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Finite-Dimensional Irreducible Representations of the Quantum Superalgebra $U_q[gl(n/1)]$

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Abstract. It is shown that every finite-dimensional irreducible module over the general linear Lie superalgebra $gl(n/1)$ can be deformed to an irreducible module of $U_q[gl(n/1)]$, a q-analogue of the universal enveloping algebra of $gl(n/1)$. The results are extended also to all Kac modules, which in the atypical cases remain indecomposible. Within each module expressions for the transformations of the Gel'fand-Zetlin basis under the action of the algebra generators are written down. An analogoue of the Poincaré-Birkhoff-Witt theorem is formulated.

1. Introduction

During the last years the quantum groups became a field of increasing interest in various branches of physics and mathematics. The concept of a quantum group was introduced by Drinfeld [6]. Its essence crystallized from the intensive development of the quantum inverse problem method [8] and the investigations related to the Yang-Baxter equation (see the collection of papers [14] and the references therein).

An important class of quantum groups are the quantized universal enveloping algebras, called also quantum algebras. A quantum algebra $U_q[G]$ associated with the algebra G is a deformation of the universal enveloping algebra $U[G]$ of G endowed with a structure of a Hopf algebra. In all applications we know these are one-parameter deformations (see, however, [28]). The first example of a Hopf algebra of this kind was given for $G = sl(2)$ [23]. The generalization to any Kac-Moody Lie algebra with a symmetrizable generalized Cartan matrix is due to Drinfeld [7] and Jimbo [15]. An example of a quantum superalgebra, i.e., of a quantum algebra associated with a Lie superalgebra, namely the orthosymplectic Lie superalgebra (LS) $osp(1/2)$ was considered by Kulish [24]. The corresponding construction for an arbitrary Kac-Moody superalgebra with a symmetrizable generalized Cartan matrix was reported in [39]; an independent approach for the basic Lie superalgebras [18] was developed in [2].

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The representation theory of the quantum algebras has been also an object of intensive studies. An important result in this frame was the proof that for generic values of q ($=q$ is not a root of unity) any finite-dimensional module of $A_n \equiv sl(n+1)$, $n \in \mathbf{N}$, can be deformed in an irreducible module of $U_q[A_n]$ [16] and that one obtains in this way all finite-dimensional irreducible modules of $U_q[A_n]$ [34]. This result has been generalized by Lusztig [26] for all integrable modules over Kac-Moody algebras with symmetrizable generalized Cartan matrix. In [17] Jimbo gave explicit expressions for the deformed $U_q[A_n]$ modules in terms of the "undeformed" Gel'fand-Zetlin basis. Similar results for $U_q[so(n)]$ have been reported in [11] without specifying however whether the deformation is a Hopf algebra. Available are also various results for the representations of some lower rank quantum (super)algebras [22, 25, 36] and for other, mainly oscillator representations of the quantum algebras associated with all classical Lie algebras [4, 12] and some of the basic Lie superalgebras [2, 3, 5, 9, 10]. The latter are obtained through realization of these algebras in terms of q-deformed Bose and Fermi operators [1, 4, 27].

In the present paper we show as a main result that for generic values of q every finite-dimensional irreducible module over the general linear Lie superalgebra $gl(n/1)$ can be turned into an irreducible module of $U_q[gl(n/1)]$. To this end we use the results from [39] for the basic LS $sl(n/1) \equiv A(n-1/0)$ [18]. In order to simplify the transformation relations [as this is usually done also for $sl(n)$] we extend $sl(n/1)$ by an one-dimensional center to $gl(n/1)$. Similarly for $gl(n)$ [17] we write down explicit relations for the transformations of the $U_q[gl(n/1)]$ modules in terms of the Gel'fand-Zetlin basis, introduced in [30, 31].

The Lie superalgebras $sl(n/1)$ and more generally $sl(n/m)$ belong to the class of the simple complex Lie superalgebras (LS's), classified by Kac [19–21] and by Scheunert et al. [37, 38]. More precisely they belong to the subclass of the basic LS's [18]. Kac showed that the irreducible finite-dimensional modules of any basic LS G fall into two classes, referred as to typical and atypical. All of them are highest weight modules. In particular each irreducible $sl(n/1)$ module [and hence irreducible $gl(n/1)$ module] $W(\Lambda)$ with a highest weight Λ can be obtained from a $sl(n/1)$ module $\bar{V}(\Lambda)$ ([18], p. 613) induced out of an irreducible module of the even subalgebra with the same highest weight Λ. Following the terminology of [40] we call all such modules $\bar{V}(\Lambda)$ Kac modules. In case of a typical representation $W(\Lambda)=\bar{V}(\Lambda)$. In the atypical case each Kac module $\bar{V}(\Lambda)$ is reducible and indecomposible; it contains a unique maximal proper submodule J, such that

$$W(\Lambda)=\bar{V}(\Lambda)/J\,. \qquad (1.1)$$

We shall see that in the quantum case these properties still hold: the deformed module $\bar{V}(\Lambda)$ remains reducible and indecomposible, J is deformed onto itself, so that (1.1) also holds and gives a q-deformed atypical module. All modules we consider are simultaneously irreducible, reducible or indecomposible both with respect to $U_q[sl(n/1)]$ and $U_q[gl(n/1)]$. Therefore we use throughout the more convenient $gl(n/1)$ notation.

We recall for further reference that the universal enveloping algebra $U[gl(n/1)] \equiv U$ can be defined as a $\mathbf{Z}_2$-graded associative algebra with unity, generated by the indeterminants e_{ij}, $i, j = 1, \ldots, n+1$ under the relations

$$\begin{aligned}[e_{ij}, e_{kl}] &\equiv e_{ij}e_{kl} - (-1)^{\theta_{ij}\theta_{kl}} e_{kl}e_{ij} \\ &= \delta_{jk}e_{il} - (-1)^{\theta_{ij}\theta_{kl}} \delta_{li}, e_{kj}; \end{aligned} \qquad (1.2)$$

here and everywhere in the sequel

$$\theta_{ij}=\theta_i+\theta_j \tag{1.3}$$

and

$$\theta_i=0 \quad \text{for} \quad i\leqq n, \qquad \theta_i=1 \quad \text{for} \quad i>n. \tag{1.4}$$

The $\mathbf{Z}_2$-grading on $U_q[gl(n/1)]$ is imposed from the requirement that e_{ij} is even (respectively odd), if θ_{ij} is an even (respectively odd) number.

The paper is organized as follows. In Sect. 2 we give a definition of $U_q[gl(n/1)]$. In Sect. 3 we introduce an analogue of the canonical generators e_{ij} of U and write down the supercommutation relations between the generators, which hold also for $U_q[gl(n/m)]$ (to our knowledge such relatively compact relations have not been written even for $U_q[gl(n)]$); see, for instance the commutation relations in [13]. We formulate also the analogue of the Poincaré-Birkhoff-Witt theorem (P.B.W. Theorem). Section 4 is devoted to the representation theory of $U_q[gl(n/1)]$ according to what we have said above.

2. The Quantum Algebra $U_q[gl(n/1)]$

$U_q[gl(n/1)]\equiv U_q$ is a free associative algebra with unity **1** generated by e_i, f_i, $k_j\equiv q^{h_j/2}$, $i=1,\dots,n$, $j=1,\dots,n+1$, and the relations (unless otherwise stated the indices i, j below run over all possible values):

1) $$k_ik_j=k_jk_i, \qquad k_ik_i^{-1}=k_i^{-1}k_i=1; \tag{2.1}$$

2) $$k_ie_jk_i^{-1}=q^{\frac{1}{2}(\delta_{ij}-\delta_{i,j+1})}e_j, \qquad k_if_jk_i^{-1}=q^{\frac{1}{2}(\delta_{i,j+1}-\delta_{ij})}f_j; \tag{2.2}$$

3) $$e_if_j-f_je_i=\delta_{ij}(q-q^{-1})^{-1}(k_i^2k_{i+1}^{-2}-k_{i+1}^2k_i^{-2}), \qquad i=1,\dots,n-1, \tag{2.3}$$

$$e_nf_n+f_ne_n=(q-q^{-1})^{-1}(k_n^2k_{n+1}^2-k_n^{-2}k_{n+1}^{-2}); \tag{2.4}$$

4) $$e_ie_j=e_je_i, \quad f_if_j=f_jf_i, \quad \text{if} \quad |i-j|\neq 1, \tag{2.5}$$

$$e_n^2=f_n^2=0; \tag{2.6}$$

5) $$e_i^2e_{i+1}-(q+q^{-1})e_ie_{i+1}e_i+e_{i+1}e_i^2=0, \tag{2.7}$$

$$f_i^2f_{i+1}-(q+q^{-1})f_if_{i+1}f_i+f_{i+1}f_i^2=0, \tag{2.8}$$

$$e_{i+1}^2e_i-(q+q^{-1})e_{i+1}e_ie_{i+1}+e_ie_{i+1}^2=0, \qquad i=1,\dots,n-2, \tag{2.9}$$

$$f_{i+1}^2f_i-(q+q^{-1})f_{i+1}f_if_{i+1}+f_if_{i+1}^2=0, \qquad i=1,\dots,n-2. \tag{2.10}$$

The $\mathbf{Z}_2$ grading in $U_q[gl(n/1)]$ is uniquely defined by the requirement that the only odd generators are e_n, f_n:

$$\begin{aligned}&\deg(e_i)=\deg(f_i)=0, \quad i=1,\dots,n-1, \quad \deg(e_n)=\deg(f_n)=1,\\&\deg(k_i)=0, \quad i=1,\dots,n+1.\end{aligned} \tag{2.11}$$

It is straightforward to show that U_q is a Hopf superalgebra with respect to a counity ε, a comultiplication Δ and an antipode S defined as

$$\varepsilon(e_i)=\varepsilon(f_i)=\varepsilon(h_i)=0, \tag{2.12}$$

$$\begin{aligned}
&\Delta(k_i)=k_i\otimes k_i\,,\\
&\Delta(e_i)=e_i\otimes k_ik_{i+1}^{-1}+k_i^{-1}k_{i+1}\otimes e_i\,, \qquad i=1,\dots,n-1\,,\\
&\Delta(f_i)=f_i\otimes k_ik_{i+1}^{-1}+k_i^{-1}k_{i+1}\otimes f_i\,, \qquad i=1,\dots,n-1\,,\\
&\Delta(e_n)=e_n\otimes k_nk_{n+1}+k_n^{-1}k_{n+1}^{-1}\otimes e_n\,,\\
&\Delta(f_n)=f_n\otimes k_nk_{n+1}+k_n^{-1}k_{n+1}^{-1}\otimes f_n\,,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&S(k_i)=k_i^{-1}\,,\\
&S(e_i)=-qe_i\,, \quad S(f_i)=-q^{-1}e_i\,, \quad i=1,\dots,n-1\,,\\
&S(e_n)=-e_n\,, \quad S(f_n)=-f_n\,.
\end{aligned} \tag{2.14}$$

Note [29] that in a Hopf superalgebra A the multiplication $m^{\otimes}$ on $A\otimes A$ is a graded one: for any homogeneous elements a, b, c, $d\in A$

$$m^{\otimes}(a\otimes b\otimes c\otimes d)\equiv(a\otimes b)(c\otimes d)=(-1)^{\deg(b)\deg(c)}ac\otimes bd\,. \tag{2.15}$$

Moreover $\Delta:A\to A\otimes A$ is a morphism in the sense of graded algebras.

3. An Analogue of P.B.W. Theorem

Let $H=\text{lin.env.}\{h_i\equiv e_{ii}\,|\,i=1,\dots,n+1=N\}$ be the Cartan subalgebra of the nonquantum ($\equiv$classical) $gl(n/1)$ (see (1.2) or, for instance, [32]), H^* – the dual to H space with a bases

$$\varepsilon^1,\varepsilon^2,\dots,\varepsilon^N, \qquad \varepsilon^i(h_j)=\delta^i_j\,. \tag{3.1}$$

Define a nondegenerate hermitian form on H^* as

$$(\varepsilon^i,\varepsilon^j)=(-1)^{\theta_i}\delta_{ij}\,. \tag{3.2}$$

On the Cartan subalgebra of $sl(n/1)$ the form (3.2) is proportional to the Killing form. Moreover $(e_i=e_{i,i+1}, f_i=e_{i+1,i})$

$$[h,e_i]=(\varepsilon^i-\varepsilon^{i+1})(h)e_i\,, \quad [h,f_i]=(-\varepsilon^i+\varepsilon^{i+1})(h)f_i \quad \forall h\in H\,, \tag{3.3}$$

i.e., $\varepsilon^i-\varepsilon^{i+1}$, $-\varepsilon^i+\varepsilon^{i+1}$ are the roots of e_i and f_i, respectively [Eqs. (3.4) follow also from (2.2) at $q\to 1$]. Denote by Q the root lattice,

$$Q=\left\{\sum_{i=1}^{n} n_i(\varepsilon^i-\varepsilon^{i+1})\,|\,n_i\in\mathbf{Z}\right\}\subset H^* \tag{3.4}$$

and associate with each e_i, f_i, $k_i\in U_q$ degrees $\varepsilon^i-\varepsilon^{i+1}$, $-\varepsilon^i+\varepsilon^{i+1}$, 0 from Q, respectively. This turns U_q into a Q-graded algebra. Thus U_q is both $\mathbf{Z}_2$-graded and Q-graded. In order to avoid possible confusions (and in agreement with the classical terminology) we call the Q-degree of any homogeneous element $a\in U_q$ a weight of a and refer to a as to a weight vector.

For any $\mathbf{Z}_2$-homogeneous weight vectors $a,b\in U_q$ define a q^{κ}-deformed supercommutator ($\kappa=\pm1$) as follows:

$$[a,b]_{q^{\kappa}}=ab-(-1)^{\deg(a)\deg(b)}q^{\kappa(\alpha,\beta)}ba\,, \tag{3.5}$$

where $\deg(a)$, $\deg(b)$ are the degrees of a and b with respect to $\mathbf{Z}_2$ and α,β are the weights, the Q-degrees of a and b, respectively. In the case $q=1$ or if $a\perp b$ (3.5) becomes usual supercommutator

$$[a,b]=ab-(-1)^{\deg(a)\deg(b)}ba\,. \tag{3.6}$$

Denote $e_{i,i+1}=e_i$, $e_{i,i+1}=f_i$, $i=1,\dots,n$ and set for each $i\neq j=1,\dots,N$,

$$e_{ij}=[\dots[e_i, e_{i+1}]_{q^{-1}}, e_{i+2}]_{q^{-1}},\dots,e_{j-1}]_{q^{-1}} \quad \text{if} \quad i<j, \tag{3.7}$$

$$e_{ij}=[\dots[f_i, f_{i-1}]_q, f_{i-2}]_q,\dots,f_{j+1}]_q \quad \text{if} \quad i>j. \tag{3.8}$$

Each $e_{ij}\in U_q$ is a weight vector with a weight $\varepsilon^i-\varepsilon^j$ and a degree $\deg(e_{ij})=\theta_i+\theta_j\equiv\theta_{ij}$. At $q=1$ e_{ij} become the canonical root vectors of $gl(n/1)$. Therefore also at $q\neq 1$ it is more natural to call e_{ij}, $i\neq j=1,\dots,N$ root vectors and their Q-degrees – roots. Moreover e_{ij} is positive, $e_{ij}>0$ (respectively negative, $e_{ij}<0$) if $i<j$ (respectively $i>j$). We consider each set Δ_+,Δ_- as totally ordered, setting

$$e_{ij}<e_{kl} \quad \text{if} \quad i<k \quad \text{or if} \quad i=k \quad \text{and} \quad j<l. \tag{3.9}$$

The computation of the "commutation relations" between the root vectors is fairly lengthy and we present only the barest outline. Denote h_i as e_{ii}, $h_i=e_{ii}$, $i=1,\dots,N$. Let

$$\theta(i_1>i_2>\dots>i_k)=\begin{cases}1 & \text{if} \quad i_1>i_2>\dots>i_k,\\ 0 & \text{otherwise}\end{cases}$$

and $\theta(i_k<\dots<i_2<i_1)=-\theta(i_1>i_2>\dots>i_k)$. Then one has:

1.
$$[q^{e_{ii}/2}, q^{e_{jj}/2}]=0; \tag{3.10}$$

2.
$$q^{e_{ii}/2}e_{jk}-q^{\frac{1}{2}(\delta_{ij}-\delta_{ik})}e_{jk}q^{e_{ii}/2}=0; \tag{3.11}$$

3. For any $e_{ij}>0$, $e_{kl}<0$,

$$\begin{aligned}[e_{ij}, e_{kl}]=&\{-\theta(j>k>i>l)(-1)^{\theta_k}(q-q^{-1})e_{kj}e_{il}\\ &-\delta_{il}\theta(j>k)(-1)^{\theta_{lk}}e_{kj}+\delta_{jk}\theta(i>l)e_{il}\}q^{(-1)^{\theta_i}e_{ii}-(-1)^{\theta_k}e_{kk}}\\ &+q^{(-1)^{\theta_j}e_{jj}-(-1)^{\theta_l}e_{ll}}\{\theta(k>j>l>i)(-1)^{\theta_j}(q-q^{-1})e_{il}e_{kj}\\ &-\delta_{il}\theta(k>j)(-1)^{\theta_{ij}}e_{kj}+\delta_{jk}\theta(l>i)e_{il}\}\\ &+\frac{\delta_{il}\delta_{jk}}{q-q^{-1}}\{q^{e_{ii}-(-1)^{\theta_{ij}}e_{jj}}-q^{-e_{ii}+(-1)^{\theta_{ij}}e_{jj}}\};\end{aligned} \tag{3.12}$$

4. For $0<e_{ij}<e_{kl}$,

$$[e_{ij}, e_{kl}]_{q^{-1}}=\delta_{jk}e_{il}-\theta(l>j>k>i)(-1)^{\theta_k}(q-q^{-1})e_{kj}e_{il}; \tag{3.13}$$

5. For $0>e_{ij}>e_{kl}$,

$$[e_{ij}, e_{kl}]_q=\delta_{jk}e_{il}+\theta(i>k>j>l)(-1)^{\theta_k}(q-q^{-1})e_{kj}e_{il}; \tag{3.14}$$

6.
$$[e_{ij}, e_{ij}]=0, \quad i\neq j. \tag{3.15}$$

The essential outcome from (3.15) is that the square of each odd root vector is zero:

$$(e_{ij})^2=0 \quad \text{if} \quad \theta_i+\theta_j=1. \tag{3.16}$$

One can unify Eqs. (3.13) and (3.14) in two different ways:

a) For $0\lessgtr e_{ij}\lessgtr e_{kl}$,

$$[e_{ij}, e_{kl}]_{q^{\mp 1}}=\delta_{jk}e_{il}-\theta(l\gtrless j\gtrless k\gtrless i)(-1)^{\theta_k}(q-q^{-1})e_{kj}e_{il}; \tag{3.17}$$

b) For $0<e_{ij}<e_{kl}$ or $e_{ij}<e_{kl}<0$,

$$[e_{ij}, e_{kl}]_{q^{-1}}=\delta_{jk}e_{il}-\delta_{il}(-1)^{\theta_{ij}\theta_{kl}}q^{\varepsilon_i^2}e_{kj} \\ -\{(-1)^{\theta_k}\theta(l>j>k>i)+(-1)^{\theta_i}\theta(k>i>l>j)\}(q-q^{-1})e_{kj}e_{il}. \tag{3.18}$$

Proposition 1. *The relations* (3.10)–(3.15) *hold if and only if the defining Eqs.* (2.1)–(2.10) *are fulfilled. Therefore* $U_q[gl(n/1)]$ *can be defined as an associative algebra with unity, generated form the indeterminants* e_{ij}, $k_i=q^{e_{ii}/2}$, $i\neq j=1,\ldots,n+1=N$ *and the relations* (3.10)–(3.15).

Remark. We have written the supercommutation relations (3.10)–(3.15) in a form which is more general than what is required by $U_q[gl(n/1)]$. For $i,j,k,l=1,\ldots,n+m$ Eqs. (3.10)–(3.15) define the quantum superalgebra $U_q[gl(n/m)]$. At $q\to 1$ these relations reduce to (1.2).

Proposition 2 (P.B.W. Theorem). *The set of all ordered monomials*

$$(e_{12})^{p_{12}}(e_{13})^{p_{13}}\ldots(e_{n,n+1})^{p_{n,n+1}}(k_1)^{p_1}\ldots(k_{n+1})^{p_{n+1}} \\ \times(e_{21})^{p_{21}}(e_{31})^{p_{31}}\ldots(e_{n+1,n})^{p_{n+1,n}}, \tag{3.19}$$

where $(p_1,\ldots,p_{n+1})\in\mathbf{Z}^{n+1}$,

$$p_{ij}\in\mathbf{Z}_2 \quad \text{if} \quad \theta_{ij}=1, \qquad p_{ij}\in\mathbf{Z}_+ \quad \text{if} \quad \theta_{ij}=0 \quad \text{or} \quad 2, \tag{3.20}$$

constitute a basis in $U_q[gl(n/1)]$.

In all essential points the proof is very similar to the one for $U_q[sl(n+1)]$ in [35]. We mark only the main steps.

1. Let U_q^+, U_q^- and U_q^0 be the subalgebras of U_q, generated by Δ_+, Δ_- and $(k_1^{\pm 1},\ldots,k_{n+1}^{\pm 1})\equiv K$, respectively. Observe that whenever $e_{kj}e_{il}$ appears in the right-hand side of (3.12)–(3.14) the multiples always supercommute:

$$e_{kj}e_{il}=(-1)^{\theta_{kj}\theta_{il}}e_{il}e_{kj}. \tag{3.21}$$

Therefore $U_q=U_q^-U_q^0U_q^+$. Since U_q^+ is a polynomial only of $e_1,\ldots,e_n$, U_q^- – of $f_1,\ldots,f_n$ and U_q^0 – of K, from $x^+x^0x^-=0$ ($x^\pm\in U_q^\pm$, $x^0\in U_q^0$) it follows that either x^+ or x^0 or $x^-=0$. This finally yields (see for more details [35]) that ($\otimes$ below is a tensor product of vector spaces)

$$U_q=U_q^-\otimes U_q^0\otimes U_q^+. \tag{3.22}$$

2. Clearly all monomials

$$(k_1)^{p_1}\ldots(k_{n+1})^{p_{n+1}}, \qquad (k_1,\ldots,k_{n+1})\in\mathbf{Z}^{n+1} \tag{3.23}$$

define a basis in U_q^0.

3. From (3.13) and (3.21) one easily concludes that any $x^+\in U_q^+$ is a (finite) linear combination of ordered monomials

$$(e_{12})^{p_{12}}(e_{13})^{p_{13}}\ldots(e_{n,n+1})^{p_{n,n+1}} \tag{3.24}$$

with p_{ij} satisfying (3.20). The linear independence of the vectors (3.24), which are monomials of only even root vectors has been proved in [34]. The monomials (3.24) of only odd positive root vectors are also linearly independent since different monomials have different weights. Both observations now give that all monomials (3.24) constitute a basis in U_q^+.

4. Similarly one concludes that all

$$(e_{21})^{p_{21}}(e_{31})^{p_{31}}\dots(e_{n+1,n})^{p_{n+1,n}}, \tag{3.25}$$

for which (3.20) holds, give a basis in U_q^-. In view of (3.22) the monomials (3.19) constitute a basis in U_q.

4. Finite-Dimensional Representations of $U_q[gl(n/1)]$

We now proceed to show that every finite-dimensional irreducible $gl(n/1)$ module can be turned into an irreducible $U_q[gl(n/1)]$ module. We recall [30, 31] that the Kac modules

$$\bar{V}([M_{1,n+1}, M_{2,n+1}, \dots, M_{n+1,n+1}]) \equiv \bar{V}([M]_{n+1})$$

of $gl(n/1)$ are in one-to-one correspondence with the set of all complex $n+1$-tuples

$$[M]_{n+1} \equiv [M_{1,n+1}, M_{2,n+1}, \dots, M_{n+1,n+1}], \tag{4.1}$$

for which

$$M_{i,n+1} - M_{j,n+1} \in \mathbf{Z}_+ \qquad \forall i<j=1,\dots,n. \tag{4.2}$$

The Gel'fand-Zetlin basis (GZ-basis) in $\bar{V}([M]_{n+1})$ consists of all patterns

$$(M) \equiv \begin{pmatrix} M_{1,n+1}, & M_{2,n+1}, & \dots, & M_{n,n+1}, & M_{n+1,n+1} \\ M_{1n}, & M_{2n}, & \dots, & M_{nn} & \\ \vdots & \vdots & & & \\ M_{12}, & M_{22} & & & \\ M_{11} & & & & \end{pmatrix} \tag{4.3}$$

which are consistent with the conditions:

$$\begin{aligned} &(1) \qquad M_{in} = M_{i,n+1} - \varphi_i, \ \varphi_1, \varphi_2, \dots, \varphi_n = 0, 1; \\ &(2) \qquad M_{i,j+1} - M_{ij} \in \mathbf{Z}_+, \quad \mathrm{M}_{ij} - \mathrm{M}_{i+1\,j+1} \in \mathbf{Z}_+ \quad \forall i<j=1,\dots,n-1. \end{aligned} \tag{4.4}$$

For an arbitrary GZ pattern (M) denote by $(M)_{\pm i,j}$ the pattern obtained from (M) by the replacement $M_{ij} \to M_{ij} \pm 1$; set $L_{ij} = M_{ij} - i$ and let $[x] = \dfrac{q^x - q^{-x}}{q - q^{-1}}$.

Proposition 3. *Each $gl(n/1)$ Kac module $\bar{V}([M]_{n+1})$ is an $U_q[gl(n/1)]$ module with respect to the following transformation of the basis ($k=1,\dots,n-1$):*

$$h_i(M) = \left(\sum_{j=1}^{i} M_{ji} - \sum_{j=1}^{i-1} M_{j,i-1}\right)(M), \qquad i=1,\dots,n+1; \tag{4.5}$$

$$f_k(M) = \sum_{j=1}^{k} \left| \frac{\prod_{i=1}^{k+1} [L_{i,k+1} - L_{jk} + 1] \prod_{i=1}^{k-1} [L_{i,k-1} - L_{jk}]}{\prod_{i \neq j=1}^{k} [L_{ik} - L_{jk} + 1][L_{ik} - L_{jk}]} \right|^{1/2} (M)_{-jk}, \tag{4.6}$$

$$e_k(M)=\sum_{j=1}^{k}\left|\frac{\prod_{i=1}^{k+1}[L_{i,k+1}-L_{jk}]\prod_{i=1}^{k-1}[L_{i,k-1}-L_{jk}-1]}{\prod_{i\neq j=1}^{k}[L_{ik}-L_{jk}][L_{ik}-L_{jk}-1]}\right|^{1/2}(M)_{jk}, \tag{4.7}$$

$$f_n(M)=\sum_{i=1}^{n}(1-\varphi_i)(-1)^{i-1}(-1)^{\varphi_1+\dots+\varphi_{i-1}}[L_{i,n+1}+L_{n+1,n+1}+2n+1]^{1-p}$$
$$\times\left|\frac{\prod_{k=1}^{n-1}[L_{k,n-1}-L_{i,n+1}]}{\prod_{k\neq i=1}^{n}[L_{k,n+1}-L_{i,n+1}]}\right|^{1/2}(M)_{-in}, \tag{4.8}$$

$$e_n(M)=\sum_{i=1}^{n}\varphi_i(-1)^{i-1}(-1)^{\varphi_1+\dots+\varphi_{i-1}}[L_{i,n+1}+L_{n+1,n+1}+2n+1]^{p}$$
$$\times\left|\frac{\prod_{k=1}^{n-1}[L_{k,n-1}-L_{i,n+1}]}{\prod_{k\neq i=1}^{n}[L_{k,n+1}-L_{i,n+1}]}\right|^{1/2}(M)_{in}, \tag{4.9}$$

where $p=0, \frac{1}{2}, 1$.

If

$$M_{j,n+1}+M_{n+1,n+1}\neq j-n \qquad \forall j=1,\dots,n, \tag{4.10}$$

then $\bar{V}([M]_{n+1})$ *is an irreducible* $U_q[gl(n/1)]$ *module for any* $p=0, \frac{1}{2}, 1$.

If for certain $j=1,\dots,n$

$$M_{j,n+1}+M_{n+1,n+1}=j-n, \tag{4.11}$$

then the corresponding Kac module is reducible. More precisely one has the following cases.

a) *The case* $p=0$. $\bar{V}([M]_{n+1})$ *is an indecomposible module. The maximal invariant subspace* $W([M]_{n+1})$ *is irreducible. It is a linear span of all* GZ-*patterns* (4.3) *for which* $\varphi_j=0$. *The factor space*

$$\bar{V}([M]_{n+1})/W([M]_{n+1}) \tag{4.12}$$

is an irreducible U_q *module isomorphic to the subspace* $W([M]_{-j,n+1})$ *spanned on all* GZ-*patterns corresponding to* $\varphi_j=1$.

b) *The case* $p=1$. $\bar{V}([M]_{n+1})$ *is indecomposable. The maximal invariant subspace is* $W([M]_{-j,n+1})$ *and it is irreducible. The irreducible* U_q *module*

$$\bar{V}([M]_{n+1})/W([M]_{-j,n+1}) \tag{4.13}$$

is isomorphic to $W([M]_{n+1})$.

c) *The case* $p=1/2$. *Now* $\bar{V}([M]_{n+1})$ *is a direct sum of the irreducible* U_q *modules* $W([M]_{n+1})$ *and* $W([M]_{-j,n+1})$,

$$\bar{V}([M]_{n+1})=W([M]_{n+1})\oplus W([M]_{-j,n+1}). \tag{4.14}$$

The above proposition together with case a), if (4.11) holds, indicates that every finite dimensional irreducible $gl(n/1)$ module can be deformed to an irreducible U_q module. We formulate this result as a separate statement.

Proposition 4. *Each finite-dimensional irreducible $gl(n/1)$ module $W([M]_{n+1})$, i.e., the linear span of all* GZ*-patterns* (4.3) *consistent with the conditions*

$$(1)\qquad M_{in}=M_{i,n+1}-\varphi_i, \varphi_1,\ldots,\varphi_n=0,1\,;$$

$$(2)\qquad M_{i,j+1}-M_{ij}\in\mathbf{Z}_+\,,\quad M_{ij}-M_{i+1,j+1}\in\mathbf{Z}_+\quad \forall i<j=1,\ldots,n-1\,;\qquad (4.15)$$

$$(3)\ \textit{if}\ M_{j,n+1}+M_{n+1,n+1}=j-n,\ \textit{then}\ \varphi_j=0,$$

is deformed to an irreducible $U_q[gl(n/1)]$ module (for generic values of q) by the transformation relations (4.5)–(4.9) *for p being either* 0, *or* 1/2.

5. Discussion

The main result of the paper, formulated in Proposition 4, was proved by writing down explicit relations for the transformation of the basis under the action of the generators of $U_q[gl(n/1)]$ [see Eqs. (4.5)–(4.9)]. To this end we have used essentially the classical transformation formulae, achieving representation of the quantum superalgebra U_q simply by deforming the matrix elements in an appropriate way. The basis within each module remains the same, the Gel'fand-Zetlin basis, for both $U[gl(n/1)]$ and its deformation $U_q[gl(n/1)]$. Unfortunately similar results are unavailable at present for all finite-dimensional irreducible representation of $gl(n/m)$ (partial results have been reported in [33]) or for the other basic LS's. The situation is even worse. Although the finite-dimensional irreducible modules are completely classified [18] (this is not the case with the indecomposible modules) at present even the dimensions of certain atypical modules are unknown. This is the reason why we have not considered here the more general quantum superalgebra $U_q[gl(n/m)]$. The present paper is not answering also the question whether the deformed finite-dimensional irreducible modules of the (classical) $gl(n/1)$ exhaust all such modules of the quantum algebra. We have also not touched the representation theory of $U_q[gl(n/1)]$ for q being a root of unity.

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