Commun. Math. Phys. 141, 1–8 (1991)

Communications in
Mathematical
Physics

Metamorphoses: Sudden Jumps in Basin Boundaries*

Kathleen T. Alligood[1], **Laura Tedeschini-Lalli**[2], **and James A. Yorke**[3]

[1] Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA
[2] Dipartimento di Matematica, Universita di Roma "La Sapienza", I-00185 Rome, Italy
[3] Department of Mathematics and, Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA

Received August 14, 1986; in revised form April 15, 1990

Abstract. In some invertible maps of the plane that depend on a parameter, boundaries of basins of attraction are extremely sensitive to small changes in the parameter. A basin boundary can jump suddenly, and, as it does, change from being smooth to fractal. Such changes are called *basin boundary metamorphoses.* We prove (under certain non-degeneracy assumptions) that a metamorphosis occurs when the stable and unstable manifolds of a periodic saddle on the boundary undergo a homoclinic tangency.

Dynamical systems in the plane can have many coexisting attractors. In order to be able to predict long-term or asymptotic behavior in such systems, it is important to be able to recognize to which attractor (final state) a given trajectory will tend. The set of initial conditions whose trajectories are asymptotic to a particular attractor is called the *basin of attraction* of that attractor. In some systems that depend on a parameter, it has been observed that the boundaries of these basins are extremely sensitive to small changes in the parameter. Not only can a boundary jump suddenly, but it can also change from being smooth to being fractal. These changes, called *boundary metamorphoses*, are studied at length in [GOY]. In this paper we prove a theorem, originally stated in [GOY], which characterizes the jumps in basin boundaries.

The Hénon map $f(x,y)=(A-x^2-Jy,x)$ provides an example of this phenomenon. We fix $J=0.3$ and vary A, resulting in a one-parameter, invertible map of the plane. The Jacobian of f is J; hence, f is area contracting for all A. We will be looking specifically at the boundary of the basin of attraction of infinity. (The basin of infinity is the set of all points (x,y) such that $|f^n(x,y)|\to\infty$ as $n\to\infty$.) Figures 1a and 1b show the basin of infinity in black for $A=1.314$ and $A=1.320$, respectively. In Fig. 1b we see that the basin of infinity contains points which were previously (at $A=1.314$) well within the white region. This new set of black points has not

* This research was supported in part by grants and contracts from the Defense Advanced Research Projects Agency, the Consiglio Nazionale delle Ricerche (Comitato per le Matematiche), and the National Science Foundation

Fig. 1a and b. The figure shows the basin of attraction of infinity in black for the Henon map

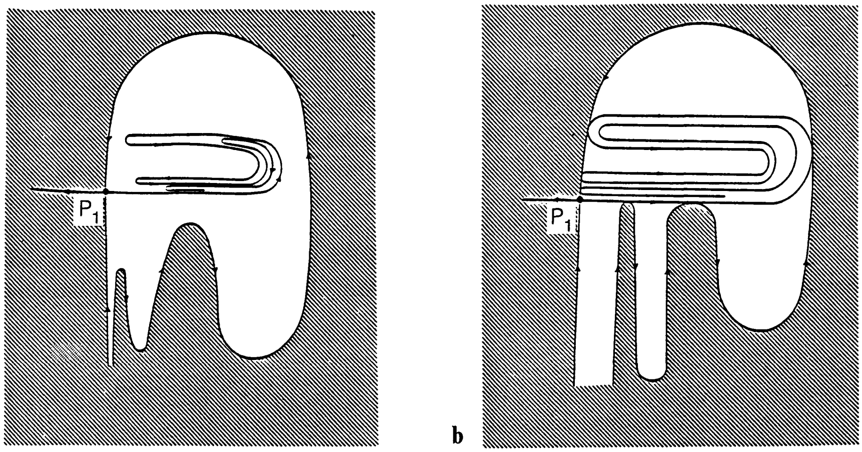
$$f(x, y) = (A - x^2 - Jy, x).$$

We fix $J = 0.3$. **a** A is 1.314, and in **b** A is increased to 1.32. The change in the basin of infinity illustrates a basin boundary jump

gradually moved in from the boundary of the white region. Rather, beyond a certain critical value $A = A^* \approx 1.3145$, black points suddenly begin appearing deep in the interior of the white region. As A increases, the thin bands thicken. This is a discontinuous change in the basin of infinity.

In order to understand this phenomena, we must examine the dynamical behavior on the basin boundary. At $A = 1.314$ (Fig. 1a) the boundary is observed numerically to consist of a saddle fixed point p_1, and its stable manifold $W^s(p_1)$. (The *stable manifold* $W^s(p)$ of a fixed point p is the set of points (x, y) such that $f^n(x, y) \to p$ as $n \to \infty$. More generally, the stable manifold $W^s(p_k)$ of a periodic point p_k of period k is the set of points (x, y) such that $f^{nk}(x, y) \to p_k$ as $n \to \infty$. Analogously, the *unstable manifold* $W^u(p_k)$ of p_k is the set of points (x, y) such that $f^{-nk}(x, y) \to p_k$ as $n \to \infty$. Such sets can be proved to be smooth curves.) One branch of the unstable manifold of p_1 at $A = 1.314$ extends into the white region, as shown in Fig. 2a. At the critical value $A^* \approx 1.3145$, after which the basin boundary jumps into the white region, we find that $W^s(p_1)$ and $W^u(p_1)$ are tangent (Fig. 2b). Hammel and Jones [HJ] were the first to prove a theorem relating the tangency of $W^s(p_1)$ and $W^u(p_1)$ (called a *homoclinic tangency*) to basin boundary metamorphoses. Their methods are different from ours, however. We want to relate these metamorphoses to the saddle periodic orbits which are found near the points of tangency and which we describe below.

The complicated dynamical behavior which occurs at homoclinic tangencies has been studied at length in recent years, especially in the papers of Gavrilov and Silnikov [GS], Newhouse [N], and Robinson [R]. Under certain non-degeneracy assumptions, there are horseshoe maps defined on subsets of the plane near a point q_0 of tangency of $W^s(p_1)$ and $W^u(p_1)$. Figure 3 shows a rectangle B_4 and some of its iterates under f. Notice that $f^4(B_4)$ is horseshoe-shaped and intersects B_4 in two components. In fact, for n sufficiently large, there is a rectangle B_n near the point of tangency q_0 such that f^n restricted to B_n is a horseshoe map. There is necessarily a saddle orbit of period n in each of the two components of the intersection of B_n and $f^n(B_n)$ (see, for example, [R]). One of these saddles will have a "flipped" unstable manifold (i.e., $D_x f^n$ at this saddle has an eigenvalue less than -1), and the other



a b

Fig. 2a and b. The figure show the stable and unstable manifolds of a fixed point p_1 before and at tangency, respectively

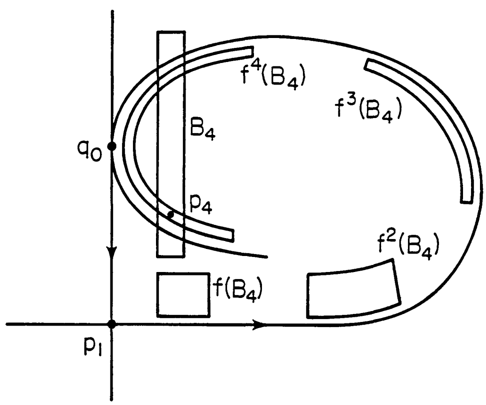


Fig. 3. This illustrates a horseshoe map. The invariant set of the horseshoe is in $B_4 \cap f^4(B_4)$

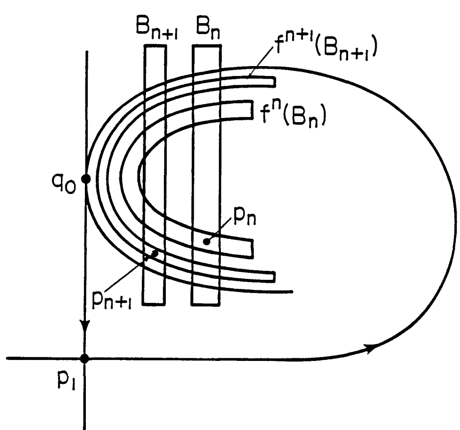


Fig. 4. It shows the relative positions of two simple Newhouse saddles p_n and p_{n+1} of periods n and $n+1$, respectively

will not. We label the unflipped saddle p_n. This orbit is called a "simple Newhouse saddle" in [TY].

The larger n is, the closer B_n will be to q_0 and $W^s(p_1)$. This corresponds to the fact that the length of time (i.e., the number of iterates of f) it takes for a point to move around the fixed point p_1 is determined by how close the point is to the stable manifold $W^s(p_1)$. What we see (Fig. 4) is an infinite family of horseshoes, and a sequence $\{p_n\}$ of simple Newhouse saddles (where p_n has period n and is in B_n) such that $\{p_n\} \to q_0$. In the following theorem, as stated in [GOY], the saddle fixed point S corresponds to p_1 in the discussion above, and the saddle orbit T corresponds to a simple Newhouse saddle p_n, for some n. The term "first non-degenerate tangency" refers to the following set (H) of hypotheses:

(i) $W^u(p_1)$ does not intersect $W^s(p_1)$ for $A < A_*$.
(ii) When $A = A_*$, $W^u(p)$ forms a tangency of "finite order" with $W^s(p)$ at a point q_0; i.e., there is a coordinate system (u, v) in a neighborhood of q_0 such that q_0 corresponds to $(0, 0)$, $W^s(p)$ corresponds to the u-axis $(v = 0)$, and $W^u(p)$ corresponds to $\{(u, v): v = h(u)\}$, where $\dfrac{d^i h}{du^i} = \begin{cases} 0 & \text{if } 1 \leqq i < n \\ a > 0 & \text{if } i = n \end{cases}$ for some $n \geqq 2$; and
(iii) $W^u(p)$ crosses $W^s(p)$ with non-zero speed. I.e., for each A in $(A_*, A_* + \delta)$, $W^u(p)$ intersects $W^s(p)$ at a point q_A (the continuation of q_0). Furthermore, for each neighborhood $\mathscr{U}$ of q_0 sufficiently small that $\mathscr{U} - W^s(p)$ is disconnected, there are points of $W^u(p_1)$ in both components of $\mathscr{U} - W^s_A(p_1)$, where $W^s_A(p_1)$ is the component of $\mathscr{U} \cap W^s(p_1)$ containing q_A.

For the application in which $W^s(p_1)$ is the boundary of the basin of infinity, condition (iii) implies that points in $W^u(p_1)$ are in the basin of infinity for each A in $(A_*, A_* + \delta)$.

Theorem. *Consider a diffeomorphism f of the plane depending on a parameter A with a saddle fixed point or periodic orbit S. We assume that the absolute value of the determinant of the Jacobian of f (or of f^n in the case of a periodic orbit of period n) is less than one at every point of the plane. Assume that f has a transition value A_* as A increases where the stable and unstable manifolds of S have a non-degenerate tangency and then cross for the first time. Then there will be a periodic saddle T that is in the closure of the stable manifold of S for all A slightly greater than A_* but is not in it at A_*. The saddle T is a positive distance from the stable manifold of S for $A = A_*$.*

We prove this theorem with the aid of the following lemma. The existence of the sequence $\{p_n\}$ of hyperbolic saddles was first demonstrated in the case of quadratic tangency by Gavrilov and Silnikov [GS], and later by techniques of Robinson [R] where arbitrary finite order tangencies are considered. For n sufficiently large, the horseshoe map f^n restricted to B_n persists for parameter values beyond tangency, i.e., for A in the interval $[A_*, A_* + \delta)$, where $\delta > 0$ is very small. We let $p_k = p_k(A)$ be the continuation of the simple Newhouse saddle of period k for A in $[A_*, A_* + \delta)$.

Lemma. *Let p_k be a simple Newhouse saddle of period k (as described above) for A in $[A_*, A_* + \delta)$.*

(i) *There exists $N_0 > 0$ independent of A in $[A_*, A_* + \delta)$ such that $W^u(p_n)$ crosses $W^s(p_{n+1})$ for all $n \geqq N_0$.*
(ii) *For each A in $(A_*, A_* + \delta)$ there exists $N_1(A) \geqq 0$ such that $W^u(p_m)$ crosses $W^s(p_1)$, for all $m \geqq N_1(A)$.*

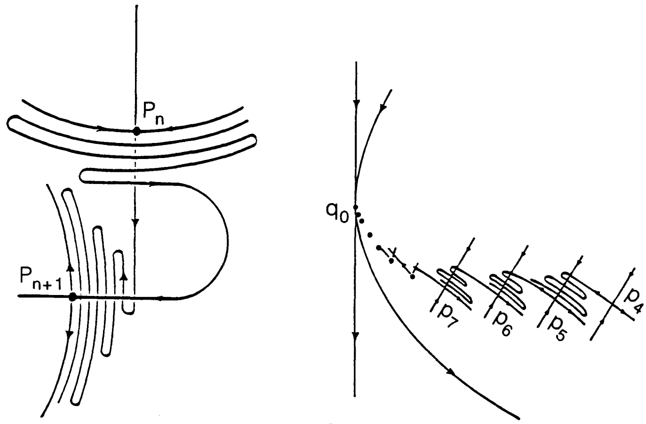


Fig. 5a and b. This indicates that the closure of $W^u(p_{n+1})$ is contained in the closure of $W^u(p_n)$. **b** indicates that the point of tangency q_0 is in the closure of the unstable manifolds of infinity many simple Newhouse saddles

We postpone the proof of this lemma due to its technical nature and proceed to show how the theorem follows. By part (i) of the lemma, we assume that there exists $N_0>0$ independent of A in $[A_*, A_*+\delta)$ such that $W^u(p_n)$ crosses $W^s(p_{n+1})$ for all $n \geqq N_0$, A in $[A_*, A_*+\delta)$. The crossing of $W^u(p_n)$ and $W^s(p_{n+1})$ implies that the forward iterates of any segment of $W^u(p_n)$ containing x will eventually contain all of $\overline{W^u(p_{n+1})}$ (the closure of $W^u(p_{n+1})$) in its set of limit points (see Fig. 5a). For a discussion of this result in the case of a transverse crossing (the λ-Lemma), see, for example, the exposition in [GH]. Hence $\overline{W^u(p_{n+1})} \subset \overline{W^u(p_n)}$. Proceeding inductively, we have

(i) $\overline{W^u(p_m)} \subset \overline{W^u(p_n)}$, (see Fig. 5b), for every $m \geqq n$, $n \geqq N_0$. By part (ii) of the lemma, there exists a number $N_1(A)>0$ such that $W^u(p_m)$ crosses $W_1^s(p)$ for $m_1 \geqq N(A)$, $A \in (A_*, A_*+\delta)$. Thus
(ii) $\overline{W^u(p_1)} \subset \overline{W^u(p_m)}$, for every $m \geqq N_1(A)$, $A \in (A_*, A_*+\delta)$. Putting together (i) and (ii), we have
(iii) $\overline{W^u(p_1)} \subset \overline{W^u(p_n)}$, for all $n \geqq N_0$, $A \in (A_*, A_*+\delta)$. For $A>A_*$, $W^u(p_1)$ crosses $W^s(p_1)$. By (iii), $W^u(p_n)$ must also cross $W^s(p_1)$, for all $n \geqq N_0$. Thus $\overline{W^s(p_n)} \subset \overline{W^s(p_1)}$, for all $n \geqq N_0$, and p_{N_0} is in the closure of $W^s(p_1)$, for all A in $[A_*, A_*+\delta)$.

Remarks.

(1) At $A=A_*$, each of the saddles p_n is a positive distance from the boundary $W^s(p_1)$ of the basin of infinity. For every A slightly larger than A_*, the theorem says that there is an N_0 such that p_{N_0} is in the closure of $W^s(p_1)$. Thus there is a jump in the boundary at $A=A_*$.

(2) Since the Hénon map is analytic, the tangency at $A_* \approx 1.314$ ($J=0.3$) is of finite order and techniques of [R] would apply. At this tangency N_0 appears to be 4 (see [GOY]). This is supported by computer evidence that for A slightly greater than 1.314, the saddle p_4 is on the boundary of the basin of infinity.

(3) The proof of the theorem characterizes the boundary after tangency by showing that there are infinitely many saddles and their stable manifolds contained in $\overline{W^s(p_1)}$. The fact that there is a jump in the boundary is, of course,

implied by this characterization. The existence of such a jump can be demonstrated by a simpler, topological argument. For any path I connecting the left and right sides of B_n (cf. Fig. 4), $f^n(I)$ extends through the horseshoe image $f^n(B_n)$. If $f^n(B_n)$ crosses B_{n+1} (as shown in Fig. 4), a portion of $f^n(I)$ connects the left and right side of B_{n+1}. If, at tangency ($A = A_*$), $f^r(B_r)$ so crosses B_{r+1} for all r, $r \geqq n$, then $\overline{\bigcup_{r \geqq n} f^r(I)}$ contains q_0. For $A > A_*$, some forward iterate of I will then cross $W^s(p_1)$.

Proof of Lemma. Following the construction of [R, TY, and GH] (Sect. 6.6), we assume the following:

(i) $DF(p_1)$ has eigenvalues ν and λ which satisfy $0 < \nu < 1$, $\lambda > 1$, and $\nu\lambda < 1$, for all A near A_*.
(ii) There exists a neighborhood U of p_1 in which the map f is linear up to smooth changes of coordinates; i.e., $f(x, y, A) = (\lambda x, \nu y)$ for (x, y) in U, all A near A_*. (Here we need an additional non-resonance assumption – namely, that ν and λ are not integer multiples of each other.) Hence, locally, $W^s(p_1)$ is given by the y-axis and $W^u(p_1)$ is given by the x-axis.
(iii) There is a non-degenerate tangency of $W^s(p_1)$ and $W^u(p_1)$. Specifically, there exist points $(p_0, 0)$ in $W^u(p_1) \cap U$ and $(0, q_0)$ in $W^s(p_1) \cap U$ such that $f^k(p_0, 0, A_*) = f^k_{A^*}(p_0, 0) = (0, q_0)$ and $W^s(p_1)$ and $W^u(p_1)$ near $(0, q_0)$ satisfy (H). Let $V = [p_0 - \varepsilon, p_0 + \varepsilon] \times [0, \varrho]$ for some $\varepsilon > 0$ and $\varrho > 0$, and let W be a trapezoidal neighborhood with vertices $f^k_{A^*}(p_0 - \varepsilon, 0)$, $f^k_{A^*}(p_0 + \varepsilon, 0)$, and the projections of these points on the y-axis, $W^s(p_1)$. (See Fig. 6.) We assume that V and W are in $\mathscr{U}$.

For n sufficiently large and $A = A_*$, $f^{-n+k}(V)$ stretches across W. For such n, let $B_n = f^{-n+k}(V) \cap W$. Actually, since $f^{-n+k}(V)$ may wind around a lot, we let B_n be the connected component of $f^{-n+k}(V) \cap W$ which is nearest $W^s(p_1)$. Under hypotheses (H), we know that f^n restricted to B_n is a horseshoe map, in the sense of Smale [S]. (See, for example, [GH] in the case of quadratic tangency and [R] for the case of finite order tangency.) Specifically, we use the following facts about such maps:

(1) B_n and $f^n(B_n)$ intersect in two components, $W_{1,n}$ and $W_{2,n}$. The saddle p_n is contained in $W_{1,n}$ and is the only fixed point of f^n in $W_{1,n}$. Furthermore, p_n is the only point in $W_{1,n}$ which stays in $W_{1,n}$ under all forward and backward iterates of f^n.

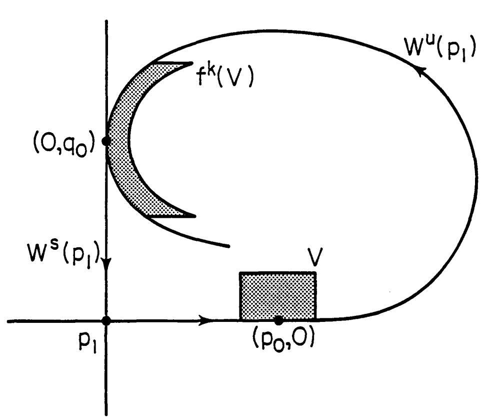


Fig. 6. The figure illustrates definitions used in the proof of the Lemma

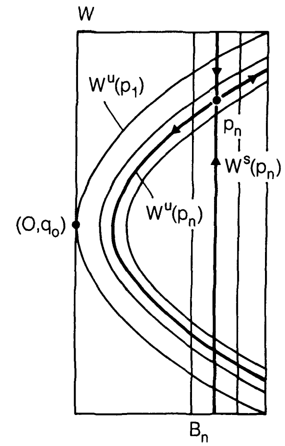


Fig. 7. *It shows parts of the stable and unstable manifolds of the simple Newhouse saddle* p_n

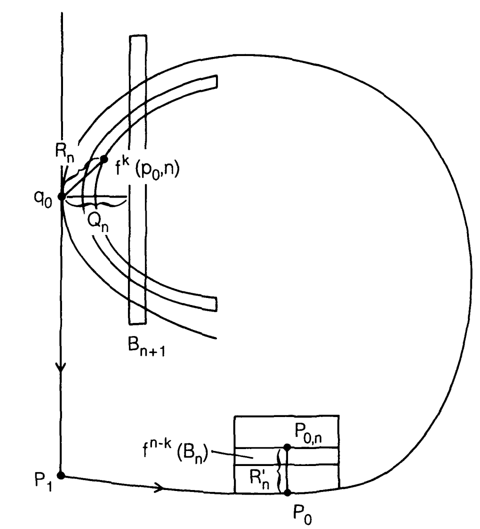


Fig. 8. This illustrates definitions used in the proof of the Lemma

(2) The only points which stay in $W_{1,n}$ under all forward (respectively, backward) iterates of f^n are in $W^s(p_n)$ (respectively $W^u(p_n)$).

We argue that the stable manifold of p_n extends (vertically) through B_n (see Fig. 7). Let L_0 be any horizontal segment in B_n. It is easily seen that $f^n(L_0)$ is a parabola which extends through $f^n(B_n)$. Recursively, let $L_i = f^n(L_{i-1}) \cap W_{1,n}$ for $i = 1, 2, 3, \ldots$. Then $f^{-ni}(L_i)$, $i \geqq 1$, are nested intervals converging to a point z_0. Since $f^{mn}(z_0)$ is in $W_{1,n}$ for all $m > 1$, z_0 must be in $W^s(p_n)$. This argument shows that $W^s(p_n)$ intersects the top and bottom of B_n and first leaves B_n through these sides. A similar argument (using iterates of f^{-1}) shows that $W^u(p_n)$ extends through the horseshoe $f^n(B_n)$, first leaving the horseshoe through the "feet." (See Fig. 7.)

In order to prove that $W^u(p_n)$ intersects $W^s(p_{n+1})$, we will show that the horseshoe $f^n(B_n)$ containing $W^u(p_n)$ crosses through B_{n+1} for n sufficiently large (see Fig. 8). When $A = A_*$ let Q be the distance from $(0, q_0)$ to B_{n+1}, let R' be the (vertical distance from p_0 to the point $p_{0,n}$ at the top of the rectangle $f^{n-k}(B_n)$, and let R_n be the distance from $q_0 = f^k(p_0)$ to $f^k(p_{0,n})$, as shown in Fig. 8.

By our assumption that V and W are in U, Q is given by $(p_0 - \varepsilon)(\lambda^{-(n-k+1)})$, and R'_n is given by $(q_0 + \delta)\mu^{(n-k)}$. Thus $\dfrac{R'_n}{Q_n} = \dfrac{\lambda(q_0+\delta)}{(p_0-\varepsilon)}(\mu\lambda)^{(n-k)}$. Since $\mu\lambda < 1$, $\dfrac{R'_n}{Q_n} \to 0$ as $n \to \infty$. Similarly, $R_n/Q_n \to 0$ as $n \to \infty$, since R_n is the distance from $q_0 = f^k(p_0)$ to $f^k(p_{0,n})$ and f, hence f^k, is Lipschitz; (i.e., there exists a constant K, independent of n, such that $R_n = KR'_n$). Thus there exists $N_0 > 0$ such that $R_n/Q_n < 1$ for $n \geqq N_0$. When $R_n < Q_n$, $W^u(p_n)$ crosses $W^s(p_{n+1})$. Part (i) of the lemma follows from the observation that for $A \in [A_*, A_* + \delta)$, the horseshoe $f^n(B_n)$ is pulled even further through B_{n+1} for $n \geqq N_0$, implying that $W^u(p_n)$ continues to cross $W^s(p_{n+1})$.

For the proof of part (ii), notice that $f_A(p_{0,n}) \to f_A(p_0)$ for all A, since $p_{0,n} \to p_0$. For each $m > N_0$, the image $f_A^m(B_m)$ contains part of $W^u(p_m)$. Thus since $f_A(p_{0,m}) \to f_A(p_0)$, there exists a sequence $\{r_m\}$ such that r_m is on $W^u(p_m)$ and $r_m \to f_A(p_0)$. We assume without loss of generality that, for A in $(A_*, A_* + \delta)$, $f_A(p_0)$ is on the opposite side of $W^s(p_1)$ from B_n and that the distance $\varrho(A)$ from $f_A(p_0)$ to q_0 is positive (by assumption (iii)). (Assumption (iii) does not say specifically that $f_A(p_0)$ is on the opposite side of $W^s(p_1)$, but that some points of $W^u(p_1)$ in a neighborhood of $f_A(p_0)$ are. The number R'_n, however, measures the distance from $W^u(p_1)$ (the x-axis) to the rectangle $f^{n-k}(B_n)$ and thus is independent of the point p'_0 chosen on $W^u(p_1)$ in a neighborhood of p_0.) Let $N_1(A) > 0$ be sufficiently large that $|r_m - f_A(p_0)| < \varrho(A)$, for all $m \geqq N_1(A)$. Then for $A \in (A_*, A_* + \delta)$, $W^u(p_m)$ crosses $W^s(p_1)$ for all $m \geqq N_1(A)$.

References

[GH] Guckenheimer, J., Holmes, P.: Nonlinear oscillations, dynamical systems, and bifurcation of vector fields. Berlin, Heidelberg, New York: Springer 1983

[GOY] Grebogi, C., Ott, E., Yorke, J.: Basin boundary metamorphoses: Changes in accessible boundary orbits. Physica **24**D, 243–262 (1987)

[GS] Gavrilov, N., Silnikov, L.: On three-dimensional dynamical systems close to systems with structurally unstable homoclinic curve I. Math. USSR Sbornik **17**, 467–485 (1972); and Part II. Math. USSR Sbornik **19**, 139–156 (1973)

[HJ] Hammel, S., Jones, C.: Jumping stable manifolds for dissipative maps of the plane. Physica **35**D, 87–106 (1989)

[N] Newhouse, S.: Diffeomorphisms with infinitely many sinks. Topology **13**, 9–18 (1974)

[R] Robinson, C.: Bifurcation to infinitely many sinks. Commun. Math. Phys. **90**, 433–459 (1983)

[S] Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)

[TY] Tedeschini-Lalli, L., Yorke, J.: How often do simple dynamical processes have infinitely many sinks? Commun. Math. Phys. **106**, 635–657 (1986)

Communicated by J. N. Mather