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Stark Wannier Ladders

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Abstract. We study the Schrödinger equation for an electron in a one dimensional crystal submitted to a constant electric field. We prove the existence of ladders of resonances, the imaginary part of which is exponentially small with the field.

The Schrödinger equation for electrons in a crystal submitted to an external constant electrical field has attracted much attention [13] since it is a first step in understanding conductivity in solids. A recent review on the subject can be found in [11].

For several decades, the experimental evidence of resonance states (called also Bloch oscillators), was questioned. In fact, it was only recently, that their effect clearly appeared in the electro-optical properties of semiconductor superlattices (man-made crystals in which layers of two distinct semi-conductors alternate, the period in the perpendicular direction to the layers can be of the order of hundreds of normal lattice periods) [4, 12]. As it will be shown, resonant states live in regions whose length is proportional to the spectral band widths of the Bloch Hamiltonian and inversely proportional to the external field. So, occurrence of small energy bands near the Fermi energy, as in superlattices, favour their observation.

Mathematically, existence of resonances for the one dimensional Hamiltonian $-\frac{h^2}{2m}\frac{d^2}{dx^2} + V_P(x) + Fx$ has been rigorously proven for large external electric fields by Agler and Froese [1] in the case $V_P(x)$ is a Fourier series with a finite number of terms (F is the product of the particle charge by the electrical field). Nothing was said in this paper about the resonance widths which were expected to be exponentially small with respect to the electrical field (see the numerical treatment of the semi infinite Kronig-Penney model [2]).

In this paper, we give a new proof for the existence of the resonances, establish the link between their widths and the spectral properties of the Bloch Hamiltonian, and prove their exponential behavior. The localisation of the resonance states is understood in the scope of the tilted bands picture introduced by Zener. We shall exploit the similarity of the situation with the one which appears when there are potential barriers and shape resonances occur. In this case, the wave function, in classically forbidden regions, has exponential behavior while in the wells it oscillates. In our problem, in regions where $E-Fx$ belongs to a gap the wave function has exponential behavior, while in regions where $E-Fx$ belongs to a band, it oscillates with an amplitude which remains quite stable [see Fig. 1]. By analogy, we shall call "Zener barrier regions" the first ones and "Zener allowed regions" the last ones. In the case where the periodic potential is analytic, the number of gaps is generically infinite and the gap widths decrease very rapidly as the energy increases, so we are faced with a problem similar to the one with an infinite number of barriers whose height is decreasing.

In Part I, we describe the transformation under which the hamiltonian is converted into a non-self-adjoint operator, the eigenvalues of which are the resonances of the former problem. Subsequently, all the study will be done on this new operator.

In Part II, we use ideas borrowed from the papers of Briet-Combes-Duclos [4] and Helffer-Sjöstrand [8] on multiple wells operators and shape resonances. They introduce single well Hamiltonians obtained by "filling" all the wells, except one. Each of these operators has discrete spectrum. They, then use a formula which links the resolvent of the original Hamiltonian with the resolvents of the single well operators. Thus, they link the resonances of the original problem to the eigenvalues of the single well operators. Like them, we introduce partitions of unity, and define new operators, H_i, whose potential coincide with the initial potential only in a region, outside this region, the potential is simply the periodic one. We will also define an operator, H_{N+1}, which is the only one to be affected by the analytic transformation, and which plays a special role in our analysis.

In Part III, we will study the spectrum of the H_i $(i=1,\dots N)$, show that, to the contrary of H, the H_i have eigenvalues and that the corresponding Green functions decrease exponentially in the "Zener barrier regions," as does the Green function of the multiple well problem in the classically forbidden regions.

In Part IV, we shall prove that H_{N+1} does not have eigenvalues in some energy regions and again, that the corresponding Green function decreases exponentially. Using the formula which links the resolvent of the non-self-adjoint operator to the resolvents of the H_i, we get the resolvent expansion for the resonances. In particular, we get an upper bound for the resonance widths.

We have become acquainted with the works of Combes, Hislop [6] and Buslaev, Dimitrieva [5]. They cover different electrical field regimes and the ideas behind their proofs are different. Let us emphasize that we are not considering a multiple well problem as in the Combes-Hislop approach. In particular, even if E is larger than $V_P(x)+Fx$, we can be in a "Zener barrier region." In our paper, we are not performing strictly a semi-classical limit: some of the results are proven considering the limit $\varepsilon\to 0$ in $-\dfrac{\varepsilon^2 h_0^2}{2m}\dfrac{d^2}{dx^2}+V_P(x)+\varepsilon^r F_0 x$, $r>1$ that is taking simultaneously $h\to 0$ and $F\to 0$ in $-\dfrac{h^2}{2m}\dfrac{d^2}{dx^2}+V_P(x)+Fx$, in such a way $\dfrac{F}{h}\to 0$. Notice that if h is sufficiently small, F can be taken arbitrarily small. The choice, $r\geqq 1$, has been done in order to get large "Zener barrier regions" for ε small enough, since their width is proportional to the gaps, which are, in the energy regions of interest of order ε or $\varepsilon(-\log\varepsilon)^{-1}$, and inversely proportional to F.

In Parts I, II, III the discussion is independent of ε. To simplify the notations we will write:

$$H = -\frac{d^2}{dx^2} + V_P(x) + Fx.$$

Hypothesis. $V_P(x)$ is a periodic potential (of period a), symmetric about the origin, analytic in the strip $|\mathrm{Im}\, z| < A$, and for some $E_0 > V_M$ [maximum value of $V_P(x)$] and *all E satisfying $E_0 > E > V_M$, $V(x) = E$ has two simple roots $iy(E)$ and $-iy(E)$*, ($y(E) \in \mathbb{R}$), which are closer to the real axis than any other roots. (H.1)

I. Local Deformation

We construct an analytic family of operators using the following space transformation:

$$t_b : x \in \mathbb{R} \to t_b(x) = x + ibf(x); \qquad 0 < b < A,$$

where f is a real C^3 function whose graph is represented below

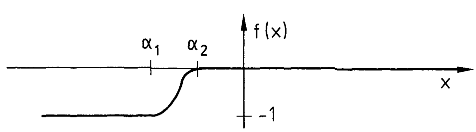


Fig. 1

It is constant outside interval $[\alpha_1, \alpha_2]$ which will be made precise later. We define a transformation U_b on $L^2(R)$ by:

$$U_b : g(x) \to (U_b g)(x) = \sqrt{1 + ibf'(x)}\, g(x + ibf(x)).$$

Under this transformation our hamiltonian $H = -\frac{d^2}{dx^2} + V_P(x) + Fx$ becomes:

$$H(b) := U_b H U_b^{-1} = \frac{1}{1+ibf'(x)}\left(-\frac{d^2}{dx^2}\right)\frac{1}{1+ibf'(x)} + V_p(x + ibf(x))$$

$$+ F(x + ibf(x)) + \frac{1}{1+ibf'(x)}\, S_b(x)\, \frac{1}{1+ibf'(x)},$$

where $S_b(x)$ is the Schwarzian: $S_b(x) = \frac{1}{2}\left[\frac{t_b'''}{t_b'} - \frac{3}{2}\left(\frac{t_b''}{t_b'}\right)^2\right]$.

Remark. If the support of ϕ is included in $[\alpha_2, +\infty)$ then

$$(H(b)\phi)(x) = -\frac{d^2}{dx^2}\phi(x) + V_p(x)\phi(x) + Fx\phi(x).$$

$H(b)$ is a non-self-adjoint operator whose eigenvalues correspond to the resonances of the original operator.

II. Partition

As was mentioned in the introduction, the idea behind the partition we adopt, is based on the Zener picture of the tilted bands. If F is small, it was believed since Zener that on some small interval centered at x_i, Fx could be approximated by Fx_i. Then, locally, solutions of the differential equation $\left(-\frac{d^2}{dx^2}+V_P(x)+Fx\right)\phi=E\phi$ could be well approximated by a linear combination of Bloch waves corresponding to the energy $E-Fx_i$, called effective energy.

Recall that Bloch waves are solutions of equation $H_B\psi_\pm=\left(-\frac{d^2}{dx^2}+V_P(x)\right)\psi_\pm=E\psi_\pm$ with the property: $\psi_\pm(x+a)=e^{\pm ik(E)a}\psi_\pm(x)$. If we call $\Delta(E)$ the trace of the monodromy matrix (see a more complete discussion in Part III), $k(E)$ is given, modulo $\frac{2\pi}{a}$, by $2\cos k(E)a=\Delta(E)$. Consider real E, if $-2<\Delta(E)<2$, $k(E)$ is real and E belongs to the spectrum of H_B; if $\Delta(E)>2$, $k(E)=i\kappa(E)$ with $\kappa(E)\in\mathbb{R}^+$; if $\Delta(E)<-2$, $k(E)=\frac{\pi}{a}+i\kappa(E)$ with $\kappa(E)\in\mathbb{R}^+$, in these last two cases E belongs to the resolvent set. The spectrum is constituted generically by an infinite number of intervals called bands, separated by intervals (E_i, E_i') of width Γ_i, $i=1,2,\ldots$ called gaps. E_0' will denote the infimum of the spectrum.

If F is small, it was believed that the solution of $\left(-\frac{d^2}{dx^2}+V_P(x)+Fx\right)\phi=E\phi$, at points spaced by a, or they oscillate with an amplitude which is nearly constant if the "effective energy," $E-Fx$ belongs to a band or they behave exponentially, if the "effective energy" belongs to a gap. This belief has been confirmed by numerical computations; Fig. 2 below gives an example. Curves represent the real part of solutions of the Schrödinger equation corresponding to different values of E.

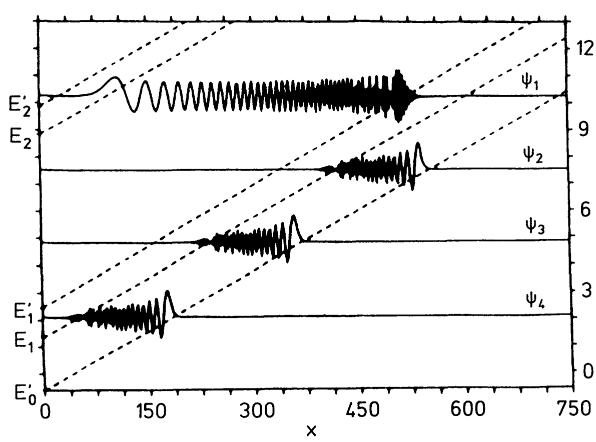


Fig. 2

To define the unity partitions we need first to define some intervals on $\mathbb{R}$. We denote by N_0 the number of bands entirely inside (V_m, V_M), where V_m, V_M are respectively the minimum and maximum value of $V_p(x)$. We take now: $N=N_0+1$.

We denote by $W_p(x)$, the saw-tooth-function: $W_p(x)=x$ if $0 \leqq x \leqq a$, $W_p(x+a) = W_p(x)$ and define,

$$\tilde{H}_B = -\frac{d^2}{dx^2} + V_p(x) + F W_p(x) = -\frac{d^2}{dx^2} + \tilde{V}_p(x).$$

To simplify notations concerning spectral values for $\tilde{H}_B$ we shall forget the $\sim$, for instance we shall denote (E_i, E_i') the i^{th} spectral gap interval for $\tilde{H}_B$. We denote by E_i^m, $i=1,2\ldots$ the values of E for which the derivative of the discriminant, $\Delta(E)$, is zero (point E_i^m is near the middle of the i^{th} gap) and $\kappa_i^m := \kappa(E_i^m)$.

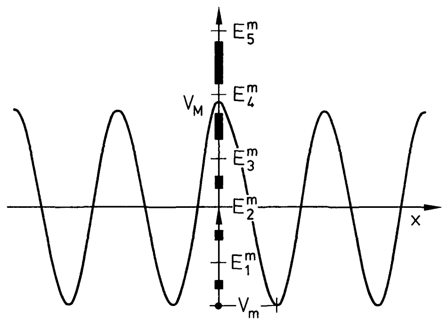


Fig. 3

We denote by $[x]$ the entire part of real number x, and define:

$$\mu_i' = \left[\frac{E_N^m - E_i'}{Fa}\right], \quad \mu_i = \left[\frac{E_N^m - E_i}{Fa}\right] \quad i=1,\ldots,N$$

(notice that $\mu_N' < 0 < \mu_N < \ldots < \mu_1' < \mu_1 < \mu_0'$, see Fig. 4)

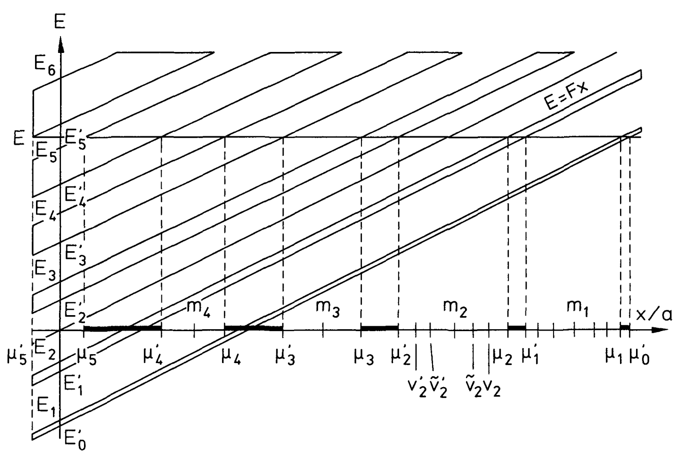


Fig. 4

Tilted bands are framed.

$$m_i = \left[\frac{E_N^m - E_i^m}{Fa}\right].$$

We denote by:

$$\mathscr{E}_i, \mathscr{E}'_i, \quad \text{the values for which,} \quad \kappa(\mathscr{E}_i) = \kappa(\mathscr{E}'_i) = \frac{1}{\sqrt{3}}\,\kappa_i^m,$$

$$\tilde{\mathscr{E}}_i, \tilde{\mathscr{E}}'_i, \quad \text{the values for which,} \quad \kappa(\tilde{\mathscr{E}}_i) = \kappa(\tilde{\mathscr{E}}'_i) = \frac{1}{\sqrt{2}}\,\kappa_i^m,$$

and such that $\mathscr{E}_i < \tilde{\mathscr{E}}_i < E_i^m < \tilde{\mathscr{E}}'_i < \mathscr{E}'_i$.

We define:

$$v_i = \left[\frac{E_N^m - \mathscr{E}_i}{Fa}\right], \quad v'_i = \left[\frac{E_N^m - \mathscr{E}'_i}{Fa}\right],$$

$$\tilde{v}_i = \left[\frac{E_N^m - \tilde{\mathscr{E}}_i}{Fa}\right], \quad \tilde{v}'_i = \left[\frac{E_N^m - \tilde{\mathscr{E}}'_i}{Fa}\right].$$

If $E < E'_0$, $\frac{d\Delta(E)}{dE} < 0$ and $\Delta(E) \to \infty$ as $E \to \infty$, so for $i = 0$, we need special definitions. We define E_0^m such that $\Delta(E_0^m) = \Delta(E_1^m)$, then v'_0, $\tilde{v}'_0$, m_0 will be defined as before, while $\tilde{v}_0$, v_0 are the symmetric of $\tilde{v}'_0$ and v'_0 with respect to m_0.

Notice that:

$$\begin{aligned} &\mu'_N < v'_N < \tilde{v}'_N < m_N = 0 < \tilde{v}_N < v_N < \mu_N < \dots \\ &\qquad < \mu'_i < v'_i < \tilde{v}'_i < m_i < \tilde{v}_i < v_i < \mu_i < \dots \\ &\qquad < \mu'_0 < v'_0 < \tilde{v}'_0 < m_0 < \tilde{v}_0 < v_0 \quad \text{(see Fig. 4, where } N = 5). \end{aligned}$$

Let $\chi(I)$ be the characteristic function of interval I.

Define the following set of operators whose potential coincide, in some intervals, with the potential in $H(b)$:

$$H_0 = -\frac{d^2}{dx^2} + \tilde{V}_p(x) + \chi(-\infty, v'_0 a) F v'_0 a + \chi(v'_0 a, +\infty) F[x],$$

$$\begin{aligned} H_i = &-\frac{d^2}{dx^2} + \tilde{V}_p(x) + \chi(-\infty, v'_i a) F v'_i a + \chi(v'_i a, v_{i-1} a) F[x] \\ &+ \chi(v_{i-1} a, +\infty) F v_{i-1} a \quad \text{for} \quad i = 1, \dots, N. \end{aligned}$$

$$\begin{aligned} H_{N+1} = &\frac{1}{1 + ibf'(x)}\left(-\frac{d^2}{dx^2}\right)\frac{1}{1 + ibf'(x)} + \frac{1}{1 + ibf'(x)} W_b(x) \frac{1}{1 + ibf'(x)} \\ &+ \chi(-\infty, v_N a) \cdot [V_p(x + ibf(x)) + F(x + ibf(x))] \\ &+ \chi(v_N a, +\infty)(\tilde{V}_p(x) + F v_N a). \end{aligned}$$

In the same spirit as the method proposed, for the shape resonances, by Briet, Combes, and Duclos [4] and Helffer and Sjöstrand [8], we want to study the resolvent of $H(b)$ in terms of the resolvents of H_i ($i = 0, 1, \dots, N+1$).

We define a partition of unity in the following manner:

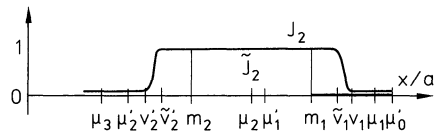
$\tilde{J}_{N+1}$ is the characteristic function of $(-\infty, m_N = 0)$,

$\tilde{J}_i$ is the characteristic function of (m_i, m_{i-1}),

$\tilde{J}_0$ is the characteristic function of $(m_0, +\infty)$.

Another set of functions, named J_i, are defined in the following manner: J_i is a C_0^∞ function which takes the value 1 on $(\tilde{v}_i', a, \tilde{v}_{i-1}a)$, its support is $(v_i'a, v_{i-1}a)$ for $i=1$ to N. The support of J_{N+1} is $(-\infty, v_N a)$, it takes the value 1 on $(-\infty, \tilde{v}_N a)$. We impose the following conditions on their derivatives: $\frac{d^n J_i}{dx^n} < \alpha_n \left(\frac{F}{\Gamma_i}\right)^n$, $n=1,2$ (α_n are constants).

Fig. 5



Now, we establish the link between the resolvent of $H(b)$ and the resolvents of H_i. Let us denote:

$$R_i = (H_i - z)^{-1}, \qquad \tilde{R} = \sum_{i=0}^{N+1} J_i R_i \tilde{J}_i$$

then:

$$(H(b)-z)\tilde{R} = \sum_{i=0}^{N+1} J_i (H(b)-z) R_i \tilde{J}_i + \sum_{i=0}^{N+1} \left[-\frac{d^2}{dx^2}, J_i \right] R_i \tilde{J}_i .$$

As $H(b)$ and H_i coincide on the support of $J_i : J_i(H(b)-z) = J_i(H_i - z)$, using $\sum_{i=0}^{N+1} J_i \tilde{J}_i = 1$ we get:

$$(H(b)-z)\tilde{R} = 1 + \sum_{i=0}^{N+1} \left[-\frac{d^2}{dx^2}, J_i \right] R_i \tilde{J}_i .$$

Denoting: $M_i = \left[-\frac{d^2}{dx^2}, J_i \right]$ and $K_i = M_i R_i \tilde{J}_i$, we obtain:

$$R(b) = (H(b)-z)^{-1} = \left(\sum_{i=0}^{N+1} J_i R_i \tilde{J}_i \right) \left(1 + \sum_{i=0}^{N+1} K_i \right)^{-1} . \tag{II.1}$$

To prove that $H(b)$ eigenvalues are at distance $e^{-\alpha/F}$ from the eigenvalues of H_i we need to prove that the $\left\| \sum_{i=0}^{N+1} K_i \right\|$ becomes smaller than 1, as z becomes distant from eigenvalues of H_i by a quantity which is exponentially small with respect to F.

The kernel of K_i is:

$$-\frac{d^2 J_i}{dx^2} G_i(x, y; z) \tilde{J}_i(y) - 2 \frac{dJ_i}{dx} \frac{dG_i}{dx} (x, y; z) \tilde{J}_i(y) .$$

In the subsequent section we shall prove that $G_i(x, y; z)$ and $\frac{dG_i}{dx}(x, y; z)$, when x and y belong to the same ZBR, contain a term which is exponentially small with respect to F.

III. Study of the H_i and K_i

First we will consider the spectra of operators H_i for $i=1$ to N. Afterwards we will consider the operators H_{N+1} and H_0, which have a distinct definition and play a distinct role.

The potential term in H_i is constituted of three parts, it is equal to: $\tilde{V}_p(x)+Fv_i'a$, on interval $(-\infty, v_i'a)$, it is equal to $\tilde{V}_p(x)+F[x]$ on interval $(v_i'a, v_{i-1}a)$ and it is equal to $\tilde{V}_p(x)+Fv_{i-1}a$, on interval $(v_{i-1}a, +\infty)$.

Intervals $(E_j'+Fv_i'a, E_{j+1}+Fv_i'a)$, $j=0,\dots,\infty$ are contained in the continuous part of the spectrum of H_i, since continuous part of the spectrum of the operator:

$$H_i^{\mathrm{left}} = -\frac{d^2}{dx^2}+\tilde{V}_p(x)+Fv_i'a,$$

defined on $L^2(-\infty, v_i'a)$, with Dirichlet condition at $v_i'a$ is:

$$\bigcup_{j=0}^{\infty}[E_j'+Fv_i'a, E_{j+1}+Fv_i'a].$$

Similarly $(E_j'+Fv_{i-1}a,\ E_{j+1}+Fv_{i-1}a)$, $j=0,\dots,\infty$ are also contained in the continuous part of the spectrum of H_i since the continuous part of the spectrum of

$$H_i^{\mathrm{right}} = -\frac{d^2}{dx^2}+\tilde{V}_p(x)+Fv_{i-1}a,$$

defined on $L^2(v_{i-1}a, +\infty)$ with Dirichlet condition at $v_{i-1}a$ is:

$$\bigcup_{j=0}^{\infty}[E_j'+Fv_{i-1}a, E_{j+1}+Fv_{i-1}a].$$

The choice of v's has been done in such a way that:

$$A_i=\left\{\bigcup_{j=0}^{\infty}[E_j'+Fv_i'a, E_{j+1}+Fv_i'a]\right\}\cup\left\{\bigcup_{j=0}^{\infty}[E_j'+Fv_{i-1}a, E_{j+1}+Fv_{i-1}a]\right\}$$

is not all $\mathbb{R}$, in particular, interval $I_i=[E_N^m-(\mathscr{E}_{i-1}-E_{i-1}),\ E_N^m+(E_i'-\mathscr{E}_i')]$ is included in the complement of A_i. Furthermore by construction, $I=\bigcap_{i=1}^{N} I_i$ is non-void.

Now we look at the spectrum of H_i in interval I.

Proposition 1. *In interval I, the spectra of H_i $(i=1,\dots,N)$ is composed of eigenvalues spaced by $Fa+0(F^2)$.*

If $v_i'a<x<y<v_ia$ and $E\in I$ the Green function corresponding to H_i satisfies:

$$|G_i(x,y;E)|\leqq c_i\frac{e^{-a\sum\limits_{j=\{x\}}^{\{y\}}\kappa(E-Fai)}}{\operatorname{dist}(E,\sigma(H_i))}\leqq c_i\frac{e^{-\frac{1}{\sqrt{3}}\kappa_i^m|x-y|}}{\operatorname{dist}(E,\sigma(H_i))}$$

$$\left|\frac{dG_i}{dx}(x,y;z)\right|\leqq c_i'\kappa_i^m\frac{e^{-\frac{1}{\sqrt{3}}\kappa_i^m|x-y|}}{\operatorname{dist}(E,\sigma(H_i))},$$

where $\{x\}$ denotes $\left[\frac{x}{a}\right]$, c_i and c_i' are constants independent of F.

Proof. Take E inside I_i, the Green function is given by:

$$G_i(x,y;E)=\frac{\psi^+(x)\psi^-(y)}{W(\psi^-;\psi^+)},\quad \text{if}\quad x<y,$$

where $\psi^+\in L^2(-\infty;0)$, $\psi^-\in L^2(0;+\infty)$ and satisfy $H_i\psi^\pm=E\psi^\pm$.

In each interval $[(j-1)a,ja]$, ψ^+ and ψ^- will be expressed as linear combinations of functions ϕ_j^1 which satisfy:

$$\phi_j^1[(j-1)a]=1,\quad \frac{d\phi_j^1}{dx}[(j-1)a]=0,$$

$$\phi_j^2[(j-1)a]=0,\quad \frac{d\phi_j^2}{dx}[(j-1)a]=1.$$

and are solutions of:

$$\left(-\frac{d^2}{dx^2}+\tilde{V}_p(x)+Faj\right)\phi=E\phi \qquad \text{if}\quad v_i'<\mathrm{j}<v_i,$$

$$\left(-\frac{d^2}{dx^2}+\tilde{V}_p(x)+Fv_i'a\right)\phi=E\phi \qquad \text{if}\quad j<v_i',$$

$$\left(-\frac{d^2}{dx^2}+\tilde{V}_p(x)+Fv_{i-1}a\right)\phi=E\phi \quad \text{if}\quad j>v_i.$$

Denoting:

$$\psi^\pm=a_j^\pm\phi_j^1+b_j^\pm\phi_j^2 \tag{III.2}$$

and:

$$A_j=\phi_j^1(ja),\quad C_j=\frac{d\phi_j^1}{dx}(ja),\quad B_j=\phi_j^2(ja),\quad D_j=\frac{d\phi_j^2}{dx}(ja),$$

it is easy to show that:

$$\begin{pmatrix}a_j\\ b_j\end{pmatrix}=\begin{pmatrix}A_{j-1} & B_{j-1}\\ C_{j-1} & D_{j-1}\end{pmatrix}\begin{pmatrix}a_{j-1}\\ b_{j-1}\end{pmatrix}\equiv\mathbb{M}_{j-1}\begin{pmatrix}a_{j-1}\\ b_{j-1}\end{pmatrix}.$$

We shall denote by $\mathbb{M}_{j-1}$ the monodromy matrix $\begin{pmatrix}A_{j-1} & B_{j-1}\\ C_{j-1} & D_{j-1}\end{pmatrix}$, its determinant is equal to 1.

If $j\leqq v_i'$, $\mathbb{M}_j$ is constant and equal to $\mathbb{M}_{v_i'}$, its eigenvalues are $e^{\pm ik(E-Fv_i'a).a}$, where k satisfies $2\cos k(E-Fv_i'a).a=\operatorname{Tr}\mathbb{M}_{v_i'}$ ($\operatorname{Tr}\mathbb{M}_{v_i'}$, called discriminant, was denoted Δ in paragraph II). Since by construction, $E-Fv_i'a$ belongs to a gap, $\operatorname{Im}k(E-Fv_i'a)\neq 0$. Then, on $(-\infty, v_i'a)$, the solution ψ^+, which decreases at $-\infty$, is a Bloch wave and satisfies:

$$\psi^+((v_i'+j)a)=e^{-ik(E-Fv_i'a)ja}\psi^+(v_i'a),\qquad (\operatorname{Im}k>0,\, j<0).$$

We normalize it taking $\psi^+(v_i'a)=1$.

If $v_i'<j<v_{i-1}$, eigenvalues of $\mathbb{M}_j$ are $e^{\pm ik(E-Fja).a}$ where k satisfies: $2\cos k(E-Fja).a=\operatorname{Tr}\mathbb{M}_j$.

If $v_i'<j\leqq\mu_i$, $E-Fja$ belongs to a gap, so $\operatorname{Im}k(E-Fja)\neq 0$.

If $\mu_i<j\leqq\mu_{i-1}'$, $E-Fja$ belongs to a gap, so $\operatorname{Im}k(E-Fja)=0$.

If $\mu_{i-1}'<j\leqq v_{i-1}$, $E-Fja$ belongs to a gap, so $\operatorname{Im}k(E-Fja)\neq 0$.

If $j>v_{i-1}$, $\mathbb{M}_j$ is constant, as $E-Fv_{i-1}a$ belongs to a gap, $\operatorname{Im}k(E-Fv_{i-1}a)\neq 0$.

Then, on $(v_{i-1}a, +\infty)$, the solution ψ^-, which decreases at $+\infty$, is a Bloch wave and satisfies:

$$\psi^-(v_{i-1}+j)a)=e^{+ik(E-Fv_{i-1}a)ja}\psi^-(v_{i-1}a), \qquad (\operatorname{Im}k>0, j>0).$$

We normalize it taking $\psi^-(v_{i-1}a)=1$.

In Appendix A1 we diagonalize $\mathbb{M}_j$: $\mathbb{M}_j=\mathbb{S}_j^{-1}\mathbb{D}_j\mathbb{S}_j$ and show that $\begin{pmatrix} u_j^+ \\ v_j^+ \end{pmatrix}$: $=\mathbb{S}_j\begin{pmatrix} a_j^+ \\ b_j^+ \end{pmatrix}$ increases exponentially as j goes from v_i' to v_i, while, $\frac{v_j^+}{u_j^+}$ remains small, so the direction of vector $\begin{pmatrix} a_j^+ \\ b_j^+ \end{pmatrix}$ remains approximately constant and furthermore does not depend on E.

The behaviour of ψ^- in the region $(v_i'a, v_ia)$ is more intricate as we have to start on the right of $v_{i-1}'a$ with the Bloch wave which decreases at $+\infty$, then going to the left of $\mu_{i-1}'a$, we cross a region where the eigenvalues of $\mathbb{M}_j$ are purely imaginary.

In Appendix A1 we show that, if $\left|\frac{A_j+D_j}{2}\right|<1$, one can write $\mathbb{M}_j^{-1}=\mathbb{T}_j\mathbb{R}_\theta\mathbb{T}_j^{-1}$, where $\mathbb{R}_\theta$ is the rotation matrix corresponding to angle $\theta=k(E-Fja).a$, and the vector $\begin{pmatrix} a_j^- \\ b_j^- \end{pmatrix}$ essentially rotates by an angle $\theta=k(E-Fja).a$ every time we apply $\mathbb{M}_j^{-1}$. $k(E-Fja).a$ goes from 0 to π as j goes from μ_{i-1}' to μ_i. Furthermore when we vary E by the quantity Fa the total angle the vector rotates varies by a quantity near π. This means that in an interval of length Fa, exists a value for E such that vectors $\begin{pmatrix} a_j^- \\ b_j^- \end{pmatrix}$ and $\begin{pmatrix} a_j^+ \\ b_j^+ \end{pmatrix}$ get the same direction, then, ψ^- and ψ^+ become proportional. This value is an eigenvalue of H_i.

Replacing in (III.1) $\psi^\pm$ by the expressions (III.2) we get for $x<y$,

$$G(x;y;E)=\frac{[a_{\{x\}}^+\phi_{\{x\}}^1(x)+b_{\{x\}}^+\phi_{\{x\}}^2(x)]\,[a_{\{y\}}^-\phi_{\{y\}}^1(y)+b_{\{y\}}^-\phi_{\{y\}}^2(y)]}{a_{\{y\}}^+b_{\{y\}}^- - a_{\{y\}}^-a_{\{y\}}^+}$$

and:

$$G([x]a;[y]a;E)=\frac{a_{\{x\}}^+a_{\{y\}}^-}{a_{\{y\}}^+b_{\{y\}}^- - a_{\{y\}}^-b_{\{y\}}^+}=\frac{\frac{a_{\{x\}}^+}{a_{\{y\}}^+}}{\frac{b_{\{y\}}^-}{a_{\{y\}}^-}-\frac{b_{\{y\}}^+}{a_{\{y\}}^+}}.$$

In this expression the denominator is an analytic function of E whose zeros E_i are spaced by $Fa+O(F^2)$. For E close to E_i, it has a lower bound of the form: $\beta|E-E_i|$ for E in a neighborhood of E_i, (β increases as F decreases).

Using the fact that a^+ increases exponentially on the interval $(v_i'a, v_ia)$, one gets, if x and y belong to it, that the numerator behaves like $e^{-a\sum_{j=\{x\}}^{\{y\}}\kappa(E-Fai)}$. The same is true for $\frac{d}{dx}G_i(x,y;E)$. Q.E.D.

Now we look for bounds on the norm of $K_i=M_iR_i\tilde{J}_i$. Let us remark that the supports of M_i and $\tilde{J}_i$ are disjoint, but support of $K_i=M_iR_i\tilde{J}_i$ is not entirely contained in a gap, so we cannot use directly Proposition 1. The natural way would be to control the behavior of ψ^+ in the "Zener allowed region." Unfortunately we dont know how to do that and are obliged to introduce a new "decomposition" of H_i.

Let us introduce:

I_i^+ is a C_0^∞ function whose support is $(-\infty, v_i a)$ and takes value 1 on $(-\infty, \tilde{v}_i a)$.

I_i^0 is a C_0^∞ function whose support is $(\tilde{v}_i a, \tilde{v}'_{i-1} a)$ and takes value 1 on $(v_i a, v'_{i-1} a)$.

I_i^- is a C_0^∞ function whose support is $(v'_{i-1} \mathrm{a}, +\infty)$ and takes value 1 on $(\tilde{v}'_{i-1} a, +\infty)$.

We impose also:

$$(I_i^+)^2 + (I_i^0)^2 + (I_i^-)^2 = 1 . \tag{III.3}$$

Now define the operators:

$$H_i^+ = -\frac{d^2}{dx^2} + \tilde{V}_p(x) + \chi(-\infty, v'_i a) F v'_i a + \chi(v'_i a, v_i a) F[x] + \chi(v_i a, \infty) F v_i a ,$$

$$\begin{aligned} H_i^0 = -\frac{d^2}{dx^2} &+ \tilde{V}_p(x) + \chi(-\infty, \tilde{v}_i a) F \tilde{v}_i a + \chi(\tilde{v}_i a, \tilde{v}'_{i-1} a) F[x] \\ &+ \chi(\tilde{v}'_{i-1} a, \infty) F \tilde{v}'_{i-1} , \end{aligned}$$

$$\begin{aligned} H_i^- = -\frac{d^2}{dx^2} &+ \tilde{V}_p(x) + \chi(-\infty, v'_{i-1} a) F v'_{i-1} a + \chi(v'_{i-1} a, v_{i-1} a) F[x] \\ &+ \chi(v_{i-1} a, \infty) F v_{i-1} a . \end{aligned}$$

H_i^s coincide with H_i on the support of I_i^s, $s = -+, 0, -$.

Denoting:

$$R_i^s = (H_i^s - E)^{-1} , \qquad s = +, 0, - ,$$

we get:

$$\begin{aligned} &[I_i^+ R_i^+ I_i^+ + I_i^0 R_i^0 I_i^0 + I_i^- R_i^- I^-] (H_i - E) \\ &= (I_i^+)^2 + (I_i^0)^2 + (I_i^-)^2 - I_i^+ R_i^+ [H_i, I_i^+] \\ &- I_i^+ R_i^+ [H_i^0, I_i^0] - I_i^- R_i^- [H_i^-, I_i^-] . \end{aligned}$$

Denoting:

$$N_i^+ = [H_i, I_i^+] , \qquad N_i^0 = [H_i^0, I_i^0] , \qquad N_i^- = [H_i^-, I_i^-] ,$$

we obtain:

$$\begin{aligned} R_i = & I_i^+ R_i^+ I_i^+ + I_i^0 R_i^0 I_i^0 + I_i^- R_i^- I_i^- \\ & + I_i^+ R_i^+ N_i^+ R_i + I_i^0 R_i^0 N_i^0 R_i + I_i^- R_i^- N_i^- R_i \end{aligned}$$

and:

$$\begin{aligned} K_i = M_i R_i \tilde{J}_i = & M_i I_i^+ R_i^+ I_i^+ \tilde{J}_i + M_i I_i^- R_i^- I_i^- \tilde{J}_i \\ & - M_i I_i^+ R_i^+ N_i^+ R_i \tilde{J}_i - M_i I_i^- R_i^- N_i^- R_i \tilde{J}_i . \end{aligned}$$

Introduce χ_i^+ the characteristic function of $(\tilde{v}_i a, v_i a)$ which is the support of N_i^+ and χ_i^- the characteristic function of $(v'_{i-1} a, \tilde{v}'_{i-1} a)$ which is the support of N_i^-. Then:

$$\begin{aligned} \|K_i\| \leqq & \|M_i I_i^+ R_i^+ I_i^+ \tilde{J}_i\| + \|M_i I_i^+ R_i^+ \chi_i^+\| \, \|N_i^+ R_i\| \\ & + \|M_i I_i^- R_i^- I_i^- \tilde{J}_i\| + \|M_i I_i^- R_i^- \chi_i^-\| \, \|N_i^- R_i\| . \end{aligned}$$

Let us consider the first term, $K_i^+ := M_i I_i^+ R_i^+ I_i^+ \tilde{J}_i$ and estimate its norm by:

$$\|K_i^+\| < \sup_x \int |K_i^+(x,y)|dy + \sup_y \int |K_i^+(x,y)|dx .$$

As $\frac{dJ_i}{dx} \cdot \frac{dI_i^+}{dx} = 0$ its kernel, $K_i^+(x,y)$ is:

$$\left\{ -\frac{d^2 J_i}{dx^2} I_i^+(x) G_i^+(x,y;E) - 2\frac{dJ_i}{dx} I_i^+(x) \frac{dG_i^+}{dx}(x,y;E) \right\} I_i^+(y)\tilde{J}_i(y) .$$

Supports of $\frac{dJ_i}{dx} I_i^+$ or $\frac{d^2 J_i}{dx^2} I_i^+$ and $I_i^+ \tilde{J}_i$ are in the same ZBR and are disjoint. Using $\frac{d^n J_i}{dx^n} < \alpha_n \left(\frac{F}{\Gamma_i}\right)^n$, $n=1,2$ we remain only with the problem of the estimates of $G_i^+(x,y;E)$ and $\frac{dG_i^+}{dx}(x,y;E)$ with x and y separated by a distance, larger than:

$$l_i := \operatorname{dist}\left(\operatorname{supp} \frac{dJ_i}{dx} I_i^+, \operatorname{supp} I_i^+ \tilde{J}_i\right) = (m_i - \tilde{v}_i')a .$$

The estimate is obtained as in Proposition 1, except, it can easily be seen that, H_i^+ has no eigenvalues in a neighborhood of E_N^m. So we get:

$$\|K_i^+\| < \left[\frac{1}{W_{1i}} \kappa_i^m \frac{F}{\Gamma_i} + \frac{1}{W_{i2}} \frac{F^2}{\Gamma_i^2}\right] e^{-\frac{\kappa_i^m l_i}{\sqrt{3}}} ,$$

where W_{i1} and W_{i2} are two constant energies.

The second term in $\|K_i\|$ contains $\|M_i I_i^+ R_i^+ \chi_i^+\|$ which is of the same form as the previous one and term $\|N_i^+ R_i\|$ whose bound is found using the fact $\frac{d^2}{dx^2}$ is relatively bounded with respect to H_i. So this term is bounded by:

$$\left[\frac{1}{W_{i3}} \frac{1}{a^2} + \frac{1}{W_{i4}} \frac{F}{\Gamma_i a} + \frac{1}{\operatorname{dist}(E;\sigma(H_i))} \left(c_{i1} \frac{1}{a^2} + c_{i2} \frac{F}{\Gamma_{ia}} + c_{i3} \frac{F^2}{\Gamma_{i_i}^2}\right)\right] e^{-\frac{\kappa_i^m l_i}{\sqrt{3}}} ,$$

where W_{i3}, W_{i4} are constant energies and c_{i1}, c_{i2}, c_{i3} are constants.

The following terms in $\|K_i\|$ are estimated in the same way. Introducing,

$$l_{i-1} := \operatorname{dist}\left(\operatorname{supp} \frac{dJ_i}{dx} I_i^-, \operatorname{supp} I_i^- \tilde{J}_i\right) = (\tilde{v}_{i-1} - m_{i-1})a$$

finally we get:

Proposition 2. *Two polynomials* P_i^1, P_i^2 *of degree four exist in* $\frac{F}{\Gamma_i}$ *whose terms of degree zero are absent and whose coefficients all include* $\frac{1}{\operatorname{dist}(E;\sigma(H_i))}$, *and such that,*

$$\|K_i(E)\| \leq P_i^1\left(\frac{F}{\Gamma_i}\right) e^{-\frac{\kappa_i^m l_i}{\sqrt{3}}} + P_i^2\left(\frac{F}{\Gamma_i}\right) e^{-\frac{\kappa_{i-1}^m l_{i-1}}{\sqrt{3}}} , \tag{III.4}$$

where $l_i = (m_i - \tilde{v}_i')a$ *and* $l_{i-1} = (\tilde{v}_{i-1} - m_{i-1})a$.

Remark. l_i and l_{i-1} are proportional to $\frac{\Gamma_i}{F}$ so $\|K_i(E)\|$ becomes exponentially small as F goes to 0.

IV. Study of H_{N+1} and K_{N+1}

We study now the non-self-adjoint operator H_{N+1} and in particular, its resolvent in a narrow rectangular subset of the complex plane which contains E_N^m. The aim is to prove that H_{N+1} has no spectrum in this domain, and so, controlling K_{N+1} to prove that resonances appear as perturbations of the H_i eigenvalues ($i=1, \ldots, N$).

We use once more the technique described in Part III.

α_1, α_2 which enter in the definition of function f (see Fig. 1) are chosen in this way: $\alpha_1 < \alpha_2 < \mu'_N a$.

To study $R_{N+1} = (H_{N+1} - E)^{-1}$ we introduce again a new partition of unity: I_{N+1}^0 and I_{N+1}^- are C_0^∞ functions which take value 1 respectively on $(-\infty, \tilde{v}_N a)$ and $(v_N a, +\infty)$ and whose supports are respectively $(-\infty, v_N a)$ and $(\tilde{v}_N a, +\infty)$ and such that $(I_{N+1}^0)^2 + (I_{N+1}^-)^2 = 1$.

Now define the operators:

$$H_{N+1}^0 = \frac{1}{1+ibf'(x)} \left(-\varepsilon^2 \frac{h_0^2}{2m} \frac{d^2}{dx^2} \right) \frac{1}{1+bf'(x)} + V_p(x+ibf(x))$$

$$+ \varepsilon^r F_0(x+ibf(x)) + \frac{1}{1+ibf'(x)} S_b(x) \frac{1}{1+ibf'(x)},$$

$$H_{N+1}^- = -\varepsilon^2 \frac{h_0^2}{2m} \frac{d^2}{dx^2} + \tilde{V}_p(x) + \chi(-\infty, v'_N a)\varepsilon^r F_0 v'_N a$$

$$+ \chi[v'_N a, v_N a]\varepsilon^r F_0[x] + \chi(v_N a, \infty)\varepsilon^r F_0 v_N a\,.$$

H_{N+1}^- coincide with H_{N+1} on the support of I_{N+1}^-.

Denoting $R_{N+1}^s = (H_{N+1}^s - E)^{-1}$, $s = 0, -$, and doing again the same calculus as in Part III where i is replaced by $N+1$, we obtain:

$$\|K_{N+1}\| \leqq \|M_{N+1} I_{N+1}^- R_{N+1}^- I_{N+1}^- \tilde{J}_{N+1}\|$$
$$+ \|M_{N+1} I_{N+1}^- R_{N+1}^- \chi_{N+1}^-\| \, \|N_{N+1}^- R_{N+1}\|\,.$$

H_{N+1}^- has no eigenvalues at least in a neighborhood of E_N^m.

Using the same technique as in Proposition 2, we get the same kind of result:

$$\|K_{N+1}(E)\| \leqq \left[P_{N+1}\left(\frac{F}{\Gamma_N}\right)\right] e^{-\frac{\kappa_N^m l_N}{\sqrt{3}}},$$

where P_{N+1} is a polynomial of degree four whose term of degree 0 is absent and whose coefficients all include $\|R_{N+1}\|$.

To study the resolvent of H_{N+1}, we will construct an operator $\hat{H}_{N+1}$ the Green function of which is explicitly known, and show that $(\hat{H}_{N+1} - H_{N+1})\,\hat{R}_{N+1}$ is a bounded operator whose norm is smaller that 1 as long as ε is sufficiently small. First we use a technique inspired by the one proposed by Herbst and Howland in [9].

Definition of H_{N+1}. Let us recall that

$$H_{N+1}=\frac{1}{t_b'(x)}\left(-\varepsilon^2\frac{d^2}{dx^2}+S_b(x)\right)\frac{1}{t_b'(x)}$$
$$+\chi(-\infty,v_Na)\cdot(V_p(t_b(x))+\varepsilon^r F_0 t_b(x))$$
$$+\chi(v_Na,+\infty)(\tilde{V}_p(x)+\varepsilon^r F_0 v_N a),$$

where $t_b(x)=x+ibf(x)$.

Let us now introduce a new space transformation, $t: x\in\mathbb{R}\to t(x)\in\mathbb{R}$, which leaves invariant $\chi(v_Na,+\infty)$ and will be defined below. Let us denote U the transformation on $L^2(\mathbb{R})$,

$$U: g(x)\to(Ug)(x)=\sqrt{t'(x)}\,g(t(x)).$$

Under this transformation $-\dfrac{d^2}{dx^2}$ becomes

$$U\left(-\frac{d^2}{dx^2}\right)U^{-1}=\frac{1}{t'(x)}\left(-\frac{d^2}{dx^2}+S(x)\right)\frac{1}{t'(x)},$$

where S is the Schwarzian, and $g(x)$ becomes $U(g(x))U^{-1}=g(t(x))$, so H_{N+1} becomes

$$U(H_{N+1})U^{-1}=\frac{1}{t_b'(t(x))}U\left(-\varepsilon^2\frac{d^2}{dx^2}\right)U^{-1}\frac{1}{t_b'(t(x))}+\frac{1}{t_b'(t(x))^2}S_b(t(x))$$
$$+\chi(-\infty,v_Na)[V_p(t_b(t(x)))+\varepsilon^r F_0 t_b(t(x))]$$
$$+\chi(v_Na,+\infty)(\tilde{V}_p(x)+\varepsilon^r F_0 v_N a)$$
$$=\frac{1}{t_b'(t(x))}\frac{1}{t'(x)}\left(-\varepsilon^2\frac{d^2}{dx^2}+\chi(-\infty,v_Na)\cdot[V_p(t_b(t(x)))\right.$$
$$\left.+F t_b(t(x))](t_b'(t(x))t'(x))^2\right)\cdot\frac{1}{t_b'(t(x))}\frac{1}{t'(x)}$$
$$+\frac{1}{t_b'(t(x))^2}\frac{1}{t'(x)^2}S(x)+\frac{1}{t_b'(t(x))^2}S_b(t(x))$$
$$+\chi(v_Na,+\infty)(\tilde{V}_p(x)+\varepsilon^r F_0 v_N a).$$

Let us take an energy in the N^{th} gap and such that $E-\varepsilon^r F_0 v_N a - V_M>0$.

For $x\in(-\infty,v_Na)$ we will choose $t(x)$ such that:

$$(V_p(t_b(t(x)))+\varepsilon^r F_0 t_b(t(x))-E)(t_b'(t(x))t'(x))^2=V_0+\varepsilon^r F_0 x-E,$$

where V_0 is the mean value of V_p. Denoting $\tau(x)=t_b(t(x))$, this expression can be written in the form:

$$(V_p(\tau(x))+\varepsilon^r F_0\tau(x)-E)(\tau'(x))^2=V_0+\varepsilon^r F_0 x-E.$$

So for $x<v_Na$, $\tau(x)$ will be given by:

$$\int_{v_Na}^{\tau}\sqrt{E-V_p(u)-\varepsilon^r F_0 u}\,du=\int_{v_Na}^{x}\sqrt{E-V_0-\varepsilon^r F_0 x}\,dx$$

and for $x>v_Na$, by $\tau(x)=x$. So now, $t(x)=t_b^{-1}(\tau(x))$ and in particular $t(v_Na)=v_Na$.

Let us come back to H_{N+1},

$$H_{N+1}-E=U^{-1}\frac{1}{t_b'(t(x))}\frac{1}{t'(x)}\left(-\varepsilon^2\frac{d^2}{dx^2}+\chi(-\infty,v_Na)(V_0+\varepsilon^rF_0x-E)\right)$$
$$\times\frac{1}{t_b'(t(x))}\frac{1}{t'(x)}U+U^{-1}\frac{1}{t_b'(t(x))^2}\frac{1}{t'(x)^2}S(x)U$$
$$+U^{-1}\frac{1}{t_b'(t(x))^2}S_b(t(x))U+\chi(v_Na,+\infty)(\tilde{V}_p(x)+Fv_Na).$$

We will write $H_{N+1}-E=\hat{H}_{N+1}-E+Q(x)$, with:

$$\hat{H}_{N+1}-E=U^{-1}\frac{1}{t_b'(t(x))}\frac{1}{t'(x)}\left(-\varepsilon^2\frac{d^2}{dx^2}+\chi(-\infty,v_Na)(V_0+\varepsilon^rF_0x-E)\right)$$
$$\times\frac{1}{t_b'(t(x))}\frac{1}{t'(x)}U+\chi(v_Na,+\infty)(\tilde{V}_p(x)+Fv_Na),\tag{IV.1}$$

and

$$Q(x)=U^{-1}\frac{1}{t_b'(t(x))^2}\frac{1}{t'(x)^2}S(x)U+U^{-1}\frac{1}{t_b'(t(x))^2}S_b(t(x))U$$
$$=\frac{1}{t_b'(x)^2}\frac{1}{t'(t^{-1}(x))^2}S(t^{-1}(x))+\frac{1}{t_b'(x)^2}S_b(x).$$

Using the expressions for the Schwarzians, we get:

$$Q(x)=\chi(-\infty,v_Na)\varepsilon^2\frac{h_0^2}{2m}$$
$$\times\left(\frac{1}{4}\frac{V_p''(t_b(x))}{E-V_p(t_b(x))-\varepsilon^rF_0t_b(x)}+\frac{5}{16}\frac{(V_p'(t_b(x))+\varepsilon^rF_0)^2}{(E-V_p(t_b(x))-\varepsilon^rF_0t_b(x))^2}\right.$$
$$\left.-\frac{5\varepsilon^{2r}F_0^2}{16}\frac{E-V_p(t_b(x))-\varepsilon^rF_0t_b(x)}{(E-V_0-\varepsilon^rF_0t^{-1}(x))^3}\right).\tag{IV.2}$$

Let us remark two facts: Q decreases at $-\infty$ like $\frac{1}{|x|}$; because, $E-V_P^{\mathrm{Max}}-\varepsilon^rF_0v_Na>0$, $Q(x)$ has no singularities as long as b is sufficiently small, this remains true even if E has a small imaginary part.

Study of $(H_{N+1}-E)^{-1}$.

As Airy functions are solutions for $-\frac{d^2}{dx^2}\phi+(V_0+\varepsilon^rF_0x-E)\phi=0$, from (IV.1) we can deduce that:

$$\left(\frac{dt^{-1}}{dx}\right)^{-1/2}t_b'(x)Ai\left(\varepsilon^{(-2+r)/3}F_0^{1/3}\left(t^{-1}(x)-\frac{E-V_0}{\varepsilon^rF_0}\right)\right),$$
$$\left(\frac{dt^{-1}}{dx}\right)^{-1/2}t_b'(x)Bi\left(\varepsilon^{p-2+r)/3}F_0^{1/3}\left(t^{-1}(x)-\frac{E-V_0}{\varepsilon^rF_0}\right)\right)$$

are solutions for $(\hat{H}_{N+1}-E)\phi=0$ on interval $(-\infty,v_Na)$.

As $x \to -\infty$,

$$\phi^+(x) := \left\{ Ai\left(\varepsilon^{(-2+r)/3} F_0^{1/3}\left(t^{-1}(x) - \frac{E-V_0}{\varepsilon^r F_0}\right)\right) + Bi\left(\varepsilon^{(-2+r)/3} F_0^{1/3}\left(t^{-1}(x) - \frac{E-V_0}{\varepsilon^r F_0}\right)\right)\right\}\left(\frac{dt^{-1}}{dx}\right)^{-1/2} t_b'(x)$$

behaves like:

$$\varepsilon^{1/6-r/12} F_0^{-1/12}\left(\frac{E-V_0}{\varepsilon^r F_0} - t^{-1}(x)\right)^{-1/4} \times \exp i\left(2/3\varepsilon^{-1+r/2} F_0^{1/2}\left(\frac{E-V_0}{\varepsilon^r F_0} - t^{-1}(x)\right)^{3/2} + \pi/4\right),$$

so it decreases exponentially at $-\infty$, while,

$$\psi^-(x) := \left\{ Ai\left(\varepsilon^{(-2+r)/3} F_0^{1/3}\left(t^{-1}(x) - \frac{E-V_0}{\varepsilon^r F_0}\right)\right) - Bi\left(\varepsilon^{(-2+r)/3} F_0^{1/3}\left(t^{-1}(x) - \frac{E-V_0}{\varepsilon^r F_0}\right)\right)\right\}\left(\frac{dt^{-1}}{dx}\right)^{-1/2} t_b'(x)$$

behaves like

$$\varepsilon^{1/6-r/12} F_0^{-1/12}\left(\frac{E-V_0}{\varepsilon^r F_0} - t^{-1}(x)\right)^{-1/4} \times \exp i\left(2/3\varepsilon^{-1+r/2} F_0^{1/2}\left(\frac{E-V_0}{\varepsilon^r F_0} - t^{-1}(x)\right)^{3/2} + \pi/4\right)$$

so, increases exponentially at $-\infty$.

Call ϕ^- the $L^2(0, +\infty)$-function, solution of $\hat{H}_{N+1}\phi = E\phi$. On $(+\nu_N a, +\infty)$, ϕ^- is a Bloch function, decreasing exponentially at $+\infty$. It can be written in the form:

$$\phi^- = \phi^1_{\nu_N} + m(E - F\nu_N a)\phi^2_{\nu_N},$$

where

$$m(E) = \frac{\dfrac{D(E)-A(E)}{2} - \sqrt{\left(\dfrac{A(E)+D(E)}{2}\right)^2 - 1}}{B(E)}.$$

Notice that: $m(E - F\nu_N a) = \dfrac{d\phi^-}{dx}(\nu_N a)$.

For some $\Lambda_N \in [E_N, E_N']$; $B(\Lambda_N) = 0$ (see Eastham [7] p. 37)$\Rightarrow m(\Lambda_N)$ is not defined. Denote:

$$\Gamma_N^1 = \left\{ E \,\middle|\, |E - \Lambda_N| > \frac{E_N' - E_N}{10} \right\}, \quad \Gamma_N^2 = \left[E_N^m - \frac{E_N' - E_N}{4}, E_N^m + \frac{E_N' - E_N}{4} \right]$$

and $\Gamma_N^0 = \Gamma_N^1 \cap \Gamma_N^2$;

Lemma 1. $\exists \varepsilon_0$ *such that for all* $\varepsilon < \varepsilon_0$, *such that:* $D_N = \{z \in \mathbb{C} \mid \operatorname{Re} z \in \Gamma_N^0, |\operatorname{Im} z| < \varepsilon^r F_0 b\}$ *is in the resolvent set of* $\hat{H}_{N+1}$.

Proof. The Green function for $\hat{H}_{N+1}$ is given by:

$$G_{N+1}(x, y; E) = \frac{2m}{\varepsilon^2 h_0^2} \frac{\phi^+(x)\phi^-(y)}{W(\phi^+, \phi^-)}. \tag{IV.3}$$

We calculate the wronskian at point $v_N a$, then we divide numerator and denominator by $\phi^+(v_N a)\ \phi^-(v_N a)$. As we choose $\phi^-(v_N a) = 1$ we get for the new denominator,

$$w(E) := m(E + F v_N a) - \frac{\phi^{+\prime}(v_N a)}{\phi^+(v_N a)}.$$

$\phi^{+\prime}(v_N a)$ and $\phi^+(v_N a)$ are explicit since $t^{-1}(v_N a) = v_N a$. Furthermore as $\varepsilon \to 0$, we can replace the Airy functions by their asymptotic expression. We get that $\phi^+(v_N a)$ behaves like:

$$\varepsilon^{1/6 - r/12} F_0^{-1/12} \left(\frac{E - V_0}{\varepsilon^r F_0} - v_N a \right)^{-1/4}$$
$$\times \exp i \left(2/3 \varepsilon^{-1 + r/2} F_0^{1/2} \left(\frac{E - V_0}{\varepsilon^r F_0} - v_N a \right)^{3/2} + \pi/4 \right)$$

and $\phi^{+\prime}(v_N a)$ behaves like,

$$i\varepsilon^{-5/6 + 5r/12} F_0^{+5/12} \left(\frac{E - V_0}{\varepsilon^r F_0} - v_N a \right)^{+1/4}$$
$$\times \exp i \left(2/3 \varepsilon^{-1 + r/2} F_0^{1/2} \left(\frac{E - V_0}{\varepsilon^r F_0} - v_N a \right)^{3/2} + \pi/4 \right).$$

So it is easy to see that $\dfrac{\phi^{+\prime}(v_N a)}{\phi^+(v_N a)}$ contain an imaginary part which increases as $\varepsilon \to 0$. As E is real and belongs to a gap,

$$\sqrt{\left(\frac{A(E) + D(E)}{2} \right)^2 - 1}$$

is real and so is $m(E)$. Then, $w(E)$ is non-zero.

Furthermore $|w(E)|$ is larger than $M(E)\varepsilon^{-1}$, where $M(E)$ is a positive constant. If $E \in \Gamma_N^0$, $A(E)$, $B(E)$, $D(E)$ are analytic functions, the derivative $\dfrac{dw}{dE}$ exists and its modulus is bounded by $M'(E)\varepsilon^{-1}$, [where $M'(E)$ is a positive constant]. So $w(E + i\beta)$ do not vanish if β is smaller than $\dfrac{M(E)}{M'(E)}$. Taking $\beta_0 = \inf\limits_{E \in \Gamma^0} \dfrac{M(E)}{M'(E)}$, the wronskian is non-zero for all E such that $\operatorname{Re} E \in \Gamma_N^0$ and $|\operatorname{Im} E| < \beta_0$.

Since

$$\|\chi(-\infty, v_N a)(\hat{H}_{N+1} - E)^{-1} \chi(-\infty, v_N a)\|$$
$$< \sup_{x \in (-\infty, v_N a)} \int_{-\infty}^{v_N a} |G(x, y; E)| dy + \sup_{y \in (-\infty, v_N a)} \int_{-\infty}^{v_N a} |G(x, y; E)| dx,$$

using the explicit expressions for ϕ^+ and ϕ^- in the Green formula (IV.3), we get for $|\operatorname{Im} E| < \varepsilon^r F_0 b$, if ε is sufficiently small in such a way $\varepsilon^r F_0 b < \beta_0$, the following upper bound:

$$\|\chi(-\infty, v_N a)(\hat{H}_{N+1} - E)^{-1}\chi(-\infty, v_N a)\| \leqq \frac{C_1}{b\varepsilon^r F_0 + \operatorname{Im} E},$$

where C_1 is a constant.

Lemma 2. $\exists \varepsilon_0 > 0$ *such that if* $\varepsilon < \varepsilon_0$, *then* D_N *is in the resolvent set for* H_{N+1}.

Proof.
$$H_{N+1} = \hat{H}_{N+1} - \chi(-\infty, v_N a)Q.$$

We want to prove that

$$\|A\| := \|\chi(-\infty, v_N a)Q^{1/2}(\hat{H}_{N+1} - E)^{-1}Q^{1/2}\chi(-\infty, v_N a)\| < 1$$

for $E \in D_N$ and ε sufficiently small. Let us analyze the behaviour of the different terms with respect to ε. As E belongs to the N^{th} gap, its distance to the top of the periodic potential is of order of $-\varepsilon(\log\varepsilon)^{-1}$. In fact, as $\varepsilon \to 0$, Weinstein and Keller prove in [14] that, in the neighborhood of V_M, "bands" and "gaps" have quite the same width, while the first bands (i.e. near V_m) are exponentially narrow and the first gaps of the order of ε. Recently, März [10] improve this result showing that near V_M the gaps and the bands behave like $\varepsilon(-\log\varepsilon)^{-1}$. Then, the first term in Q is bounded by a term proportional to $-\varepsilon\log\varepsilon$. Apparently the second term has a worse behaviour but calculating its maxima it appears that it behaves also like $-\varepsilon\log\varepsilon$. The third term is of higher order in ε.

$G(x, y; E) = \dfrac{2m}{\varepsilon^2 h_0^2} \dfrac{\phi^+(x)\phi^-(y)}{\phi^+(v_N a)\phi^-(v_N a)w(E)}$ behaves like ε^{-1}, since $w(E)$ behaves like ε^{-1}.

The Hilbert-Schmidt norm of $\chi(-\infty, v_N a)Q^{1/2}$

$$\chi(-\infty, v_N a)Q^{1/2}(\hat{H}_{N+1} - E)^{-1}Q^{1/2}\chi(-\infty, v_N a),$$

exists but we do not get any decrease as $\varepsilon \to 0$, because taking brutally the modulus of G we kill the oscillations of the Airy functions. So, we have to use a clever technique which consists in considering $\operatorname{Tr}(AA^*AA^*)$, (this technique was inspired by [11, p.23]).

So we get the integral:

$$\int_{-\infty}^{v_N a} dx \int_{-\infty}^{v_N a} dy \int_{-\infty}^{v_N a} du \int_{-\infty}^{v_N a} dv$$
$$\times Q(x)Q(y)Q(u)Q(v)G(x, y)\overline{G(u, y)}G(u, v)\overline{G(x, v)}.$$

In sector $v < u < y < x$, for instance, the product of the four Green functions give us

$$|\phi^-(x)|^2\phi^+(y)\phi^-(y)\phi^+(u)\phi^-(u)|\phi^+(v)|^2/\phi^+(v_N a)^4\phi^-(v_N a)^4 w(E)^4.$$

$\phi^+(y)\phi^-(y)$ contains a term which oscillates like

$$\exp i\left(2/3\varepsilon^{-1+r/2}F_0^{r/2}\left(\frac{E - V_0}{\varepsilon^r F_0} - s^{-1}(x)\right)^{3/2} + \pi/4\right).$$

One can use integration by parts and explicit formulas for integrals of Airy functions to prove that $\mathrm{Tr}(AA^*AA^*)$ goes to zero as ε decreases.

Now we can collect the sparse results to prove the main theorem.

V. Main Result

Theorem. *Given h_0 and F_0 and $r>1$, if V_p satisfies* (H.1), *ε_0 exists, such that if $\varepsilon<\varepsilon_0$, there are at least $N(\varepsilon)$ ladders of resonances for $-\varepsilon^2\frac{h_0^2}{2m}\frac{d^2}{dx^2}+V_p(x)+\varepsilon^r F_0$, where $N(\varepsilon)-1$ is the number of bands of $-\varepsilon^2\frac{h_0^2}{2m}\frac{d^2}{dx^2}+V_p(x)$, strictly contained in the interval (V_m, V_M). Their width is exponentially small with respect to F_0 and ε.*

Proof. To look at the spectrum of $H(b)$, we use formula (II.1). We want to show that for ε sufficiently small and E not too close to the H_i eigenvalues, $\sum_{i=0}^{N+1} K_i(E)$ has a norm which is smaller than 1.

Let us denote by $D_i:=(f\in L^2(R)\,|\,\text{support }(f)\subset\text{support }\tilde{J}_i)$. As K_i sends D_i on $D_{i+1}\oplus D_{i-1}$, we emphasize this fact, denoting $K_i:=K_{i,j+1}+K_{i,i-1}$. It is easy to see that

$$\left\|\sum_{i=1}^{N(\varepsilon)} K_i\right\| <2\sup_i\,(\|K_{i,i+1}\|+\|K_{i,i-1}\|)<2\sqrt{2}\sup_i\,\|K_i\|\,.$$

We have first to observe the effects of the introduction of ε in the formulas giving $\|K_i(E)\|$. *In Proposition* 2, *including* $\varepsilon^2\frac{h_0^2}{2m}$ in front of $-\frac{d^2}{dx^2}$ in the definition of H_i, we have to multiply the right-hand term in (III.4) by $\varepsilon^2\frac{h_0^2}{2m}$ (which has the dimension of an energy times a length to the power two). In (III.4) we have also to replace F by $\varepsilon^r F_0$ and Γ_i by a gap width, $\Gamma_i(\varepsilon)$ of order ε for small i and of order $\varepsilon(-\log\varepsilon)^{-1}$ for $i=N(\varepsilon)$.

Since the eigenvalues of H_i are distant by $\varepsilon^r F_0 a+O(\varepsilon^{2r}F_0^2)$, in an interval included in Γ_N^0 of width $\varepsilon^r F_0 a$, the number of eigenvalues coming from the distinct H_i, $(i=1\ldots N(\varepsilon))$ is approximately $N(\varepsilon)$. As $N(\varepsilon)$ is of the order of ε^{-1}, [let us write $N(\varepsilon)=d\varepsilon^{-1}$] it exists in this interval an eigenvalue, let us say, λ_j^0, from H_j, whose distance to the other eigenvalues is larger than $d^{-1}\varepsilon^{1+r}F_0 a$.

Choosing E on the circle (C) of radius $d^{-1}\varepsilon^{1+r}F_0 a/2$, centered at λ_j^0, we will evaluate now $\|K_i(E)\|$,

$$\|K_i(E)\|\leqq\varepsilon^2 P_{i1}\left(\frac{\varepsilon^r F_0}{\Gamma_i}\right)e^{-\frac{\kappa_i^m l_i}{\sqrt{3}}}+\varepsilon^2 P_{i2}\left(\frac{\varepsilon^r F_0}{\Gamma_i}\right)e^{-\frac{\kappa_{i-1}^m l_{i-1}}{\sqrt{3}}}\qquad i=1,\ldots,N(\varepsilon)\,. \tag{V.1}$$

Remember that polynomials P_{i1}, P_{i2} have no constant term and that their coefficients contain $\frac{1}{\mathrm{dist}(E,\sigma(H_i))}$. As $\mathrm{dist}(E,\sigma(H_i))=d^{-1}\varepsilon^{1+r}F_0 a/2$ it appears that terms in front of the exponentials are of the order ε^0 for small i and of the order $-\log\varepsilon$ for $i=N(\varepsilon)$.

In Appendix B using the formula given by Weinstein and Keller [14], for the discriminant, we show (see Lemma B1) that κ_i^m decreases as i goes from 1 to $N(\varepsilon)$ and κ_N^m behaves like $\frac{(E_N^m - V_M)}{\varepsilon}$. Since gaps and bands near V_M behave like $\varepsilon(-\log\varepsilon)^{-1}$ as ε goes to zero, κ_N^m behaves like $(-\log\varepsilon)^{-1}$. So the exponential terms in (V.1) are bounded by $e^{-\frac{c\varepsilon^{1-r}(-\log\varepsilon)^{-2}}{F_0\sqrt{3}}}$, while the polynomial terms are bounded by quantities of order $-\log\varepsilon$. Then $\left\|\sum_{i=1}^{N(\varepsilon)} K_i\right\|$ is consequently of the order of $\log\varepsilon\cdot e^{-\frac{c\varepsilon^{1-r}(-\log\varepsilon)^{-2}}{F_0\sqrt{3}}}$, and goes to zero as ε goes to zero.

Using now $(H(b)-z)^{-1}=\left(\sum_{i=0}^{N(\varepsilon)+1} J_i R_i \tilde{J}_i\right)\left(1+\sum_{i=0}^{N(\varepsilon)+1} K_i\right)^{-1}$ it is clear that the resolvent is defined on the circle (C), then we can assert that the resonance exists.

We can take a circle C' of radius much smaller, for instance of order $\log\cdot e^{-\frac{c\varepsilon^{1-r}(-\log\varepsilon)^{-2}}{F_0\sqrt{3}}}$. It is again true that $\left\|\sum_{i=0}^{N(\varepsilon)+1} K_i\right\|<1$ if $E\in C'$, so we can deduce that the resonance width is smaller than: $c_1\cdot\log\varepsilon\cdot e^{-\frac{c\varepsilon^{1-r}(-\log\varepsilon)^{-2}}{F_0\sqrt{3}}}$.

Appendix A

Study of Product of Matrices $\mathbb{M}_j$. The matrices $\mathbb{M}_j=\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$ defined in Part III have determinant equal to 1.

Their eigenvalues $\lambda_j^+, \lambda_j^-=(\lambda_j^+)^{-1}$ can be real or complex depending on $\frac{A_j+D_j}{2}$,

$$\lambda_j^{\pm}=\frac{A_j+D_j}{2}\pm\sqrt{\left(\frac{A_j+D_j}{2}\right)^2-1}\,.$$

a) If $\frac{A_j+D_j}{2}>1$ then denoting $\frac{A_j+D_j}{2}=ch\kappa_j a$ we get $\lambda_j^+=e^{+\kappa_j a}$.

b) If $\frac{A_j+D_j}{2}<-1$ then denoting $\frac{A_j+D_j}{2}=-ch\kappa_j a$ we get $\lambda_j^+=-e^{-\kappa_j a}$.

c) If $-1<\frac{A_j+D_j}{2}<1$ then denoting $\frac{A_j+D_j}{2}=\cos k_j a$ we get $\lambda_j^+=e^{+ik_j a}$.

In cases a) and b) $\mathbb{M}_j$ can be written as $\mathbb{S}_j\begin{pmatrix}\lambda_j^+ & 0\\ 0 & (\lambda_j^+)^{-1}\end{pmatrix}\mathbb{S}_j^{-1}$.

In case c) $\mathbb{M}_j=\mathbb{T}_j\mathbb{R}(k_j a)\mathbb{T}_j^{-1}$, where $\mathbb{R}(k_j a)$ is the rotation matrix by the angle $k_j a$.

As A_j, B_j, C_j, D_j depend analytically on E, for small F, $\mathbb{M}_j$ and $\mathbb{M}_{j+1}$ differ by a quantity of the order of Fa and $\mathbb{S}_j^{-1}\mathbb{S}_{j-1}$ or $\mathbb{T}_j^{-1}\mathbb{T}_{j-1}$ can be written as the sum of the identity plus a matrix the norm of which is $0(Fa)$. In fact:

$$\mathbb{S}_j\mathbb{S}_{j-1}^{-1}=\mathbb{1}+\frac{B_j}{\lambda_j^- -\lambda_j^+}\left(\begin{array}{c|c}\frac{\lambda_j^+-A_j}{B_j}-\frac{\lambda_{j-1}^+-A_{j-1}}{B_{j-1}} & \frac{\lambda_j^--A_j}{B_j}-\frac{\lambda_{j-1}^--A_{j-1}}{B_{j-1}}\\ \frac{\lambda_{j-1}^+-A_{j-1}}{B_{j-1}}-\frac{\lambda_j^+-A_j}{B_j} & \frac{\lambda_{j-1}^--A_{j-1}}{B_{j-1}}-\frac{\lambda_j^--A_j}{B_j}\end{array}\right)$$

$$=\mathbb{1}+\Delta_j\,.$$

Lemma A1. *The vector* $\mathbb{M}_j \dots \mathbb{M}_\nu \mathbb{S}_0^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *rotates by an angle smaller than* δFa *as long as* $\nu < j < \nu'$ *and its norm increase exponentially.*

Proof. As $\nu < j < \nu'$ we are in case a) or b) then we can write:

$$\begin{aligned}\mathbb{M}_j \dots \mathbb{M}_1 \mathbb{M}_\nu \mathbb{S}_\nu^{-1} &= \mathbb{S}_j \mathbb{D}_j \mathbb{S}_j^{-1} \dots \mathbb{S}_i \mathbb{D}_i \mathbb{S}_i^{-1} \mathbb{S}_{i-1} \mathbb{D}_{i-1} \mathbb{S}_{i-1} \dots \\ &\quad \dots \mathbb{D}_{\nu+1} \mathbb{S}_{\nu+1} \mathbb{S}_\nu^{-1} \\ &= \mathbb{S}_j \mathbb{D}_j (\mathbb{1} + \mathbb{A}_j) \dots \mathbb{D}_i (\mathbb{1} + \mathbb{A}_i) \dots \\ &\quad \dots \mathbb{D}_{\nu+1} (\mathbb{1} + \mathbb{A}_{\nu+1}) .\end{aligned}$$

Denote:

$$\gamma_j \begin{pmatrix} 1 \\ \eta_j \end{pmatrix} = \mathbb{D}_j (\mathbb{1} + \mathbb{A}_j) \dots D_i (\mathbb{1} + \mathbb{A}_i) \dots D_1 (1 + \mathbb{A}_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

$$\begin{aligned} & D_{j+1} (\mathbb{1} + \mathbb{A}_{j+1}) \gamma_j \begin{pmatrix} 1 \\ \eta_j \end{pmatrix} \\ &= \gamma_j \begin{pmatrix} \lambda_{j+1}^+ & 0 \\ 0 & \lambda_{j+1}^- \end{pmatrix} \begin{pmatrix} 1 + \delta_{j+1}^{11} & \delta_{j+1}^{12} \\ \delta_{j+1}^{21} & 1 + \delta_{j+1}^{22} \end{pmatrix} \begin{pmatrix} 1 \\ \eta_j \end{pmatrix} \\ &= \gamma_j \lambda_{j+1}^+ (1 + \delta_{j+1}^{11} + \eta_j \delta_{j+1}^{12}) \begin{pmatrix} 1 \\ \dfrac{\lambda_{j+1}^- (\delta_{j+1}^{21} + \eta_j (1 + \delta_{j+1}^{22}))}{\lambda_{j+1}^+ (1 + \delta_{j+1}^{11} + \eta_j \delta_{j+1}^{12})} \end{pmatrix} .\end{aligned}$$

By identification:

$$\gamma_{j+1} = \gamma_j \lambda_{j+1}^+ (1 + \delta_{j+1}^{11} + \eta_j \delta_{j+1}^{12}) ,$$
$$\eta_{j+1} = \frac{\lambda_{j+1}^- (\delta_{j+1}^{21} + \eta_j (1 + \delta_{j+1}^{22}))}{\lambda_{j+1}^+ (1 + \delta_{j+1}^{11} + \eta_j \delta_{j+1}^{12})} .$$

Now consider the sequence η_j. As long as $\dfrac{\lambda_{i+1}^- (1 + \delta Fa)}{\lambda_{i+1}^+ (1 - \delta Fa)} < \varrho < 1$, for all $i < j$, using the fact $\dfrac{\lambda_{j+1}^- \delta_{j+1}^{21}}{\lambda_{j+1}^+ (1 + \delta_{j+1}^{11})} < \varrho \delta Fa$ and $\eta_0 = 0$, one can prove that $\eta_{j+1} < \dfrac{\delta Fa}{1 - \varrho}$. One can find F sufficiently small in such a way $\dfrac{\lambda_{j+1}^- (1 + \delta Fa)}{\lambda_{j+1}^+ (1 - \delta Fa)} < \varrho < 1$ for all j such that $\nu < j < \nu'$.

Remark. The lemma is also valid on the other side of the "band" as $\mu'_{i-1} < j < \mu_{i-1}$.

Lemma A2. *If* $\mathbb{M}_{\mu_i}^{-1} \dots \mathbb{M}_i^{-1} \dots \mathbb{M}_{\mu_{i-1}}^{-1}$ *is applied to a vector; this one rotates by an angle:* $\sum_{i = \mu'_{i-1}}^{\mu_i} (k_i a + F a \alpha_i)$, *where the* α_i *are uniformly bounded quatities.*

Proof. As $\mu_i < i < \mu'_{i-1}$, we are in case c) then:

$$\begin{aligned}\mathbb{M}_{\mu_i}^{-1} \dots \mathbb{M}_i^{-1} \dots \mathbb{M}_{\mu_{i-1}}^{-1} &= \mathbb{T}_{\mu_i} \mathbb{R}(-k_{\mu_i} a) \mathbb{T}_{\mu_i}^{-1} \dots \mathbb{T}_i \mathbb{R}(-k_i a) \mathbb{T}_i^{-1} \dots \\ &\quad \dots \mathbb{T}_{\mu'_{i-1}} \mathbb{R}(-k_{\mu'_{i-1}} a) \mathbb{T}_{\mu'_{i-1}}^{-1} \\ &= \mathbb{T}_{\mu_i} \mathbb{R}(-k_{\mu_i} a) (\mathbb{1} + \tilde{\delta}_{\mu_i}) \dots \mathbb{R}(-k_i a) (\mathbb{1} + \tilde{\delta}_i) \dots \\ &\quad \dots \mathbb{R}(-k_{\mu'_{i-1}} a) (\mathbb{1} + \tilde{\delta}_{\mu'_{i-1}})\end{aligned}$$

Denote

$$\tilde{\gamma}_i \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} = \mathbb{R}(-k_i a)(1+\tilde{\delta}_i)\ldots\mathbb{R}(-k_{\mu_i'-1}a)(1+\tilde{\delta}_{\mu_i'-1})\begin{pmatrix}1\\0\end{pmatrix}.$$

Then:

$$\theta_{i+1} = \theta_i - k_{i+1}a + Fa\alpha_{i+1}, \quad \text{where} \quad \alpha_{i+1} = 0(1).$$

So the total rotation angle will be $\theta_{\mu_i} = -\sum_{i=\mu_i'-1}^{\mu_i}(k_i a + Fa\alpha_i)$.

When we vary E, θ_{μ_i} varies by a quantity,

$$d\theta_{\mu_i} = -\sum_{i=\mu_i'-1}^{\mu_i} a\frac{dk_i}{dE}dE + Fa\frac{d\alpha_i}{dE}dE.$$

Suppose the variation is $dE = \xi Fa$, $0<\xi<1$, as k_i is a function of Fai:

$$d\theta_{\mu_i} = -\xi a \cdot \sum_{i=\mu_i'-1}^{\mu_i} \frac{dk(E-Fai)}{dE}Fa + F\frac{d\alpha(E-Fai)}{dE}Fa.$$

As $$F\to 0, \quad d\theta_{\mu_i} \to \xi a \int \frac{dk}{dE}dE = \xi.\pi.$$

We notice that this quantity is small and goes to zero as $F\to 0$.

So the total angle variation as E is increased by Fa is near π.

Remark. Adding Fa to E just translate all the indices, so, the only difference appears at the extremities of the product and we get an extra rotation of π (0 at one extremity, π at the other).

Appendix B

Lemma B1. *If $V(x)$ is real analytic and for some $E_0 > V_M$ and all E satisfying $E_0 > E > V_M$, $V(x) = E$ has two simple roots $iy(E)$ and $-iy(E)$ which are closer to the real axis than any other roots, two constants A and A' exist such that:*

$$\kappa_i^m > \frac{2\sqrt{2m}(V_M - E_i^m}{\varepsilon h_0 a\sqrt{A}}, \quad \text{for} \quad i = 1, 2\ldots N(\varepsilon)-1,$$

$$\kappa_N^m > \frac{2\sqrt{2m}(E_N^m - V_M)}{\varepsilon h_0 a\sqrt{A'}}.$$

Proof. Hypothesis on V is hypothesis (P3) in the work by Weinstein and Keller. In [14, formula (4.6)], they give the following expression for the discriminant:

$$\Delta(E) \sim -2(e^{2\Pi\alpha(E)\lambda}+1)^{1/2}\sin\left[\frac{\pi}{2} + \varepsilon^{-1}\int_0^a [E - V_P(x)]_+^{1/2}dx\right],$$

where: $\alpha(E) = \frac{1}{\pi}\int_{x_0}^{x_1}[V_P(x)-E]^{1/2}dx$ with $V_P(x_0) = V_P(x_1) = E$, $\lambda = \frac{\sqrt{2m}}{\varepsilon h_0}$ and $[f]_+ := \max(f;0)$. Notice that if $E < V_M$, $\alpha(E)$ is positive and that the maxima of $\Delta(E)$ appear approximately when

$$\sin\left[\frac{\pi}{2} + \varepsilon^{-1}\int_0^a [E - V_P(x)]^{1/2}dx\right] = -1,$$

so we can check that their distance behaves like ε.

To estimate $\alpha(E)$ we replace the periodic potential in the integral by the parabola $V_M + \frac{d^2 V_M}{dx^2}(0)x^2$. Supposing $V(x) > V_M + \frac{d^2 V_M}{dx^2}(0)x^2$, [if it is not the case, we replace $\frac{d^2 V_M}{dx^2}(0)$ by another constant $-A$ in such a way the inequality becomes true] we will get a lower bound for $\alpha(E_i^m)$, for $i = 1, 2 \ldots N(\varepsilon) - 1$. In fact if $\tilde{x}_0$ and $\tilde{x}_1$ are defined by $V_M - A\tilde{x}_0^2 - E_i^m = 0$, $V_M - A\tilde{x}_1^2 - E_i^m = 0$, then:

$$\alpha(E_i^m) > \frac{1}{\pi} \int_{\tilde{x}_0}^{\tilde{x}_1} [V_M - Ax^2 - E_i^m]^{1/2} dx = \frac{V_M - E_i^m}{\pi\sqrt{A}}.$$

Using the fact: $\Delta(E) = 2ch\kappa a$ and $\Delta(E_i^m) \sim 2(e^{2\Pi\alpha(E)\lambda} + 1)^{1/2}$ one obtains:

$$\kappa_i^m > \frac{2\sqrt{2m}(V_M - E_i^m)}{\varepsilon h_0 a\sqrt{A}}.$$

For $E_0 > E > V_M$, by hypothesis the solutions of $V(x) = E$ are pure imaginary points. Let us denote $x_0 = iy_0$ and $x_1 = iy_1$ the solutions of $V(x) = E_N^m$. It appears that $\alpha(E) = -\frac{1}{\pi} \int_{y_0}^{y_1} [E - V_P(iy)]^{1/2}\, dy$ is negative. In the neighborhood of the origin, one can choose A' such that $V(iy) > V_M + A'y^2$. Defining by $\tilde{y}_0$, $\tilde{y}_1$ the solutions of $E_N^m - V_M - A'y^2 = 0$, we get:

$$\alpha(E_N^m) > -\frac{1}{\pi} \int_{\tilde{y}_0}^{\tilde{y}_1} [E_N^m - V_M - A'y^2]^{1/2} dy = \frac{V_M - E_N^m}{\pi\sqrt{A'}}.$$

Then one obtains $\kappa_N^m > \dfrac{2\sqrt{2m}(E_N^m - V_M)}{\varepsilon h_0 a\sqrt{A'}}$.

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