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Periodic and Partially Periodic Representations of $SU(N)_q$

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Abstract. The Gelfand–Zetlin basis is adapted to $SU(N)_q$ for q a root of unit. Extra parameters are incorporated in the matrix elements of the generators to obtain all the invariants corresponding to the augmented center. A crucial identity is derived and proved, which guarantees the periodicity of the action of the generators. Full periodicity is relaxed by stages, some raising and lowering operators remaining injective while others become nilpotent with corresponding changes in the dimension of the representation. In the extreme case of highest weight representations, all the raising and lowering operators are nilpotent. As an alternative approach an auxiliary algebra giving all the periodic representations is presented. An explicit solution of this system for $N=3$, while fully equivalent to the G.–Z. basis, turns out to be much simpler.

1. Introduction

The Gelfand–Zetlin basis was proposed [1] for the classical ($q=1$) $U(N)$ and $O(N)$ groups. This was extended [2] to non-semisimple $IU(N)$ and $IO(N)$ groups, which have an Abelian subalgebra in semidirect product with the homogeneous $U(N)$ and $O(N)$ subalgebras respectively. The q-analogues of both the cases, homogeneous [3] and inhomogeneous [4] were then proposed for the unitary case and for q not a root of unity. It will be shown in Sect. 2 that the G.–Z. basis works also for q a root of unity if the domains of the parameters involved are chosen suitably. In particular periodicity requirements can be imposed systematically. For $SU(2)$ and $SU(3)$ periodic representations [5, 6] were classified elsewhere. Their relations with the corresponding representation in the G.–Z. basis will be given. But the G.–Z. basis, adapted to the root of unity case works canonically for any N. One can impose full periodicity in all the parameters. As proved in Sect. 3, one can also relax these constraints by stages. To each stage corresponds a typical domain of the parameters involved. Study of such constraints is rewarding. They probe and display fully the possibilities of the formalism. An alternative study is developed in Sect. 4, which reduces the classification of periodic representation of $\mathcal{U}(SU(N))_q$ to that of an auxiliary algebra $\mathcal{A}$, with a reduction of the dimension by a factor m^{N-1} if $q^m = 1$. In Sect. 5, the G.–Z. presentation is finally shown to reproduce the classification of [6] in the case of $\mathcal{U}(SU(3))_q$.

The quantum group $\mathcal{U}(SU(N))_q$ is defined by the generators

$$\left.\begin{array}{r} q^{\pm h_i/2}, \\ e_i, \\ \text{and } f_i \end{array}\right\} \quad (i = 1,\ldots,N)$$

and the relations

$$\left\{\begin{array}{l}
q^{h_i/2}.q^{-h_i/2} = q^{-h_i/2}.q^{h_i/2} = 1, \\
q^{h_i/2}.q^{h_j/2} = q^{h_j/2}.q^{h_i/2}, \\
q^{h_i/2}.e_j.q^{-h_i/2} = q^{a_{ij}/2} e_j, \\
q^{h_i/2}.f_j.q^{-h_i/2} = q^{-a_{ij}/2} f_j, \\
[e_i, f_j] = \delta_{ij} \dfrac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \equiv \delta_{ij}[h_i], \\
[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for } |i-j| \geqq 2 \\
e_i^2 . e_{i\pm 1} - (q + q^{-1}) e_i . e_{i\pm 1} . e_i + e_{i\pm 1} . e_i^2 = 0 \\
f_i^2 . f_{i\pm 1} - (q + q^{-1}) f_i . f_{i\pm 1} . f_i + f_{i\pm 1} . f_i^2 = 0
\end{array}\right. \tag{1}$$

where $(a_{ij})_{i,j=1,..N-1}$ is the cartan matrix of $SU(N)$, i.e.

$$\left\{\begin{array}{l}
a_{ii} = 2 \\
a_{i i\pm 1} = -1 \\
a_{ij} = 0 \quad \text{for } |i-j| \geqq 2.
\end{array}\right.$$

The last two relations are called the Serre relations.

As usual, we shall only consider representations of the algebra structure of the quantum group. The coalgebra structure allows the composition of representations, whereas the R matrix, when an expression can be found for it, is an intertwiner between the differently ordered tensor products.

We suppose in the following that m is the smallest integer such that $q^m = \mathbf{1}$, and we denote, as in [6]

$$m^* = \left\{\begin{array}{ll} m & \text{if } m \text{ is odd} \\ \dfrac{m}{2} & \text{if } m \text{ is even.} \end{array}\right.$$

Let M be a finite dimensional simple module over $\mathcal{U}(SU(N))_q$. As usual, the generators q^{h_i} are simultaneously diagonalizable on M. As a consequence of the commutation relations, $(q^{h_i})^m$, $(e_i)^m$, and $(f_i)^m$, belong to the center of the algebra. They are thus scalar on M.

2. The Gelfand Zetlin Basis and Periodic Representations

The G.–Z. basis is defined as follows: let M be vector space generated by the set of

$$|h\rangle = \left| \begin{array}{ccccccc} h_{1N} & & h_{2N} & \cdots & h_{N-1,N} & & h_{NN} \\ & h_{1N-1} & & \cdots & & h_{N-1,N-1} & \\ & & \ddots & \cdots & \iddots & & \\ & & h_{12} & & h_{22} & & \\ & & & h_{11} & & & \end{array} \right\rangle$$

where the h_{jN}'s, $j=1,\ldots,N$ of the top line are constant on M. We define $\lambda \equiv \sum_{i=1}^{N} h_{iN}$.

In the generic case (q not a root of unity), the indices h_{ij} are integers and satisfy

$$h_{i,j+1} \geqq h_{i,j} \geqq h_{i+1,j+1}, \quad i \leqq j = 1,\ldots,N-1.$$

On the contrary, let us choose here non-integer or complex values for the h_{ij}'s. We consider them modulo m and we denote them, for convenience

$$h_{ij} = \zeta_{ij} + l_{ij} + i,$$

where $l_{ij} \in \mathbb{Z}/m\mathbb{Z}$ and $\zeta_{ij} \in [0,1[\times i\mathbb{R}$. In the following, we call ζ_{ij} the "fractional part" of h_{ij}. The restriction to real h's would be a necessary condition for a unitary representation.

Each h_{ij} will be changed by 1, 0 or -1 by the action of the generators of $\mathcal{U}(SU(N))_q$, so that its fractional part ζ_{ij} will be constant on an irreducible representation.

The action of the generators of $\mathcal{U}(SU(N))_q$ on this basis is given, as described in [4], by

$$q^{h_k} = q^{A_k^k - A_{k+1}^{k+1}}, \quad (k = 1,\ldots,N-1),$$

where

$$q^{A_1^1}|h\rangle = q^{h_{11}}|h\rangle,$$
$$q^{A_k^k}|h\rangle = q^{\left(\sum_{i=1}^{k} h_{ik} - \sum_{i=1}^{k-1} h_{i\,k-1}\right)}|h\rangle \quad (k = 2,\ldots,N),$$

and by

$$f_k|h\rangle = A_{k+1}^k|h\rangle = \sum_{j=1}^{k} c_{jk}\left(-\frac{P_1(jk,h)\,P_2(jk,h)}{P_3(jk,h)}\right)^{1/2}|h_{jk}-1\rangle,$$

$$e_k|h\rangle = A_k^{k+1}|h\rangle = \sum_{j=1}^{k}\frac{1}{c_{jk}}\left(-\frac{P_1(jk,h_{jk}+1)\,P_2(jk,h_{jk}+1)}{P_3(jk,h_{jk}+1)}\right)^{1/2}|h_{jk}+1\rangle,$$

where $|h_{jk}\pm 1\rangle$ is obtained from $|h\rangle$ by changing only h_{jk} by $h_{jk}\to h_{jk}\pm 1$.

The parameters c_{jk} have been introduced to achieve periodicity with the full quota of invariants, namely N^2-1, the dimension of the underlying classical algebra. Finally,

$$P_1(jk,h) = \prod_{i=1}^{k+1}[h_{i,k+1} - h_{jk} - i + j + 1],$$

$$P_2(jk,h)=\prod_{i=1}^{k-1}[h_{i,k-1}-h_{jk}-i+j],$$
$$P_3(jk,h)=\prod_{\substack{i=1\\ i\neq 1}}^{k}[h_{ik}-h_{jk}-i+j+1][h_{ik}-h_{jk}-i+j].$$

These expressions are coherent with $h_{jk}\in\mathbb{C}/m\mathbb{Z}$. To avoid zero's of P_3 when $q^m=1$, $(m=3,4,\dots)$, each difference $(h_{ik}-h_{jk})$ will first be non-integer for even m, and non also half integers for odd m. That is

$$\forall k,\quad \forall i,j\leqq k\quad \zeta_{ik}-\zeta_{jk}\notin\mathbb{Z}\quad (\text{respectively }\notin\tfrac{1}{2}\mathbb{Z}\text{ if } m \text{ is odd}).$$

Theorem. *All the relations defining $\mathscr{U}(SU(N))_q$ are satisfied on M characterized by $\{h_{jN},\zeta_{jk},c_{jk}\}$ and hence M is a module over $\mathscr{U}(SU(N))_q$.*

The dimension of this module is (for the fully periodic case with $q^m=1$)

$$m^{(N(N-1))/2},$$

where m is the number of allowed values for each h_{ij}, which varies by steps of 1 or -1 and is defined modulo m, and where $\frac{N(N-1)}{2}$ is the number of h_{jk} $(1\leqq j\leqq k<N)$. (The indices h_{jN}, $j=1,\dots,N$ on the top are fixed and partially characterize the module.)

Proposition. *The module characterized by $\{h'_{jN}=h_{\sigma_N(j)N}-\sigma_N(j)+j,\ \zeta'_{jk}=\zeta_{\sigma_k(j)k},\ c'_{jk}=c_{\sigma_k(j)k}\}$, where σ_k is a permutation of k elements, is equivalent to M.*

Proof. The intertwiner is given by

$$h_{jk}\longmapsto h'_{jk}=h_{\sigma_k(j),k}-\sigma_k(j)+j.$$

Remarks. The representation of $\mathscr{U}(SU(N))_q$ defined above in the G.–Z. basis with non-integer or complex values of the h_{ik}'s is not invariant under the action of the Weyl group. The simplest way to check this is to note that the eigenvalues of the generators h_k have non-related fractional part. In other words, the set of weights is not invariant under Weyl reflections. In the case of partially periodic representations, some, but not all the raising and lowering operators will actually become nilpotent. The action of the Weyl group on a representation will nevertheless provide another representation.

In the case $\lambda=0$, the first two invariants of $\mathscr{U}(SU(N))_q$ can be written as [4]

$$C_{(\varepsilon)}=\sum_{j>i}A^{j}_{i(\varepsilon)}A^{i}_{j(\varepsilon)}q-\varepsilon(A^i_i+A^j_j+2j)$$
$$+(q-q^{-1})^{-2}\left(\sum_{i=1}^{N}q^{-\varepsilon(2A^i_i+2i+1)}-Nq^{-\varepsilon(N+2)}\right)$$

with $(\varepsilon=\pm 1)$ and where, by definition,

$$A^{j+1}_{j-p(\varepsilon)}=A^{j+1-p}_{j-p}\,A^{j+1}_{j+1-p(\varepsilon)}-q^{\varepsilon}\,A^{j+1}_{j+1-p(\varepsilon)}\,A^{j+1-p}_{j-p},$$
$$A^{j-p}_{j+1(\varepsilon)}=A^{j+1-p}_{j+1(\varepsilon)}\,A^{j-p}_{j+1-p}-q^{\varepsilon}\,A^{j-p}_{j+1-p}\,A^{j+1-p}_{j+1(\varepsilon)},$$

with $A^{j+1}_{j(\varepsilon)}\equiv A^{j+1}_{j}$ and $p=1,\dots,j-1$.

For $\lambda \neq 0$, one should replace each A^i_i by

$$\hat{A}^i_i \equiv A^i_i - \frac{1}{N}(A^1_1 + \cdots + A^N_N).$$

Now the Casimirs related to the first two (quadratic and third order) classical invariants take the values $\left(\lambda \equiv \sum_i h_{iN} = 0\right)$

$$C_{(\varepsilon)}|h\rangle = \frac{1}{(q-q^{-1})^2}\left(\sum_{i=1}^{N} q^{-\varepsilon(2h_{N+1-i,N}+2i+1)} - Nq^{-\varepsilon(N+2)}\right)|h\rangle .$$

They are invariant since each $h_{i,N}$ is invariant.

For $N > 3$, there are more C's which generalize the fourth to the $(N-1)^{\text{th}}$ order classical invariants. They will not be explicitly introduced here. However the $(N-1)$ independent invariant parameters h_{iN} ($i = 1,\dots,N$ with $h_{1N} + \cdots h_{NN} = 0$ or $h_{NN} = 0$) play their role in a much more simple and direct way. The numbers $(h_{iN} - h_{NN})$ correspond in the classical case to the lengths of the rows of the Young tableaux.

Two questions arise now:

–Is this module simple?

–Does this description reproduce the whole set of simple modules over $\mathcal{U}(SU(N))_q$?

Our answer is the following: we check that the Casimirs $(q^{h_i})^m$, $(e_i)^m$, $(f_i)^m$ and $C_{(\varepsilon)}$ are scalar or M (modulo a quotient when m is even), and then prove the simplicity of M. In the last section, we shall prove that the periodic modules over $\mathcal{U}(SU(3))_q$ defined by these relations (i.e. those on which e_i and f_i are injective) reproduce exactly the periodic representations of $\mathcal{U}(SU(3))_q$ described in [6], where a complete classification was established.

It is first immediate that $(q^{h_k})^m$ is scalar on M:

$$(q^{h_k})^m|h\rangle = q^{m\left(-\sum_{i=1}^{k+1} h_{i\,k+1} + 2\sum_{i=1}^{k} h_{ik} - \sum_{i=1}^{k-1} h_{i,k-1}\right)}|h\rangle.$$

Let us now give the expression of the action of $(A^k_{k+1})^r$ on $|h\rangle$. The coefficient of $(A^k_{k+1})^r|h\rangle$ on the element $|h'\rangle$ of the basis, which we denote

$$\langle h'|(A^k_{k+1})^r|h\rangle$$

satisfies the recursion relation

$$\langle h'|(A^k_{k+1})^r|h\rangle = \sum_{l=1}^{k} c_{lk}\left(-\frac{P_1(lk, h'_{lk}+1)\;P_2(lk, h'_{lk}+1)}{P_3(lk, h'_{lk}+1)}\right)^{1/2}\langle h'_{lk}+1|(A^k_{k+1})^{r-1}|h\rangle. \tag{rec}$$

This looks like the recursion relation satisfied by the multinomial coefficients

$$\begin{pmatrix} p_1 + p_2 \cdots + p_k \\ p_1 \quad p_2 \cdots \quad p_k \end{pmatrix} = \frac{(p_1 + p_2 + \cdots p_k)!}{p_1!\,p_2!\cdots p_k!}$$

but here the coefficients are not 1. It will actually provide a q-deformed multinomial coefficient,

$$\begin{bmatrix} p_1 + p_2 \cdots + p_k \\ p_1 \quad p_2 \cdots \quad p_k \end{bmatrix} = \frac{[p_1 + p_2 + \cdots p_k]!}{[p_1]!\,[p_2]!\cdots[p_k]!},$$

i.e. with factorials replaced by q-factorials

$$[p]! = [p][p-1]\ldots[2][1]$$

which vanish for p multiple of m^*.

In the development of $\langle h'|(A^k_{k+1})^r|h\rangle$, each path from $|h\rangle$ to $|h'\rangle$ provides different contribution, but the product of numerators $\prod(-P_1 P_2)^{1/2}$ only depends on $|h\rangle$ and $|h'\rangle$ and hence factorize.

Proposition. *We have, for* $|h'\rangle = |h_{1k}-p_1, h_{2k}-p_2,\ldots,h_{kk}-p_k\rangle$ *and* $r = p_1 + p_2 + \cdots + p_k$,

$$\langle h'|(A^k_{k+1})^r|h\rangle =$$
$$\frac{N_k(h')}{N_k(h)}\begin{bmatrix} p_1+\cdots+p_k \\ p_1\cdots p_k \end{bmatrix}\prod_{j=1}^{k} c_{jk}^{p_j}\prod_{p=1}^{p_j}\left(-\frac{P_1(jk,h_{jk}-p+1)P_2(jk,h_{jk}-p+1)}{P'_3(jk,h_{jk}-p+1)}\right)^{1/2},$$

where

$$N_k(h) = \prod_{i\neq j}[h_{ik}-h_{jk}-i+j]^{1/4},$$

$$N_k(h') = \prod_{i\neq j}[h_{ik}-p_i-h_{jk}+p_j-i+j]^{1/4},$$

$$P'_3(jk,h) = \prod_{\substack{i=1\\ i\neq j}}^{k}[h_{ik}-h_{jk}-i+j+1]^2.$$

This expression actually satisfies the recursion relation (rec), which is enough to prove the proposition, provided the following identity holds:

Lemma. *For arbitrary p's and a's, and if* $[a_i - a_l]\neq 0$ *for* $i\neq l$,

$$\sum_{l=1}^{k}[p_l]\prod_{i\neq l}\frac{[a_i-a_l+p_i]}{[a_i-a_l]} = [p_1+\cdots+p_k].$$

Proof. With $x_i = q^{2a_i}$, the proof of this identity reduces to the following one: Let

$$Q^{(k)} \equiv \prod_{1\leqq i<j\leqq k}(x_j - x_i)$$

and let

$$Q_l^{(k)} \equiv Q^{(k)}(x_1, x_2,\ldots,-x_l,\ldots,x_r),$$

only x_l changing sign. Then the initial identity can be shown to follow from

$$\sum_{l=1}^{k}Q_l^{(k)} = \frac{1}{2}(1-(-1)^k)Q^{(k)}.$$

Now symmetry properties under interchange of x_i, x_j (any pair) and induction from lowest values of k are sufficient to prove this relation.

So we know explicitly the action of $(A^k_{k+1})^r$ on $|h\rangle$, whatever the value of q. Now if q is a m^{th} root of unity, most of the multinomial coefficients will vanish for $r = m$:

–If m is odd, all the coefficients $\begin{bmatrix} & m & \\ p_1 & p_2 & \cdots & p_k \end{bmatrix}$ but those for which one of the p_i is equal to m vanish. Since we identify $|h\rangle$ and $|h_{ik}-m\rangle$, (consistently with the fact that the generators acting on these two states give exactly the same matrix elements) this proves that $(A^k_{k+1})^m$ is diagonal on the G.–Z. basis. Furthermore,

$$(A^k_{k+1})^m|h\rangle = \sum_{j=1}^{k} c^m_{jk} \prod_{p=1}^{m} \left(- \frac{P_1(jk, h_{jk}-p)\, P_2(jk, h_{jk}-p)}{P'_3(jk, h_{jk}-p)} \right)^{1/2} |h_{jk}-m\rangle$$

(with $|h_{jk}-m\rangle = |h\rangle$) and since all the products run from 1 to m, each diagonal value only depends on the fractional parts of the h_{ik}'s, which are constant on the module. $(A^k_{k+1})^m$ is hence scalar on M.

Finally, since $(A^{k+1}_k)^m$ obviously satisfies the same properties, we conclude that $(q^{h_i})^m$, $(e_i)^m$, $(f_i)^m$, and $C_{(\varepsilon)}$ are scalar on the module M defined by the G.–Z. basis if m is odd. In this case,

$$\dim M = m^{(N(N-1)/2)}.$$

–If m is even, $\begin{bmatrix} & m & \\ p_1 & p_2 & \cdots & p_k \end{bmatrix}$ is zero unless either one p_i is equal to m, or two p_i and p_j are equal to m^*. We have then to quotient the module M defined by the G.–Z. basis, (since we want $(A^k_{k+1})^m$ to be scalar), by the relation

$$|h_{ik}-m^*, h_{jk}-m^*\rangle \equiv |h\rangle. \qquad \text{(quot)}$$

This is again consistent with the fact that the actions of the generators are the same. The matrix elements coincide on these two states. Identical behaviour of two states $|h\rangle$ and $|h_{1k}-p_1,\ldots,h_{kk}-p_k\rangle$ with $p_1+\cdots+p_k=m$ under the action of all generators turns out to be a necessary and sufficient condition for a non-zero value of $\begin{bmatrix} & m & \\ p_1 & p_2 & \cdots & p_k \end{bmatrix}$. The dimension is then divided by 2^{k-1} for each k from 2 to $N-1$. It is then reduced to

$$\frac{m^{(N(N-1)/2)}}{2^{((N-1)(N-2)/2)}} = (m^*)^{((N-1)(N-2)/2)} . m^{(N-1)}.$$

$(A^k_{k+1})^m$ is now scalar on the quotient.

If m is even, the elements $(q^{h_i})^m$, $(e_i)^m$, $(f_i)^m$, and $C_{(\varepsilon)}$ of the center of $\mathcal{U}(SU(N))_q$ are scalar on M quotiented by (quot).

Proposition. M (*quotiented by* (quot) *if m is even*) *defines a simple module over* $\mathcal{U}(SU(N))_q$.

Proof. Let us consider an induction on N. For $N=2$, M reproduces the representations of [5], which are irreducible. As a vector space, from the very definition of the G.–Z. basis,

$$M \equiv \bigoplus_{|h\rangle} \mathbb{C}|h\rangle = \bigoplus_{l_{jN-1}} M_{N-1}(\{l_{jN-1}+\zeta_{jN-1}+j, \zeta_{jk}, c_{jk}\}),$$

where each term of the direct sum is the subspace of M with a fixed second line $h_{jN-1} = l_{jN-1}+\zeta_{jN-1}+j$. Each term is now a module over $\mathcal{U}(SU(N-1))_q \subset$

$\mathscr{U}(SU(N))_q$, and corresponds *precisely* to the definition of the G.–Z. presentation of periodic module over $\mathscr{U}(SU(N-1))_q$. So we suppose it is simple in the induction. We assume that the set of eigenvalues of the first to the N^{th} order Casimirs of $\mathscr{U}(SU(N-1))_q$ exactly characterizes $\{l_{jN-1}\}$ for given values of $\{\zeta_{jN-1}\}$, so that for each $\{l_{jN-1}\}$, there exists in $\mathscr{U}((SU(N-1))_q$ a projector on $M_{N-1}(\{l_{jN-1}+\zeta_{jN-1}+j, \zeta_{jk}, c_{jk}\})$. Combining such projectors with the action of e_{N-1} and f_{N-1}, and provided that all the ζ_{jk} on the same line are different, it is possible to find, for every $x, y \in M$, with $x \neq 0$, an element $\mathcal{O}$ of $\mathscr{U}(SU(N))_q$ such that $\mathcal{O}x = y$. Hence M is simple.

In the two cases m odd or even, the dimension of fully periodic representation of $\mathscr{U}(SU(N))_q$ is equal to

$$(m^*)^{((N-1)(N-2)/2)} \cdot m^{(N-1)}.$$

3. Partially Periodic Representations

In the generic case of periodic representation, we had

$$h_{ij} - h_{i'j} \notin \mathbb{Z} \quad \text{if} \quad i \neq i' \quad (2 \leqq j \leqq N).$$

On the other hand regular highest weight representation can also be obtained for q a root of unity. When q is not a root of unity the necessary and sufficient conditions on the h_{ij}'s of a simple highest weight module in the G.–Z. basis are

$$h_{ij+1} - h_{ij} \in \mathbb{Z}^+,$$
$$h_{ij} - h_{i+1\,j+1} \in \mathbb{Z}^+.$$

When $q^m = 1$, the h_{ij} are defined in $\mathbb{Z}/m\mathbb{Z}$. They can be written

$$h_{ij} = \zeta + l_{ij} + i$$

and these constraints of ordering become, for so-called regular highest weights irreducible representations

a) In the first line, the l_{iN} are cyclically ordered and distinct, and their maximal difference cannot exceed m^*.

b) $l_{i'j} \in I_{ij}$ if, and only if $i = i'$. where $I_{ij} = \{l_{i+1,j+1}+1, l_{i+1,j+1}+2, \ldots, l_{i,j+1}\}$ i.e. the numbers of the rows $j+1$ separate the numbers of the row j.

As a consequence of b), if some consecutive h_{iN}'s of the first row are equal, or equivalently if the corresponding l_{iN}'s differ only by one, all the h_{ij}'s in the subtriangle under them are fixed to the same value. No e_i or f_i can change them. Furthermore, *all* the denominators involving them cancel with numerators involving h's of the same subtriangle, and only $h_{i_0 j_0}$, the one at the bottom of the subtriangle, remains as a parameter in some numerators. All these fixed h's can be removed from the definition of the G.–Z. basis, only $h_{i_0 j_0}$ being kept as a parameter instead of those of the first line. The restriction $N \leqq m^*$ which is a priori a consequence of a), becomes $N - N_1 + 1 \leqq m^*$ if the subtriangle is of size N_1 and is on the boundary, or $N - N_1 - N_2 + 2 \leqq m^*$ if there are two such subtriangles on both corners.

When transposed into the theory of representations of Kac–Moody algebras, the constraint $h_{1N} - h_{NN} + N \leqq m^*$, consequence of a), corresponds to the integrability condition. Note that this last inequality is well defined in $\mathbb{Z}$ when the h's of the *highest weight* representation are considered in $\mathbb{Z}$ instead of $\mathbb{Z}/m\mathbb{Z}$ with triangular inequalities as in the classical case.

Between the fully periodic case and the last one corresponding to highest weight representations, (with all e_i's and f_i's nilpotent), one has different stages of partial periodicity, when integer differences are restricted to subtriangles

$$\begin{array}{ccccccccc}
h_{p+1,N} & & h_{p+2,N} & & \cdots & & h_{p+N_1-1,N} & & h_{p+N_1,N} \\
& h_{p+1,N-1} & & & \cdots & & & h_{p+N_1-1,N-1} & \\
& & \ddots & & \cdots & & ⋰ & & \\
& & & h_{p+1,N-N_1+2} & & h_{p+2,N-N_1+2} & & & \\
& & & & h_{p+1,N-N_1+1} & & & &
\end{array}$$

In this subtriangle each h has the same fractional part (say ζ) and the h_{ij} in this subtriangle satisfy the conditions a) and b) of regular highest weight modules of $\mathscr{U}(SU(N_1))_q$.

The triangular structure itself is imposed by regularity condition. The constraint is in fact the following: if the set of indices h_{jk} with the same fractional part has $n > 1$ elements on line $k < N$, it must have exactly $n + 1$ elements on line $k + 1$ and $n - 1$ elements on line $k - 1$. By virtue of the remark of Sect. 2 on possible permutations of the h_{jk}'s on the same line, the set of h_{jk} with fractional value ζ can be gathered into a subtriangle as described above. In such a triangle zeroes in the numerator prevent divergences due to zeroes in the denominator.

A subtriangle by itself corresponds to a certain highest weight representation, say $\Pi_1^{(N_1)}$, of $\mathscr{U}(SU(N_1))_q$. One can introduce several such subtriangles of h_{ij}'s

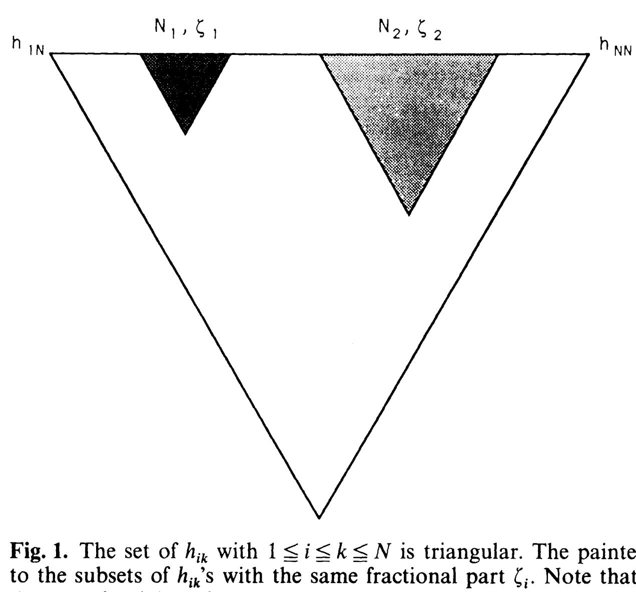

Fig. 1. The set of h_{ik} with $1 \leqq i \leqq k \leqq N$ is triangular. The painted subtriangles N_i correspond to the subsets of h_{ik}'s with the same fractional part ζ_i. Note that the figures represent neither the sets of weights of representations of $SU(3)$ nor the values of h_{ik} but the set of ik, indices of the h_{ik} defining the Gelfand Zetlin basis

with constant ζ's (up to permutations σ_k in the rows). These triangles have to be non-intersecting and with distinct ζ's. (See Fig. 1.)

Proposition. *The dimension of the partially periodic module with subtriangles N_i and conditions a) and b) in each subtriangle is*

$$\prod_i \left(\frac{\dim \Pi_i^{(N_i)}}{(m^*)^{(N_i(N_i-1)/2)}} \right) (m^{*(N-1)(N-2)/2} m^{N-1})$$

and this module is simple.

Proof. This is a consequence of the results on the dimension and the irreducibility of the corresponding highest weight representations $\Pi_i^{(N_i)}$.

With a given structure of subtriangles, the permutations in the rows that preserve the subtriangles do not preserve the conditions of ordering of the integer parts of the h_{ij}'s. More precisely, they are in one to one correspondence with the possible orderings. The first line of indices of a subtriangle N_i has fixed values. Its ordering in $\mathbb{Z}/m\mathbb{Z}$ leads to N_i possible permutations of the first line. The second line has then $(N_i-1)!$ possible orderings. This leads then to $\prod_i N_i(N_i-1)!\,(N_i-2)!\ldots(2)!$ choices of sets of inequalities between the integer parts of the h_{ij}'s.

The truncation of periodicity and hence of the dimension is associated with nilpotency of certain operators but not, in general, of the generators e_i, f_i themselves. One should rather look at the lowering (L) and raising (R) operators studied in the classical case long before [7] and recently generalized to the quantum one [8]. They can be conveniently defined recursively. To study them adequately here would require the development of another elaborate formalism. So we merely indicate that they can provide insight concerning the systematics of the partially periodic cases.

Example: Flat Representations. Let us consider the following partially periodic module: suppose that all the h_{jk}'s with $j \geqq 2$ are equal. There is actually one single subtriangle $N_1 = N-1$ frozen to a single value h_{NN}, as depicted on Fig. 2. The module we obtain was briefly mentioned in [6]. It has dimension $(m^*)^{(N-1)}$ since $\dim \Pi_1^{(N_1)} = 1$, so one can just consider the first column. All the weights have multiplicity one. (This case corresponds to the symmetric tensors of the classical $SU(N)$.) Let us denote $h_k \equiv h_{1k}$ for $k = 1, \ldots N$. The expressions of the generators are very simple on this module, since there are no denominators,

$$f_k\,|h\rangle = A_{k+1}^{k}\,|h\rangle = c_k(-[h_{k+1}-h_k+1][h_{k-1}-h_k])^{1/2}\,|h_k-1\rangle,$$
$$e_k\,|h\rangle = A_k^{k+1}\,|h\rangle = \frac{1}{c_k}(-[h_{k+1}-h_k][h_{k-1}-h_k-1])^{1/2}\,|h_k+1\rangle$$

with by convention $h_0 = h_{NN}$.

This is the simplest example of periodic representation of $\mathcal{U}(SU(N))_q$. It corresponds to trivial representations of the auxilary algebra defined in the following section, i.e. with all the operators β scalar.

When now some consecutive h_k have the same fractional part, some of the generators e_i and f_i become nilpotent. They are all nilpotent when all the h_k have the same fractional part. This case generalizes the "flat" representations of

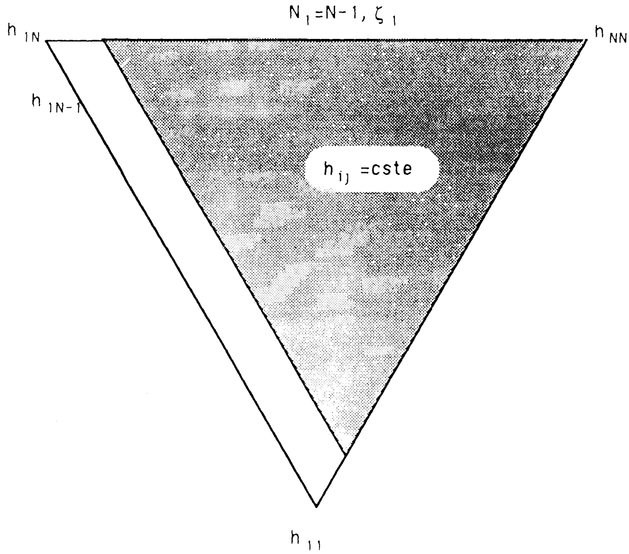

Fig. 2. For the simplest periodic representations of $\mathscr{U}(SU(N))_q$, all the h_{ik} are frozen to the same value but one column, as for symmetric tensors

$\mathscr{U}(SU(3))_q$ introduced in [6]. These representations are highest weight representations, and they are either symmetric tensors or the irreducible parts of the symmetric tensors which become too large (in fact rather irreducible factors of the symmetric tensors, since these are indecomposable).

4. Alternative Method and Auxiliary Algebra $\mathscr{A}$

We present here an alternative method for the study of the representations of $\mathscr{U}(SU(N))_q$. This method reduces the problem of the classification of periodic irreducible representations of $\mathscr{U}(SU(N))_q$ to the classification of m^{N-1} times smaller irreducible representations of an auxiliary algebra $\mathscr{A}$. As in [6], we first suppose that all the generators e_i and f_i are injective on a module M. Let the q^{h_i} be simultaneouly diagonalized. Since the set

$$\{e_{2i};\ 2 \leqq 2i \leqq N-1\} \cup \{f_{2i+1};\ 1 \leqq 2i+1 \leqq N-1\}$$

only contains mutually commuting operators, there exist correlated bases on the common eigenspaces M_P of q^{h_i} such that each e_{2i}, f_{2i+1} can be seen as a scalar operator $\alpha_{2i}.\mathrm{Id}$ or $\alpha_{2i+1}.\mathrm{Id}$ times an operator that translates the basis of M_P into the basis of another $M_{P'}$. In other words, if we identify all the M_P's to the same $\mathscr{M}$, then $e_{2i} = \alpha_{2i}.\mathrm{Id}$ and $f_{2i+1} = \alpha_{2i+1}.\mathrm{Id}$ on $\mathscr{M}$. This can be written as

$$M = \left[\bigoplus_P \mathbb{C}|P\rangle\right] \otimes \mathscr{M},$$

$$e_{2i}\cdot|P\rangle \otimes |x\rangle = \alpha_{2i}\ |p_{2i}-1\rangle \otimes |x\rangle,$$

$$f_{2i+1}\cdot|P\rangle \otimes |x\rangle = \alpha_{2i+1}\ |p_{2i+1}+1\rangle \otimes |x\rangle,$$

where $P = (p_1, \ldots, p_{N-1})$ characterizes the set of eigenvalues of the q^{h_i}'s.

Then there are $N-1$ complex parameters μ_i such that

$$q^{h_i}\cdot|P\rangle\otimes|x\rangle = q^{\mu_i-2p_i+p_{i-1}+p_{i+1}}|P\rangle\otimes|x\rangle$$

for $i=1,\dots N-1$, where by convention $p_0=p_N=0$. The set P characterizes exactly the set of eigenvalues of the q^{h_i}'s provided m and N are coprime. (This is the condition of invertibility of the Cartan matrix of $SU(N)$ in $\mathbb{Z}_m$.) As proved in [6] in the case $N=3$, this constraint has no major significance and can be removed.

As a consequence of the commutation relations, the expressions of the other generators e_{2i+1} and f_{2i} are given by

$$e_{2i+1}\cdot|P\rangle\otimes|x\rangle = \frac{1}{\alpha_{2i+1}(q-q^{-1})^2}|p_{2i+1}-1\rangle$$
$$\otimes\left[-q^{\mu_{2i+1}-2p_{2i+1}+p_{2i}+p_{2i+2}+1}-q^{-\mu_{2i+1}+2p_{2i+1}-p_{2i}-p_{2i+2}-1}+\beta^{2i+1}_{p_{2i},p_{2i+2}}\right]|x\rangle,$$

$$f_{2i}\cdot|P\rangle\otimes|x\rangle = \frac{1}{\alpha_{2i}(q-q^{-1})^2}|p_{2i}+1\rangle$$
$$\otimes\left[-q^{\mu_{2i}-2p_{2i}+p_{2i-1}+p_{2i+1}-1}-q^{-\mu_{2i}+2p_{2i}-p_{2i-1}-p_{2i+1}+1}+\beta^{2i}_{p_{2i-1},p_{2i+1}}\right]|x\rangle,$$

where the operator $\beta^{2i+1}_{p_{2i},p_{2i+2}}$ depends only on p_{2i}, p_{2i+2} and $\beta^{2i}_{p_{2i-1},p_{2i+1}}$ depends only on p_{2i-1} and p_{2i+1}. Furthermore, some Serre relations imply that this dependence is

$$\begin{cases} \beta^{2i+1}_{p_{2i},p_{2i+2}} = \sum\limits_{\substack{\varepsilon=+,-\\ \varepsilon'=+,-}} q^{\varepsilon p_{2i}+\varepsilon' p_{2i+2}}\beta^{2i+1}_{\varepsilon\varepsilon'}, \\ \beta^{2i}_{p_{2i-1},p_{2i+1}} = \sum\limits_{\substack{\varepsilon=+,-\\ \varepsilon'=+,-}} q^{\varepsilon p_{2i-1}+\varepsilon' p_{2i+1}}\beta^{2i}_{\varepsilon\varepsilon'}. \end{cases}$$

The classification of periodic irreducible representations of $\mathcal{U}(SU(N))_q$ is now equivalent to the classification of the irreducible representations of the auxiliary algebra $\mathcal{A}$ generated by the operators $\beta_{\varepsilon\varepsilon'}$. A simple module over $\mathcal{A}$ of dimension k will be in correspondence with a simple module over $\mathcal{U}(SU(N))_q$ of dimension $k\cdot m^{N-1}$.

The relations satisfied by the $\beta_{\varepsilon\varepsilon'}$'s follow from (1). The commutation of generators e_i or f_i and e_j or f_j for $|i-j|\geqq 2$ imply

$$[\beta^i_{\varepsilon_1\varepsilon_2},\beta^j_{\varepsilon_3\varepsilon_4}]=0 \quad \text{for } |i-j|\geqq 3,$$
$$[\beta^i_{\varepsilon_1\varepsilon_2},\beta^{i+2}_{\varepsilon_2\varepsilon_3}]=0,$$
$$[\beta^i_{\varepsilon_1\varepsilon_2},\beta^{i+2}_{-\varepsilon_2\varepsilon_3}]+[\beta^i_{\varepsilon_1-\varepsilon_2},\beta^{i+2}_{\varepsilon_2\varepsilon_3}]=0.$$

The Serre relations and $[e_{2i+1},f_{2i+2}]=0$ for each pair of neighbouring points $2i+1$, $2i+2$ of the Dynkin diagram imply the following relations between the $\beta^i_{\varepsilon\varepsilon'}$'s

$$[\beta^{2i+1}_{\varepsilon\pm},\beta^{2i+2}_{\pm\varepsilon'}]_q = \pm q^{\mp(\mu_{2i+1}+\mu_{2i+2})}\delta_{(\varepsilon,\mp)}\delta_{(\varepsilon',\mp)}(q-q^{-1}), \qquad (\pm\pm)$$

$$[\beta^{2i+1}_{\varepsilon\mp},\beta^{2i+2}_{\pm\varepsilon'}] = (q-q^{-1})(\mp\delta_{(\varepsilon,\mp)}q^{\mp\mu_{2i+1}}\beta^{2i+2}_{\mp\varepsilon'}\pm\delta_{(\varepsilon',\pm)}q^{\pm\mu_{2i+2}}\beta^{2i+1}_{\varepsilon,\pm}), \quad (\pm\mp)$$

$$\begin{cases}[\beta^{2i+1}_{+-},\beta^{2i+1}_{-+}]_q+[\beta^{2i+1}_{--},\beta^{2i+1}_{++}]_q=q-q^{-1}\\ \quad [\beta^{2i+1}_{+-},\beta^{2i+1}_{++}]_q=[\beta^{2i+1}_{--},\beta^{2i+1}_{-+}]_q=0\end{cases}, \tag{E}$$

$$\begin{cases}[\beta^{2i+2}_{+-},\beta^{2i+2}_{-+}]_q+[\beta^{2i+2}_{++},\beta^{2i+2}_{--}]_q=q-q^{-1}\\ \quad [\beta^{2i+2}_{++},\beta^{2i+2}_{-+}]_q=[\beta^{2i+2}_{+-},\beta^{2i+2}_{--}]_q=0\end{cases}, \tag{F}$$

where $[A,B]_q=qAB-q^{-1}BA$. The equation $(-+)$ is not independent and is a consequence of $(++)$, $(+-)$ and (E). These relations between β^{2i+1} and β^{2i+2} are valid for each value of i such that $1<2i+1<2i+2<N-1$. We obtain the corresponding relations between $\beta^{2i+1}_{\varepsilon_1\varepsilon_2}$ and $\beta^{2i}_{\varepsilon_3\varepsilon_4}$ for $1<2i<2i+1<N-1$ by the correspondence

$$\beta^{2i}_{\varepsilon_3\varepsilon_4}\leftrightarrow\beta^{2i+2}_{\varepsilon_4\varepsilon_3}\qquad \mu_{2i}\leftrightarrow\mu_{2i+2}$$
$$\beta^{2i+1}_{\varepsilon_1\varepsilon_2}\leftrightarrow\beta^{2i+1}_{\varepsilon_2\varepsilon_1}.$$

On the boundary, since $p_0=p_N=0$, we have in fact only

$$\beta^1_\varepsilon\equiv\beta^1_{+,\varepsilon}+\beta^1_{-,\varepsilon},$$
$$\beta^{N-1}_\varepsilon\equiv\beta^{N-1}_{\varepsilon,+}+\beta^{N-1}_{\varepsilon,-},$$

and relations between $\beta^1_{\varepsilon_1}$ and $\beta^2_{\varepsilon_2\varepsilon_3}$ are obtained by adding some of the equations of each group $(++)$, or $(--)$...in order to reconstruct $\beta^1_{\varepsilon_1}$ instead of $\beta^1_{\pm,\varepsilon_1}$. This is similarly true at the other edge of the Dynkin diagram for the relations involving β^{N-1}_ε. The particular case of $\mathscr{U}(SU(3))_q$ is detailed in the following section and the classification of the representation of $\mathscr{A}$ if shown to be equivalent to the set of representations provided by the G.–Z. basis.

5. Equivalence with the Gelfand–Zetlin Basis for Periodic Representations of $\mathscr{U}(SU(3))_q$

In the particular case of $\mathscr{U}(SU(3))_q$, the only pair of neighbouring points on the Dynkin diagram is (1, 2) and the algebra $\mathscr{A}$ is reduced to the simple form given in [6], and also in [12], that is

$$q^{\mu_1+\mu_2}(q\beta^1_+\beta^2_+-q^{-1}\beta^2_+\beta^1_+)=q-q^{-1}, \tag{$++$}$$
$$q^{-\mu_1-\mu_2}(q\beta^2_-\beta^1_--q^{-1}\beta^1_-\beta^2_-)=q-q^{-1}, \tag{$--$}$$
$$q\beta^1_-\beta^1_+-q^{-1}\beta^1_+\beta^1_-=q-q^{-1}, \tag{E}$$
$$q\beta^2_+\beta^2_--q^{-1}\beta^2_-\beta^2_+=q-q^{-1}, \tag{F}$$
$$q^{-\mu_1}\beta^2_--q^{\mu_2}\beta^1_++\frac{1}{q-q^{-1}}[\beta^1_-,\beta^2_+]=0, \tag{$+-$}$$
$$q^{\mu_1}\beta^2_+-q^{-\mu_2}\beta^1_--\frac{1}{q-q^{-1}}[\beta^1_+,\beta^2_-]=0. \tag{$-+$}$$

Now that μ_1 and μ_2 can be eliminated from these relations by a rescaling of $\beta^{1,2}_\pm$. The rescaled $\beta^{1,2}_\pm$ of this paper correspond to u, u', v and v' of [6]. The representations of $\mathscr{A}$ have been classified in [6]. Their dimension k can be

$1 \leqq k \leqq m^*$. But no simple explicit expression for the action of the generators on a given representation of $\mathscr{U}(SU(3))_q$ was derived from this study.

To do this, we consider a module $\mathscr{M}$ on the auxiliary algebra $\mathscr{A}$, and a basis on M such that the operators β^1_+ and β^1_- have non-zero matrix elements on the diagonal and on the line just under the diagonal, whereas β^2_+ and β^2_- are on the diagonal and on the first line over the diagonal. In [6], u or β^1_+ was diagonalized. This was well suited for demonstrating the completeness of the solutions. The different choice mentioned above is equivalent and moreover leads to the simple and elegant forms given below. We define

$$a_1 = \tfrac{1}{3}(2\mu_1 + \mu_2) - p_1,$$
$$a_2 = \tfrac{1}{3}(\mu_1 + 2\mu_2) - p_2$$

in order to combine the integer part and non-integer part of the eigenvalues of q^{h_1} and q^{h_2}. For definiteness of the third roots, we suppose here that m is not a multiple of 3. It was however proved in [6] that no particularity happens in this case. A periodic module over $\mathscr{U}(SU(3))_q$ has then the general form:

$$q^{1/3(2h_1+h_2)}|a_1, a_2, i\rangle = q^{a_1}|a_1, a_2, i\rangle,$$
$$q^{1/3(h_1+2h_2)}|a_1, a_2, i\rangle = q^{a_2}|a_1, a_2, i\rangle,$$
$$f_1|a_1, a_2, i\rangle = \alpha_1|a_1 - 1, a_2, i\rangle,$$
$$e_2|a_1, a_2, i\rangle = \alpha_2|a_1, a_2 + 1, i\rangle,$$
$$\begin{aligned} e_1|a_1, a_2, i\rangle &= \frac{1}{\alpha_1}[a_1 - b_1 - i][a_2 - a_1 - b_1 - i - 1]|a_1 + 1, a_2, i\rangle \\ &\quad + \frac{\rho}{\alpha_1}[a_2 - b_2 - i][c - b_2 - i - 1]|a_1 + 1, a_2, i + 1\rangle, \\ f_2|a_1, a_2, i\rangle &= \frac{1}{\alpha_2}[a_2 - b_2 - i][a_1 - a_2 - b_2 - i + 1]|a_1, a_2 - 1, i\rangle \\ &\quad + \frac{1}{\rho\alpha_2}[a_1 - b_1 - i][-c - b_1 - i + 1]|a_1, a_2 - 1, i - 1\rangle, \end{aligned}$$

a_1 and a_2 are in the generic case non-integer indices, defined modulo m. The index i is integer and can take k values ($k = 1, \ldots, m^*$).

The values of the Casimirs $C_{(\varepsilon)}$ on this representation are

$$C_{(\varepsilon)} = (q - q^{-1})^{-2}(q^{2\varepsilon(b_1 - b_2 + 1)} + q^{2\varepsilon(c-1)} + q^{-2\varepsilon(c + b_1 - b_2)} - 3).$$

We now give an explicit correspondence between this basis and the Gelfand Zetlin basis defined in Sect. 2.

Let

$$|a_1, a_2, i\rangle = \sum_{h_{11}, h_{12}, h_{22}} T^{h_{11} h_{12} h_{22}}_{a_1, a_2, i} \left| \begin{matrix} h_{13} & & h_{23} & & h_{33} \\ & h_{12} & & h_{22} & \\ & & h_{11} & & \end{matrix} \right\rangle. \qquad (T)$$

Clearly, q^{h_i} determines the correspondence between a_1 and h_{11}, and between a_2, h_{12} and h_{22} by

$$T^{h_{11}h_{12}h_{22}}_{a_1 a_2 i} = \delta_{a_1, h_{11}} \delta_{a_2, h_{12} + h_{22}} T^{h_{12}h_{22}}_{a_1 a_2 i}.$$

Identification of the action of the generators on left-hand side and right-hand side of (T) provides the following constraints:

$$f_1:\quad T^{h_{12},h_{22}}_{a_1-1,a_2 i}=\frac{c_{11}}{\alpha_1}(-P_1(1,1,h))^{1/2}\,T^{h_{12},h_{22}}_{a_1,a_2 i},\tag{α}$$

$$\begin{aligned}e_2:\quad T^{h_{12},h_{22}}_{a_1,a_2+1,i}&=\frac{1}{\alpha_2 c_{12}}\left(-\frac{P_1P_2}{P_3}(1,2,h)\right)^{1/2}T^{h_{12}-1,h_{22}}_{a_1,a_2 i}\\&\quad+\frac{1}{\alpha_2 c_{22}}\left(-\frac{P_1P_2}{P_3}(2,2,h)\right)^{1/2}T^{h_{12},h_{22}-1}_{a_1,a_2,i},\end{aligned}\tag{β}$$

$$e_1:\quad T^{h_{12},h_{22}}_{a_1,a_2,i+1}=\frac{-1}{\rho}\frac{[h_{12}-b_1-i][h_{22}-b_1-i-1]}{[a_2-b_2-i][c-b_2-i-1]}T^{h_{12},h_{22}}_{a_1,a_2,i}.\tag{γ}$$

The constraint (δ) provided by f_2 is a bit more complicated and less useful, so we do not derive it here.

Combining β and γ leads to

$$\begin{aligned}&\frac{1}{c_{12}}\left(-\frac{P_1P_2}{P_3}(1,2,h)\right)^{1/2}[b_2-b_1-h_{22}][h_{22}-b_1-i-1]\,T^{h_{12}-1,h_{22}}_{a_1,a_2,i}\\&+\frac{1}{c_{22}}\left(-\frac{P_1P_2}{P_3}(2,2,h)\right)^{1/2}[b_2-b_1-h_{22}-1][h_{12}-b_1-i]\,T^{h_{12},h_{22}-1}_{a_1,a_2,i}=0\end{aligned}\tag{γ'}$$

and

$$T^{h_{12},h_{22}}_{a_1,a_2+1,i}=\frac{-1}{\alpha_2 c_{12}}\left(-\frac{P_1P_2}{P_3}(1,2,h)\right)^{1/2}\frac{[h_{12}-h_{22}+1][h_{22}+h_{12}-b_1-i]}{[b_2-b_1-h_{12}-1][h_{12}-b_1-i]}T^{h_{12}-1,h_{22}}_{a_1,a_2,i}.\tag{γ''}$$

It is clear that the knowledge of one of the coefficients T is enough to constrain all the others. We have now to check the compatibility of these relations, and identify the parameters of the module.

The identification follows from the comparison between the values of the elements of the center of $\mathscr{U}(SU(3))_q$.

We already had $h_{11}=a_1$ and $h_{12}+h_{22}=a_2$.

Remark. Requiring that the values of the Casimirs $(e_i)^m$ and $(f_i)^m$ are the same in both modules provides the same constraints as requiring periodicity of T when m is added to one of the indices a_1 or a_2.

Now

$$T^{h_{12},h_{22}}_{a_1,a_2,i+m}=T^{h_{12},h_{22}}_{a_1,a_2,i}$$

provides a new constraint on h_{12} and h_{22}, which are then fixed. This constraint is indeed

$$\left(-\frac{1}{\rho}\right)^m\prod_{i=0}^{m-1}\frac{[h_{12}-b_1-i][h_{22}-b_1-i-1]}{[a_2-b_2-i][c-b_2-i-1]}=1.$$

Using the following identity

$$\begin{aligned}\prod_{k=0}^{n-1}\sin\left(x+\frac{2k}{n}\pi\right)&=\frac{(-1)^{(n-1)/2}}{2^{n-1}}\sin nx\quad\text{(for } n\text{ odd)}\\&=\frac{(-1)^{n/2}}{2^{n-1}}(1-\cos nx)\quad\text{(for } n\text{ even)}\end{aligned}$$

we deduce

$$\left(-\frac{1}{\rho}\right)^m \frac{[m(h_{12}-b_1)][m(h_{22}-b_1)]}{[m(a_2-b_2)][m(c-b_2)]} = 1 \quad \text{for } m \text{ odd}$$

or

$$\left(-\frac{1}{\rho}\right)^m \frac{\left[\frac{m}{2}(h_{12}-b_1)\right]^2 \left[\frac{m}{2}(h_{22}-b_1)\right]^2}{\left[\frac{m}{2}(a_2-b_2)\right]^2 \left[\frac{m}{2}(c-b_2)\right]^2} = 1 \quad \text{for } m \text{ even.}$$

Since $h_{12}+h_{22}=a_2$, this equation can be solved as a quadratic equation in $q^{m^*(h_{12}-h_{22})}$ in terms of a_2, b_1, b_2 and c. One can then take some $m^{*\,\text{th}}$ root. The fractional parts of h_{12}, h_{22}, h_{11} are thus all fixed in terms of the parameters of the other basis.

Now

$$T^{h_{12},h_{22}}_{a_1-m,a_2,i} = T^{h_{12},h_{22}}_{a_1,a_2,i}$$

fixes c_{11} and

$$T^{h_{12}-m,h_{22}+m}_{a_1,a_2,i} = T^{h_{12}-m,h_{22}}_{a_1,a_2-m,i} = T^{h_{12},h_{22}}_{a_1,a_2,i}$$

fixes c_{12} and c_{22}.

The identification values of the Casimirs $C_{(\varepsilon)}$ provides (for example, since there are six possibilities):

$$\begin{aligned} h_{13} &= b_2 - b_1 - 2, \\ h_{23} &= -b_2 + b_1 + c, \\ h_{33} &= -c + 2. \end{aligned}$$

Finally, a tedious computation shows that the constraint (δ) provided by the identification of the action of f_2 is satisfied when the above identifications of the parameters hold.

We have then established the correspondence between the fully periodic representations given by the G.–Z. basis and the m^3-dimensional representations of [6]. The $k.m^2$-dimensional representations of [6] are obtained when c and b_1-b_2 are integers (and $k=2c+b_1-b_2-1$). They correspond to partially periodic representations described by the introduction of a subtriangle $N_1=2$ of h_{ik} with the same fractional part. If this subtriangle is made of h_{23}, h_{33} and h_{22}, then h_{22} can take

$$h_{23}-h_{33}+1 = 2c+b_1-b_2-1 = k$$

values, so the dimension is also $k.m^2$.

We have then a correspondence between the classification of periodic representations of [6] for $N=3$ and the periodic representations given by the G.–Z. basis. The correspondence with the classification of [5] for $N=2$ is immediate. So the G.–Z. basis is adapted to describe all the periodic irreducible representations of $\mathcal{U}(SU(N))_q$ for $N=2,3$, and we expect that this is also true for an arbitrary N.

Conclusion

We have presented two different approaches to the construction of all periodic and partially periodic representations of $SU_q(N)$. The q-deformed matrix elements of the G.–Z. basis has been shown to incorporate, remarkably enough, the necessary mechanism for periodicity constraints. The relaxation of periodicity by stages (injective operators becoming nilpotent one by one) can also be carried out elegantly on this basis. Our aim has been to construct the most general solution including all the possibilities. Certain subclasses can posses particularly desirable properties. A detailed analysis of such subclasses and of their composition, following the discussion of Refs. [10, 14] would be of interest. Combining the techniques of this paper with those of Ref. [4], the inhomogeneous case can be studied for q a root of unity. The classical Abelian subalgebra ("translations") will now acquire, along with partial non-commutativity, periodic and hence lattice-like structure. This will be studied elsewhere.

The periodic representations of $\mathscr{U}(SU(2))_q$ were used in [15] and [16] to generalize the chiral Potts model. The extension to $\mathscr{U}(SU(3))_q$ is described in [12].

The alternative matrix algebra presented for the periodic case has its own interests; it seems to have a fairly direct connection with the construction of certain classes of statistical models [12]. The solution for $SU(3)$ is presumably, the simplest possible form exhibiting all the eight invariants. The denominators typical of the G.–Z. basis are absent. Construction of such explicit solutions for $N > 3$ presents an interesting challenge.

There are several recent studies of representations when q is a root of unity [9 to 14]. The distinguishing feature of our formalism is the explicit construction of matrix elements showing how they can lead to different stages of periodicity. Such explicit results will be essential for evaluation of all invariants and the Clebsch–Gordan coefficients.

Before sending the revised version of this paper, we received the preprints [17] and [18]. In [17], the periodic representations of $\mathscr{U}_q(sl(n+1, \mathbb{C}))$ are given, for generic values of the parameters. Date et al. express $\mathscr{U}_q(sl(n+1, \mathbb{C}))$ in terms of a higher dimensional Weyl algebra. But partially periodic representations do not appear naturally with this technique. In [18], the minimal periodic representations of $\mathscr{U}_q(gl(n, \mathbb{C}))$ are related to the generalization of the chiral Potts model, extending then to $n > 3$ the results of [12].

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Note added in proofs. Although the G.–Z. basis provides the correct number of parameters for maximal dimension, it appears that some parameters are lost in smaller dimensions. We thank V. Kac for having pointed out this fact. This is probably related to a reminiscence of the unitarity of the classical G.–Z. basis.

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