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Bifurcation Frequency for Unimodal Maps

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Abstract. We consider some natural one-parameter unfoldings f_μ, of a unimodal map f_0 whose periodic points are hyperbolic and whose critical point is non-degenerate and eventually periodic. Among other facts, it follows from our theorems that, if the Julia set of f_0 does not contain intervals, the relative measure of the bifurcation set is zero at zero.

Introduction

It is extremely surprising that in such a simple space as an interval, there should exist important and rich dynamical systems. However many fascinating types of interval dynamics have been discovered. They are of interest in their own right as well as being useful mathematical models, and frequently as being part of higher dimensional systems.

Informally, we think of a dynamical system as a "system in movement;" as time goes by each point in a phase space evolves according to some deterministic law. An important feature of a dynamical system is its limit set; the set where the orbits accumulate. The dynamical behaviour inside the limit set can be "simple" or "complex." The results of this article support the view that systems with "simple" limit set are very frequent.

Here we deal with interval dynamics generated by iteration of unimodal maps. One simple case, the axiom A case, is when the periodic points are hyperbolic and the critical value lies inside the basin of a periodic sink. In this case we have a hyperbolic dynamic which is structurally stable and can be reduced to the dynamic of simpler symbolic models. There exist other cases in which the dynamics are described by absolutely continuous ergodic invariant probability measures.

We consider some natural one-parameter unfoldings f_μ ($\mu \geqq 0$), of a unimodal map f_0 whose periodic points are hyperbolic and whose critical point is eventually periodic. We have two cases depending on the topological structure of the Julia set Σ_0 of f_0 (the complement of the basin of the periodic sinks of f_0): if Σ_0 does not contain intervals, among other facts, it follows from our theorems that f_μ is axiom A with frequency one at $\mu=0$. If Σ_0 contains intervals, it follows from Yakobson [8] that f_μ has an absolutely continuous ergodic invariant probability measure with frequency one at $\mu=0$.

Assuming the property of negative Schwarzian, if Σ_0 does not contain intervals then it is a hyperbolic set and f_μ is axiom A for all small μ. For us the interesting case is when Σ_0 is not a hyperbolic set and does not contain intervals. This situation may appear for example when we unfold an axiom A map; without destroying the periodic sinks we move the critical value out of the basin.

For a qualitative understanding of the dynamical complexity of interval maps, it is important to know how frequent the axiom A case is among the members of an one-parameter family. Many other similar questions have been asked and some of them answered; we mention Yakobson [8], Collet and Eckmann [1], and Guckenheimer [2]. Also, Newhouse and Palis [5] and Palis and Takens [6] have answered similar questions in the context of diffeomorphisms of a compact surface. The tools used in our case are completely different: in [5] and [6] the important thing is the relative position of two "rigid" Cantor sets, for us what is important is the relative position of the point $f_\mu(0)$ and the Julia set Σ_μ of f_μ. One of the difficulties that we have is that the Julia set Σ_μ may contain intervals for many values of the parameter.

1. Main Theorems and Basic Facts

Definition. A C^r $(r \geqq 2)$ interval map $f:[-1,1] \circlearrowleft$ is called unimodal if $f(-1) = f(1) = -1$ and f has only one critical point, zero.

Definition. The basin B of a unimodal map f is the interior of the set of points whose forward orbit converges to a periodic point. The immediate basin B_0 is the union of the connected components of B which contain periodic points in their closure. The Julia set Σ is the complement of B.

By the kneading theory of Milnor and Thurston [4], we know that the topological structure of the dynamics generated by a unimodal map f is determined by the forward orbit of the critical point. The complexity and stability of the dynamics depends on the relative position of the critical value $f(0)$ and the basin B. When $f(0)$ is in B the dynamic is well understood and much simpler. We consider certain one parameter families f_μ of unimodal maps, and estimate how frequently $f_\mu(0)$ is in the Julia set Σ_μ of f_μ.

To state the theorems precisely we fix an arbitrary unimodal map f such that: all the periodic points are hyperbolic and the critical point is non-degenerate (i.e. $f''(0) \neq 0$) and eventually periodic. We choose the smallest $k \geqq 0$ such that $f^k(0)=p$, where p is a periodic repeller. We consider the set of C^r $(r \geqq 2)$ unimodal maps with the C^2 topology and denote by $\mathscr{G}^s(f)$, $(s \geqq 1)$ the set of C^s families $\{f_\mu\}_{\mu\in[0,1]}$ of C^r $(r \geqq 2)$ unimodal maps, such that:

$$f_0 = f,\ \partial_\mu f_\mu^k(0)|_{\mu=0} \neq 0 \quad \text{and} \quad \partial_\mu f_\mu^k(0)|_{\mu=0} \neq \partial_\mu p_\mu|_{\mu=0};$$

where $\mu \mapsto p_\mu$ (small μ) is the function implicitly defined by $f_\mu^l(p_\mu) = p_\mu$ (l is the period of p) and $p_0 = p$.

Given some family $\{f_\mu\}_{\mu\in[0,1]}$ in $\mathscr{G}^1(f)$ we denote the Julia set of f_μ by Σ_μ. We define the following set:

$$\mathscr{U} := \{\mu \in [0,1]; f_\mu(0) \in \Sigma_\mu\}.$$

For parameters μ in $\mathscr{U}$ the corresponding dynamic of f_μ is "complex." It will be easy to see that $\mathscr{U}$ contains almost all (in the Lebesgue sense) the bifurcation set of the family, $\mathscr{U}$ contains almost all the set of parameters μ such that f_μ has an absolutely continuous invariant measure, and $\mathscr{U}$ also contains almost all the set of parameters such that f_μ is not axiom A. Our theorems about the density of $\mathscr{U}$ have immediate consequences about the density of these sets.

We would like to prove that $\mathscr{U}$ is a small set but unfortunately sometimes this is not the case. It follows from Yakobson [8] that, if Σ_0 contains intervals, the Lebesgue density of $\mathscr{U}$ is one at zero. We prove that, if Σ_0 does not contain intervals, the Lebesgue density of $\mathscr{U}$ is zero at zero.

We denote the Lebesgue measure of a Lebesgue measurable set A by $|A|$. Let us state our theorems.

Theorem A. *Let $\{f_\mu\}_{\mu\in[0,1]}$ be in $\mathscr{G}^1(f)$ such that the Julia set Σ_0 of $f_0=f$ does not contain intervals. Then:*

a) *The Lebesgue density of $\mathscr{U}$ is smaller than one at zero* $\left(i.e.\ \limsup_{\varepsilon\to 0^+}\frac{|\mathscr{U}\cap[0,\varepsilon]|}{\varepsilon}\leqq\alpha<1\right)$.

b) *If the critical point of f_0 is not an accumulation point of Σ_0, the Lebesgue density of $\mathscr{U}$ is zero at zero (i.e. $\alpha=0$).*

Theorem B. *Let $\{f_\mu\}_{\mu\in[0,1]}$ be in $\mathscr{G}^2(f)$ such that f_μ is C^4 and the Julia set Σ_0 of $f_0=f$ does not contain intervals. Then, the Lebesgue density of $\mathscr{U}$ is zero at zero* $\left(i.e.\ \lim_{\varepsilon\to 0^+}\frac{|\mathscr{U}\cap[0,\varepsilon]|}{\varepsilon}=0\right)$.

Theorem A is much simpler than Theorem B. The proof of Theorem B is by induction, Theorem A being the first step. For this first step we need only that the family is a C^1 family of C^2 unimodal maps, but for Theorem B we need a C^2 family of C^4 unimodal maps f_μ.

Now we introduce some notation and some preliminary facts. They will be important in the study of the relative motion of the critical value $f_\mu(0)$ and the Julia set Σ_μ.

Notation. We consider a C^2 unimodal map f all of whose periodic points are hyperbolic, and whose critical point is non-degenerate and eventually periodic:

a) B_0 denotes the immediate basin and Σ denotes the Julia set of f.
b) J denotes a central interval, that is: an interval bounded by a point q and by the unique point q^* symmetric to q (i.e. $f(q^*)=f(q)$). We consider only the points q such that $J\cap\left(\bigcup_{j=1}^{\infty} f^j(q)\right)=\emptyset$, and for some $m>0$, $f^m(q)$ is a periodic point. We will state explicitly, when necessary, whether J is open or closed.
c) $E_n:=\{x\in[-1,1];f^j(x)\notin B_0\cup J, j=0,\dots,n-1\}$, $n\geqq 1$.

In fact we are not interested in considering the central intervals J and the sets E_n for an individual map f; but for a family $\{f_\mu\}_{\mu\in[0,1]}$ in $\mathscr{G}^1(f)$.

Remark. Let $\{f_\mu\}_{\mu\in[0,1]}$ be in $\mathscr{G}^1(f)$.

a) Corresponding to the C^1 variation f_μ of $f_0=f$, we have the C^1 variations J_μ and $B_{0\mu}$ of J and B_0 respectively. They are C^1 variations in the sense that the boundary

points of J_μ and $B_{0\mu}$ are C^1 functions implicitly defined in some interval $[0,\delta_0]$. For this, the hyperbolicity of the periodic points of $f_0=f$ is important.
b) $B_{0\mu}$ as defined above may be smaller than the immediate basin of f_μ.
c) $E_{n\mu}:=\{x\in[-1,1]; f_\mu^j(x)\notin B_{0\mu}\cup J_\mu, j=0,\ldots,n-1\}, n\geqq 1$. We remark that the connected components of $E_{n\mu}\backslash E_{(n+1)\mu}$ are mapped diffeomorphically by f_μ^n onto connected components of $B_{0\mu}\cap J_\mu$. Therefore the boundary points of $E_{n\mu}$ are also C^1 functions, defined in the same interval $[0,\delta_0]$.

Lemma (1.1). *Let $\{f_\mu\}_{\mu\in[0,1]}$ be in $\mathscr{G}^1(f)$. Given an open central interval J_μ there exist δ in $(0,\delta_0)$, $C_1>0$, $\lambda_1>1$, and $R_1<\infty$ such that; for all $\mu\leqq\delta$ and x in $E_{n\mu}$ $(n\geqq 1)$ we have:*

a) $|\partial_x f_\mu^n(x)|\geqq C_1\lambda_1^n$,
b) $\dfrac{|\partial_{xx} f_\mu^n(x)|}{|\partial_x f_\mu^n(x)|^2}\leqq R_1$,
c) *The Lebesgue measure $|E_{n\mu}|$ converges exponentially fast to zero as n goes to infinity.*

Proof. *a)* First of all we observe that $E_{n\mu}$ $(n\geqq 1$ and $\mu\leqq\delta_0)$ is a decreasing sequence of compact sets, $f_\mu^k(E_{n\mu})\subset E_{(n-k)\mu}$ $(0<k<n)$ and $\bigcap_{j=1}^{\infty} E_{j\mu}$ is a non-empty compact invariant set of f_μ. The set $\bigcap_{j=1}^{\infty} E_{j0}$ does not contain attractive periodic points nor non-hyperbolic periodic points, nor critical points. From Mãné [3] we conclude that $\bigcap_{j=1}^{\infty} E_{j0}$ is a hyperbolic set for f_0. Therefore we can choose some k and some $\bar\lambda>1$ such that $|\partial_x f_0^k(x)|\geqq\bar\lambda$, for all x in $\bigcap_{j=1}^{\infty} E_{j0}$. By continuity we can diminish $\bar\lambda$ a little and take N big enough to ensure $|\partial_x f_0^k(x)|\geqq\bar\lambda$ for all x in $E_{N0}=\bigcap_{j=1}^{N} E_{j0}$. Now, if we diminish $\bar\lambda$ a little more, by remark (c) we can choose δ in $(0,\delta_0)$ such that $|\partial_x f_\mu^k(x)|\geqq\bar\lambda$, for all x in $E_{N\mu}=\bigcap_{j=1}^{N} E_{j\mu}$ and $\mu\leqq\delta$. Now, given $n>N$ we write $n-N=jk+l$, where $0\leqq l<k$. Therefore:

$$|\partial_x f_\mu^n(x)|=|\partial_x f_\mu^{N+l}(f_\mu^{jk}(x))|\,|\partial_x f_\mu^{jk}(x)|.$$

For x in $E_{n\mu}$, the first factor is uniformly bounded away from zero and the second is bigger than $(\bar\lambda)^j$. Part (a) now follows immediately.
b) We observe that:

$$\frac{\partial_{xx} f_\mu^n(x)}{(\partial_x f_\mu^n(x))^2}=\sum_{j=0}^{n-1}\frac{1}{\partial_x f_\mu^{n-1-j}(f_\mu^{j+1}(x))}\cdot\frac{\partial_{xx} f_\mu(f_\mu^j(x))}{(\partial_x f_\mu(f_\mu^j(x)))^2},$$

and the factor $\dfrac{\partial_{xx} f_\mu}{(\partial_x f_\mu)^2}$ is bounded in the complement of J_μ. Now part (b) follows from part (a).
c) Let $K_{n\mu}$ be a connected component of $E_{n\mu}$. We choose the smallest $j\geqq 0$ such that

$$(B_{0\mu}\cup J_\mu)\cap f_\mu^{n+j}(K_{n\mu})\neq\emptyset,$$

j is uniformly bounded by some $M<\infty$. Part (b) implies that f_μ^{n+j} restricted to $K_{n\mu}$ is almost linear. It follows that the proportion of $K_{n\mu}$ mapped by f_μ^{n+j} into $B_{0\mu}\cup J_\mu$

is uniformly positive. It follows that $|E_{n\mu}|$ decreases exponentially with n and part (c) follows. □

Lemma (1.2). *Let $\{f_\mu\}_{\mu\in[0,1]}$ be in $\mathcal{G}^1(f)$. Given an open central interval J_μ there exist $R_2<\infty$ and δ in $(0,\delta_0)$ such that; for all y in J_μ such that $f_\mu([0,y])\subset E_{(n-1)\mu}$ and $\mu\leqq\delta$ we have:*

a) $\dfrac{S_{n\mu}}{R_2}|y|\leqq|\partial_x f_\mu^n(y)|\leqq R_2 S_{n\mu}|y|$, *where* $S_{n\mu}:=|\partial_{xx}f_\mu^n(0)|\neq 0$.

b) $\dfrac{S_{n\mu}}{2R_2}y^2\leqq|f_\mu^n(y)-f_\mu^n(0)|\leqq\dfrac{R_2}{2}S_{n\mu}y^2$.

Proof. We omit this proof which is a straightforward consequence of Lemma (1.1) and the non-degeneracy of the critical point. □

2. Bifurcation Frequency

This section is dedicated to proving Theorem A. We will use the notation and remarks of Sect. (1).

We recall that $\mathcal{U}:=\{\mu\in[0,1];f_\mu(0)\in\Sigma_\mu\}$, where Σ_μ is the Julia set of f_μ. Theorem A has immediate consequences about the bifurcation frequency at zero, that is; about the Lebesgue density at zero of the bifurcation set of the family considered. This is the reason for the title of this section.

The main idea in the proof of Theorem A is that we can decompose the phase interval $[-1,1]$ into two sets: For some special open central interval J_μ, we consider the union of the pre-images of $B_{0\mu}\cup J_\mu$ and its complement. When the critical value traverses the phase interval it meets the union of the pre-images of $B_{0\mu}\cup J_\mu$ for almost all the parameters. We will choose some special J_μ which contains a piece of basin attached to its boundary. This will force the critical value to cross the basin every time it crosses some pre-image of $B_{0\mu}\cup J_\mu$. One technical fact that we will need is some control over the derivatives of the C^1 functions of μ defined by the boundary points of $E_{n\mu}$. This will control the relative motion of the critical value and pre-images of $B_{0\mu}\cup J_\mu$, and is the central point of this section. This is the role of the following lemma. To state the lemma we consider a general open central interval J_μ and denote a boundary point of $E_{n\mu}$ by x_μ^n. We recall that p_μ denotes the periodic repeller of f_μ such that $f_0^k(0)=p_0$ (see Sect. 1). We also remark once more that (by the hyperbolicity of the periodic points of f_0) x_μ^n and p_μ are C^1 functions defined for μ in some interval $[0,\delta_0]$.

Throughout this section and the following one a symbol like $\partial_\mu f_\mu^n(p_\mu)$ means the derivative of f_μ^n with respect to μ at the point p_μ. The argument of a function will never be differentiated.

Fundamental Lemma (2.1). *Given an open central interval J_μ there exists δ in $(0,\delta_0)$ such that, for all $\xi>0$ there exists $\varrho>0$ such that: if $\mu\leqq\delta$ and $|x_\mu^n-p_\mu|<\varrho$, then $|\partial_\mu x_\mu^n-\partial_\mu p_\mu|<\xi$.*

Proof. The strategy of the proof is simple. We calculate the derivatives $\partial_\mu p_\mu$ and $\partial_\mu x_\mu^n$ and afterwards we estimate the difference between them. One basic principle is that $\partial_\mu p_\mu$ and $\partial_\mu x_\mu^n$ are related to $\dfrac{\partial_\mu f_\mu^n}{\partial_x f_\mu^n}$.

We recall that l is the period of the periodic repeller p_μ. Then for any $j \geqq 1$ we have:

$$\partial_\mu p_\mu = \frac{\partial_\mu f_\mu^{jl}(p_\mu)}{1-\partial_x f^{jl}(p_\mu)}.$$

To calculate $\partial_\mu x_\mu^n$ we observe that $f_\mu^n(x_\mu^n)$ is some point a_μ in the forward orbit of the boundary of $B_{0\mu} \cup J_\mu$. Since this forward orbit is finite the derivative $\partial_\mu a_\mu$ is bounded in $[0, \delta]$ for some δ in $(0, \delta_0)$. We have:

$$\partial_\mu x_\mu^n = \frac{\partial_\mu a_\mu}{\partial_x f_\mu^n(x_\mu^n)} - \frac{\partial_\mu f_\mu^n(x_\mu^n)}{\partial_\mu f_\mu^n(x_\mu^n)}.$$

In order to compare $\partial_\mu p_\mu$ and $\partial_\mu x_\mu^n$ we rewrite them as:

$$\partial_\mu p_\mu = \frac{\partial_x f_\mu^{jl}(p_\mu)}{1-\partial_x f_\mu^{jl}(p_\mu)} \cdot \frac{\partial_\mu f_\mu^{jl}(p_\mu)}{\partial_x f_\mu^{jl}(p_\mu)}$$

and

$$\partial_\mu x_\mu^n = \frac{\partial_\mu a_\mu}{\partial_x f_\mu^n(x_\mu^n)} - \frac{1}{\partial_x f_\mu^{jl}(x_\mu^n)} \cdot \frac{\partial_\mu f_\mu^i(f_\mu^{jl}(x_\mu^n))}{\partial_x f_\mu^i(f_\mu^{jl}(x_\mu^n))} - \frac{\partial_\mu f_\mu^{jl}(x_\mu^n)}{\partial_x f_\mu^{jl}(x_\mu^n)},$$

where $i = n - jl \geqq 0$.

We observe that $\dfrac{\partial_\mu f_\mu^t(x)}{\partial_x f_\mu^t(x)}$ is bounded independently of x in $E_{t\mu}$, $t \geqq 1$ and μ in $[0, \delta]$; this fact is a consequence of Lemma (1.1) together with the boundedness of $\dfrac{\partial_\mu f_\mu}{\partial_x f_\mu}$ outside J_μ, and the following formula:

$$\frac{\partial_\mu f_\mu^t(x)}{\partial_x f_\mu^t(x)} = \sum_{k=0}^{t-1} \frac{1}{\partial_x f_\mu^k(x)} \cdot \frac{\partial_\mu f_\mu(f_\mu^k(x))}{\partial_x f_\mu(f_\mu^k(x))}.$$

Now, given any $\xi > 0$ we choose $\varrho > 0$ such that for all x_μ^n such that $|x_\mu^n - p_\mu| < \varrho$ we have n very big. Then we can fix j big enough so that $jl \leqq n$ and

$$\left| \partial_\mu p_\mu + \frac{\partial_\mu f_\mu^{jl}(p_\mu)}{\partial_x f_\mu^{jl}(p_\mu)} \right| < \xi/3,$$

and

$$\left| \partial_\mu x_\mu^n + \frac{\partial_\mu f_\mu^{jl}(x_\mu^n)}{\partial_x f_\mu^{jl}(x_\mu^n)} \right| < \xi/3.$$

Once j is fixed we diminish $\varrho > 0$, if necessary, and by continuity we have

$$\left| \frac{\partial_\mu f_\mu^{jl}(p_\mu)}{\partial_x f_\mu^{jl}(p_\mu)} - \frac{\partial_\mu f_\mu^{jl}(x_\mu^n)}{\partial_x f_\mu^{jl}(x_\mu^n)} \right| \leqq \xi/3.$$

The lemma follows. □

Proof of Theorem (A). Given $\{f_\mu\}_{\mu \in [0,1]}$ in $\mathscr{G}^1(f)$, firstly we choose a special open central interval J_μ. We have two cases: in the first case the Julia set Σ_0 of f_0 does not accumulate at the critical point, and in the second case Σ_0 accumulates at the critical point.

If Σ_0 does not accumulate at the critical point (zero), we choose the smallest j_0 such that the closure of $B_{j_0 0} := \bigcup_{i=0}^{j_0} f_0^{-i}(B_{00})$ contains zero. $B_{j_0 0}$ has two symmetric connected components attached to zero. We take their union with zero to define our open central interval J_0.

If Σ_0 accumulates at zero, given some j_0 we define the open central interval J_0; the smallest open central interval containing the two symmetric connected components of $B_{j_0 0}$ which are the nearest to zero. We choose j_0 big enough to have the distance between J_0 and $\bigcup_{i=1}^{\infty} f_0^i(0)$ positive.

Once $B_{j_0 0}$ and J_0 have been chosen, $B_{j_0\mu}$ and J_μ $(\mu<\delta_0)$ are respectively the C^1 variations of J_0 and of $B_{j_0 0}$ associated to the family considered. We remark that a uniformly positive proportion of J_μ corresponds to the connected components of the basin of f_μ attached to the boundary of J_μ. In the first case this proportion tends to one as μ goes to zero.

Now we define:

$$E_{n\mu}:=\{x\in[-1,1]; f_\mu^j(x)\notin B_{j_0\mu}\cup J_\mu, j=0,\ldots,n-1\}, \qquad n\geqq 1.$$

Though $B_{0\mu}$ was replaced by $B_{j_0\mu}$ in the definition of $E_{n\mu}$ the facts and lemmas that we proved are still true, for the same reason. We decompose the phase interval as:

$$[-1,1]=B_{j_0\mu}\cup J_\mu\cup\left(\bigcup_{n=1}^{\infty}(E_{n\mu}\cap f_\mu^{-n}(J_\mu))\right)\cup\left(\bigcap_{j=1}^{\infty}E_{j\mu}\right),$$

and define the set

$$U_\delta:=\left\{\mu\in[0,\delta]; f_\mu^k(0)\in\bigcup_{n=1}^{\infty}(E_{n\mu}\cap f_\mu^{-n}(J_\mu))\right\},$$

where k is such that $f_0^k(0)$ is the periodic point p_0.

Now we claim that for some δ, U_δ contains almost all $\mathscr{U}\cap[0,\delta]$. In fact: by Lemma (2.1) the boundary points of $E_{n\mu}\cap f_\mu^{-n}(J_\mu)$ which are near to the periodic point p_μ (C^1 variation of p_0) have almost the same μ-derivative as p_μ, by hypothesis we have that $\partial_\mu f_\mu^k(0)|_{\mu=0}\neq\partial_\mu p_\mu|_{\mu=0}$ and by Lemma (1.1) $|E_{n\mu}|$ decreases exponentially as n goes to infinity. The claim follows.

Each connected component of $E_{n\mu}\cap f_\mu^{-n}(J_\mu)$ is mapped diffeomorphically by f_μ^n onto a connected component of $B_{j_0\mu}\cup J_\mu$. By Lemma (1.1) it follows that the proportion of $E_{n\mu}\cap f_\mu^{-n}(J_\mu)$ contained in the basin is comparable to the proportion of J_μ contained in the basin. By Lemma (2.1) and the hypothesis that $\partial_\mu f_\mu^k(0)|_{\mu=0}\neq\partial_\mu p_\mu|_{\mu=0}$, we can conclude that the proportion of parameters μ in $[0,\delta]$ such that $f_\mu^k(0)$ is in the basin of f_μ is positive. When Σ_0 does not accumulate at zero, the proportion of such parameters in $[0,\varepsilon]$ tends to one when ε tends to zero. The theorem follows. □

3. More About Bifurcation Frequency

This section is dedicated to proving Theorem B. As in Yakobson [8], Collet and Eckmann [1], and Guckenheimer [2] we will define an inductive process. In general it is necessary to eliminate parameters to assure metric control throughout the induction. In our case we follow closely Guckenheimer [2] in the definition of the inductive process, but we will exploit the existence of a persistent basin to improve the process without elimination of parameters.

Theorem B is a direct consequence of Theorem A and Theorem C, stated below.

Theorem (C). *Let $\{f_\mu\}_{\mu\in[0,1]}$ be in $\mathcal{G}^2(f)$ such that each f_μ is C^4 and the Julia set Σ_0 of $f_0=f$ does not contain intervals and accumulates at the critical point. Then the Lebesgue density of $\mathcal{U}$ is zero at zero.*

Poincaré maps will be the main tool in the proof of Theorem C. To define the inductive process referred to above, we will choose a special closed central interval J_μ^0 bounded by some connected components of the basin. The Poincaré map $\mathcal{P}_\mu$ of J_μ^0 will permit us to decompose the phase interval and afterwards the parameter interval: we will choose the central connected component J_μ^1 of the domain of $\mathcal{P}_\mu$ to be a new closed central interval, and will consider the union of pre-images of J_μ^1 under $\mathcal{P}_\mu$. For almost all μ in $\mathcal{U}\cap[0,\varepsilon_1]$, some ε_1, we will have $\mathcal{P}_\mu(0)$ in this union. Each connected component of this union will have a definite proportion of basin attached to its boundary. Then when $\mathcal{P}_\mu(0)$ crosses such a component, the proportion of parameters μ for which $\mathcal{P}_\mu(0)$ lies in the basin is uniformly positive. In a second induction step we will consider the Poincaré map $\bar{\mathcal{P}}_\mu$ of J_μ^1 and the analogous procedure. We will need metric control in the phase and parameter intervals.

Given a family $\{f_\mu\}_{\mu\in[0,1]}$ as in Theorem C we start our construction. We define a closed central interval J_0^0: it is the closed connected component of the complement of $B_{j_0 0}=\bigcup_{i=0}^{j_0} f_0^{-i}(B_{00})$ which contains zero. We choose j_0 big enough to have the distance between J_0^0 and $\bigcup_{i=1}^{\infty} f_0^i(0)$ positive. Now we consider J_μ^0 and $B_{j_0\mu}$ ($\mu<\delta_0$), the C^1 variation of J_0^0 and $B_{j_0 0}$, respectively. We also define:

$$E_{n\mu}^0:=\{x\in[-1,1];f_\mu^j(x)\notin B_{j_0\mu}\cup J_\mu^0, j=0,\dots,n-1\}\,,\qquad n\geqq 1\,.$$

Then we decompose the phase interval as:

$$[-1,1]=B_{j_0\mu}\cup J_\mu^0\cup\left(\bigcup_{n=1}^{\infty}(E_{n\mu}^0\cap f_\mu^{-n}(J_\mu^0))\right)\cup\left(\bigcap_{j=1}^{\infty}E_{j\mu}^0\right).$$

As in the proof of Theorem A we define:

$$U_\varepsilon:=\left\{\mu\in[0,\varepsilon];f_\mu^k(0)\in\bigcup_{n=1}^{\infty}(E_{n\mu}^0\cap(f_\mu^{-n}(J_\mu^0))\right\},$$

and for the same reason as in the proof of Theorem A there exists $\varepsilon_0>0$ such that U_{ε_0} contains almost all of $\mathcal{U}\cap[0,\varepsilon_0]$. We will prove that the density of U_{ε_0} is zero at zero.

For μ in U_{ε_0} we consider the Poincaré map $\mathcal{P}_\mu$ of J_μ^0. Its domain is a countable union of closed intervals. One of those intervals, J_μ^1, contains the critical point (zero), we call it the central interval. The other intervals, L_μ^0, we call lateral intervals.

Fundamental Lemma (3.1). *Given $\gamma>2$ there exists a closed central interval J_μ^0, ε in $(0,\varepsilon_0)$ and A, $M<\infty$ such that for all μ in U_ε the Poincaré map $\mathcal{P}_\mu$ of J_μ^0 satisfies:*

a) $|\partial_x\mathcal{P}_\mu(y)|\geqq\gamma$ *and* $\dfrac{|\partial_{xx}\mathcal{P}_\mu(y)|}{|\partial_x\mathcal{P}_\mu(y)|^2}\leqq M$, *if y belongs to any lateral interval L_μ^0.*

b) $\left|\dfrac{\partial_{xx}\mathcal{P}_\mu(z)}{\partial_x\mathcal{P}_\mu(z)}-\dfrac{1}{z}\right|\dfrac{1}{|\partial_x\mathcal{P}_\mu(z)|}\leqq A$, *for all $z\neq 0$ in the central interval J_μ^1.*

Proof. a) The proof has two parts: In the first part, we consider the individual map f_0 and define a Poincaré map $\mathcal{P}_0$ which satisfies part (a). It does not have central interval. In the second part we vary the parameter μ and prove part (a) for $\mathscr{P}_\mu$. When we vary the parameter some central intervals appear but condition (b) is nevertheless satisfied.

Let J_0^0 be the central interval defined before and $\mathscr{P}_0$ its Poincaré map. Let L_0^0 be a lateral interval where $\mathscr{P}_0 = f_0^m$. We consider the maximal interval T which contains L_0^0 and such that f_0^m is monotone in T and $f_0^m(T) \cap B_{00} = \emptyset$. We observe that $f_0^m(T)$ contains J_0^0 and the two connected components of $B_{j_0 0}$ attached to the boundary of J_0^0. By Strien [7] $\sum_{i=0}^{m-1} |f_0^i(T)|$ is uniformly bounded, therefore we can use Köbe's lemma (see Strien [7]) to conclude that f_0^m is almost linear in L_0^0, that is: given x, y in L_0^0 $\frac{|\partial_x f_0^m(x)|}{|\partial_x f_0^m(y)|}$ is uniformly bounded. Now we observe that for a small J_0^0, the minimum possible m is big and consequently T is small. Then we must have a point x in T such that $|\partial_x f_0^m(x)|$ is big. In fact by Köbe's lemma, x is in L_0^0 and we conclude that $|\partial_x \mathscr{P}_0|$ can taken to be uniformly bigger than a given $\gamma > 2$.

Now we consider (as before) $\varepsilon_0 > 0$, J_μ^0 and the Poincaré map $\mathscr{P}_\mu$ of J_μ^0 $(\mu \leqq \varepsilon_0)$. Let us prove part (a): Let y be in some lateral interval L_μ^0 and $\mathscr{P}_\mu(y) = f_\mu^{m+1}(y)$. Then by Lemma (1.2) we have:

$$|\partial_x \mathscr{P}_\mu(y)| \geqq |\partial_x f_\mu^m(f_\mu(y))| \left(\frac{2S_{1\mu}}{R_2^3} |f_\mu(y) - f_\mu(0)| \right)^{1/2}.$$

For μ in U_{ε_0} the points $f_\mu(y)$ and $f_\mu(0)$ define an interval which contains connected components of $E_{m\mu}^0 \cap f_\mu^{-m}(B_{j_0\mu})$. These components are mapped by f_μ^m onto connected components of $B_{j_0\mu}$, by Lemma (1.1), with bounded distortion. Therefore, there exists $C > 0$ such that:

$$|\partial_x f_\mu^m(f_\mu(y))| \, |f_\mu(y) - f_\mu(0)| \geqq C,$$

and if m is big we have $|\partial_x \mathscr{P}_\mu(y)| \geqq \gamma$ as we want. We are left with finitely many lateral intervals with small m. But as $|\partial_x \mathscr{P}_0|$ is bigger than γ, we can vary μ in a small interval $[0, \varepsilon]$ and maintain $|\partial_x \mathscr{P}_\mu|$ bigger than γ in those intervals.

To prove the boundedness of $\frac{\partial_{xx} \mathscr{P}_\mu}{(\partial_x \mathscr{P}_\mu)^2}$ in L_μ^0 we observe that:

$$\frac{\partial_{xx} \mathscr{P}_\mu}{(\partial_x \mathscr{P}_\mu)^2} = \frac{1}{\partial_x f_\mu^m \circ f_\mu} \cdot \frac{\partial_{xx} f_\mu}{(\partial_x f_\mu)^2} + \frac{\partial_{xx} f_\mu^m \circ f_\mu}{(\partial_x f_\mu^m \circ f_\mu)^2},$$

and, by the same argument as before, we have that $|\partial_x f_\mu^m \circ f_\mu| \, |\partial_x f_\mu|^2$ is bounded away from zero in L_μ^0. Then the first term in the above sum is uniformly bounded. By Lemma (1.1) the second term is also uniformly bounded. Part (a) follows.

b) Let $z \neq 0$ be in the central interval J_μ^1 where $\mathscr{P}_\mu = f_\mu^{j+1}$. Then we have:

$$\left| \frac{\partial_{xx} \mathscr{P}_\mu(z)}{\partial_x \mathscr{P}_\mu(z)} - \frac{1}{z} \right| \frac{1}{|\partial_x \mathscr{P}_\mu(z)|} \leqq \frac{|\partial_{xx} f_\mu^j(f_\mu(z))|}{|\partial_x f_\mu^j(f_\mu(z))|^2}$$
$$+ \left| \frac{\partial_{xx} f_\mu(z)}{\partial_x f_\mu(z)} - \frac{1}{z} \right| \frac{1}{|\partial_x f_\mu(z)|} \cdot \frac{1}{|\partial_x f_\mu^j(f_\mu(z))|}.$$

By the non-degeneracy of the critical point we can assume that f_μ is symmetric (e.g. $f_\mu(-z)=f_\mu(z)$), and since f_μ is C^4 we can conclude that there exists $d<\infty$ such that:

$$\left|\frac{\partial_{xx} f_\mu(z)}{\partial_x f_\mu(z)}-\frac{1}{z}\right| \frac{1}{|\partial_x f_\mu(z)|} \leqq d.$$

Part (b) follows from this and Lemma (1.1). □

Besides the metric control in the phase interval obtained in Lemma (3.1) we need metric control in the parameter interval. One important point is to estimate how fast the critical value $\mathscr{P}_\mu(0)$ crosses pre-images of the central interval J_μ^1. For this we need estimates for $\frac{\partial_\mu \mathscr{P}_\mu}{\partial_x \mathscr{P}_\mu}$ and $\partial_\mu \mathscr{P}_\mu$. Before we state the lemmas concerning these estimates, we introduce some notation:

Notation. a) $\tilde{J}_\mu^0$ and $\tilde{J}_\mu^1$ denote the maximal open central intervals such that $\tilde{J}_\mu^0 \backslash J_\mu^0$ and $\tilde{J}_\mu^1 \backslash J_\mu^1$ are contained in the basin of f_μ.
b) y_μ^0 denotes a boundary point of $\tilde{J}_\mu^1$. We remark that $|\partial_x \mathscr{P}_\mu(y_\mu^0)|$ tends to infinity when μ is in U_ε and tends to zero.

Lemma (3.2). *There exist a closed central interval J_μ^0, ε in $(0, \varepsilon_0)$, $T<\infty$ and $\beta>0$ such that for all μ in a connected component U of U_ε we have:*

a) $\dfrac{|\partial_\mu \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|} \leqq \dfrac{T}{|\partial_x \mathscr{P}_\mu(y_\mu^0)|} \cdot \dfrac{|J_\mu^0|}{|U|}$, *in each lateral interval L_μ^0.*

b) $|\partial_\mu \mathscr{P}_\mu| \geqq \beta \dfrac{|J_\mu^0|}{|U|}$, *in the central interval $\tilde{J}_\mu^1$.*

Proof. a) Let μ be in U_{ε_0} (ε_0 as before) and L_μ^0 be a lateral interval where $\mathscr{P}_\mu = f_\mu^{m+k}$. Since $\partial_\mu f_\mu^k(0)|_{\mu=0} \neq 0$ we can choose J_μ^0 small enough to have $|\partial_\mu f_\mu^k|$ bounded away from zero in $\tilde{J}_\mu^0$. We know that

$$\frac{|\partial_\mu \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|} \leqq \frac{|\partial_\mu f_\mu^k|}{|\partial_x f_\mu^k|}\left(1+\frac{1}{|\partial_\mu f_\mu^k|} \cdot \frac{|\partial_\mu f_\mu^m \circ f_\mu^k|}{|\partial_x f_\mu^m \circ f_\mu^k|}\right),$$

and by the proof of Lemma (2.1), $\dfrac{\partial_\mu f_\mu^m \circ f_\mu^k}{\partial_x f_\mu^m \circ f_\mu^k}$ is uniformly bounded in L_μ^0. Therefore there exists $\tilde{T}<\infty$ such that:

$$\frac{|\partial_\mu \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|} \leqq \tilde{T} \frac{|\partial_\mu f_\mu^k|}{|\partial_x f_\mu^k|}.$$

By Lemma (2.1) and the hypothesis that $\partial_\mu f_\mu^k(0)|_{\mu=0} \neq \partial_\mu p_\mu|_{\mu=0}$, we know that there exists ε in $(0, \varepsilon_0)$ such that: for all μ in a connected component U of U_ε, $|\partial_\mu f_\mu^k|$ in L_μ^0 is comparable to $\dfrac{|\Gamma_\mu|}{|U|}$, where Γ_μ is the connected component of the pre-image by f_μ of J_μ^0 which contains $f_\mu^k(0)$.

By Lemma (1.1) we also know that $|\Gamma_\mu|$ is comparable to $\dfrac{|J_\mu^0|}{|\partial_x f_\mu^m(f_\mu^k(y_\mu^0))|}$. Now to conclude part (a) we observe that, by the non-degeneracy of the critical point, $|\partial_x f_\mu^k|$ in L_μ^0 is bigger than $C\,|\partial_x f_\mu^k(y_\mu^0)|$, for some $C>0$. Part (a) follows.
b) Let μ be in a connected component U of U_ε and $\mathscr{P}_\mu = f_\mu^{j+k}$ in the central interval

$\tilde{J}^1_\mu$. Since $|\partial_\mu f^k_\mu|$ is bounded away from zero we have:

$$\partial_\mu \mathscr{P}_\mu = \partial_x f^j_\mu \circ f^k_\mu \cdot \partial_\mu f^k_\mu \left(1 + \frac{1}{\partial_\mu f^k_\mu} \cdot \frac{\partial_\mu f^j_\mu \circ f^k_\mu}{\partial_x f^j_\mu \circ f^k_\mu}\right).$$

If $\varepsilon > 0$ is small enough, $\dfrac{\partial_\mu f^j_\mu \circ f^k_\mu}{\partial_x f^j_\mu \circ f^k_\mu}$ in $\tilde{J}^1_\mu$ is very near to $\partial_\mu p_\mu$ (see proof of Lemma (2.1)). By the hypothesis that $\partial_\mu f^k_\mu(0)|_{\mu=0} \neq \partial_\mu p_\mu|_{\mu=0}$ there exists $\alpha < 1$ such that in $\tilde{J}^1_\mu$ we have:

$$\frac{|\partial_\mu f^j_\mu \circ f^k_\mu|}{|\partial_x f^j_\mu \circ f^k_\mu|} \leqq \alpha |\partial_\mu f^k_\mu| \quad \text{or} \quad |\partial_\mu f^k_\mu| \leqq \alpha \frac{|\partial_\mu f^j_\mu \circ f^k_\mu|}{|\partial_x f^j_\mu \circ f^k_\mu|},$$

therefore in $\tilde{J}^1_\mu$ we have:

$$|\partial_\mu \mathscr{P}_\mu| \geqq (1-\alpha)\, |\partial_x f^j_\mu \circ f^k_\mu|\, |\partial_\mu f^k_\mu|.$$

For the same reason as in part (a) we can conclude part (b). □

We need one more technical lemma concerning mixed derivatives.

Lemma (3.3). *There exist a closed central interval J^0_μ, ε in $(0, \varepsilon_0)$ and d_3, d_4, d_5, $d_6 < \infty$ such that, for all μ in a connected component U of U_ε we have:*

a) $\dfrac{|\partial_{\mu x} \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|^2} \leqq d_3 \dfrac{|J^0_\mu|}{|U|}$, *in a lateral interval L^0_μ.*

b) $\dfrac{|\partial_{\mu\mu} \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|^2} \leqq d_5 \dfrac{|J^0_\mu|^2}{|U|^2}$, *in a lateral interval L^0_μ.*

c) $\dfrac{|\partial_{\mu x} \mathscr{P}_\mu|}{|\partial_\mu \mathscr{P}_\mu|\, |\partial_x \mathscr{P}_\mu|} \leqq d_4$ *and* $\dfrac{|\partial_{\mu\mu} \mathscr{P}_\mu|}{|\partial_\mu \mathscr{P}_\mu|^2} \leqq d_6$, *in* $\tilde{J}^1_\mu \backslash \{0\}$.

Proof. The proof is straightforward and involves no new ideas. We therefore omit it. □

Now, Lemmas (3.1), (3.2), and (3.3) provide all the information we need to start our inductive process.

Induction (First Step). Given $\gamma > 2$ there exist a closed central interval J^0_μ and $\varepsilon_1 > 0$ such that, for all μ in a connected component U of U_{ε_1}, the Poincaré map $\mathscr{P}_\mu$ of J^0_μ satisfies the following recursive properties:

RP1) In each lateral interval L^0_μ we have:

a) $|\partial_x \mathscr{P}_\mu| \geqq \gamma$ and $\dfrac{|\partial_{xx} \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|^2} \leqq \dfrac{D_1}{|\tilde{J}^0_\mu|}$, for some $D_1 < \infty$,

b) $\dfrac{|\partial_{\mu x} \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|^2} \leqq D_3 \dfrac{|J^0_\mu|}{|U|}$ and $\dfrac{|\partial_{\mu\mu} \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|^2} \leqq D_5 \dfrac{|J^0_\mu|^2}{|U|^2}$, for some $D_3, D_5 < \infty$.

RP2) In the central interval $\tilde{J}^1_\mu$ we have:

a) $\left|\dfrac{\partial_{xx} \mathscr{P}_\mu(z)}{\partial_x \mathscr{P}_\mu(z)} - \dfrac{1}{z}\right| \dfrac{1}{|\partial_x \mathscr{P}_\mu(z)|} \leqq \dfrac{D_2}{|\tilde{J}^0_\mu|}$, for all z in $\tilde{J}^1_\mu \backslash \{0\}$ and some $D_2 < \infty$.

b) $\dfrac{|\partial_{\mu x} \mathscr{P}_\mu|}{|\partial_\mu \mathscr{P}_\mu|\, |\partial_x \mathscr{P}_\mu|} \leqq \dfrac{D_4}{|\tilde{J}^0_\mu|}$ and $\dfrac{|\partial_{\mu\mu} \mathscr{P}_\mu|}{|\partial_\mu \mathscr{P}_\mu|^2} \leqq \dfrac{D_6}{|\tilde{J}^0_\mu|}$, for some $D_4, D_6 < \infty$.

RP3)

a) $\frac{|\partial_\mu \mathscr{P}_\mu|}{|\partial_x \mathscr{P}_\mu|} \leqq \frac{T}{|\partial_x \mathscr{P}_\mu(y_\mu^0)|} \frac{|J_\mu^0|}{|U|}$ in each lateral interval L_μ^0, for some $T < \infty$.

b) $|\partial_\mu \mathscr{P}_\mu| \geqq \beta \frac{|J_\mu^0|}{|U|}$ in the central interval $\tilde{J}_\mu^1$, for some $\beta > 0$.

These properties are the metric control that we need to make a first estimate of the proportion of parameters in U_{ε_1} for which the critical value $\mathscr{P}_\mu(0)$ lies in the basin. In the second inductive step we will make a second estimate and so on. We will need this proportion to be positive independently of the inductive step.

Before making any estimate we define the second inductive step and prove that the recursive properties in the first inductive step are preserved. Afterwards we will prove that the proportion referred to above is positive independently of the inductive step. In order for the induction to continue we need to consider parameter intervals closer and closer to zero.

Remark. a) We remark that like J_μ^0 the central interval, J_μ^1 has two connected components of the basin attached to its boundary. We have defined $\tilde{J}_\mu^1$ the maximal open central interval such that $\tilde{J}_\mu^1 \setminus J_\mu^1$ is contained in the basin.
b) Let F_μ be the set of points in J_μ^0 where $\mathscr{P}_\mu$ is not defined. F_μ is the union of some connected components of the basin and a set with null Lebesgue measure. We define:

$$E_{n\mu}^1 := \{y \in J_\mu^0, \mathscr{P}_\mu^i(y) \notin F_\mu \cup \tilde{J}_\mu^1, i = 0, \ldots, n-1\}, \qquad n > 1.$$

We denote a boundary point of $E_{j\mu}^1 \cap \mathscr{P}_\mu^{-j}(\tilde{J}_\mu^1)$ by y_μ^j, $j \geqq 1$. We remember that y_μ^0 denotes a boundary point of $\tilde{J}_\mu^1$.
c) Given x, y in the same connected component of $E_{n\mu}^1$ it follows from (RP1. a) that:

$$\frac{|\partial_x \mathscr{P}_\mu^n(x)|}{|\partial_x \mathscr{P}_\mu^n(y)|} \leqq e^{2\frac{D_1}{|\tilde{J}_\mu^0|}|\mathscr{P}_\mu^n(x) - \mathscr{P}_\mu^n(y)|}.$$

We will use many times this property of bounded distortion. One important point is that the constant D_1 grows during the induction, but the factor $|\mathscr{P}_\mu^n(x) - \mathscr{P}_\mu^n(y)|$ is smaller than $|\tilde{J}_\mu^1|$ and guarantees the control of distortion as the induction goes on.
d) Given z in the central interval J_μ^1 we have from (RP2. a) that:

$$\omega_\mu |z| e^{-\frac{D_2}{|\tilde{J}_\mu^0|}|\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)|} \leqq |\partial_x \mathscr{P}_\mu(z)| \leqq \omega_\mu |z| e^{\frac{D_2}{|\tilde{J}_\mu^0|}|\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)|},$$

where $\omega_\mu := |\partial_{xx} \mathscr{P}_\mu(0)|$. Hence

$$\omega_\mu \frac{z^2}{2} e^{-\frac{D_2}{|\tilde{J}_\mu^0|}|\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)|} \leqq |\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)| \leqq \omega_\mu \frac{z^2}{2} e^{\frac{D_2}{|\tilde{J}_\mu^0|}|\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)|}.$$

In the first inductive step the constant D_2 may be big, and make it difficult to estimate the size of the connected components of the basin, which are attached to the boundary of J_μ^1. We will make this estimate using f_μ directly and Lemma (1.1). We will be able to choose this constant as small as we like from the second inductive step onwards.

The following lemma is an obvious consequence of (RP3. a) and gives us control of the relative motion of the pre-image $\mathscr{P}_\mu^{-j}(\tilde{J}_\mu^1)$ and $\mathscr{P}_\mu(0)$.

Lemma (3.4). *There exist ε in $(0,\varepsilon_1)$ $\beta>0$ and $H<1/2$ such that, for all μ in a connected component U of U_ε, we have:*

a) $|\partial_\mu y^j_\mu| \leqq \beta H \frac{|J^0_\mu|}{|U|}, j \geqq 0.$

b) $|\partial_\mu \mathscr{P}_\mu(0)| \geqq \beta \frac{|J^0_\mu|}{|U|}.$

Proof. All the points y^j $(j \geqq 1)$ are mapped by $\mathscr{P}^j_\mu$ to some point y^0_μ in the boundary of $\tilde{J}^1_\mu$. The point y^0_μ is mapped by $\mathscr{P}_\mu$ (in fact by the extension of $\mathscr{P}_\mu$ to $\tilde{J}^1_\mu$) to a boundary point x^0_μ of $\tilde{J}^0_\mu$. Therefore we have:

$$\partial_\mu y^0_\mu = \frac{\partial_\mu x^0_\mu}{\partial_x \mathscr{P}_\mu(y^0_\mu)} - \frac{\partial_\mu \mathscr{P}_\mu(y^0_\mu)}{\partial_x \mathscr{P}_\mu(y^0_\mu)},$$

but $|\partial_x \mathscr{P}_\mu(y^0_\mu)|$ tends to infinity when μ is in U_ε and tends to zero, and $\partial_\mu x^0_\mu$ is bounded. We also have for $j \geqq 1$ that:

$$\partial_\mu y^j_\mu = \frac{\partial_\mu y^0_\mu}{\partial_x \mathscr{P}^j_\mu(y^j_\mu)} - \frac{\partial_\mu \mathscr{P}^j_\mu(y^j_\mu)}{\partial_x \mathscr{P}^j_\mu(y^j_\mu)}.$$

Now the lemma follows immediatelly from the recursive property (RP3). □

Lemma (3.4) together with the fact that the Lebesgue measure of $E^1_{n\mu}$ decreases exponentially, when n tends to infinity, imply that the set:

$$V_{\varepsilon_2} := \left\{ \mu \in U_{\varepsilon_2}; \mathscr{P}_\mu(0) \in J^1_\mu \cup \left(\bigcup_{j=1}^{\infty} (E_{j\mu} \cap \mathscr{P}_\mu^{-j}(J^1_\mu)) \right) \right\}$$

contains almost all of the set $\mathscr{U} \cap [0, \varepsilon_2]$, for some $0 < \varepsilon_2 \leqq \varepsilon_1$ (ε_1 given by the first induction step).

For μ in V_{ε_2} we consider the Poincaré map $\bar{\mathscr{P}}_\mu$ of J^1_μ. Its domain is a countable union of closed intervals. One of those intervals, J^2_μ, contains the critical point (zero). We call it the central interval. The other intervals, L^1_μ, we call lateral intervals.

We remark that J^2_μ has two connected components of basin attached to its boundary. We define the central interval $\tilde{J}^2_\mu$; the maximal open interval such that $\tilde{J}^2_\mu \backslash J^2_\mu$ is contained in the basin. We denote by z^0_μ a boundary point of $\tilde{J}^2_\mu$.

Now we state the second inductive step.

Induction (Second Step). Given $\bar{\gamma} > 2$ and $\bar{D}_2 > 0$ there exists ε_2 in $(0, \varepsilon_1)$ such that for all μ in a connected component V of V_{ε_2}, the Poincaré map $\bar{\mathscr{P}}_\mu$ of J^1_μ satisfies the following recursive properties:

$\overline{\text{RP1}}$) In a lateral interval L^1_μ we have:

a) $|\partial_x \bar{\mathscr{P}}_\mu| \geqq \bar{\gamma}$ and $\frac{|\partial_{xx} \bar{\mathscr{P}}_\mu|}{|\partial_x \bar{\mathscr{P}}_\mu|^2} \leqq \frac{\bar{D}_1}{|\tilde{J}^1_\mu|}$, for some $\bar{D}_1 < \infty$.

b) $\frac{|\partial_{\mu x} \bar{\mathscr{P}}_\mu|}{|\partial_x \bar{\mathscr{P}}_\mu|^2} \leqq \bar{D}_3 \frac{|J^1_\mu|}{|V|}$ and $\frac{|\partial_{\mu\mu} \bar{\mathscr{P}}_\mu|}{|\partial_x \bar{\mathscr{P}}_\mu|^2} \leqq \bar{D}_5 \frac{|J^1_\mu|^2}{|V|^2}$, for some $\bar{D}_3, \bar{D}_5 < \infty$.

$\overline{\text{RP2}}$) In the central interval $\tilde{J}_\mu^2$ we have:

a) $\left|\dfrac{\partial_{xx}\bar{\mathscr{P}}_\mu(\omega)}{\partial_x\bar{\mathscr{P}}_\mu(\omega)} - \dfrac{1}{\omega}\right| \dfrac{1}{|\partial_x\bar{\mathscr{P}}_\mu(\omega)|} \leqq \dfrac{\bar{D}_2}{|\tilde{J}_\mu^1|}$, for all ω in $\tilde{J}_\mu^2\backslash\{0\}$.

b) $\dfrac{|\partial_{\mu x}\bar{\mathscr{P}}_\mu|}{|\partial_\mu\bar{\mathscr{P}}_\mu|\,|\partial_x\bar{\mathscr{P}}_\mu|} \leqq \dfrac{\bar{D}_4}{|\tilde{J}_\mu^1|}$ and $\dfrac{|\partial_{\mu\mu}\bar{\mathscr{P}}_\mu|}{|\partial_\mu\bar{\mathscr{P}}_\mu|^2} \leqq \dfrac{\bar{D}_6}{|\tilde{J}_\mu^1|}$ for some $\bar{D}_4, \bar{D}_6 < \infty$.

$\overline{\text{RP3}}$)

a) $\dfrac{|\partial_\mu\bar{\mathscr{P}}_\mu|}{|\partial_x\bar{\mathscr{P}}_\mu|} \leqq \dfrac{\bar{T}}{|\partial_x\bar{P}_\mu(z_\mu^0)|}\,\dfrac{|J_\mu^1|}{|V|}$ in each lateral interval L_μ^1, for some $\bar{T} < \infty$.

b) $|\partial_\mu\bar{\mathscr{P}}_\mu| \geqq \bar{\beta}\dfrac{|J_\mu^1|}{|V|}$ in the central interval $\tilde{J}_\mu^2$, for some $\bar{\beta} > 0$.

Proof. Throughout this proof we will consider μ in a connected component V of V_{ε_1}.

$\overline{\text{RP1}}$. a): Let z be in a lateral interval L_μ^1, where $\bar{\mathscr{P}}_\mu = \mathscr{P}_\mu^{m+1}$. From (RP2. a) we have that:

$$|\partial_x\bar{P}_\mu(z)| \geqq |\partial_x\mathscr{P}_\mu^m(\mathscr{P}_\mu(z))|(2\omega_\mu|\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)|)^{1/2}e^{-D_2},$$

where $\omega_\mu := |\partial_{xx}\mathscr{P}_\mu(0)|$. For μ in V_{ε_1} the points $\mathscr{P}_\mu(z)$ and $\mathscr{P}_\mu(0)$ define an interval which contains a connected component of basin. This component is mapped by $\mathscr{P}_\mu^m$, with bounded distortion by (RP1. a), onto a connected component C_μ^1 of $\tilde{J}_\mu^1\backslash J_\mu^1$. Then there exists $C > 0$ such that:

$$|\partial_x\bar{\mathscr{P}}_\mu(z)| \geqq C(\omega_\mu|\partial_x\mathscr{P}_\mu^m(\mathscr{P}_\mu(z))|\,|C_\mu^1|)^{1/2}.$$

Now we estimate $\omega_\mu|C_\mu^1|$: we observe that $\mathscr{P}_\mu$ maps C_μ^1 onto a connected component C_μ^0 of $\tilde{J}_\mu^0\backslash J_\mu^0$. Then there exists t_μ in C_μ^1 such that $|\partial_x\mathscr{P}_\mu(t_\mu)|\,|C_\mu^1| = |C_\mu^0|$ but if we take j such that $\mathscr{P}_\mu = f_\mu^{j+1}$ in J_μ^1 we have:

$$\omega_\mu|C_\mu^1| = \frac{|\partial_x f_\mu^j(f_\mu(0))|}{|\partial_x f_\mu^j(f_\mu(t_\mu))|}\cdot\frac{|\partial_{xx}f_\mu(0)|\,|C_\mu^0|}{|\partial_x f_\mu(t_\mu)|}.$$

By Lemma (1.1) and the non-degeneracy of the critical point we conclude that $\omega_\mu|C_\mu^1| \geqq A_1|\tilde{J}_\mu^1|^{-1}$, for some $A_1 > 0$. It follows from (RP2. a) that there exists $A_2 > 0$ such that:

$$|\partial_x\bar{\mathscr{P}}_\mu(z)| \geqq A_2\omega_\mu^{1/4},$$

since ω_μ tends to infinity when μ tends to zero we can choose ε_2 in $(0, \varepsilon_1)$ such that $|\partial_x\bar{\mathscr{P}}_\mu(z)| \geqq \bar{\gamma}$, for all μ in V_{ε_2}.

As for the boundedness of $\dfrac{\partial_{xx}\bar{\mathscr{P}}_\mu(z)}{(\partial_x\bar{\mathscr{P}}_\mu(z))^2}$, for z in L_μ^1 we observe that:

$$\frac{|\partial_{xx}\bar{\mathscr{P}}_\mu(z)|}{|\partial_x\bar{\mathscr{P}}_\mu(z)|^2} \leqq \frac{|\partial_{xx}\mathscr{P}_\mu^m(\mathscr{P}_\mu(z))|}{|\partial_x\mathscr{P}_\mu^m(\mathscr{P}_\mu(z))|^2} + \frac{1}{|\partial_x\mathscr{P}_\mu^m(\mathscr{P}_\mu(z))|}\left(\frac{D_2}{|\tilde{J}_\mu^0|} + \frac{1}{|\partial_x\mathscr{P}_\mu(z)|\,|z|}\right)$$

and by (RP2. a)

$$|\partial_x\mathscr{P}_\mu(z)|\,|z| \geqq 2|\mathscr{P}_\mu(z) - \mathscr{P}_\mu(0)|\,e^{-\frac{D_2}{|\tilde{J}_\mu^0|}}.$$

By the same argument as before there exists $A_3 > 0$ such that

$$|\partial_x\mathscr{P}_\mu^m(\mathscr{P}_\mu(z))|\,|\partial_x\mathscr{P}_\mu(z)|\,|z| \geqq A_3|C_\mu^1|,$$

by the non-degeneracy of the critical point and Lemma (1.1) we know that $\frac{|C_\mu^1|}{|\tilde{J}_\mu^1|}$ is bounded away from zero. Now we can conclude that there exists $d<\infty$ such that:

$$\frac{|\partial_{xx}\bar{\mathscr{P}}_\mu(z)|}{|\partial_x\bar{\mathscr{P}}_\mu(z)|^2} \leqq \frac{2D_1+D_2}{|\tilde{J}_\mu^0|}+\frac{d}{|\tilde{J}_\mu^1|}.$$

Now we observe that $\frac{|\tilde{J}_\mu^1|}{|\tilde{J}_\mu^0|}$ tends to zero when μ tends to zero and there exists ε_2 in $(0,\varepsilon_1)$ such that $(\overline{\text{RP1}}.\text{a})$ is satisfied.

$\overline{\text{RP3}}$. a): Let z be in a lateral interval L_μ^1, where $\bar{\mathscr{P}}_\mu=\mathscr{P}_\mu^{m+1}$,

$$\frac{\partial_\mu\bar{\mathscr{P}}_\mu}{\partial_x\bar{\mathscr{P}}_\mu}=\frac{\partial_\mu\mathscr{P}_\mu}{\partial_x\mathscr{P}_\mu}\left(1+\frac{1}{\partial_\mu\mathscr{P}_\mu}\cdot\frac{\partial_\mu\mathscr{P}_\mu^m\circ\mathscr{P}_\mu}{\partial_x\mathscr{P}_\mu^m\circ\mathscr{P}_\mu}\right).$$

We recall that $|\partial_x\mathscr{P}_\mu(y_\mu^0)|$ tends to infinity when μ tends to zero. Then by (RP3) it is easy to see that there exist ε_2 in $(0,\varepsilon_1)$ and $H<1/2$ such that for all μ in V_{ε_2} we have:

$$\frac{|\partial_\mu\bar{\mathscr{P}}_\mu(z)|}{|\partial_x\bar{\mathscr{P}}_\mu(z)|}\leqq(1+H)\frac{|\partial_\mu\mathscr{P}_\mu(z)|}{|\partial_x\mathscr{P}_\mu(z)|}.$$

By (RP2. b) $\partial_\mu\mathscr{P}_\mu(z)$ is comparable to $\partial_\mu\mathscr{P}_\mu(0)$. By Lemma (3.4), when μ crosses the component V, the critical value $\mathscr{P}_\mu(0)$ crosses a connected component Γ_μ of $E_{m\mu}^1 \cap\mathscr{P}_\mu^{-m}(J_\mu^1)$ and from (RP2. b) we conclude that there exists $K<\infty$ such that

$$|\partial_\mu\mathscr{P}_\mu(z)|\leqq K\frac{|\Gamma_\mu|}{|V|}.$$

By (RP1. a) we know that $|\Gamma_\mu|$ is comparable to $|J_\mu^1|\,|\partial_x\mathscr{P}_\mu^m(\mathscr{P}_\mu(z_\mu^0))|^{-1}$ and by (RP2. a) we also know that $|\partial_x\mathscr{P}_\mu(z)|\geqq|\partial_x\mathscr{P}_\mu(z_\mu^0)|e^{-2D_2}$. It follows that there exists $\bar{T}<\infty$ such that:

$$\frac{|\partial_\mu\bar{\mathscr{P}}_\mu(z)|}{|\partial_x\bar{\mathscr{P}}_\mu(z)|}\leqq\frac{\bar{T}}{|\partial_x\bar{\mathscr{P}}_\mu(z_\mu^0)|}\cdot\frac{|J_\mu^1|}{|V|},$$

for all z in a lateral interval L_μ^1 and μ in V_{ε_2}.

$\overline{\text{RP3}}$. b). Let ω be in the central interval $\tilde{J}_\mu^2$, where $\bar{\mathscr{P}}_\mu=\mathscr{P}_\mu^{j+1}$,

$$\partial_\mu\bar{\mathscr{P}}_\mu=\partial_x\mathscr{P}_\mu^j\circ\mathscr{P}_\mu\cdot\partial_\mu\mathscr{P}_\mu\left(1+\frac{1}{\partial_\mu\mathscr{P}_\mu}\cdot\frac{\partial_\mu\mathscr{P}_\mu^j\circ\mathscr{P}_\mu}{\partial_x\mathscr{P}_\mu^j\circ\mathscr{P}_\mu}\right).$$

For the same reason as in the proof of $(\overline{\text{RP3}}.\text{a})$ we know that there exists ε_2 in $(0,\varepsilon_1)$ such that for all μ in V_{ε_2} we have:

$$|\partial_\mu\bar{\mathscr{P}}_\mu(\omega)|\geqq\tfrac{1}{2}|\partial_x\mathscr{P}_\mu^j(\mathscr{P}_\mu(\omega))|\cdot|\partial_\mu\mathscr{P}_\mu(\omega)|,$$

and there exists $\bar{\beta}>0$ such that:

$$|\partial_\mu\bar{\mathscr{P}}_\mu(\omega)|\geqq\bar{\beta}\frac{|J_\mu^1|}{|V|}.$$

RP2. a). For $\omega \neq 0$ in the central interval $\tilde{J}_\mu^2$, where $\bar{\mathscr{P}}_\mu = \mathscr{P}_\mu^{j+1}$ we have:

$$\left|\frac{\partial_{xx}\bar{\mathscr{P}}_\mu(\omega)}{\partial_x\bar{\mathscr{P}}_\mu(\omega)} - \frac{1}{\omega}\right| \frac{1}{|\partial_x\bar{\mathscr{P}}_\mu(\omega)|} \leqq \frac{|\partial_{xx}\mathscr{P}_\mu^j(\mathscr{P}_\mu(\omega))|}{|\partial_x\mathscr{P}_\mu^j(\mathscr{P}_\mu(\omega))|^2} + \left|\frac{\partial xx\mathscr{P}_\mu(\omega)}{\partial_x\mathscr{P}_\mu(\omega)} - \frac{1}{\omega}\right| \frac{1}{|\partial_x\mathscr{P}_\mu(\omega)|} \cdot \frac{1}{|\partial_x\mathscr{P}_\mu^j(\mathscr{P}_\mu(\omega))|}.$$

The first term is bounded by $\frac{2D_1}{|\tilde{J}_\mu^0|}$ and the second term is term by $\frac{D_2}{|\tilde{J}_\mu^0|}$. Since $\frac{|\tilde{J}_\mu^1|}{|\tilde{J}_\mu^0|}$ tends to zero when μ tends to zero there exists ε_2 in $(0, \varepsilon_1)$ such that $(\overline{\text{RP2}}.\text{a})$ is satisfied.

$\overline{\text{RP2}}$. b). We omit the proof which is straightforward.

$\overline{\text{RP1}}$. b). In a lateral interval L_μ^1, where $\bar{\mathscr{P}}_\mu = \mathscr{P}_\mu^{m+1}$ we have:

$$\frac{\partial\mu x\bar{\mathscr{P}}_\mu}{(\partial_x\bar{\mathscr{P}}_\mu)^2} = \frac{\partial_{xx}\mathscr{P}_\mu^m \circ \mathscr{P}_\mu}{(\partial_x\mathscr{P}_\mu^m \circ \mathscr{P}_\mu)^2} \cdot \frac{\partial_\mu\mathscr{P}_\mu}{\partial_x\mathscr{P}_\mu} + \frac{1}{\partial_x\mathscr{P}_\mu^m \circ \mathscr{P}_\mu} \cdot \frac{\partial_{\mu x}\mathscr{P}_\mu}{(\partial_x\mathscr{P}_\mu)^2} + \frac{1}{\partial_x\mathscr{P}_\mu} \frac{\partial_{\mu x}\mathscr{P}_\mu^m \circ \mathscr{P}_\mu}{(\partial_x\mathscr{P}_\mu^m \circ \mathscr{P}_\mu)^2}.$$

We claim that the first term of this sum is bounded by some constant times $\frac{|J_\mu^1|}{|V|}$. To prove the boundedness of the other terms and the boundedness of $\frac{\partial_{\mu\mu}\bar{\mathscr{P}}_\mu}{(\partial_x\bar{\mathscr{P}}_\mu)^2}$, the argument is the same. Let us prove the claim. It follows from the proof of $(\overline{\text{RP3}}.\text{a})$ that there exists $M < \infty$ such that:

$$\frac{|\partial_{xx}\mathscr{P}_\mu^m \circ \mathscr{P}_\mu|}{|\partial_x\mathscr{P}_\mu^m \circ \mathscr{P}_\mu|^2} \frac{|\partial_\mu\mathscr{P}_\mu|}{|\partial_x\mathscr{P}_\mu|} \leqq \frac{\gamma}{\gamma - 1} \frac{D_1}{|\tilde{J}_\mu^0|} \frac{M}{|\partial_x\bar{\mathscr{P}}_\mu(z_\mu^0)|} \frac{|J_\mu^1|}{|V|}.$$

We also know from $(\overline{\text{RP2}}.\text{a})$ that:

$$2|\partial_x\bar{\mathscr{P}}_\mu(z_\mu^0)| \geqq \bar{\omega}_\mu |\tilde{J}_\mu^2| e^{-D_2},$$

and there exists q_μ in $\tilde{J}_\mu^2$ such that:

$$\bar{\omega}_\mu |\tilde{J}_\mu^2| \geqq \frac{\omega_\mu}{|\partial_x\mathscr{P}_\mu(q_\mu)|} |\bar{\mathscr{P}}_\mu(\tilde{J}_\mu^2)| e^{-D_1},$$

once $|\bar{\mathscr{P}}_\mu(\tilde{J}_\mu^2)| \geqq |C_\mu^1|$ and $\omega_\mu |C_\mu^1|$ is comparable to $|\tilde{J}_\mu^1|^{-1}$ we conclude that there exists $C > 0$ such that:

$$\bar{\omega}_\mu |\tilde{J}_\mu^2| \geqq \frac{C}{|\tilde{J}_\mu^2|\,|\tilde{J}_\mu^1|}.$$

Our claim and $(\overline{\text{RP1}}.\text{b})$ follow. □

Now before the proof of Theorem B we remark once again that the constants $\bar{D}_i$ $(i = 2, 3, 4, 5, 6)$ can be chosen to be arbitrarily small. In fact, reducing $\varepsilon_2 > 0$ in the second inductive step we can make the metric control as good as we want. The Poincaré map $\bar{\mathscr{P}}_\mu$ can be made very expansive and linear in the lateral intervals and very quadratic in the central interval. Another point that we want to make concerns the properties of $\omega_\mu |C_\mu^1|$, $\frac{|\tilde{J}_\mu^1|}{|\tilde{J}_\mu^0|}$ and $\frac{|C_\mu^1|}{|\tilde{J}_\mu^1|}$, that we have used during the proof of the second inductive step. It is easy to see that if C_μ^2 is one of the connected components of basin attached to the boundary of J^2_μ and μ is in V we have that: $\bar{\omega}_\mu |C^2_\mu|$ tends to infinity, $\frac{|\tilde{J}^1_\mu|}{|\tilde{J}^2_\mu|}$ tends to zero and $\frac{|C^2_\mu|}{|\tilde{J}^2_\mu|}$ is bounded away from zero when V tends to zero. These facts guarantee that the induction does not stop.

Proof of Theorem C. We take a connected component U of U_{ξ_1} (some $\xi_1>0$) and make a first estimate of the proportion of parameters in U, for which the critical value lies in the basin.

By Lemma (3.4) there exists $\xi_1>0$ such that for μ in U the critical value $\mathscr{P}_\mu(0)$ traverses the interval J^0_μ very fast; at least twice as fast as the boundary points y^j_μ of $E^1_{j\mu}\cap\mathscr{P}_\mu^{-j}(\tilde{J}^1_\mu)$ with the relative speed of $\mathscr{P}_\mu(0)$ and y^j_μ being almost linear, more precisely:

$$|\partial_\mu \mathscr{P}_\mu(0)| \geqq 2|\partial_\mu y^j_\mu|, \quad j \geqq 1,$$

and for μ_1, μ_2 in U we have:

$$\frac{|\partial_\mu(\mathscr{P}_\mu(0)-y^j_\mu)|_{\mu=\mu_1}|}{|\partial_\mu(\mathscr{P}_\mu(0)-y^j_\mu)|_{\mu=\mu_2}|} \leqq \frac{1+H}{1-H} e^{2D_6}.$$

Since $|E^1_{j\mu}|$ decreases exponentially when j goes to infinity we conclude once again that

$$V_{\xi_1} := \left\{\mu\in[0,\xi_1];\, \mathscr{P}_\mu(0)\in J^1_\mu \cup \left(\bigcup_{j=1}^{\infty} (E^1_{j\mu}\cap\mathscr{P}_\mu^{-j}(J^1_\mu))\right)\right\}$$

contains almost all the set $\mathscr{U}\cap[0,\xi_1]$.

Let us make a first estimate of the proportion of $\mathscr{U}$ inside U. We prove that the proportion $\frac{|U\cap V_{\xi_1}|}{|U|}$ is smaller than one. We recall that the connected components C^1_μ of $\tilde{J}^1_\mu\setminus J^1_\mu$ are contained in the basin. Then we consider connected components V_{jl} ($j\geqq 0$) of V_{ξ_1} and the maximal open intervals $\tilde{V}_{jl}$ such that for all μ in $\tilde{V}_{jl}\setminus V_{jl}$ we have $\mathscr{P}_\mu(0)$ in $E^1_{j\mu}\cap\mathscr{P}_\mu^{-j}(\tilde{J}^1_\mu\setminus J^1_\mu)$ if $j\geqq 1$, or in $\tilde{J}^1_\mu\setminus J^1_\mu$ if $j=0$. We have that:

$$|U\cap V_{\xi_1}| \leqq |U| - \sum_{j,l} |\tilde{V}_{jl}\setminus V_{jl}|,$$

and we define

$$m_{jl} := \inf\left\{\frac{|C^1_\mu|}{|\tilde{J}^1_\mu|};\, \mu\in\tilde{V}_{jl}\right\}.$$

It follows that:

$$|U\cap V_{\xi_1}| \leqq |U| - \frac{1-H}{1+H} e^{-D_6} \sum_{j,l} m_{jl} |\tilde{V}_{jl}|,$$

and from the non-degeneracy of the critical point and Lemma (1.1), there exists $\alpha>0$ such that $m_{jl}\geqq\alpha\frac{|C^1_\mu|}{|\tilde{J}^1_\mu|}$, for all μ in $\tilde{V}_{jl}$. We can assume that $\left|\bigcup_{j,l}\tilde{V}_{jl}\right| = |U|$ and therefore:

$$\sum_{j,l} m_{jl}|\tilde{V}_{jl}| \geqq \alpha \int_U \frac{|C^1_\mu|}{|\tilde{J}^1_\mu|} d\mu,$$

from (RP2. a) we know that:

$$\frac{|C_\mu^1|}{|\tilde{J}_\mu^1|} \geqq \frac{|C_\mu^0|}{4|\mathscr{P}_\mu(\tilde{J}_\mu^1)|} e^{-2D_2}.$$

By (RP2. b) $\frac{|C_\mu^0|}{|\mathscr{P}_\mu(\tilde{J}_\mu^1)|}$ is an almost linear function in U and varies between one and its minimum. Then we conclude that

$$\int_U \frac{|C_\mu^1|}{|\tilde{J}_\mu^1|} d\mu \geqq |U| \frac{e^{-2D_2}}{8},$$

and for some $\varrho < 1$ we have

$$|U \cap V_{\xi_1}| \leqq \varrho |U|.$$

The constants involved in the above estimates do not depend on the component U of U_{ξ_1}. They depend only on the constants of the metric control in the first inductive step. We conclude that

$$|V_{\xi_1}| < \varrho |U_{\xi_1}|.$$

One important fact that we have remarked upon is that the metric control becomes better and better as the induction goes on, so that ϱ is uniformly smaller than one and the theorem follows. □

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References

1. Collet, P., Eckmann, J.-P.: On the abundance of aperiodic behaviour for maps on the interval. Commun. Math. Phys. **73**, 115–160 (1980)
2. Guckenheimer, J.: Renormalization of one-dimensional mappings and strange attractors, Lefstchetz Centennial Conference, Part III, 1984. Vergovsky, A. (ed.). Contemp. Math. **58**, 143–160 (1982)
3. Mañé, R.: Hyperbolicity, sinks and measures in one-dimensional dynamics. Commun. Math. Phys. **100**, 495–524 (1985)
4. Milnor, J., Thurston, W.: On iterated maps on the interval. I, II. Preprint (1976)
5. Newhouse, S., Palis, J.: Cycles and bifurcation theory. Astérisque **3**, 44–140 (1976)
6. Palis, J., Takens, F.: Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms. Inv. Math. **82**, 397–422 (1985)
7. van Strien, S.Y.: Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition. Commun. Math. Phys. **128**, 437–495 (1990)
8. Yakobson, M.V.: Absolutely continuous invariant measures for one parameter families of one-dimensional maps. Commun. Math. Phys. **81**, 39–88 (1981)

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