The interesting case is that in which the compact support of φ_ω is contained in V_ω; the $\varphi_{\omega i}$ are then not needed, or one can take $\varphi_{\omega i} \equiv \varphi_\omega$

We shall apply this to the case when $\psi = \psi_\omega$ and $\varphi = \varphi_{\omega i}$. We have then

$$\begin{aligned}\mathrm{Var}\,[\varphi\cdot(\Phi\circ\psi)] &\leqq \int dxdy|\varphi|\cdot|\partial^2(\Phi\circ\psi)| \\ &\quad + 2\int dxdy|\partial\varphi|\cdot|\partial(\Phi\circ\psi)| + \int dxdy|\partial^2\varphi|\cdot|\Phi\circ\psi| \\ &\leqq \mathrm{Var}\,\Phi\left[\|\varphi\|_0 + \left(\frac{1}{2}\int dxdy\left|\varphi\frac{\psi''}{\psi'}\right|^2\right)^{1/2}\right] \\ &\quad + 2\left(\int dxdy|\partial\varphi|^2\right)^{1/2}\left(\int dxdy|\partial\Phi|^2\right)^{1/2} + \mathrm{Var}\,\varphi\cdot\|\Phi\|_0 \\ &\leqq \mathrm{Var}\,\Phi\left(\frac{3}{2}\mathrm{Var}\,\varphi + \left(\frac{1}{2}\int dxdy\left|\varphi\frac{\psi''}{\psi'}\right|^2\right)^{1/2}\right). \end{aligned} \tag{1.1}$$

Taking the limit where $\varphi \to \varphi_\omega$ and summing over ω we get

$$\|\mathscr{M}\| \leqq \tfrac{3}{2}K + L$$

which concludes the proof of Proposition 1.1.

1.2. Proposition. *$\mathscr{B}$ is a Banach algebra with respect to the norm $\frac{3}{2}\mathrm{Var}$.*

This is a corollary of the above proof since

$$\tfrac{3}{2}\mathrm{Var}\,(\varphi_1\varphi_2) \leqq \tfrac{3}{2}\mathrm{Var}\,\varphi_1\cdot\tfrac{3}{2}\mathrm{Var}\,\varphi_2$$

as a special case of (1.1).

1.3. Remark. If we assume that

$$K' = \sum_\omega \sup_i \mathrm{Var}\,\varphi_{\omega i}\circ\psi_\omega^{-1} < \infty \tag{1.2}$$

we have complete symmetry between $\mathscr{M}$ and $\mathscr{M}'$, and in particular $\mathscr{M}'$ is a bounded operator on $\mathscr{B}$. The assumption (1.2) is not needed, however, for what follows.

2. Main Results

Associated with $\mathscr{M}$ there is an operator $|\mathscr{M}|$ acting on the Banach space of bounded continuous functions $\mathbb{C}\mapsto\mathbb{C}$ with the uniform norm:

$$(|\mathscr{M}|\Phi)(z) = \sum_\omega |\varphi_\omega(z)|\,\Phi(\psi_\omega z).$$

The norm of $|\mathscr{M}|$ is

$$\||\mathscr{M}|\|_0 = \sup_z (|\mathscr{M}|1)(z)$$

and the spectral radius of $|\mathscr{M}|$ will be denoted by

$$R = \lim_{m\to\infty}(\||\mathscr{M}|^m\|_0)^{1/m} = \lim_{m\to\infty}(\||\mathscr{M}|^m 1\|_0)^{1/m}.$$

Similarly we have an operator $|\mathscr{M}'|$ such that

$$(|\mathscr{M}'|\Phi)(z) = \sum_\omega |\varphi_\omega(\psi_\omega^{-1}z)|\,\Phi(\psi_\omega^{-1}z)$$

and its spectral radius will be denoted by

$$R' = \lim_{m\to\infty} (\| |\mathcal{M}'|^m \|_0)^{1/m} = \lim_{m\to\infty} (\| |\mathcal{M}'|^m 1 \|_0)^{1/m}.$$

We have $R, R' \leqq \Sigma \| \varphi_\omega \|_0 \leqq K/4$, so that $|\mathcal{M}|, |\mathcal{M}'|$ are bounded operators even if (1.2) is not satisfied.

2.1. Theorem.
(a) *The spectral radius of* $\mathcal{M}$, *acting on* $\mathcal{B}$, *is* $\leqq \max(R, R')$.
(b) *If* $R' < R$, *the essential spectral radius of* $\mathcal{M}$ *is* $\leqq (RR')^{1/2}$.

2.2. Theorem. *If* $\varphi_\omega \geqq 0$ *for all* ω, *then the spectral radius of* $\mathcal{M}$ *is* $\geqq R$. *If furthermore* $R' < R$, *then* R *is an eigenvalue of* $\mathcal{M}$, *and there is a corresponding eigenfunction* $\Phi_R \geqq 0$.

Suppose now that the ψ_ω are holomorphic. Let $F(\omega_1, \ldots, \omega_m)$ be the set of fixed points of $\psi_{\omega_1} \circ \cdots \circ \psi_{\omega_m}$, and define

$$\zeta_m = \sum_{\omega_1 \cdots \omega_m} \sum_{x \in F(\omega_1, \ldots, \omega_m)} \varphi_{\omega_m}(x) \cdot \varphi_{\omega_{m-1}}(\psi_{\omega_m} x) \cdots \varphi_{\omega_1}(\psi_{\omega_2} \cdots \psi_{\omega_m} x).$$

2.3. Conjecture. *The power series*

$$\zeta(z)^{-1} = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \zeta_m$$

converges for $|z| < (RR')^{-1/2}$ *and its zero are precisely the inverses* λ^{-1} *of the eigenvalues* λ *of* $\mathcal{M}$ *with the same multiplicities.*

The above statements are related to the work of Denker, Urbański [2] and Przytycki [9]. This previous work was concerned with Hölder continuous functions, and used Ljubich's ideas on almost periodic operators [5], which only give information on eigenvalues λ with modulus $|\lambda|$ equal to the spectral radius. The present approach, using the space $\mathcal{B}$, gives a more detailed description of the spectrum of the transfer operator. It presents one-dimensional complex analogues of the results and conjectures in [11] for the one-dimensional real case. The zeta function for piecewise monotone maps in one dimension is analyzed by Baladi and Keller [1]. This paper contains references to the earlier work of Hofbauer, Keller and others on the transfer operator in one dimension. For more general background on dynamical zeta functions see in particular Haydn [4], Pollicott [7, 8], Ruelle [10] and references given there.

We prove Theorem 2.1 (a) in Sect. 3, Theorem 2.1 (b) in Sect. 4, and Theorem 2.2 in Sect. 5. Section 6 is dedicated to an important extension to multiple-valued functions.

3. The Spectral Radius of $\mathcal{M}$

In the estimates of this section, we write $\int$ instead of $\int dx dy$, and from now on the limit $\varphi_{\omega i} \to \varphi_\omega$ is understood but not explicitly performed. We have

$$\mathcal{M}^m \Phi(z) = \sum_{\omega_1 \cdots \omega_m} \varphi_{\omega_m}(z) \varphi_{\omega_{m-1}}(\psi_{\omega_m} z) \cdots \varphi_{\omega_1}(\psi_{\omega_2} \circ \cdots \circ \psi_{\omega_m} z) \Phi(\psi_{\omega_1} \circ \cdots \circ \psi_{\omega_m} z).$$

Therefore

$$\operatorname{Var}\mathscr{M}^m\Phi = \int|\partial^2\mathscr{M}^m\Phi| \leqq 2\sum_{k<\ell} A_{k\ell} + 2\sum_{\ell} B_\ell + \sum_k C_k + D, \tag{3.1}$$

where

$$A_{k\ell} = \sum_{\omega_1\cdots\omega_m} \int|\varphi_{\omega_m}\cdots\partial(\varphi_{\omega_\ell}\circ\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m})\cdots\partial(\varphi_{\omega_k}\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_m})$$
$$\cdots(\varphi_{\omega_1}\circ\psi_{\omega_2}\circ\cdots\circ\psi_{\omega_m})(\Phi\circ\psi_{\omega_1}\circ\cdots\circ\psi_{\omega_m})|,$$

$$B_\ell = \sum_{\omega_1\cdots\omega_m} \int|\varphi_{\omega_m}\cdots\partial(\varphi_{\omega_\ell}\circ\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m})$$
$$\cdots(\varphi_{\omega_1}\circ\psi_{\omega_2}\circ\cdots\circ\psi_{\omega_m})\partial(\Phi\circ\psi_{\omega_1}\circ\cdots\circ\psi_{\omega})|,$$

$$C_k = \sum_{\omega_1\cdots\omega_m} \int|\varphi_{\varphi_m}\cdots\partial^2(\varphi_{\omega_k}\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_m})$$
$$\cdots(\varphi_{\omega_1}\circ\psi_{\omega_2}\circ\cdots\circ\psi_{\omega_m})(\Phi\circ\psi_{\omega_1}\circ\cdots\circ\psi_{\omega_m})|,$$

$$D = \sum_{\omega_1\cdots\omega_m} \int|\varphi_{\omega_m}\cdots(\varphi_{\omega_1}\circ\psi_{\omega_2}\circ\cdots\circ\psi_{\omega_m})\cdot\partial^2(\Phi\circ\psi_{\omega_1}\circ\cdots\circ\psi_{\omega_m})|.$$

To estimate the terms in (3.1) we shall use the Schwartz inequality and changes of variables of integration. (The basic fact is that if we integrate a product of two derivatives and make a conformal change of variable, the factors in the area element cancel the factors coming from the derivatives.) With these remarks one can handle the ∂ and ∂^2 occurring in $A_{k\ell}B_\ell$, C_k, D. The reader should not worry too much about the numerical coefficients in the following estimates since they are unimportant for our later purposes.

We have

$$A_{k\ell} \leqq \|\Phi\|_0 \sum_{\omega_1\cdots\omega_m} \int|\varphi_{\omega_m}\circ\psi_{\omega_m}^{-1}\circ\cdots\circ\psi_{\omega_\ell}^{-1}|\cdots|\partial(\varphi_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1})|$$
$$\cdot|\varphi_{\omega_{\ell-1}}|\cdots|\partial(\varphi_{\omega_k}\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})|\cdots|\varphi_{\omega_1}\circ\psi_{\omega_2}\circ\cdots\circ\psi_{\omega_{\ell-1}}|$$
$$\leqq \||\mathscr{M}'|^{m-\ell}\|_0\||\mathscr{M}|^{k-1}\|_0\|\Phi\|_0 A_{k\ell}^*,$$

where

$$A_{k\ell}^* = \sum_{\omega_k}\sum_{\omega_\ell}\int|\partial(\varphi_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1})| \sum_{\omega_{k+1}\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}}|\cdots|\partial(\varphi_{\omega_k}\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})|$$
$$\leqq \sum_{\omega_\ell}\left[\int|\partial(\varphi_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1})|^2\right]^{1/2}\sum_{\omega_k}\left[\int\left[\sum_{\omega_{k+1}\cdots\omega_{\ell-1}}|\varphi_{\omega_{\ell-1}}|\cdots|\partial(\varphi_{\omega_k}\circ\cdots\circ\psi_{\omega_{\ell-1}})|\right]^2\right]^{1/2}$$
$$= \frac{K}{\sqrt{2}}\sum_{\omega_k}\left[\int[\cdots]^2\right]^{1/2},$$

where

$$\int[\cdots]^2 \leqq \int \sum_{\omega_{k-1}\cdots\omega_{\ell-1}} |\cdots| \sum_{\omega'_{k+1}\cdots\omega'_{\ell-1}} |\cdots|$$
$$\cdot\tfrac{1}{2}\left[|\partial(\varphi_{\omega_k}\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})|^2 + |\partial(\varphi_{\omega_k}\circ\psi_{\omega'_{k+1}}\circ\cdots\circ\psi_{\omega'_{\ell-1}})|^2\right]$$

$$\leqq \| |\mathcal{M}|^{\ell-k-1} \|_0 \int \sum_{\omega_{k+1}\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}}| \cdots |\varphi_{\omega_{k+1}} \circ \psi_{\omega_{k+2}} \circ \cdots \circ \psi_{\omega_{\ell-1}}|$$

$$\cdot |\partial(\varphi_{\omega_k} \circ \psi_{\omega_{k+1}} \circ \cdots \circ \psi_{\omega_{\ell-1}})|^2$$

$$= \| |\mathcal{M}|^{\ell-k-1} \|_0 \int \sum_{\omega_{k+1}\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}} \circ \psi^{-1}_{\omega_{\ell-1}} \circ \cdots \circ \psi^{-1}_{\omega_{k+1}}|$$

$$\cdots |\varphi_{\omega_{k+1}} \circ \psi^{-1}_{\omega_{k+1}}| \cdot |\partial \varphi_{\omega_k}|^2$$

$$\leqq \| |\mathcal{M}|^{\ell-k-1} \|_0 \| |\mathcal{M}'|^{\ell-k-1} \|_0 \int |\partial \varphi_{\omega_k}|^2,$$

so that

$$A^*_{k\ell} \leqq \frac{K^2}{2} (\| |\mathcal{M}|^{\ell-k-1} \|_0 \| |\mathcal{M}'|^{\ell-k-1} \|_0)^{1/2}$$

and

$$A_{k\ell} \leqq \frac{K^2}{2} \| |\mathcal{M}'|^{m-\ell} \|_0 (\| |\mathcal{M}'|^{\ell-k-1} \|_0 \| \mathcal{M}|^{\ell-k-1} \|_0)^{1/2} \| |\mathcal{M}|^{k-1} \|_0 \| \Phi \|_0.$$

We have similarly

$$B_l \leqq \| |\mathcal{M}'|^{m-\ell} \|_0 B^*_\ell,$$

where

$$B^*_\ell = \sum_{\omega_\ell} \int |\partial(\varphi_{\omega_\ell} \circ \psi^{-1}_{\omega_\ell})| \sum_{\omega_1\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}}| \cdots |\varphi_{\omega_1} \circ \psi_{\omega_2} \circ \cdots \circ \psi_{\omega_{\ell-1}}| \cdot |\partial(\Phi \circ \psi_{\omega_1} \circ \cdots \circ \psi_{\omega_{\ell-1}})|$$

$$\leqq \sum_{\omega_\ell} [\int |\partial(\varphi_{\omega_\ell} \circ \psi^{-1}_{\omega_\ell})|^2]^{1/2} \left[\int \left[\sum_{\omega_1\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}}| \cdots |\partial(\Phi \circ \psi_{\omega_1} \circ \cdots \circ \psi_{\omega_{\ell-1}})| \right]^2 \right]^{1/2}$$

$$\leqq \frac{K}{\sqrt{2}} [\| |\mathcal{M}|^{\ell-1} \|_0 \| |\mathcal{M}'|^{\ell-1} \|_0 \int |\partial \Phi|^2]^{1/2}$$

$$\leqq \frac{K}{2} (\| |\mathcal{M}|^{\ell-1} \|_0 \| |\mathcal{M}'|^{\ell-1} \|_0)^{1/2} \operatorname{Var} \Phi,$$

and therefore

$$B_\ell \leqq \frac{K}{2} \| |\mathcal{M}'|^{m-\ell} \|_0 (\| \mathcal{M}'|^{\ell-1} \|_0 \| |\mathcal{M}|^{\ell-1} \|_0)^{1/2} \operatorname{Var} \Phi.$$

We have

$$C_k \leqq \| |\mathcal{M}|^{k-1} \|_0 \| \Phi \|_0 C^*_k,$$

where

$$C^*_k = \int \sum_{\omega_k\cdots\omega_m} \int |\varphi_{\omega_m} \cdots \partial^2(\varphi_{\omega_k} \circ \psi_{\omega_{k+1}} \circ \cdots \circ \psi_{\omega_m})|$$

$$\leqq \int \sum_{\omega_{k+1}\cdots\omega_m} |(\psi_{\omega_m} \circ \psi^{-1}_{\omega_m} \circ \cdots \circ \psi^{-1}_{\omega_{k+1}}) \cdots (\varphi_{\omega_{k+1}} \circ \psi^{-1}_{\omega_{k+1}})| \sum_{\omega_k} |\partial^2 \varphi_{\omega_k}|$$

$$+ \int \sum_{\omega_k\cdots\omega_m} |\varphi_{\omega_m} \cdots (\varphi_{\omega_{k+1}} \circ \psi_{\omega_{k+2}} \circ \cdots \circ \psi_{\omega_m})| \cdot |(\partial \varphi_{\omega_k}) \circ \psi_{\omega_{k+1}} \circ \cdots \circ \psi_{\omega_m}|$$

$$\cdot|((\psi'_{\omega_{k+1}}\circ\psi_{\omega_{k+2}}\circ\cdots\circ\psi_{\omega_m})\cdots\psi'_{\omega_m})'|$$
$$\leqq \| |\mathcal{M}'|^{m-k}\|_0 K + C_k^{**},$$

where

$$C_k^{**} = \sum_{\ell=k+1}^{m} \int \sum_{\omega_k\cdots\omega_m} |\varphi_{\omega_m}\cdots(\varphi_{\omega_{k+1}}\circ\psi_{\omega_{k+2}}\circ\cdots\circ\psi_{\omega_m})|$$
$$\cdot|(\partial\varphi_{\omega_k})\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_m}|\cdot|(\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})'\circ(\psi_{\omega_\ell}\circ\cdots\circ\psi_{\omega_m})|$$
$$\cdot|\psi''_{\omega_\ell}\circ\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m}|\cdot|(\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m})'|^2.$$

Since

$$|\psi''_{\omega_\ell}\circ\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m}|\cdot|(\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m})'|^2 = \frac{|\psi''_{\omega_\ell}\circ\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m}|}{|\psi'_{\omega_\ell}\circ\psi_{\omega_{\ell+1}}\circ\cdots\circ\psi_{\omega_m}|^2}\cdot|(\psi_{\omega_\ell}\circ\cdots\circ\psi_{\omega_m})'|^2,$$

we have

$$C_k^{**} \leqq \sum_{\ell=k+1}^{m} \| |\mathcal{M}'|^{m-\ell}\|_0 \int \sum_{\omega_k\cdots\omega_\ell} |\varphi_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1}|\cdot|\varphi_{\omega_{\ell-1}}\cdots(\varphi_{\omega_{k+1}}\circ\psi_{\omega_{k+2}}\circ\cdots\circ\psi_{\omega_{\ell-1}})|$$
$$\cdot|(\partial\varphi_{\omega_k})\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}}|\cdot|(\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})'|\cdot\frac{|\psi''_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1}|}{|\psi'_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1}|^2}$$
$$\leqq \sum_{\ell=k+1}^{m} \| |\mathcal{M}'|^{m-\ell}\|_0 \sum_{\omega_\ell}\left[\int |\varphi_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1}|^2 \frac{|\psi''_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1}|^2}{|\psi'_{\omega_\ell}\circ\psi_{\omega_\ell}^{-1}|^4}\right]^{1/2}$$
$$\cdot\sum_{\omega_k}\left[\int\left[\sum_{\omega_{k-1}\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}}\cdots(\varphi_{\omega_{k+1}}\circ\psi_{\omega_{k+2}}\circ\cdots\circ\psi_{\omega_{\ell-1}})|\right.\right.$$
$$\left.\left.\cdot|(\partial\varphi_{\omega_k})\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}}|\cdot|(\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})'|\right]^2\right]^{1/2}$$
$$= \sum_{\ell=k+1}^{m} \| |\mathcal{M}'|^{m-\ell}\|_0\cdot\sqrt{2}L\sum_{\omega_k}[\int[\cdots]^2]^{1/2},$$

where

$$\int[\cdots]^2 \leqq \| |\mathcal{M}|^{\ell-k-1}\|_0 \int \sum_{\omega_{k+1}\cdots\omega_{\ell-1}} |\varphi_{\omega_{\ell-1}}\cdots(\varphi_{\omega_{k+1}}\circ\psi_{\omega_{k+2}}\circ\cdots\circ\psi_{\omega_{\ell-1}})|$$
$$\cdot|(\partial\varphi_{\omega_k})\circ\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}}|^2\cdot|(\psi_{\omega_{k+1}}\circ\cdots\circ\psi_{\omega_{\ell-1}})'|^2$$
$$\leqq \| |\mathcal{M}|^{\ell-k-1}\|_0 \int \sum_{\omega_{k+1}\cdots\omega_{\ell-1}} |(\varphi_{\omega_{\ell-1}}\circ\psi_{\omega_{\ell-1}}^{-1}\circ\cdots\circ\psi_{\omega_{k+1}}^{-1})$$
$$\cdots(\varphi_{\omega_{k+1}}\circ\psi_{\omega_{k+1}}^{-1})|\cdot|\partial\varphi_{\omega_k}|^2$$
$$\leqq \| |\mathcal{M}|^{\ell-k-1}\|_0 \| |\mathcal{M}'|^{\ell-k-1}\|_0 \int |\partial\varphi_{\omega_k}|^2$$

so that

$$C_k^{**} \leqq K\cdot L \sum_{\ell=k+1}^{m} \| |\mathcal{M}'|^{m-\ell}\|_0 (\| |\mathcal{M}'|^{\ell-k-1}\|_0\cdot\| |\mathcal{M}|^{\ell-k-1}\|_0)^{1/2}$$

and

$$C_k \leqq K\left[\| |\mathcal{M}'|^{m-k} \|_0 + L \sum_{\ell=k+1}^{m} \| |\mathcal{M}'|^{m-\ell} \|_0 \right.$$

$$\left. \cdot (\| |\mathcal{M}'|^{\ell-k-1} \|_0 \cdot \| |\mathcal{M}|^{\ell-k-1} \|_0)^{1/2} \right] \| |\mathcal{M}|^{k-1} \|_0 \cdot \| \Phi \|_0.$$

We have similarly

$$D \leqq \left[\| |\mathcal{M}'|^m \|_0 + L \sum_{\ell=1}^{m} \| |\mathcal{M}'|^{m-\ell} \|_0 \cdot (\| |\mathcal{M}'|^{\ell-1} \|_0 \cdot \| |\mathcal{M}|^{\ell-1} \|_0)^{1/2} \right] \operatorname{Var} \Phi.$$

Finally, the above bounds for $A_{k\ell}, B_\ell, C_k, D$ show that

$$\lim_{m \to \infty} (\operatorname{Var} \mathcal{M}^m \Phi / \operatorname{Var} \Phi)^{1/m} \leqq \max(R, R')$$

and the spectral radius of $\mathcal{M}$ is thus $\leqq \max(R, R')$.

4. The Essential Spectral Radius of $\mathcal{M}$

Without loss of generality, we take $X = \{x + iy : 0 < x < 1,\ 0 < y < 1\}$. Choosing an integer $N > 0$, we write

$$Q(\kappa, \lambda) = \left\{ x + iy : \frac{\kappa - 1}{N} \leqq x \leqq \frac{\kappa}{N}, \frac{\lambda - 1}{N} \leqq y \leqq \frac{\lambda}{N} \right\}$$

for $\kappa, \lambda \in \{1, \ldots, N\}$. A linear operator $T_N : \mathcal{B} \mapsto \mathcal{B}$ is then defined by the condition that, for all κ, λ,

$$T_N \Phi(x + iy) = a_{\kappa\lambda} + b_{\kappa\lambda} x + c_{\kappa\lambda} y + d_{\kappa\lambda} xy$$

on $Q(\kappa, \lambda)$ where the coefficients $a_{\kappa\lambda}, \ldots, d_{\kappa\lambda}$ are such that $T_N \Phi(x + iy) = \Phi(x + iy)$ on the 4 corners of $Q(\kappa, \lambda)$.

4.1. Lemma

(a) $$\operatorname{Var} T_N \Phi \leqq 8 \operatorname{Var} \Phi,$$

(b) $$\sum_{\kappa\lambda} \max_{z \in Q(\kappa, \lambda)} |\Phi(z) - T_N \Phi(z)| \leqq 64 \operatorname{Var} \Phi.$$

First, note that

$$\int_{Q(\kappa,\lambda)} \left| \frac{\partial^2 T_N \Phi}{\partial x \partial y} \right| = \left| \Phi\left(\frac{\kappa}{N} + i\frac{\lambda}{N} \right) - \Phi\left(\frac{\kappa}{N} + i\frac{\lambda - 1}{N} \right) \right.$$

$$\left. - \Phi\left(\frac{\kappa - 1}{N} + i\frac{\lambda}{N} \right) + \Phi\left(\frac{\kappa - 1}{N} + i\frac{\lambda - 1}{N} \right) \right|$$

$$= \left| \int_{Q(\kappa,\lambda)} \frac{\partial^2 \Phi}{\partial x \partial y} \right| \leqq \int_{Q(\kappa,\lambda)} \left| \frac{\partial^2 \Phi}{\partial x \partial y} \right|$$

so that

$$\int\left|\frac{\partial^2 T_N \Phi}{\partial x \partial y}\right| \leqq \int\left|\frac{\partial^2 \Phi}{\partial x \partial y}\right|. \tag{4.1}$$

We may write

$$\begin{aligned}
\int\left|\frac{\partial^2 T_N \Phi}{\partial x^2}\right| dx\,dy &\leqq N^{-1} \sum_{\lambda=1}^{N-1} \int\left|\frac{\partial^2 T_N \Phi}{\partial x^2}\left(x+i\frac{\lambda}{N}\right)\right| dx \\
&= \sum_{\kappa\lambda}\left|\Phi\left(\frac{\kappa+1}{N}+i\frac{\lambda}{N}\right)-2\Phi\left(\frac{\kappa}{N}+i\frac{\lambda}{N}\right)+\Phi\left(\frac{\kappa-1}{N}+i\frac{\lambda}{N}\right)\right| \\
&= \sum_{\kappa\lambda}\left|\Psi_\kappa\left(\frac{\lambda}{N}\right)\right|,
\end{aligned}$$

where we have defined

$$\begin{aligned}
\Psi_\kappa(y) &= \Phi\left(\frac{\kappa+1}{N}+iy\right)-2\Phi\left(\frac{\kappa}{N}+iy\right)+\Phi\left(\frac{\kappa-1}{N}+iy\right) \\
&= \int_{\kappa/N}^{(\kappa+1)/N} \partial_x \Phi(x+iy)dx + \int_{\kappa/N}^{(\kappa-1)/N} \partial_x \Phi(x+iy)dx \\
&= \int_{\kappa/N}^{(\kappa+1)/N} dx \int_{\kappa/N}^{x} \partial^2_{\xi\xi} \Phi(\xi+iy)d\xi + \int_{\kappa/N}^{(\kappa-1)/N} dx \int_{\kappa/N}^{x} \partial^2_{\xi\xi} \Phi(\xi+iy)d\xi.
\end{aligned}$$

Therefore

$$|\Psi_\kappa(y)| \leqq \frac{1}{N}\int_{(\kappa-1)/N}^{(\kappa+1)/N} |\partial^2_{xx}\Phi(x+iy)|dx,$$

and we may choose $y \in \left[\frac{\lambda-1}{N}, \frac{\lambda}{N}\right]$ such that

$$|\Psi_\kappa(y)| \leqq \int_{(\lambda-1)/N}^{\lambda/N} d\eta \int_{(\kappa-1)/N}^{(\kappa+1)/N} dx\,|\partial^2_{xx}\Phi(x+i\eta)|.$$

We also have

$$\begin{aligned}
\frac{1}{2}\left|\Psi_\kappa\left(\frac{\lambda-1}{N}\right)-\Psi_\kappa(y)\right|+\frac{1}{2}\left|\Psi_\kappa\left(\frac{\lambda}{N}\right)-\Psi_\kappa(y)\right| &\leqq \frac{1}{2}\int_{(\lambda-1)/N}^{\lambda/N} d\eta \int_{\kappa/N}^{(\kappa+1)/N} dx\,|\partial^2_{xy}\Phi(x+i\eta)| \\
&\quad + \frac{1}{2}\int_{(\lambda-1)/N}^{\lambda/N} d\eta \int_{(\kappa-1)/N}^{\kappa/N} dx\,|\partial^2_{xy}\Phi(x+i\eta)|.
\end{aligned}$$

Hence

$$\int|\partial^2_{xx} T_N \Phi| \leqq \sum_{\kappa\lambda}\left(\frac{1}{2}\left|\Psi_\kappa\left(\frac{\lambda}{N}\right)\right|+\frac{1}{2}\left|\Psi_\kappa\left(\frac{\lambda-1}{N}\right)\right|\right) \leqq 2\int|\partial^2_{xx}\Phi|+\int|\partial^2_{xy}\Phi|. \tag{4.2}$$

Finally, (a) results from (4.1) and (4.2):

$$\int|\partial^2 T_N\Phi| \leqq \int[|\partial^2_{xx}T_N\Phi| + 2|\partial^2_{xy}T_N\Phi| + |\partial^2_{yy}T_N\Phi|]$$
$$\leqq 2\int[|\partial^2_{xx}\Phi| + 2|\partial^2_{xy}\Phi| + |\partial^2_{yy}\Phi|] \leqq 8 \operatorname{Var}\Phi.$$

The proof of (b) will be in several steps, and it will simplify our notation to assume that the function Φ has its support in

$$\left\{x+iy: \frac{1}{N} \leqq x \leqq 1-\frac{1}{N}, \frac{1}{N} \leqq y \leqq 1-\frac{1}{N}\right\}.$$

Suppose that $0 \leqq \xi \leqq 1/N$, $0 \leqq \eta \leqq 1/N$, and write $Q'(\kappa,\lambda) = \xi + i\eta + Q(\kappa,\lambda)$. Let T'_N be defined like T_N, but with $Q(\kappa,\lambda)$ replaced by $Q'(\kappa,\lambda)$. We shall first show that, for suitable choice of ξ, η,

$$\sum_{\kappa\lambda} \max_{z\in Q'(\kappa,\lambda)} |\Phi(z) - T'_N\Phi(z)| \leqq 4 \operatorname{Var}\Phi. \tag{4.3}$$

If $\eta + \frac{\lambda-1}{N} \leqq y \leqq \eta + \frac{\lambda}{N}$, we have

$$\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right) - N\left(\eta+\frac{\lambda}{N}-y\right)\Phi\left(\xi+\frac{\kappa-1}{N}+i\left(\eta+\frac{\lambda-1}{N}\right)\right)$$
$$-N\left(y-\left(\eta+\frac{\lambda-1}{N}\right)\right)\Phi\left(\xi+\frac{\kappa-1}{N}+i\left(\eta+\frac{\lambda}{N}\right)\right)$$
$$= N\left(\eta+\frac{\lambda}{N}-y\right)\int_{\eta+(\lambda-1)/N}^{y} dv\,\partial_v\Phi\left(\xi+\frac{\kappa-1}{N}+iv\right)$$
$$-N\left(y-\left(\eta+\frac{\lambda-1}{N}\right)\right)\int_{y}^{\eta+\lambda/N} dv\,\partial_v\Phi\left(\xi+\frac{\kappa-1}{N}+iv\right)$$
$$= N\left(\eta+\frac{\lambda}{N}-y\right)\int_{\eta+(\lambda-1)/N}^{y} dv\int_y^v dw\,\partial^2_{ww}\Phi\left(\xi+\frac{\kappa-1}{N}+iw\right)$$
$$-N\left(y-\left(\eta+\frac{\lambda-1}{N}\right)\right)\int_{y}^{\eta+\lambda/N} dv\int_y^v dw\,\partial^2_{ww}\Phi\left(\xi+\frac{\kappa-1}{N}+iw\right).$$

Taking absolute values, and writing

$$\left|\int_y^v dw\,\partial^2_{ww}\Phi\right| \leqq \int_{\eta+(\lambda-1)/N}^{\eta+\lambda/N} dw\,|\partial^2_{ww}\Phi|,$$

we obtain

$$\max_{y\in[\eta+(\lambda-1)/N,\eta+\lambda/N]}\left|\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)\right.$$
$$-N\left(\eta+\frac{\lambda}{N}-y\right)\Phi\left(\xi+\frac{\kappa-1}{N}+i\left(\eta+\frac{\lambda-1}{N}\right)\right)$$

$$-N\left(y-\left(\eta+\frac{\lambda-1}{N}\right)\right)\Phi\left(\xi+\frac{\kappa-1}{N}+i\left(\eta+\frac{\lambda}{N}\right)\right)\Bigg|$$
$$\leqq\frac{1}{N}\int_{\eta+(\lambda-1)/N}^{\eta+\lambda/N}dv\left|\partial_{vv}^2\Phi\left(\xi+\frac{\kappa-1}{N}+iv\right)\right|.$$

If we sum over κ and λ, this is

$$\cdots\leqq\frac{1}{N}\sum_{\kappa}\int_0^1 dy\left|\partial_{yy}^2\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)\right|$$

and the average over $\xi\in[0,1/N]$ gives

$$\cdots\leqq\int_0^1 dx\int_0^1 dy\,|\partial_{yy}^2\Phi(x+iy)|.$$

We may therefore choose $\xi\in[0,1/N]$ such that

$$\sum_{\kappa\lambda}\max_{y\in[\eta+(\lambda-1)/N,\eta+\lambda/N]}\left|\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)-T'_N\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)\right|$$
$$\leqq\int dxdy\,|\partial_{yy}^2\Phi(x+iy)|. \tag{4.4}$$

Similarly, we may choose $\eta\in[0,1/N]$ such that

$$\sum_{\kappa\lambda}\max_{x\in[\xi+(\kappa-1)/N,\xi+\kappa/N]}\left|\Phi\left(x+i\left(\eta+\frac{\lambda-1}{N}\right)\right)-T'_N\Phi\left(x+i\left(\eta+\frac{\lambda-1}{N}\right)\right)\right|$$
$$\leqq\int dxdy\,|\partial_{xx}^2\Phi(x+iy)|. \tag{4.5}$$

In general if $x+iy\in Q'(\kappa,\lambda)$, we have

$$\Phi(x+iy)-\Phi\left(x+i\left(\eta+\frac{\lambda-1}{N}\right)\right)-\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)$$
$$+\Phi\left(\xi+\frac{\kappa-1}{N}+i\left(\eta+\frac{\lambda-1}{N}\right)\right)=\int_{\xi+(\kappa-1)/N}^{x}du\int_{\eta+(\lambda-1)/N}^{y}dv\,\partial_{uv}^2\Phi,$$

and therefore, using (4.1),

$$|\Phi(x+iy)-T'_N\Phi(x+iy)|\leqq\left|\Phi\left(x+i\left(\eta+\frac{\lambda-1}{N}\right)\right)-T'_N\Phi\left(x+i\left(\eta+\frac{\lambda-1}{N}\right)\right)\right|$$
$$+\left|\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)-T'_N\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)\right|$$
$$+2\int_{Q'(\kappa,\lambda)}dudv\,|\partial_{uv}^2\Phi|. \tag{4.6}$$

Finally, (4.3) results from (4.4), (4.5), (4.6).

In the above proof, we have chosen ξ such that the expression $\sum_{\kappa}\int_0^1 dy\left|\partial_{yy}^2\Phi\left(\xi+\frac{\kappa-1}{N}+iy\right)\right|$ is less than its average over ξ. If this expression is independent of ξ, the choice $\xi=0$ is therefore allowed. In fact, it is easily seen that this situation prevails when Φ is replaced by $T'_N\Phi$. We may then similarly take $\eta = 0$, and (4.3) becomes

$$\sum_{\kappa\lambda} \max_{z\in Q(\kappa,\lambda)} |(T'_N\Phi)(z) - (T_N T'_N\Phi)(z)| \leqq 4\,\mathrm{Var}\,(T'_N\Phi) \leqq 32\,\mathrm{Var}\,\Phi. \tag{4.7}$$

Using (4.3) and (4.7) we have

$$\begin{aligned}
&\sum_{\kappa\lambda} \max_{z\in Q(\kappa,\lambda)} |\Phi(z) - T_N\Phi(z)| \\
&\quad \leqq 4\sum_{\kappa\lambda} \max_{z\in Q'(\kappa,\lambda)} |\Phi(z) - T'_N\Phi(z)| \\
&\qquad + \sum_{\kappa\lambda} \max_{z\in Q(\kappa,\lambda)} (|T'_N\Phi(z) - T_N T'_N\Phi(z)| + |T_N T'_N\Phi(z) - T_N\Phi(z)|) \\
&\quad \leqq 48\,\mathrm{Var}\,\Phi + \sum_{\kappa\lambda} \max_{z\in Q(\kappa,\lambda)} |T_N\Phi(z) - T_N T'_N\Phi(z)|,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{\kappa\lambda} \max_{z\in Q(\kappa,\lambda)} |T_N\Phi(z) - T_N T'_N\Phi(z)| &\leqq \sum_{\kappa\lambda} \max_{z\in Q(\kappa,\lambda)} |\Phi(z) - T'_N\Phi(z)| \\
&\leqq 4\sum_{\kappa\lambda} \max_{z\in Q'(\kappa,\lambda)} |\Phi(z) - T'_N\Phi(z)| \leqq 16\,\mathrm{Var}\,\Phi.
\end{aligned}$$

From this, (b) follows immediately.

For finite $\Lambda \subset \Omega$ we define $\mathscr{M}_\Lambda$ by

$$\mathscr{M}_\Lambda\Phi(z) = \sum_{\omega\in\Lambda} \varphi_\omega(z)\Phi(\psi_\omega z).$$

Theorem 2.1 (b) then results from Nussbaum's essential spectral radius formula [6][3] and the following estimate.

4.2. Proposition. *If $R' \leqq R$, we may choose $\Lambda = \Lambda(m)$ and $N = N(m)$ such that*

$$\lim_{m\to\infty} \|\mathscr{M}^m - \mathscr{M}^m_\Lambda T_N\|^{1/m} \leqq (RR')^{1/2}.$$

We take $\Lambda(m)$ such that

$$\|\mathscr{M}^m - \mathscr{M}^m_\Lambda\| \leqq (\||\mathscr{M}'|^m\|_0 \||\mathscr{M}|^m\|_0)^{1/2},$$

and it will then suffice to prove that

$$\lim_{m\to\infty} \|\mathscr{M}^m_\Lambda - \mathscr{M}^m_\Lambda T_N\|^{1/m} \leqq (RR')^{1/2}.$$

We thus have to analyze

$$\mathrm{Var}\,\mathscr{M}^m_\Lambda(\Phi - T_N\Phi) = \int |\partial^2\mathscr{M}^m_\Lambda(\Phi - T_N\Phi)| \leqq 2\sum_{k<\ell} A'_{k\ell} + 2\sum_{\ell} B'_\ell + \sum_k C'_k + D',$$

where the above terms are analogous to those in (3.1): Φ simply has to be replaced

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[3] Actually, as pointed out by the referee, we do not use the full strength of the Nussbaum formula; we only need an (easy) upper bound on the essential spectral radius

by $\Phi - T_N\Phi$. Using the estimates of Sect. 3 we find immediately

$$B'_\ell \leqq \frac{K}{2} \||\mathcal{M}'|^{m-\ell}\|_0(\||\mathcal{M}'|^{\ell-1}\|_0\||\mathcal{M}|^{\ell-1}\|_0)^{1/2}\cdot 9\,\mathrm{Var}\,\Phi$$

$$D' \leqq \left[\||\mathcal{M}'|^m\|_0 + L\sum_{\ell=1}^{m}\||\mathcal{M}'|^{m-\ell}\|_0\cdot(\||\mathcal{M}'|^{\ell-1}\|_0\||\mathcal{M}|^{\ell-1}\|_0)^{1/2}\right]\cdot 9\,\mathrm{Var}\,\Phi.$$

The terms $A'_{k\ell}, C'_k$ require some more effort. We choose ε such that $\varepsilon \leqq \||\mathcal{M}'|^{k-1}\|_0/(\mathrm{card}\,\Lambda)^{k-1}$ for $k=1,\dots,m$. Having chosen ε we may, by uniform continuity, take $N(m)$ such that[4]

$$\begin{aligned}&|\varphi_{\omega_{k-1}}(\tilde z)\cdots\varphi_{\omega_1}(\psi_{\omega_2}\cdots\psi_{\omega_{k-1}}\tilde z)|\\ &\leqq \left|\varphi_{\omega_{k-1}}\left(\psi^{-1}_{\omega_{k-1}}\cdots\psi^{-1}_{\omega_1}\frac{\kappa+i\lambda}{N}\right)\cdots\varphi_{\omega_1}\left(\psi^{-1}_{\omega_1}\frac{\kappa+i\lambda}{N}\right)\right|+\varepsilon\end{aligned}$$

for all $\tilde z$ such that $\psi_{\omega_1}\cdots\psi_{\omega_{k-1}}\tilde z \in Q(\kappa,\lambda)$, whenever $\omega_1,\dots,\omega_{k-1}\in\Lambda$, $k=1,\dots,m$, and for any κ,λ. Returning to $A'_{k\ell}, C'_k$, we write

$$\begin{aligned}&\sum_{\omega_1\cdots\omega_{k-1}}|\varphi_{\omega_{k-1}}(\tilde z)\cdots\varphi_{\omega_1}(\psi_{\omega_2}\cdots\psi_{\omega_{k-1}}\tilde z)[(\Phi-T_N\Phi)(\psi_{\omega_1}\cdots\psi_{\omega_{k-1}}\tilde z)]|\\ &=\sum_{\kappa\lambda}\sum_{\omega_1\cdots\omega_{k-1}}{}'\,|\cdots|,\end{aligned}$$

where $\sum'$ extends over those $\omega_1,\dots,\omega_{k-1}\in\Lambda$ such that $\psi_{\omega_1}\cdots\psi_{\omega_{k-1}}\tilde z\in Q(\kappa,\lambda)$. We have then

$$\begin{aligned}&\sum_{\omega_1\cdots\omega_{k-1}}|\varphi_{\omega_{k-1}}(\tilde z)\cdots\varphi_{\omega_1}(\psi_{\omega_2}\cdots\psi_{\omega_{k-1}}\tilde z)[(\Phi-T_N\Phi)(\psi_{\omega_1}\cdots\psi_{\omega_{k-1}}\tilde z)]|\\ &\leqq \max_{z'}\sum_{\omega_1\cdots\omega_{k-1}\in\Lambda}(|\varphi_{\omega_{k-1}}(\psi^{-1}_{\omega_{k-1}}\cdots\psi^{-1}_{\omega_1}z')\cdots\varphi_{\omega_1}(\psi^{-1}_{\omega_1}z')|+\varepsilon)\\ &\quad\cdot\sum_{\kappa\lambda}\max_{z(\kappa,\lambda)\in Q(\kappa,\lambda)}|(\Phi-T_N\Phi)(z(\kappa,\lambda))|\\ &\leqq(\||\mathcal{M}'|^{k-1}\|_0+\varepsilon(\mathrm{card}\,\Lambda)^{k-1})\cdot 64\,\mathrm{Var}\,\Phi\\ &\leqq 128\||\mathcal{M}'|^{k-1}\|_0\,\mathrm{Var}\,\Phi.\end{aligned}$$

Inserting this in the estimates of Sect. 3 yields

$$A'_{k\ell}\leqq\frac{K^2}{2}\||\mathcal{M}'|^{m-\ell}\|_0(\||\mathcal{M}'|^{\ell-k-1}\|_0\||\mathcal{M}|^{\ell-k-1}\|_0)^{1/2}\cdot 128\||\mathcal{M}'|^{k-1}\|_0\,\mathrm{Var}\,\Phi,$$

$$\begin{aligned}C'_k\leqq K\Bigg[&\||\mathcal{M}'|^{m-k}\|_0+L\sum_{k=\ell+1}^{m}\||\mathcal{M}'|^{m-\ell}\|_0\\ &\cdot(\|\mathcal{M}'|^{\ell-k-1}\|_0\cdot\||\mathcal{M}|^{\ell-k-1}\|_0)^{1/2}\Bigg]\cdot 128\||\mathcal{M}'|^{k-1}\|_0\,\mathrm{Var}\,\Phi,\end{aligned}$$

and Proposition 4.2 follows.

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[4] Remember that the φ_ω have to be replaced by $\varphi_{\omega i}$, and that one should take at the end $\varphi_{\omega i}\to\varphi_\omega$. This creates no difficulty since the assumed inequality can be taken to hold uniformly in i when $\varphi_{\omega i}$ is close to φ_ω

5 Proof of Theorem 2.2

We follow [11] Sect. 2.5. Define $\Phi = \sum_{\omega} \varphi_\omega \in \mathscr{B}$; since $\varphi_\omega \geqq 0$, we have

$$\| \mathscr{M}^{m-1} \Phi \|_0 = \| \, |\mathscr{M}|^m \|_0 .$$

Therefore

$$\lim_{m\to\infty} \| \mathscr{M}^m \|^{1/m} \geqq \lim_{m\to\infty} (\operatorname{Var} \mathscr{M}^{m-1}\Phi)^{1/m}$$

$$\geqq \lim_{m\to\infty} (\| \mathscr{M}^{m-1}\Phi \|_0)^{1/m} = \lim_{m\to\infty} (\| \, |\mathscr{M}|^m \|_0)^{1/m} = R, \qquad (5.1)$$

so that the spectral radius of $\mathscr{M}$ is $\geqq R$. In particular, if $R' \leqq R$, Theorem 2.1(a) shows that the spectral radius of $\mathscr{M}$ is equal to R.

If $R' < R$, we may write

$$\Phi = \Psi + \sum_j \Psi_j \qquad (5.2)$$

where, for each j, λ_j is an eigenvalue of $\mathscr{M}$ with $|\lambda_j| = R$, and Ψ_j is in the corresponding generalized eigenspace; Ψ is such that

$$\lim_{m\to\infty} \frac{\operatorname{Var} \mathscr{M}^m \Psi}{\tilde{\lambda}^m} = 0 \qquad (5.3)$$

with $0 < \tilde{\lambda} < R$. In view of (5.1),

$$\lim_{m\to\infty} \frac{1}{m} \log \operatorname{Var} \mathscr{M}^m \Phi = \log R,$$

and therefore the Ψ_j do not all vanish. Write the restriction of $\mathscr{M}$ to the generalized eigenspaces corresponding to the λ_j in Jordan normal form. It is then readily seen that there is an integer $k \geqq 0$ such that

$$\lim_{m\to\infty} \frac{1}{\lambda_j^m m^k} \mathscr{M}^m \Psi_j = \Phi_j \qquad (5.4)$$

and $\mathscr{M}\Phi_j = \lambda_j \Phi_j$ for all j, with $\Phi_j \neq 0$ for some j. From (5.2) we get

$$0 \leqq \frac{\mathscr{M}^m \Phi}{R^m} = \frac{\mathscr{M}^m \Psi}{R^m} + \sum_j \left(\frac{\lambda_j}{R}\right)^m \frac{\mathscr{M}^m \Psi_j}{\lambda_j^m}.$$

Using (5.3) and (5.4) this gives

$$\sum_j \left(\frac{\lambda_j}{R}\right)^m \Phi_j \geqq -\varepsilon(m),$$

where $\varepsilon(m) \to 0$ when $m \to \infty$. Note that the sum is finite, and that $|\lambda_j/R| = 1$ for all j. Therefore R must be an eigenvalue, say $R = \lambda_0$, and $\Phi_0 \geqq 0$, Φ_0 not identically 0.

6. Extension to Multiple-Valued Functions

In Sect. 1 we introduced data $V_\omega, \psi_\omega, \varphi_\omega$ satisfying conditions (i), (ii), and (iii). We shall now also accept finitely many triples $(V_\omega, \psi_\omega, \varphi_\omega)$ satisfying the conditions (i)′, (ii)′ and (iii)′ below. (This will allow us to consider multiple-valued analytic functions; for simplicity we omit the discussion of antiholomorphic functions.) Here are the new conditions.

(i)′ V_ω is an open disk in the n_ω-sheeted Riemann surface over $\mathbb{C}$, branched at a_ω (if $n_\omega > 1$); we assume n_ω finite, $a_\omega \in V_\omega$, and let the projection π of the Riemann surface map V_ω into X.

(ii)′ $\psi_\omega: V_\omega \mapsto \psi_\omega V_\omega$ is holomorphic and invertible, where $\psi_\omega V_\omega$ is a Riemann surface with projection π^* to a subset of X; $\psi_\omega V_\omega$ is unbranched except possibly at $\psi_\omega a_\omega$; the multiplicity n^*_ω of $\psi_\omega a_\omega$ is finite.

(iii)′ $\varphi_\omega: V_\omega \mapsto \mathbb{C}$ is continuous with compact support, and

$$\operatorname{Var}\varphi_\omega = \int_{V_\omega \setminus \{a_\omega\}} dx\, dy\, |\partial^2 \varphi_\omega| < \infty .$$

The integration is over the n_ω sheets of V_ω; the second derivatives do not make sense at a_ω, but are assumed to be measures outside of this point.

We also assume that

$$\left[\frac{1}{2} \int_{V_\omega \setminus \{a_\omega\}} dx\, dy \left| \varphi_\omega \frac{(\pi^* \psi_\omega)''}{(\pi^* \psi_\omega)'} \right|^2 \right]^{1/2} < \infty .$$

Note that this implies that $\varphi_\omega(a_\omega) = 0$ unless $n_\omega = n^*_\omega = 1$. The new terms in $\mathcal{M}\Phi(z)$ and $\mathcal{M}'\Phi(z)$ have respectively the form

$$\sum_{Z \in \pi^{-1} z} \varphi_\omega(Z) \Phi(\pi^* \psi_\omega Z)$$

and

$$\sum_{Z \in \pi^{*-1} z} \varphi_\omega(\psi_\omega^{-1} Z) \Phi(\pi \psi_\omega^{-1} Z),$$

where $\pi_{a_\omega}^{-1}$ and $\pi^{*-1} \psi_\omega a_\omega$ are counted with multiplicity n_ω and n^*_ω respectively. The operator $|\mathcal{M}|$ is defined as before by the replacement $\varphi_\omega \to |\varphi_\omega|$, and R, R' are again the spectral radii of $|\mathcal{M}|$ and $|\mathcal{M}'|$ with respect to the uniform norm $\|\cdot\|_0$.

6.1. Proposition. *Proposition 1.1 and Theorems 2.1 and 2.2 remain true with the extended definitions of $\mathcal{M}, R, R'$.*

To see this it suffices to go through the original proofs of these results, making some simple changes, and using the following lemma.

6.2. Lemma. *If Φ is a continuous function with support in X, and if the second derivatives of Φ in the sense of distributions are measures on $\mathbb{C} \setminus \{a\}$ such that*

$$\int_{\mathbb{C} \setminus \{a\}} dx\, dy\, |\partial^2 \Phi| = V < \infty ,$$

then $\Phi \in \mathcal{B}$ and $\operatorname{Var}\Phi = V$.

We may take $a = 0$. For $\varepsilon > 0$ we have

$$\int dy \left| \frac{\partial \Phi}{\partial x}(\varepsilon + iy) \right| \leq \int_\varepsilon^\infty dx \int_{-\infty}^{\infty} dy \left| \frac{\partial^2 \Phi}{\partial x^2} \right| ,$$

and similarly with ε replaced by $-\varepsilon$. Therefore

$$\int_{|x|\geqq\varepsilon} dx\,dy\left(\left|\frac{\partial\Phi}{\partial x}\right|^2+\left|\frac{\partial\Phi}{\partial y}\right|^2\right)\leqq\int_{|x|\geqq\varepsilon} dx\,dy\left|\Phi\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)\right|$$

$$+\int_{x=-\varepsilon} dy\left|\Phi\frac{\partial\Phi}{\partial x}\right|+\int_{x=\varepsilon} dy\left|\Phi\frac{\partial\Phi}{\partial x}\right|\leqq 3\|\Phi\|_0 V.$$

Similarly

$$\int_{|y|\geqq\varepsilon} dx\,dy\left(\left|\frac{\partial\Phi}{\partial x}\right|^2+\left|\frac{\partial\Phi}{\partial y}\right|^2\right)\leqq 3\|\Phi\|_0 V,$$

and we conclude that

$$\int_{x+iy\neq 0} dx\,dy\left(\left|\frac{\partial\Phi}{\partial x}\right|^2+\left|\frac{\partial\Phi}{\partial y}\right|^2\right)\leqq 3\|\Phi\|_0 V. \tag{6.1}$$

Let now χ be a smooth function on $\mathbb{C}$ with values in $[0,1]$, equal to 1 in a neighborhood of 0, and to 0 for $|z|>1$. The function Φ_ε such that

$$\Phi_\varepsilon(z)=\Phi(z)(1-\chi(z/\varepsilon))+\Phi(0)\chi(z/\varepsilon)$$

tends to Φ in the sense of distributions when $\varepsilon\to 0$. Clearly $\Phi_\varepsilon\in\mathscr{B}$ and, using (6.1) we see that, for any $\delta>0$, $\operatorname{Var}\Phi_\varepsilon<V+\delta$ when ε is small enough. From this the lemma follows.

6.3. Corollary. *Let $S=\mathbb{C}\cup\{\infty\}$ be the Riemann sphere, and $\mathscr{V}$ the Banach space of functions $S\mapsto\mathbb{C}$ for which the second derivatives, in local coordinates, are measures. (We shall specify a norm on $\mathscr{V}$ later.)*

Let now $f:S\mapsto S$ be a rational function, and $\deg f\geqq 2$. We assume that the continuous function $g:S\mapsto\mathbb{C}$ is such that $|\partial^2(g\circ f^{-1})|$ is integrable on the Riemann surface associated with f^{-1}, that $z\mapsto|g(z)/(z-a)|^2$ is integrable if $a\neq\infty$ is a critical point or multiple pole of f, and that $|g(1/z)/z|^2$ is integrable near 0 if f has a critical point or multiple pole at ∞. In particular, g vanishes at the critical points and multiple poles of f.

Define $\mathscr{L}=\mathscr{V}\to\mathscr{V}$ by

$$\mathscr{L}\Phi(z)=\sum_{Z:fZ=z} g(Z)\Phi(Z)$$

and let

$$R=\lim_{m\to\infty}\max_{z\in S}\left(\sum_{Z:f^mZ=z}|g(f^{m-1}Z)\cdots g(fZ)g(Z)|\right)^{1/m}$$

$$R'=\lim_{m\to\infty}\max_{z\in S}|g(f^{m-1}z)\cdots g(fz)g(z)|^{1/m}.$$

Then, the spectral radius of $\mathscr{L}$ is $\leqq R$, and the essential spectral radius is $\leqq(RR')^{1/2}$. If $g\geqq 0$ and $R>R'$, then R is an eigenvalue of $\mathscr{L}$, and it has an eigenfunction $\Phi_R\geqq 0$.

Choose a finite open cover $(\mathcal{O}_k)_{k\in K}$ of S, holomorphic coordinates $p_k=\mathcal{O}_k\mapsto\mathbb{C}$, and a smooth partition of unity (χ_k) associated with $(\mathcal{O}_k)$. Let $\mathscr{V}$ be the space of functions $\Phi:S\mapsto\mathbb{C}$ for which the second derivatives are measures. Then $\mathscr{V}$ is a

Banach space with respect to the norm $\|\Phi\| = \sum_k \mathrm{Var}[(\chi_k \Phi)\circ p_k^{-1}]$, and different choices of $(\mathcal{O}_k), (p_k), (\chi_k)$ yield equivalent norms.

We may assume that the sets $p_k\mathcal{O}_k$ are disjoint, and let $X = \cup_k p_k\{z : \chi_k(z) > 0\}$. The space $\mathscr{B}$ is defined as earlier, and there are continuous maps $P: \mathscr{B} \mapsto \mathscr{V}$, $J: \mathscr{V} \mapsto \mathscr{B}$, defined by $P\Phi = \sum_k \Phi \circ p_k$, $J\Psi = \sum_k (\chi_k \Psi)\circ p_k^{-1}$. Since J is isometric and PJ is the identity, we may consider $\mathscr{V}$ as a subspace of $\mathscr{B}$.

For each pair (k,ℓ) let now $V_{(k,\ell)} = p_\ell(\mathcal{O}_\ell \cap f\mathcal{O}_k)$; this is a Riemann surface with projection $\pi_{(k,\ell)}$, and we may assume that it has at most one branch point. We take $\Omega = K \times K$ and, if $\omega = (k,\ell)$, we let ψ_ω be the inverse of $p_k\mathcal{O}_k \mapsto V_\omega$.

Let also

$$\varphi_\omega = (\chi_\ell \circ \pi_\omega)\cdot[(\chi_k g)\circ\psi_\omega].$$

Defining $\mathscr{M}$ as before we see that the conditions of Proposition 6.1 are satisfied. Furthermore, $\mathscr{M} = J\mathscr{L}P$ and, since $PJ = $ identity, the spectral properties of $\mathscr{L}$ follow from those of $\mathscr{M}$. (Note that here we always have $R \geqq R'$.)

6.4. Remarks.

(a) The properties required from g are satisfied for instance if

$$g(z) = \left(\frac{|f'(z)|(1+|z|^2)}{1+|f(z)|^2}\right)^\alpha G(z)$$

with $\alpha > 0$ and $G \in \mathscr{V}$.

(b) We may extend the definition of R, R' to arbitrary continuous or upper semicontinuous (u.s.c.) $g \geqq 0$. If I is the set of f-invariant probability measures on S, we have

$$R' = \exp \max_{\rho\in I} \rho(\log g).$$

(The ergodic theorem gives

$$\max_{\rho \in I} \rho(\log g) \leqq \log R'.$$

To obtain the reverse inequality, choose $m \mapsto z_m$ such that

$$\lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1} \log g(f^k z_m) = \log R'$$

and take ρ to be a weak limit of $\frac{1}{m}\sum_{k=0}^{m-1} \delta_{f^k z}$. Then $\rho \in I$ and

$$\rho(\log g) \geqq \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1} \log \rho(f^k z) = \log R').$$

Define also the pressure

$$P(\log g) = \max_{\rho\in I}(h(\rho) + \rho(\log g)), \tag{6.2}$$

where h is the entropy. We shall see that

$$R \equiv \lim_{m\to\infty} (\|\mathscr{L}^m 1\|_0)^{1/m} = \exp P(\log g). \tag{6.3}$$

According to Przytycki [9], Lemma 2 and Lemma 4, this property holds for continuous $g > 0$. To handle the general case, let (A_n) be a decreasing sequence of continuous functions tending to $\log g$. [Note that the function $\rho \mapsto h(\rho) + \rho(\log g)$ is affine u.s.c. $I \to \mathbb{R} \cup \{-\infty\}$ because h is u.s.c. [5] and $\rho \mapsto \rho(\log g)$ is the limit of the decreasing sequence of continuous functions $\rho \mapsto \rho(A_n)$. This justifies writing max in (6.2).] If ρ_n is an equilibrium state for A_n we have

$$P(A_n) = h(\rho_n) + \rho_n(A_n).$$

Taking a subsequence we may assume that $\rho_n \to \rho$ vaguely, and obtain

$$P(\log g) \leqq \lim_{n\to\infty} P(A_n) \leqq h(\rho) + \rho(A_n) \to h(\rho) + \rho(\log g) \leqq P(\log g).$$

This implies

$$\lim_{n\to\infty} P(A_n) = P(\log g). \tag{6.4}$$

Let $\mathscr{L}_n$ be the transfer operator obtained if we replace g by $\exp A_n$ in the definition of g. By Przytycki's result we have

$$\lim_{m\to\infty} (\| \mathscr{L}_n^m 1 \|_0)^{1/m} = \exp P(A_n).$$

Replacing m by 2^m, we see (by submultiplicativity) that the left-hand side is a decreasing function of m; it is also a decreasing function of n, and the limits $m \to \infty$, $n \to \infty$ may thus be exchanged. Since

$$\lim_{n\to\infty} \| \mathscr{L}_n^m 1 \|_0 = \| \mathscr{L}^m 1 \|_0,$$

we obtain

$$\lim_{n\to\infty} P(A_n) = \lim_{m\to\infty} (\| \mathscr{L}^m 1 \|_0)^{1/m} = R. \tag{6.5}$$

From (6.4) and (6.5) we obtain (6.3) as announced.

(c) It is known that f restricted to the Julia set J is *topologically mixing* ([5], Proposition 1), i.e., if $\mathcal{O}$ is an open set and $\mathcal{O} \cap J \neq \emptyset$, then $f^m \mathcal{O} \supset J$ for some m. This implies for instance that if $g \geqq 0$ and g vanishes only at a finite number of points, and if $\mathscr{L}\Phi_R = R\Phi_R$ with $\Phi_R \geqq 0$, then Φ_R can vanish only at a finite number of points on a compact neighborhood of J. In many cases the position of the zeros of g is such that necessarily $\Phi_R > 0$ on J. It then follows that R is a simple eigenvalue of $\mathscr{L}$.

Let $g = g_0 \exp \gamma$, with $\gamma \in \mathscr{B}$. If R is a simple eigenvalue of $\mathscr{L}$, then $\gamma \mapsto R$ is analytic, and the functional derivative of $\log R$ with respect to γ is $\Phi_R \mu_R$, where μ_R is the positive measure on J such that $\mathscr{L}^* \mu_R = R\mu_R$, normalized so that $\mu_R(\Phi_R) = 1$. [This is a standard argument; let $g = g_0 e^{t\gamma}$, and write $A' = dA/dt$ at $t = 0$.] We have then

$$\begin{aligned} R' &= [\mu_R(\mathscr{L}\Phi_R)]' = \mu_R'(\mathscr{L}\Phi_R) + \mu_R(\mathscr{L}'\Phi_R) + \mu_R(\mathscr{L}\Phi_R') \\ &= \mu_R(\mathscr{L}'\Phi_R) + R[\mu_R(\Phi_R)]' = R[\mu_R(\gamma\Phi_R)]. \end{aligned}$$

(d) A result similar to Corollary 6.3 holds if the map f is replaced by an (m, n)-correspondence defined by an algebraic curve (and S may be replaced by any

closed Riemann surface, i.e., we may consider an (m, n)-correspondence on an algebraic curve).
(e) The form of Conjecture 2.3 appropriate to the situation of Corollary 6.3 is the following.

Conjecture. *The power series*

$$\zeta(z)^{-1} = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{w \in \operatorname{Fix} f^m} \prod_{k=0}^{m-1} g(f^k w)$$

converges for $|z| < (RR')^{-1/2}$ *and its zeros are precisely the inverses* λ^{-1} *of the eigenvalues* λ *of* $\mathscr{L}$, *with the same multiplicities.*

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