Julia Sets and Complex Singularities in Hierarchical Ising Models

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Abstract. We study the analytical continuation in the complex plane of free energy of the Ising model on diamond-like hierarchical lattices. It is known [12, 13] that the singularities of free energy of this model lie on the Julia set of some rational endomorphism f related to the action of the Migdal–Kadanoff renorm-group. We study the asymptotics of free energy when temperature goes along hyperbolic geodesics to the boundary of an attractive basin of f. We prove that for almost all (with respect to the harmonic measure) geodesics the complex critical exponent is common, and compute it.

1. Introduction

The purpose of this article is to analyse complex singularities in temperature of the free energy $\mathscr{F}$ in the Ising model on diamond-like hierarchical lattices. According to the traditional point of view a phase transition manifests itself as a singularity of $\mathscr{F}$ as a function of thermodynamic parameters (like temperature and external magnetic field). From this point of view the theory of phase transitions should describe the domain of analyticity of $\mathscr{F}$ and the type of its singularities at points of phase transition (see [1], where diverse approaches to the first of these problems are discussed).

Since $\mathscr{F}$ is real analytic outside of points of phase transition, it can be continued into complex space with respect to the thermodynamic parameters. Description of its complex singularities is of great interest for the theory of phase transitions because it determines analytic properties of the thermodynamic function.

The celebrated Lee–Yang theory (see [2]) gives a realisation of this approach describing the singularities of the analytic continuation of the free energy in the ferromagnetic Ising model with respect to the external magnetic field. It proves that the zeroes of the grand partition function in the ferromagnetic Ising model lie on the imaginary axis, and hence complex singularities of the free energy lie on the imaginary axis as well. An important problem stated in [2] is to study the

limit distribution of zeros of the grand partition function, since the free energy can be expressed as a logarithmic potential over this distribution.

The problem of description of complex singularities of the analytic continuation of thermodynamic functions in temperature is also very interesting from different points of view. Many properties of asymptotic behavior of thermodynamic functions in the vicinity of a critical point were investigated through the Kadanoff–Wilson–Fisher renormalisation group theory (see e.g. [3,4]). It gives a local form of real critical singularities of thermodynamic functions which has a nice universal scaling structure. A problem is how are these singularities continued to complex space and what is their global structure in complex space?

Unfortunately no general theory like the Lee–Yang theory exists which describes for general models global complex singularities of thermodynamic functions in the complex temperature plane. However some exact results were obtained for the two dimensional Ising model. The main tool here is the famous Onsager solution. It turns out that in an isotropic two dimensional Ising model the zeroes of the partition function lie asymptotically on two circles $e^{-2J/T} = \pm 1 + \sqrt{2}e^{i\varphi}$ (it was conjectured by Fisher [5] and proved in [6]). Later it was shown (see [7,8] and references there) that in anisotropic two dimensional Ising models on diverse lattices the zeroes of the partition function fill some planar regions in the complex temperature plane and in some cases the density of the limit distribution of zeroes can be found explicitly.

In the present paper we consider another exactly solvable model, namely the Ising model on diamond-like hierarchical lattices (see [9–11]). In this case the Migdal–Kadanoff renormalization transformation turns out to be a rational map f on the Riemann sphere which can be found explicitly. The following nice observation was made in papers [12,13]: the set of complex points of a phase transition coincides with the Julia set of this map. In this paper we study the analytical properties of the free energy F near the singular points. We believe that clear understanding of analytical properties of thermodynamical functions in the complex plane has much to do with the physical nature of the model.

The hierarchical sequence of the diamond-like lattices depends on one natural parameter $b \geqq 2$. The lattice Γ_0 is just two "outer" sites related by a bond. In order to obtain Γ_1 we insert in between the outer sites b inner sites related by bonds with outer ones (see Fig. 0 for $b=3$). Then in order to obtain Γ_{n+1} we replace each bond in Γ_n by the lattice $\Gamma_1, n=1,2,\ldots$.

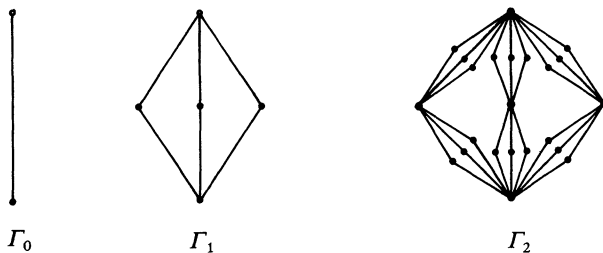$$\Gamma_0 \qquad\qquad \Gamma_1 \qquad\qquad\qquad \Gamma_2$$

Fig. 0

We will refer to papers [11–15] for the description of the Ising model on these lattices and the calculation of the thermodynamical functions. The starting point for us is the following explicit formula for the free energy:

$$\mathscr{F} = -\frac{J}{2} - \frac{T}{2} \sum_{n=0}^{\infty} (2b)^{-n} \ln(1 + t_n^b), \tag{1.1}$$

where $b = 2^{d-1}, d$ is the "dimension" of the lattice, J is the interaction constant, T is temperature, and the sequence $t_n, n \geq 0$, is given by the following recurrent equation

$$t_{n+1} = f(t_n), \quad n \geq 0, \tag{1.2}$$

where

$$f(t) = \frac{4t^b}{(1 + t^b)^2}, \tag{1.3}$$

with the initial data

$$t_0 = \exp\left(-\frac{2J}{bT}\right) \equiv G(T). \tag{1.4}$$

Equations (1.2)–(1.4) mean from the physical point of view that the map $T \mapsto G^{-1} \circ f \circ G(T)$ gives the rescaling of temperature under the Migdal–Kadanoff renorm-group transformation (see [9–15]). Note that the points $t = 0$ and $t = 1$ are superstable fixed points of the map $t \mapsto f(t)$ (low- and high-temperature fixed points of the renorm-group) and for $b > 1$ there exists the unique unstable fixed point t_c on $[0, 1]$. The critical temperature T_c is equal to $G^{-1}(t_c)$.

Formulas (1.1)–(1.4) make sense for complex values of T as well. So, we can consider the analytical continuation of free energy $\mathscr{F}(T)$ from the positive axis $T > 0$ into the complex plane. It is not hard to see that the singularities of $\mathscr{F}$ lie on the Julia set $J(f)$ [12, 13]. (For the definition of the latter, see one of the surveys [16–19].)

Let us consider now the immediate attracting basins Ω_0 and Ω_1 of points 0 and 1. One can show that $\bar{\mathbf{C}} \backslash J(f)$ is the union of preimages of these domains, and Ω_0 is a Jordan domain in $J(f)$ (see Sect. 2). In this paper we study the boundary properties of $\mathscr{F}$ in the domain Ω_0. To this end let us consider the Riemann map $\psi : \Omega_0 \to \mathbf{U}$ of Ω_0 onto the unit disk. The hyperbolic geodesics in Ω_0 are just the ψ^{-1}-images of the radii in $\mathbf{U}$. Denote by B_τ the geodesic ending at $\tau \in \partial\Omega_0$. Let us consider also the harmonic measure μ on $\partial\Omega_0$, i.e. $\mu = \psi_*^{-1}\lambda$, where λ is the Lebesgue measure on the circle $\partial\mathbf{U} \equiv \mathbf{T}$. For $t \in B_\tau$ denote by $l(t)$ the length of B_τ from t to τ (perhaps, $l(t) = \infty$).

In the present paper we prove:

(i) The derivative $\mathscr{F}'$ of free energy is continuous up to the boundary of Ω_0.

(ii) For $b > 2$ the second derivative is discontinuous in Ω, and has the following asymptotics on μ-almost all geodesics:

$$\lim_{t \to \tau, t \in B_\tau} \frac{\ln|\mathscr{F}''(t)|}{-\ln l(t)} \equiv \alpha_c = 1 - \frac{\ln 2}{\ln b} = 1 - \frac{1}{d-1}. \tag{1.5}$$

This means that for almost all geodesics the specific heat critical exponent in the region of low temperatures is universal and equal to $1 - \dfrac{1}{d-1}$.

Now let us dwell in more detail on the content of the paper.

In Sect. 2 we describe the dynamical properties of the endomorphism f. In particular, we show using the Douady–Hubbard straightening theorem that Ω_0 is a Jordan domain.

In Sect. 3 we show that $\mathscr{F}'$ is continuous in $cl\,\Omega_0$ and that $\partial\Omega_0$ is the natural boundary of analyticity of $\mathscr{F}$. The proof is based upon some amusing observations concerning f (its relation to the Koebe function and a Tchebyshev polynomial).

In Sect. 4 we discuss some technical background: the Bowen–Ruelle–Sinai thermodynamical formalism and the construction of the natural extension (the inverse limit) of f. These are the main tools (together with the ergodic theorem) for the accurate computation of the critical exponent.

In Sect. 5 we discuss the functional equation for $\mathscr{F}''$ and related spectral properties of the weighted substitution operator in the disk-algebra.

Section 6 is the central section of the paper: here we give the computation of the critical exponent, provided $\mathscr{F}''$ is not continuous up to the boundary of Ω_0.

In Sect. 7 we prove that $\mathscr{F}''$ really satisfies this property, which completes the proof of the main result.

In the last Sect. 8 we discuss some related problems.

2. Dynamics of the Map $f: t \mapsto \dfrac{4t^b}{(1+t^b)^2}$

We refer to the surveys [16–19] for the general view of the dynamics of complex rational maps. We will use some concepts and facts of this theory without extra explanations.

Let us introduce the following notations:

$f^{\circ n} = f \circ \cdots \circ f$ is the n-fold iterate of f;

$C(f)$ is the set of its critical points (a rational map of degree d has $2d-2$ critical points counting with multiplicity);

$\|\cdot\|$ is the spherical metric on $\bar{\mathbf{C}}$;

$\mathbf{U} = \{z : |z| \leq 1\}$ is the closed disk;

$\mathbf{U}^0 = \operatorname{int} \mathbf{U}$ is its interior;

$\mathbf{T} = \partial\mathbf{U}$ is the unit circle;

$B(a,r) = \{z \in \mathbf{C} : |z - a| \leq R\}$ for $a \in \mathbf{C}$;

$J(f)$ is the Julia set of f.

The function

$$f \equiv f_b : t \mapsto \frac{4t^b}{(1+t^b)^2}$$

is related to the well-known *extremal Koebe function* (see [20])

$$\mathscr{K}_0(z) = \frac{z}{(1-z)^2}.$$

Setting $K(z) = -4\mathscr{K}_0(-z)$ and $S(z) \equiv S_b(z) = z^b$ we have $f = K \circ S$.

The relation of f to the Koebe function is quite mysterious, especially if one
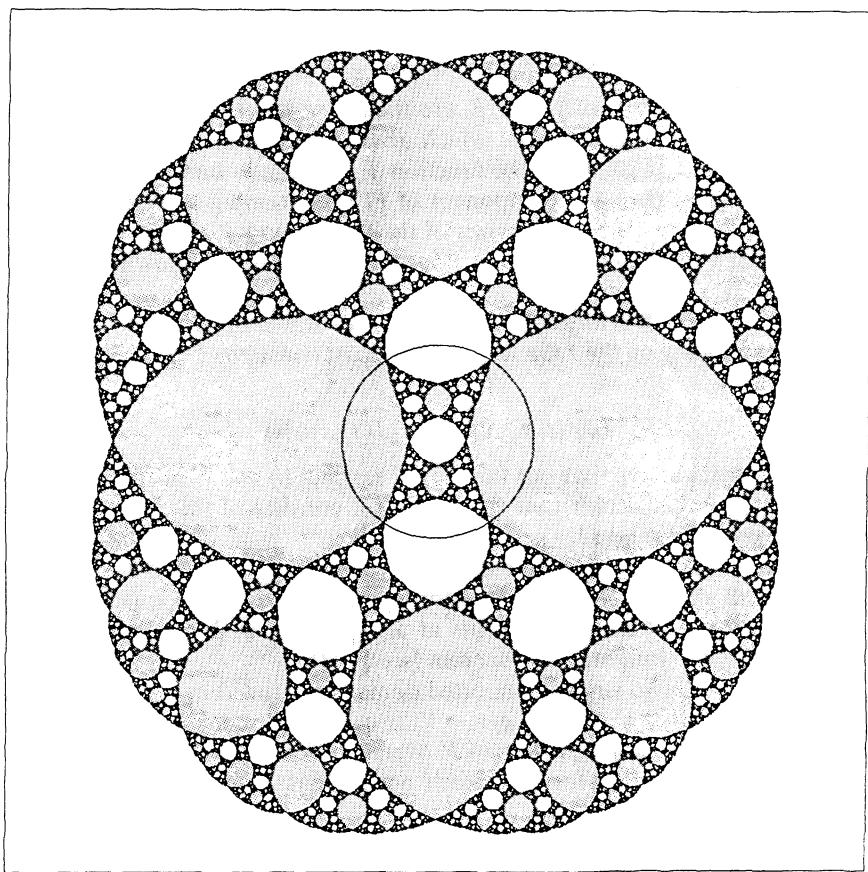
Fig. 1

relates the coefficient 4 to the Koebe constant 1/4. It becomes still more amusing if we observe that $K(t)$ is conformally conjugated to the *Tchebyshev polynomial* $T:\tau \mapsto 2\tau^2 - 1$. Indeed, the function K has two simple critical points $c_1 = 1$ and $c_2 = -1$. Moreover, $f(c_1) = c_1$, i.e. c_1 is a superstable fixed point, and $c_2 \mapsto \infty \mapsto 0$, where 0 is repelling fixed point. Up to conformal conjugation, T is the unique rational function of degree 2 possessing such properties. More specifically, $\varphi \circ K \circ \varphi^{-1} = T$, where $\varphi : t \mapsto \dfrac{1+t}{1-t}$ is the Möbius transformation mapping the triple $\{0, 1, \infty\}$ onto the triple $\{1, \infty, -1\}$. In particular, it follows that the Julia set $J(K)$ coincides with the negative semi-axis $[-\infty, 0] = \varphi^{-1}[-1, 1]$.

The power functions and Tchebyshev polynomials play a particular role in the iteration theory. They appear as the exceptions in a number of problems; e.g., only these functions have a Julia set with simple geometry. The composition $f = K \circ S$ does not possess such a property (see Fig. 1).

A rational function g is called *critically finite* if the orbits $\{g^n(c_i)\}$ of all its critical points are finite.

By the chain rule

$$C(f) = C(S) \cup S^{-1}C(K) = \{0, \infty\} \cup \{\alpha_i\}_{i=1}^b \cup \{\beta_j\}_{j=1}^b,$$

where α_i are the bth roots of 1, and β_j are the bth roots of -1. Moreover, 0 and $\alpha_1 = 1$ are superstable fixed points which absorb the orbits of all other critical points: $\alpha_i \mapsto 1$, $\beta_j \mapsto \infty \mapsto 0$. Thus, the function f is critically finite.

Denote by $\Omega_a \equiv \Omega(a)$, the component of $F(f)$ containing a. The domains Ω_0 and Ω_1 are called the *immediate basins* of the fixed points 0 and 1.

We say that a rational function g *satisfies the axiom A* if the following equivalent properties hold:

(i) The orbits of all critical points converge to stable cycles;

(ii) g is expanding on the Julia set, i.e., there exist constants $C > 0$ and $\lambda > 1$ such that

$$\| dg^n(z) \| \geqq C\lambda^n \quad (z \in J(f), \ n \in \mathbf{N}).$$

It follows from above that our function f satisifes (i) and hence satisfies axiom A. This implies in particular that the Fatou set consists of the preimages of the immediate basins Ω_0 and Ω_1.

Set $\Omega = cl\Omega_0$ and $\Gamma = \partial\Omega_0$ (these notations will be used up to the end of the paper). We will show now that Γ is a Jordan curve and even a quasicircle[1] (but by Fatou's theorem it has no tangents at any point). To this end we apply the Douady–Hubbard straightening theorem (see [21]).

Let V and V' be two simply connected domains bounded by piecewise-smooth curves, and $clV \subset V' \subset \mathbf{C}$. A map $g: V \to V'$ is called *polynomial-like of degree d* if it is a d-sheeted analytical covering of V over V' having no critical points on ∂V. By the Riemann–Hurwitz formula, such a map has $d-1$ critical points in V counting with multiplicity. Set

$$K(g) = \{z: g^n(z) \in V \ (n = 0, 1, \ldots)\}, \quad K^0(g) = \text{int } K(g).$$

The $K(g)$ is a compact subset of V.

The straightening theorem states that any polynomial-like map g is quasi-conformally conjugated to a polynomial h of the same degree, i.e., there exists a quasi-conformal homeomorphism $\psi: \mathbf{C} \to \mathbf{C}$ such that $\psi \circ g | W = h \circ \psi | W$ for some domain $W, K \subset W \subset V$. Moreover, $\psi | K^0(g)$ is conformal and

$$\psi(K(g)) = \mathbf{C} \backslash \{z: h^{\circ n}(z) \to \infty \ (n \to \infty)\}$$

is the *filled-in Julia set of h*.

Lemma 2.1. *The domain Ω_0 is Jordan, and its boundary is a quasi-circle. The restriction $f | \Omega_0$ is conformally conjugated to the power transformation $z \mapsto z^b$ of the unit disk $\mathbf{U}^0$.*

Proof. Let us construct a neighbourhood of Ω on which f is a polynomial-like map of degree b. To this end note that the function K conformally maps the disk $\mathbf{U}^0$ into the plane slitted along the semi-axis $\mathbf{R}_1 = [1, \infty)$ (this is the characteristic property of the 4-fold Koebe function). Let us consider the domain V (see Fig. 2)
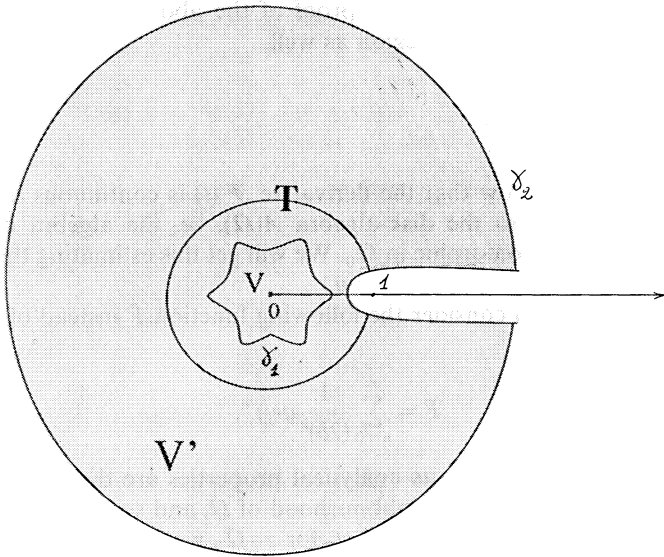
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[1] Actually, the last holds automatically

Fig. 2

bounded by the arc γ_1 of the circle $B(1, \varepsilon)$, the arc γ_2 of the circle $B(0, R)$, where $R \gg 1$ and two horizontal intervals.

Let V be the component of the inverse image $f^{-1}(V')$ containing 0. Then $clV \subset \mathbf{U}^0$, since $f(\partial \mathbf{U}) = \mathbf{R}_1$ lies outside V'. Besides, $clV \cap \gamma_1 = \varnothing$ for sufficiently small ε. Indeed, as 1 is a stable fixed point, the arc $f(\gamma_1)$ lies inside the disk $B(1, \varepsilon)$ and, hence, outside V'.

Thus, $clV \subset V'$.

Further, it is clear from $V = (K|\mathbf{U})^{-1} \circ (S^{-1}V')$ that V is simply-connected. Indeed, it is elementary that $S^{-1}V'$ is simply-connected (see e.g., [17], Lemma 1.4) while $K|\mathbf{U}$ is univalent.

We have shown that $f : V \to V'$ is a polynomial-like map. Its degree is equal to b, since $V \subset U_0$ contains the unique $(b-1)$-fold critical point 0. Clearly, $\Omega \subset K(f|V)$.

By the Straightening Theorem, $f : V \to V'$ is quasi-conformally conjugated to a polynomial h of degree b. Normalize the conjugating homeomorphism ψ in such a way that $\psi(0) = 0$. Then 0 becomes a $(b-1)$-fold critical point for h. It follows that $h(z) = Cz^b$. Normalizing ψ additionally in such a way that $\psi(t_c) = 1$, where t_c is a real fixed point lying on Γ, we get $C = 1$.

Thus, ψ conjugates $f : V \to V'$ to the power polynomial $h : z \mapsto z^b$. Consequently, $\Omega_0 = \psi^{-1}(\mathbf{U})$ is a Jordan domain bounded by a quasi-circle, and ψ conformally conjugates $f|\Omega_0$ to $h|\mathbf{U}$. The lemma is proved. ∎

Remark. Tan Lei showed us another proof of the above lemma which can be applied to the high-temperature region as well.

3. Analytic Properties of $\mathscr{F}'$

In this section we will show that the derivative $\mathscr{F}'(t)$ is continuous in the closed set Ω. So, $\mathscr{F}'$ belongs to the disk-algebra $A(\Omega)$, i.e. the algebra of functions continuous in Ω and holomorphic in Ω_0. We will get this estimating the derivative $\|Df\|_\nu$ in a special Riemannian metric μ.

From now on we will consider the following function F instead of free energy $\mathscr{F}$ (1.1):

$$F = \sum_{n=0}^{\infty} \frac{1}{(2b)^n} g \circ f^n, \tag{3.1}$$

where $g(t) = \ln(1 + t^b)$. Clearly, its analytical properties are the same as those of $\mathscr{F}$. Note that g is analytic in a neighbourhood of Ω, and so series (3.1) converges uniformly in Ω. Hence $F \in A(\Omega)$. Further, for $z \in \Omega_0$ we have

$$F'(z) = \sum_{n=0}^{\infty} \frac{1}{(2b)^n} g'(f^n(z))(f^n)'(z). \tag{3.2}$$

We want to show that this series converges uniformly in Ω, which certainly implies $F' \in A(\Omega)$. The required statement follows from the following estimate:

Lemma 3.1. $|(f^n)'(z)| \leq C(\sqrt{2b})^n$ for $z \in \Omega$.

Proof. Let us recall that $f = K \circ S$ (see Sect. 2). The power function S satisfies the functional equation $\exp(bz) = S(\exp z)$. From the dynamical viewpoint it means that $\exp$ semiconjugates the transformations $L : z \mapsto bz$ and S. Denote by σ the Euclidean metric on $\mathbf{C}$, and by $\mu = \exp_* \sigma$ its image on the punctured plane $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. We have $|d\mu| = |dt|/|t|$.

As $\|DL(z)\|_\sigma = b$ for $z \in \mathbf{C}$,

$$\|DS(t)\|_\mu = b \tag{3.3}$$

for $t \in \mathbf{C}^*$. Besides

$$\|DK(t)\|_\mu = \frac{|tK'(t)|}{|K(t)|} = \frac{|1 - t|}{|1 + t|} = \frac{1}{|\varphi(t)|}, \tag{3.4}$$

where $\varphi(t) = (1 + t)/(1 - t)$. It is surprising that exactly this function conjugates K to the Techebyshev polynomial $T : \tau \mapsto 2\tau^2 - 1$. Due to this observation it is reasonable to pass to the conjugated function

$$h = \varphi \circ f \circ \varphi^{-1} = T \circ R,$$

where $R = \phi \circ S \circ \phi^{-1}$. Consider the corresponding Riemannian metric $\nu = \phi_* \mu$. By (3.3), (3.4),

$$\|DR(\tau)\|_\nu = b, \quad \|DT(\tau)\|_\nu = \frac{1}{|\tau|}.$$

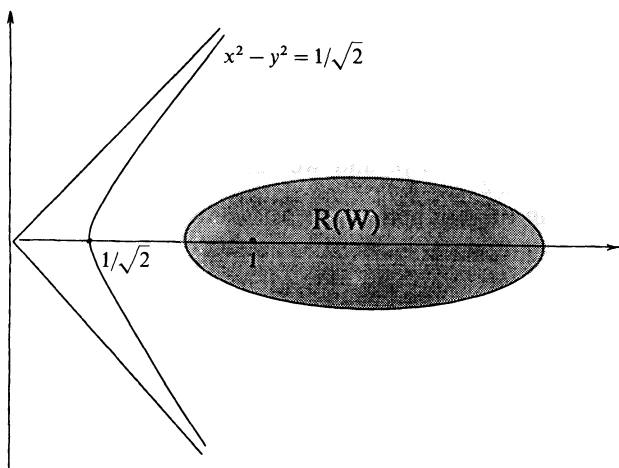$x^2 - y^2 = 1/\sqrt{2}$

$R(W)$

$1/\sqrt{2}$

1

Fig. 3

$$\|Df(t)\|_\mu = \|Dh(\tau)\|_\nu = \frac{b}{|R(\tau)|}. \tag{3.5}$$

The function h has a superstable fixed point $1 = \varphi(0)$ with the immediate attracting basin $W^0 = \varphi\Omega_0$. Since $\Omega \subset U^0$, the set $W \equiv clW^0 = \varphi(\Omega)$ lies in the right half-plane $\varphi(U^0) = P^0 = \{\tau : \text{Re } \tau > 0\}$. As W is g-invariant, $T(R(W)) \subset P^0$, and hence (see Fig. 3)

$$R(W) \subset T^{-1}(P^0) = \{\tau = x + iy : x^2 - y^2 > 1/\sqrt{2}\}.$$

Consequently, $|R(\tau)| > 1/\sqrt{2}$ for $\tau \in W$, and (3.5) implies

$$\|Df(t)\|_\mu < b\sqrt{2}, \quad t \in \Omega.$$

Hence,

$$\|Df^n(t)\|_\mu < (b\sqrt{2})^n, \quad n \in \mathbb{N}.$$

But on the boundary $\partial\Omega$ the metric μ is equivalent to the Euclidean metric and, hence, $|(f^n)'(t)| < C(\sqrt{2}b)^n$ for $t \in \partial\Omega$. By the Maximum Principle, this inequality holds for $t \in \Omega$. The lemma is proved. ∎

Now let us establish some global analytical properties of F'. It is curious that this function has no singularities at points $\beta_i = \sqrt[b]{-1}$.

Lemma 3.2. (i) *The function F' is a single-valued holomorphic function on the Fatou set $N(f)$.*

(ii) *The set Ω^0 is the maximal domain of analyticity of F'.*

Proof. Consider the multi-valued function $\sigma(t) = g(t) + \dfrac{1}{2b}g(f(t))$, where $g(t) = \ln(1 + t^b)$ as above. Let us show that it is regular near $\beta_j = \sqrt[b]{-1}, j = 1, \ldots, b$. We

have

$$\sigma(t) = \ln(1 + t^b) + \frac{1}{2b} \ln\left(1 + \frac{4^b t^{b^2}}{(1 + t^b)^{2b}}\right) = \frac{1}{2b} \ln\left((1 + t^b)^{2b} + 4^b t^{b^2}\right).$$

So, $\sigma(\beta_j) = \frac{1}{2}\ln 4 + \left(\frac{\pi}{2} + \pi n\right)i$, and we see that β_j are regular points for σ. Hence, σ is regular in the components $\Omega(\beta_j)$.

Now let V be an n-fold preimage of some $\Omega(\beta_j)$. By (3.1),

$$F(t) = \frac{1}{(2b)^n} \sigma(f^n t) + \text{regular function}.$$

Consequently, F is regular in V.

Thus, $\Omega(\infty)$ is the only component in which F is not regular, and there we have

$$F(t) = \ln(1 + t^b) + \text{regular function}.$$

Hence,

$$F'(t) = \frac{bt^{b-1}}{1 + t^b} + \text{regular function}$$

is regular in $\Omega(\infty)$.

So, F' is regular on the whole Fatou set $N(f)$, and it is obvious from (3.2) that it is single-valued. The (i) is proved.

(ii) Let us consider the functional equation for F:

$$\frac{1}{2b} F \circ f - F = g. \tag{3.6}$$

Taking the derivative, we get

$$\frac{f'}{2b} F' \circ f - F' = g' = \frac{bt^{b-1}}{1 + t^b}. \tag{3.7}$$

Provided F' can be analytically continued beyond Ω^0 into some neighbourhood U of $t \in \partial\Omega$, it follows from (3.7) that it can be continued as a meromorphic function into $f^n U$. Since $f^n U \supset J(f)$ for some n, the function F' is meromorphic on the whole sphere, i.e. rational.

But we have shown that F' has no poles in $N(f)$. If F' had a pole at a $t \in J(f)$, then by (3.7) it would have poles at all points of the grand orbit

$$\mathcal{O}(t) = \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} f^{-n}(f^m t).$$

But it is impossible since this orbit is infinite. So, F' has no poles at all, and this absurdity completes the proof. ∎

4. Symbolic Dynamics, Thermodynamical Formalism and the Natural Extension of $f|\Omega$

To study the boundary behaviour of F'' we need the Bowen–Ruelle–Sinai thermodynamical formalism (see [22, 24 or 18]). We will state the main results of this theory for our particular map $f|\Gamma$. The theory can be applied to this map without any problem because it is expanding (see Sect. 2).

Let us consider the homeomorphism $\psi:\Omega\to\mathbf{U}$ conformal on Ω_0 and reducing $f|\Omega$ to $S:z\mapsto z^b$ (Lemma 2.1). Let $\psi(t)=re^{2\pi i\theta}\in\mathbf{U}$. Associate to a point $t\in\Omega$ the pair $(r,\bar\varepsilon_+)$, where $r\in[0,1]$, and $\bar\varepsilon_+=(\varepsilon_0,\varepsilon_1,\ldots)$ is the b-adic decomposition of the $\theta\in[0,1)$. Then f turns into the transformation $(r,\bar\varepsilon_+)\mapsto(r^b,\sigma_+\bar\varepsilon_+)$, where

$$\sigma_+:(\varepsilon_0,\varepsilon_1,\ldots)\mapsto(\varepsilon_1,\varepsilon_2,\ldots)$$

is the shift on the space Σ_b^+ of all one-sided b-adic sequences. Sometimes we will identify Ω with $[0,1]\times\Sigma_b^+$, though the described correspondence is not one-to-one. Then $f|\Gamma$ will be identified with the shift σ_+.

Denote by $[\varepsilon_0,\ldots,\varepsilon_{n-1}]$ b-adic cylinders in Σ_b^+ and corresponding b-adic intervals in Γ. Let $B_\theta=\psi^{-1}\{re^{2\pi i\theta}:0\le r\le 1\}$ be *hyperbolic geodesics* in Ω, and $\Gamma_r=\psi^{-1}\{re^{2\pi i\theta}:0\le\theta\le 1\}$ be *equipotential levels* (for the Green function $\ln|\psi|$).

Let ρ be a Hölder function on Γ which is called the potential (here we pass from electrostatics to thermodynamics). Set

$$S_n\rho=\sum_{k=0}^{n}\rho\circ f^k.$$

The Gibbs measure v_ρ on Γ corresponding to the potential ρ is the measure satisfying the following estimates on cylinders:

$$v_\rho[\varepsilon_0\cdots\varepsilon_{n-1}]\asymp\exp[S_n\rho(t_{\varepsilon_0\cdots\varepsilon_{n-1}})-nP],\qquad(4.1)$$

where $t_{\varepsilon_0\cdots\varepsilon_{n-1}}$ is any point of $[\varepsilon_0\cdots\varepsilon_{n-1}]$, and $P=P_f(\rho)$ is a constant called *the pressure*, and the sign "$\alpha\asymp\beta$" means $C_1\beta\le\alpha\le C_2\beta$.

The main result of the Bowen–Ruelle–Sinai theory states that for any Hölder function ρ there exists the unique Gibbs measure v_ρ. This measure satisfies *the Variational Principle*

$$\sup_{v\in M(f)}\left(h_v(f)+\int_\Gamma f\,dv\right)=h_{v_\rho}(f)+\int_\Gamma f\,dv_\rho=P_f(\rho),\qquad(4.2)$$

where $M(f)$ denotes the compactum of all f-invariant probability measures on Γ, and $h_v(f)$ is the entropy of v.

The pressure $P_f(\rho)$ is the smooth convex functional of ρ, and its differential at ρ is the Gibbs measure v_ρ (see [23]):

$$\left.\frac{dP_f(\rho+\kappa\alpha)}{d\kappa}\right|_{\kappa=0}=\int\alpha\,dv_\rho.\qquad(4.3)$$

To the potential $\rho=0$ corresponds the unique measure of maximal entropy $\mu\equiv v_0$, namely the Bernoully measure with b equal states (in view of the model Σ_b^+). The entropy of this measure is equal to the topological entropy of $f|\Gamma$: $h_\mu(f)=h(f)=\ln b$.

The Reimann map $\psi : \Omega \to U$ transforms μ into the measure of maximal entropy for $z \mapsto z^b$, i.e. to the Lebesgue measure on $\mathbf{T}$. Hence, μ coincides with the harmonic measure on Γ corresponding to 0 (see [20]). Consequently,

$$H(0) = \int_{\Gamma} H dv$$

for any function H harmonic in Ω_0 and continuous up to the boundary.

Applying this formula to the function[2],

$$H(t) = \ln \frac{|f'(t)|}{|\psi(t)|^b},$$

we find *the characteristic exponent of μ*

$$\chi_{\mu} = \int_{\Gamma} \ln |f'| d\mu = \ln b. \qquad (4.4)$$

Let us pass now to the crucial construction of *the natural extension* (or *the inverse limit*) $\bar{f} : \bar{\Omega} \to \bar{\Omega}$ (see [25]). By definition, a point $\bar{z} \in \bar{\Omega}$ is the inverse orbit $\bar{z} = (z_0, z_{-1}, \ldots)$, i.e. $f z_{-(i+1)} = z_{-i}$, and $\bar{f} : \bar{z} \mapsto (f z_0, z_0, z_1, \ldots)$. The transformation $\bar{f}$ is invertible on $\bar{\Omega}$, and there exists the natural projection $\pi : \bar{\Omega} \to \Omega$, $\pi(z) = z_0$, semiconjugating $\bar{f}$ and f. All fibers of π except zero one are Cantor sets. Each f-invariant measure v on Γ can be uniquely lifted to the $\bar{f}$-invariant measure $\bar{v}$ on $\bar{\Gamma}$.

The symbolic dynamics for f generates the symbolic dynamics for $\bar{f}$. Namely, $\bar{\Omega}$ can be identified mod 0 with $[0,1] \times \Sigma_b$, where $\Sigma_b = \{(\cdots \varepsilon_{-1}, \varepsilon_0, \varepsilon_1 \cdots)\}$ is the space of two-sided sequences. Then

$$\bar{f} : (r, \bar{\varepsilon}) \mapsto (r^b, \sigma \bar{\varepsilon}),$$

where $\sigma : \Sigma_b \to \Sigma_b$ is the left shift, and $\bar{\mu}$ turns into the Bernoulli measure on Ω_b with equal states.

The lifts $\bar{\Gamma}_r = \pi^{-1} \Gamma_r$ of the equipotential levels will be called the *solenoids*, $\bar{\Gamma} \equiv \bar{\Gamma}_1$. The reason is that these sets can be supplied with the structure of the solenoidal group $\bar{\mathbf{T}}$, i.e. the inverse limit of the group endomorphism $z \mapsto z^b$ of $\mathbf{T}$ (see [26]). Then $\bar{f} | \bar{\Gamma}$ turns into a group automorphism, and $\bar{\mu}$ into the Haar measure on $\bar{\Gamma}$.

The space $\bar{\Omega}$ can be regarded as a continuum-sheeted Riemann surface over Ω, the "bunch of sheets" gluing together at zero. If one cuts Ω along the geodesic B_0, $\bar{\Omega}$ is foliated into the sheets $L(\bar{\varepsilon}_-)$ coded by one-sided sequences $\bar{\varepsilon}_- = (\cdots \varepsilon_{-2}, \varepsilon_1) \in \Sigma_b^-$.

The gluing of the sheets is fulfilled by the b-adic shift $A \equiv A_b : \bar{\varepsilon}_- \mapsto \bar{\varepsilon}_- + 1$ where $1 = (\cdots 0, 0, 1)$, and addition is understood in the sense of the group of b-adic numbers. Gluing together countably many sheets corresponding to an orbit $\{A^n(\bar{\varepsilon}_-)\}_{n=-\infty}^{\infty}$ of the b-adic shift, we get the logarithmic Riemann surface $W(\bar{\varepsilon}_-)$.[3] All inverse functions $f^{-n}(z)$ become single-valued on this surface.

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[2] Taking into account that $\psi(t) \sim t$, since there is the following formula for the Green function (cf. [32]):

$$\ln |\psi| = \lim_{n \to \infty} d^{-n} |f^{\circ n}|$$

[3] The inner topology of this surface differs from the topology induced from the space $\bar{\Omega}$ in which it is densely immersed

From the dynamical point of view, Riemann surfaces $W(\bar\varepsilon_-)$ are *global unstable manifolds* of f: if $\bar t, \bar\tau \in W(\bar\varepsilon_-)$, then f^{-1}-orbits of $\bar t$ and $\bar\tau$ are exponentially drawing together.

For a function $\rho(t)$ in Ω we will write $\rho(\bar t) \equiv \rho(t)$, where $t = \pi(\bar t)$; in particular, $|\bar t| \equiv |t|$. Set

$$W_\delta(\bar\varepsilon_-) = \{\bar t \in W(\bar\varepsilon_-) : |\bar t| \geq \delta\}, \quad S_{-n}\rho(\bar t) = -\sum_{k=0}^{-n+1} \rho(f^k(\bar t)). \tag{4.5}$$

The drawing together of inverse orbits originating in $W_\delta(\bar\varepsilon_-)$ is uniformly exponential. It follows in a standard way that for any function $\rho(t)$, Hölder on $\Omega^* = \Omega\setminus\{0\}$, for $\delta > 0$ and $\bar t, \bar\tau \in W_\delta(\bar\varepsilon_-)$, the following estimates hold:

$$|S_{-n}\rho(\bar t) - S_{-n}\rho(\bar\tau)| \leq C(\delta), \tag{4.6}$$

where $C(\delta)$ does not depend on n and $\bar\varepsilon_-$.

5. The Weighted Substitution Operator

Let us consider a function $\beta = A(\Omega)$ having no zeroes on $\Gamma \equiv \partial\Omega$, and $h \in A(\Omega)$. In this and the next sections we will consider the following functional equation:

$$\beta(t)U(ft) - f(t) = -h(t). \tag{5.1}$$

Let us consider the multiplicative cocycle $\beta_n(t) = \beta(t)\beta(ft)\ldots\beta(f^{n-1}t)$ associated with β. We will assume in what follows that

(i) The cocycle β has the positive Liapunov exponent:

$$\chi_\mu(\beta) \equiv \int \ln|\beta|\, d\mu > 0. \tag{5.2}$$

(ii)[4] The function $\ln|\beta|$ on Γ is not homologous to a constant. This means that there are no continuous solutions $\varphi \in C(\Gamma)$ of the equation

$$\ln|\beta| = \varphi \circ f - f + c$$

with any constant c.

Motivation. By differentiating equation (3.6), we get Eq. (5.1) for $U = F''$ with:

$$\beta(t) = \frac{1}{2b} f'(t)^2, \quad h(t) = \frac{1}{2b} F'(ft)f''(t) + g''(t). \tag{5.3}$$

The function β is holomorphic in a neighbourhood of Ω and has there the unique root at $t = 0$, and the function $h \in A(\Omega)$ (since $F' \in A(\Omega)$). Assumption (i) is valid due to (4.4):

$$\chi_\mu(\beta) = 2\chi_\mu - \ln 2b = \ln\frac{b}{2} > 0. \tag{5.4}$$

Assumption (ii) holds since $\ln|f'|$ is not homologous to a constant on T [30]. ∎

Equation (5.1) has the unique solution holomorphic in Ω_0:

$$U(t) = \sum_{n=0}^{\infty} \beta_n(t) H(f^n t), \tag{5.5}$$

where $\beta_n(t)$ is the multiplicative cocycle generated by the function β.

In this section we will explain why this solution is not, as a rule, continuous up to the boundary. The words "as a rule" means: for any $h \in A(\Omega)$ outside a set of first category. In Sect. 7 we will show that the concrete function h given by (5.3) is not excluded (so, F'' is discontinuous).

Let us consider the weighted shift operator L_β in $A(\Omega)$:

$$(L_\beta U)(t) = \beta(t) U(ft).$$

Then we can rewrite (5.1) in the following way:

$$(L_\beta - I)U = -h. \tag{5.6}$$

This leads us to the problem of spectral properties of the weighted shift operator.

It is known [27] that the spectral radius r_β of L_β in the disk-algebra can be calculated by the formula:

$$\ln r_\beta = \sup_{\nu \in M(f)} \chi_\nu(\beta), \tag{5.7}$$

and the spectrum of L_β is the unit disk: $\mathrm{spec}(L_\beta) = \{\lambda : |\lambda| \le r_\beta\}$.

If U is an eigenfunction of L_β then the function $\ln|\beta|$ is homologous to a constant:

$$\ln|\beta| = \ln|U(ft)| - \ln|U(t)| + \ln|\lambda|$$

which is not the case by Assumption (ii) above. So, the operator $L_\beta - \lambda I$ is injective for all λ. By the Banach theorem on the inverse operator (see [28]), for $\lambda \in \mathrm{spec}(L_\beta)$ the image $\mathrm{Im}(L_\beta - \lambda I)$ is the set of first Baire category. Thus, the equation $L_\beta U - \lambda U = h$ for $|\lambda| \le r_\beta$ and generic $h \in A(\Omega)$ has no solutions in $A(\Omega)$.

By (5.2), $r_\beta > 1$, and hence $1 \in \mathrm{spec}\, L_\beta$. So, Eq. (5.6) is non-solvable in $A(\Omega)$ for generic $h \in A(\Omega)$. For such an h, the analytical solution (5.4) is not continuous up to the boundary, as we have asserted.

In conclusion let us mention a rough obstruction for (5.1) to be solvable related to non-invertibility of f. Let $\bar{t} = (t_0, t_{-1}, \ldots) \in \bar{\Omega}$ be an inverse orbit of f. Iterating (5.1) we get

$$U(t) = \beta_{-n}(\bar{t}) U(t_{-n}) - \sum_{k=1}^{n} \beta_{-k}(\bar{t}) h(t_{-k}), \tag{5.8}$$

where $\beta_{-k}(\bar{t}) = [\beta(t_{-1}) \ldots \beta(t_{-k})]^{-1}$. By the ergodic theorem,

$$\lim_{n \to \infty} \frac{1}{n} \ln \beta_{-n}(\bar{t}) = -\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln \beta(\bar{f}^{-k}t) = -\int \ln|\beta| d\mu < 0 \tag{5.9}$$

for $\bar{\mu}$-almost all $\bar{t} \in \bar{\Omega}$. For such a $\bar{t}$ the sequence $\beta_{-n}(\bar{t})$ exponentially converges to zero, and we can consider the following $\bar{\mu}$-measurable function on $\bar{\Omega}$:

$$G(\bar{t}) = -\sum_{k=1}^{\infty} \beta_{-k}(\bar{t}) h(t_{-k}). \tag{5.10}$$

It follows from (5.8) that for continuous h we have $G(\bar{t}) = U(t)$, i.e. the a priori multi-valued function G turns out to be single-valued. It is the necessary condition for solvability of Eq. (5.1).

Remark that this condition is almost sufficient. Namely, if the function $G(\bar{t})$ is single-valued, then it gives the holomorphic solution in $\Omega^* = \Omega \backslash \{0\}$ which is continuous up to the boundary. However, this solution has a singularity at zero since $\beta(0) = 0$.

6. Asymptotics of $\mathscr{F}''$ Along Almost all Geodesics (Conditional Result)

In this section we assume that the function $U(t)$ given by (5.5) is not continuous up to the boundary. Under this assumption we will calculate its asymptotics along almost all (in the sense of harmonic measure) geodesics B_θ. It turns out to be the following:

$$\overline{\lim_{r \to 1}} \frac{\ln |U(re^{i\theta})|}{-\ln(1-r)} = \frac{\chi_\mu(\beta)}{\chi_\mu}. \tag{6.1}$$

Remark. By Makarov's theorem [31], $\ln l(t) \sim \ln(1-r)$ for μ-almost all geodesics. So, (6.1) coincides with the required asymptotics (1.5) (take into account formulas (4.4) and (5.4) for characteristic exponents).

First let us give a heuristic argument yielding (6.1). By (5.8), (5.10) we have

$$\beta_{-n}(\bar{t}) U(t_{-n}) \to U(t) - G(\bar{t}) \tag{6.2}$$

for almost all $\bar{t} = (t, t_{-1}, \dots) \in \bar{\Omega}$. By (5.9),

$$|\beta_{-n}(\bar{t})| \sim \left(\frac{2}{b}\right)^n,$$

where the sign "$\alpha_n \sim \beta_n$" means

$$\lim_{n \to \infty} \ln \frac{\alpha_n}{\beta_n} = 0.$$

Hence,

$$U(t_{-n}) \sim \left(\frac{b}{2}\right)^n. \tag{6.3}$$

Setting $t_{-n} = r_{-n} e^{i\theta - n}, t = re^{i\theta}$, we get $r_{-n} = r^{b^{-n}}$, and hence

$$b^n = \ln \frac{r}{r_{-n}} \sim \frac{1}{1 - r_{-n}}$$

(since r is fixed). Comparing this with (6.3), we find

$$U(t_{-n}) \sim \left(\frac{1}{1 - r_{-n}}\right)^{\alpha_c},$$

where $\alpha_c = \chi_\mu(\beta)/\chi_\mu$ is the critical exponent written above.

The main shortcoming of this calculation is that the points t_{-n} don't lie near a single radius B_θ but are wandering along $\partial\Omega$. However, they quite often approach almost every geodesic which allows us to turn our argument into a rigorous one.

Let us consider the multi-valued function $G(\bar{t})$ on the covering space $\bar{\Omega}$ given by the series (5.10). This function is correctly defined due to (5.2), and satisfies the following equation:

$$\beta(t)V(\bar{f}\bar{t}) - V(\bar{t}) = -h(t), \qquad (6.4)$$

the lift of Eq. (5.1) on $\bar{\Omega}$. In order to substantiate the asymptotics we need some analytical properties of the G.

Let us put on $\bar{\Omega} \cong [0,1] \times \Sigma_b$ measure λ, the product of the Lebesgue measure on $[0,1]$ and the Bernoulli measure $\bar{\mu}$ on Σ_b. Consider the space $A(\bar{\Omega})$ of functions measurable on $\bar{\Omega}$, analytic on almost all Riemann surfaces $W(\bar{\varepsilon}_-)$ and continuous up to the boundary in their inner topology (see Sect. 4). Remark that they probably are not defined at 0.

Lemma 6.1. *The function $G(\bar{t})$ given by (5.10) belongs to the space $A(\bar{\Omega})$.*

Proof. By the ergodic theorem, $\beta_{-n}(|\bar{t}|) \sim e^{-an}$ $(n \to \infty)$ for $\bar{\mu}$-almost all $\bar{t} = (1, \bar{\varepsilon}) \in \Gamma$. Applying (4.6) to the function $\rho = -\ln|\beta|$, we get the similar asymptotics for all $\bar{\tau} \in W(\bar{t}) \equiv W(\bar{\varepsilon}_-)$. Moreover, for all $\theta > 0$ there exists $Q(\delta, \theta)$ such that

$$|\beta_{-n}(\bar{\tau})| \le e^{C(\delta)}|\beta_{-n}(\bar{t})| \le Q(\delta, \theta)e^{-(a-\theta)n},$$

provided $|\bar{\tau}| \ge \delta > 0$. Consequently, the series (5.10) converges uniformly on $W_\delta(\bar{t})$ which yields the required statement. ∎

There is the Bernoulli measure $\bar{\mu}$ on the solenoids $\bar{\Gamma}_r \cong \Sigma_b$, and one can consider the corresponding spaces $L^\kappa(\bar{\Gamma}_r)$, $\kappa > 0$. Now let us prove the main technical lemma.

Lemma 6.2. *For sufficiently small $\kappa > 0$ and any $r \in (0, 1]$, the function $G(\bar{t})$ belongs to the space $L^\kappa(\bar{\Gamma}_r)$:*

$$\int_{\Sigma_b} |G(r, \bar{\varepsilon})|^\kappa d\bar{\mu}(\bar{\varepsilon}) < \infty.$$

Moreover, for any $\delta > 0$ the same is true for the function

$$G_\delta(\bar{\varepsilon}) = \sup_{r:\delta \le r \le 1} |G(r, \bar{\varepsilon})|.$$

Proof. Applying (4.6) to the function $\rho = \ln|\beta|$, we get for $r \ge \delta$:

$$|\beta_{-n}(r, \bar{\varepsilon})| \le Q(\delta)|\beta_{-n}(1, \bar{\varepsilon})|.$$

Consequently,

$$|G_\delta(\bar{\varepsilon})| \le (\|\psi\|_\infty Q(\delta))^\kappa \sum_{n=1}^\infty |\beta_{-n}(1, \bar{\varepsilon})|^\kappa,$$

and further

$$\int_{\bar{\Gamma}} |G_\delta|^\kappa d\bar{\mu} \le (\|\psi\|_\infty Q(\delta))^\kappa \sum_{n=1}^\infty \int_{\bar{\Gamma}} |\beta_{-n}|^\kappa d\bar{\mu}.$$

But

$$\int_{\bar{\Gamma}} |\beta_{-n}(\bar{t})|^\kappa d\bar{\mu}(\bar{t}) = \int_{\bar{\Gamma}} |\beta_n(\bar{f}^{-n}\bar{t})|^{-\kappa} d\bar{\mu}(\bar{t}) = \int_{\bar{\Gamma}} |\beta_n(\bar{t})|^{-\kappa} d\bar{\mu}(\bar{t}) = \int_{\Gamma} |\beta_n(t)|^{-\kappa} d\mu(t).$$
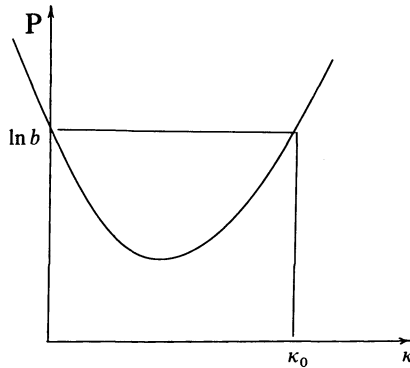
Fig. 4

Thus, it is enough to show that for sufficiently small $\kappa > 0$ the last integral exponentially tends to zero as $n \to \infty$. To this end we will apply the thermodynamical formalism.

Let v_κ be the Gibbs measure corresponding to the potential $-\kappa \ln |\beta|$, and $P_\kappa = P_f(-\kappa \ln |\beta|)$ be the corresponding pressure. By (4.1),

$$|\beta_n(t_{\varepsilon_0 \cdots \varepsilon_{n-1}})|^{-\kappa} \asymp v_\kappa[\varepsilon_0 \cdots \varepsilon_{n-1}] \exp(nP_\kappa)$$

for any point $t_{\varepsilon_0 \cdots \varepsilon_{n-1}}$ from the cylinder $[\varepsilon_0 \cdots \varepsilon_{n-1}]$.

As $\mu[\varepsilon_0 \cdots \varepsilon_{n-1}] = b^{-n}$, then

$$\int |\beta_n|^{-\kappa} d\mu \asymp b^{-n} \sum_{1 \leq \varepsilon_i \leq n} v_\kappa[\varepsilon_0 \cdots \varepsilon_{n-1}] \exp(nP_\kappa) = \exp[n(P_\kappa - \ln b)].$$

Hence, it is sufficient to check that for small enough $\kappa > 0$,

$$P_\kappa - \ln b < 0. \tag{6.5}$$

But $P_0 = h(f) = \ln b$, and by (4.3) and (5.2) we have

$$\frac{dP_\kappa}{d\kappa}\bigg|_{\kappa = 0} = -\int \ln|\beta| d\mu < 0.$$

Hence, (6.5) holds for any $\kappa \in (0, \kappa_0)$ (Fig. 4), and the lemma is proved. ∎

So, we have two solutions of Eq. (5.1), $U(\bar{t}) \equiv U(t)$ given by (5.4) and $G(\bar{t})$ given by (5.10). The first of them comes from the single-valued function on Ω_0; the second is multi-valued on Ω_0 but possesses better boundary properties. The difference $V(\bar{t}) = U(t) - G(\bar{t})$ satisfies the homogeneous equation

$$\beta(t)V(\bar{f}\bar{t}) = V(\bar{t}). \tag{6.6}$$

Since $U(t)$ is not continuous up to the boundary (see the beginning of the section), while $G \in A(\bar{\Omega})$, we have $V \neq 0$. We are going to find the asymptotics of V along $\bar{\mu}$-almost all "geodesics" $\bar{B}_{\bar{\varepsilon}} = \{(r, \bar{\varepsilon}) \in \bar{\Omega} : 0 \leq r \leq 1\}$ as $r \to 1$. Since G is continuous on almost all sheets, the same asymptotics will be valid for U.

Set

$$V_0(\bar{\varepsilon}) = \max_{1/2 \leq r \leq b/2} |V(r, \bar{\varepsilon})|.$$

By Lemma 6.2, $V_0 \in L^\varkappa$. Hence

$$\int_{\Sigma_b} \ln |V_0(\bar{\varepsilon})| d\bar{\mu}(\bar{\varepsilon}) < +\infty.$$

Now we can apply the ergodic theorem. It follows

$$\overline{\lim_{n \to \infty}} \frac{1}{n} \ln |V_0(\sigma^n \bar{\varepsilon})| \leq 0 \qquad (6.7)$$

for almost all $\bar{\varepsilon} \in \Sigma_b$. Let us consider a geodesic $\overline{B}_{\bar{\varepsilon}}$ corresponding to such an $\bar{\varepsilon}$. Let $r \geq 1/2$ and $n = n(r)$ be such a number that

$$\tfrac{1}{2} \leq r^{b^n} < (\tfrac{1}{2})^{1/b}. \qquad (6.8)$$

It follows from (6.6) and (6.7) that

$$|V(r, \bar{\varepsilon})| = |\beta_n(r, \bar{\varepsilon}_+)| \, |V(r^{b^n}, \sigma^n \bar{\varepsilon})| \prec |\beta_n(r, \bar{\varepsilon}_+)|,$$

where the sign "$\alpha \prec \beta$" means $\overline{\lim} \dfrac{1}{n} \ln (\beta_n/\alpha_n) \leq 0$. Since by the ergodic theorem, $|\beta_n(r, \bar{\varepsilon}_+)| \sim e^{na}$ for almost all $\bar{\varepsilon}_+$, we conclude

$$\overline{\lim_{r \to 1}} \frac{1}{n(r)} \ln V(r, \bar{\varepsilon}) \leq \chi_\mu(\beta).$$

From (6.8) we find

$$n(r) = -\frac{\ln(1-r)}{\ln b} + O(1).$$

Two last estimates yield

$$\overline{\lim_{r \to 1}} \frac{\ln V(r, \bar{\varepsilon})}{-\ln(1-r)} \leq \frac{\chi_\mu(\beta)}{\ln b}. \qquad (6.9)$$

In order to get the opposite estimate let us consider the set

$$X_\delta = \{\bar{\varepsilon} \in \Sigma_b : V(1/2, \bar{\varepsilon}) \geq \delta\}.$$

As $V \not\equiv 0$, $\lambda(X_\delta) > 0$ for sufficiently small $\delta > 0$. Hence, almost all orbits $\{\sigma^n \bar{\varepsilon}\}_{n=0}^\infty$ pass through X_δ infinitely many times. For such an $\bar{\varepsilon}$ let us consider a sequence $n(k) \to \infty$ for which $\sigma^{n(k)} \bar{\varepsilon} \in X_\delta$. Set $r_n = (1/2)^{b^{-n}}$. Then (6.6) implies

$$|V(r_{n(k)}, \bar{\varepsilon})| = |\beta_{n(k)}(r_{n(k)}, \bar{\varepsilon}_+)| \cdot |V(1/2, \sigma^{n(k)} \bar{\varepsilon})| \geq \delta |\beta_{n(k)}(r_{n(k)}, \bar{\varepsilon}_+)|.$$

This implies the inequality opposite to (6.9). Thus, for almost all $\bar{\varepsilon} \in \Sigma_b$ we have

$$\overline{\lim_{r \to 1}} \frac{\ln V(r, \bar{\varepsilon})}{-\ln(1-r)} = \frac{\chi_\mu(\beta)}{\ln b},$$

and the required asymptotics (6.1) is proved.

7. $\mathscr{F}''$ Is not Continuous up to the Boundary

Let us prove first that one of the derivatives is discontinuous on the interval $[0, t_c]$ at the real critical point t_c (it is a critical point from the thermodynamical viewpoint; from the dynamical viewpoint, t_c is a repelling fixed point). Denote by $\lambda = (f^n)'(t_c) > 1$ the multiplier of this point.

Lemma 7.1. Let l be a natural number for which $\lambda^l > 2b$. Then $f^{(l)}$ is discontinuous on the interval $[0, t_c]$ at t_c.

Proof. Let us linearize f at t_c:

$$\psi(fz) = \lambda\psi(z),$$

where λ is an analytic function in an neighbourhood of t_c, $\psi(t_c) = 0$, $\psi'(t_c) = 1$ (the *Königs function*). Clearly, ψ can be analytically continued on the interval $(0, t_c]$, and one-to-one maps it onto the axis $(-\infty, 0]$.
     Set

$$\tilde{F} = F \circ \psi^{-1}, \quad \tilde{g} = g \circ \psi^{-1}.$$

Then $\tilde{F}$ satisfies the following functional equation:

$$\frac{1}{2b}\tilde{F}(\lambda z) - \tilde{F}(z) = \tilde{g}(z).$$

Due to the linearization we immediately get the functional equation for $\tilde{F}^{(l)}$:

$$\frac{\lambda^l}{2b}\tilde{F}^{(l)}(\lambda z) - \tilde{F}^{(l)}(z) = \tilde{g}^{(l)}(z).$$

If $\tilde{F}^{(l)}$ is continuous on the semi-axis $(-\infty, 0]$, then it can be given on it by the following series (compare (5.10)):

$$\tilde{F}^{(l)} = -\sum_{k=1}^{\infty} \frac{(2b)^k}{\lambda^{lk}}\tilde{g}^{(l)}\left(\frac{z}{\lambda^k}\right),$$

which gives the analytical continuation of $\tilde{F}^{(l)}$ through 0. Hence F can be analytically continued through t_c contradicting Lemma 3.2. ■

Remark. The same argument can be applied to any fixed point $\alpha \in \partial\Omega$: some derivative of $F|\Omega^0$ must be discontinuous at α.

Lemma 7.2. *The function F'' is not continuous up to the boundary of Ω.*

Proof. Assuming the reverse, we will show that all derivatives $F^{(n)}$ should be continuous up to the boundary, contradicting Lemma 7.1.
     Let us consider the functional equations for the derivatives of F (cf. (3.6) and (5.3)):

$$\frac{(f')^{n+1}}{2b}F^{(n+1)} \circ f - F^{(n+1)} = -h_n, \tag{7.1}$$

where h_n can be expressed via the derivatives of F of order $\leq n$:

$$h_n = -g^{(n)} - \frac{1}{2b}\sum_{k=1}^{n-1}\left(\frac{d(f')^k}{dt} \cdot \frac{d^k F}{dt^k} \circ f\right)^{(n-k-1)} \tag{7.2}$$

It is convenient to use the following metric in Ω^0:

$$d(z, \zeta) = \inf l(\gamma),$$

where inf is taken over all rectifiable paths γ connecting z and ζ.

Assume by induction that all derivatives $F^{(k)}, k = 1, 2, \ldots, n$, are continuous in Ω ($n = 2$ is the base of induction). Then the derivatives $F^{(k)}, k = 1, 2, \ldots, n-1$, should be Lipschitz continuous with respect to the metric d. By (7.2), h_n should possess the same property. But if $F^{(n)}$ is continuous in $\Omega, n \geq 2$, then it can be given by the series (5.10):

$$F^{(n)}(t) = - \sum_{k=1}^{\infty} \frac{(2b)^k}{[(f^{\circ k})'(t_{-k})]^n} h_{n-1}(t_{-k}) \tag{7.3}$$

for a μ-typical inverse orbit $\bar{t} = \{t_0, t_{-1}, \ldots\}$. It follows from here (taking into account the expanding property of $f|\Gamma$) that $F^{(n)}$ is also Lipschitz continuous with respect to d. Hence, $F^{(n+1)}$ is bounded in Ω^0. It is enough for $F^{(n+1)}$ to be given by the series (5.10) and, hence, to be continuous up to the boundary. ∎

8. Concluding Remarks

Note that the complex critical exponent calculated in this paper differs from the usual real critical exponent α at t_c. In fact, there is the general scheme including both cases. Namely, one can associate to any invariant measure v on $\partial\Omega_0$ its own critical exponent α_v, i.e. exponent of power growth of an appropriate derivative of the free energy along v-typical geodesics. More specifically, let

$$m_v = \left[\frac{\ln 2b}{\chi_v} \right] + 1, \quad \alpha_v = 1 - \left\{ \frac{\ln 2b}{\chi_v} \right\},$$

where $[a]$ and $\{a\}$ denote the entire and the fractional part of a respectively, and χ_v is the characteristic exponent of v. Then the following general formula should be true:

$$\lim_{t \to \tau, t \in B_\tau} \frac{\ln |\mathscr{F}^{(m_v)}(t)|}{-\ln l(t)} = \alpha_v \tag{8.1}$$

for v-almost all geodesics.

Indeed, following the scheme of the present paper, we should find the first m for which the function $\beta = (f')^m / 2b$ has positive characteristic exponent:

$$\chi_v(\beta) = m\chi_v - \ln 2b > 0,$$

i.e. $m > \ln 2b / \chi_v$ (assume that $\ln 2b / \chi_v$ is non-integer). Then by formula (6.1) we get (8.1).

For the δ-measure concentrated at the critical point t_c we obtain the usual formula of the renorm-group theory (see [29, 13]):

$$\alpha = 1 - \left\{ \frac{\ln 2b}{\ln f'(t_c)} \right\} = 1 - \left\{ \frac{d\ln 2}{\ln f'(t_c)} \right\}.$$

The similar formula holds for any periodic point on $\partial\Omega$ (without any changes in the proof).

The main technical problem in proving (8.1) for general v is related to the fact that if $m > 2$, then the right-hand side of Eq. (7.1) for $F^{(m)}$ is discontinuous on the boundary. The same problem arises if one wants to calculate the critical exponent for the measure of maximal entropy on the boundary of the "high-temperature" basin Ω_1. The true formula should be $\alpha = 1 - \{\ln b / \ln 2\}$. It is interesting also to find the complex critical exponent for $b = d = 2$.

The other problem is to study the global properties of free energy on the whole Riemann sphere. They have to do with Gibbs measures on $J(f)$.

We finish the paper with the following important remark. One can show that the free energy F can be represented as the logarithmic potential of the measure of maximal entropy of f. This gives another approach to the circle of problems under consideration and a nice relation of the critical exponent to the local dimension of the measure of maximal entropy. We are grateful to P. Moussa and A. Eremenko for interesting discussions of this point.

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