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Solving the Strongly Coupled 2D Gravity: 1. Unitary Truncation and Quantum Group Structure

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Abstract. Strongly compled gravity theories with Virasoro central charges equal to 7, 13, and 19 are shown to enjoy striking properties: at these values, the subset of chiral operators with real Virasoro-weights, acting on a subspace $\mathscr{H}_{\mathrm{phys}}$, is shown to be closed by fusion and braiding, and to leave this subspace invariant. Moreover, the representation of the Virasoro algebra becomes unitary when it is restricted to $\mathscr{H}_{\mathrm{phys}}$. Strongly coupled 2D gravity with $C_{\mathrm{grav}}=7$, 13, or 19 may thus be naturally truncated obtaining a consistent conformal theory (this result is similar to the truncation that occurs for $C=1-6(p-p')^2/pp'$ with p and p' integers, where only a finite number of primary fields remains, as is well known in rational theories). The proof of this unitary truncation theorem, already summarized in a recent letter, is fully described here.

1. Introduction

This article is the third of a series [1–3] devoted to the solution of 2D gravity and minimal models by means of quantum groups. The present approach is a direct outcome of the algebraic approach to 2D critical systems which Neveu and I [4–6] introduced long ago. Its distinctive feature is that it directly deals with chiral operators that transform irreducibly under the action of the underlying quantum group, while in the more widespread type of approach [7], one works with Green functions and the link with quantum group is made at the level of group invariants and q–Clebsch Gordan or q–6-j symbols. Thus, in [7], one does not clearly see how the quantum group acts on the theory. Moreover, these approaches exclusively deal with rational theories at $C<1$. This is inappropriate for the strongly coupled gravity which we have in mind in the present paper. The special values $C_{\mathrm{grav}}=7, 13,$

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and 19 were first put forward in [6] where closure under braiding was found to hold at these values for a particular operator with real Virasoro weights. This result was extended to W_3-gravity in a recent paper [2] where contact between the argument of [6] and the quantum group structure of 2D gravity [1] was also made. Finally the following unitary truncation theorem was found to hold:

In Liouville theory, with Virasoro central charges 7, 13, and 19, the subset $\mathscr{A}_{\mathrm{phys}}$ of chiral operators with real Virasoro-weights, acting on a subspace $\mathscr{H}_{\mathrm{phys}}$, is closed by fusion and braiding, and only gives states that belong to $\mathscr{H}_{\mathrm{phys}}$. The representation of the Virasoro algebra restricted to $\mathscr{H}_{\mathrm{phys}}$ is unitary.

The present article provides the complete derivation of this theorem, summarized in a recent letter [3], which shows that 2D gravity may be formulated for the above values by only retaining operators with real Virasoro-weights. This leads to completely consistent conformal theories.

In the quantum solution of the Liouville dynamics the basic chiral conformal family of 2D gravity and minimal models appears naturally (see Appendix A for some details). Its relation with quantum groups was studied in [1] (much more about this below). For a given coupling constant γ, this conformal family involves two possible quantum modifications given by:

$$h_\pm = \frac{\pi}{4\gamma}(1-4\gamma \pm \sqrt{1-8\gamma}) = \frac{\pi}{12}(C_{\mathrm{Liou}} - 13 \pm \sqrt{(C_{\mathrm{Liou}}-25)(C_{\mathrm{Liou}}-1)}), \quad (1.1)$$

where the last identity follows from the fact that the central charge of the Liouville theory is $C_{\mathrm{Liou}} = 1 + 3/\gamma$. We use the same conventions and notations[1] as in [1]. They are summarized in Appendix A for completeness. Although it comes out from the quantization of the Liouville theory, the conformal family we are considering has an intrinsic meaning. It is characterized by the existence of two quantum-deformation parameters, which we shall denote by h and $\hat{h}$. They are related by

$$h\hat{h} = \pi^2, \qquad h + \hat{h} = \pi(C-13)/6, \quad (1.2)$$

where C is the central extension of the Virasoro algebra. In Liouville theory, $h = h_-$, $\hat{h} = h_+$, and $C = C_{\mathrm{Liou}}$. Choosing $C < 1$ one may as well describe the minimal models, however. All quantities related to $\hat{h}$ are distinguished by hatted symbols. In the weak coupling regime $\gamma < 1/8$, h and $\hat{h}$ are real. Their role is asymmetrical. For $\gamma \to 0$, $h \sim 2\pi\gamma$, while $\hat{h}$ blows up. Thus $h/2\pi$ describes quantum modifications to the chiral operators that are perturbative, while $\hat{h}$ corresponds to non-perturbative effects. In the strong coupling regime, on the contrary, $(1/8 < \gamma < \infty)$, $\hat{h} = h^*$, and both h and $\hat{h}$ must be treated on the same footing, as we shall do following [1–3]. There are four basic fields: $\psi_1(\sigma)$, $\psi_2(\sigma)$, $\hat{\psi}_1(\sigma)$, $\hat{\psi}_2(\sigma)$. The fields ψ_j (respectively $\hat{\psi}_j$) involve the quantum modification h (respectively $\hat{h}$). Since these fields are chiral, we may work on the unit circle $z = e^{i\sigma}$ $0 \leqq \sigma \leqq 2\pi$. They are solutions of a quantum Schrödinger equation which comes out of the Liouville dynamics, and is equivalent to the Ward identities that reflect the decoupling of Virasoro null vectors. See Appendix A for some pedagogical details. By operator-product expansion, the above basic fields generate a family of operators $\psi_{m\hat{m}}^{(J\hat{J})}(\sigma)$,

[1] Except that the letter ω is everywhere replaced by ϖ

$-J \leqq m \leqq J$, $-\hat{J} \leqq \hat{m} \leqq \hat{J}$, of the type $(2\hat{J}+1, 2J+1)$ in the BPZ classification. Their conformal weights are given by Kac's formula:

$$\Delta_{\mathrm{Kac}}(J, \hat{J}; C) := \frac{C-1}{24} - \frac{1}{24}((J+\hat{J}+1)\sqrt{C-1} - (J-\hat{J})\sqrt{C-25})^2, \tag{1.3}$$

(that is, $\psi_{m\hat{m}}^{(J\hat{J})}(\sigma) \times (d\sigma)^{\Delta_{\mathrm{Kac}}(J,\hat{J};C)}$ is conformally invariant). A central role is played by the zero-mode p_0 (quasi momentum) of the underlying pair of equivalent free fields (see Appendix A). It is convenient to define the rescaled variables

$$\varpi := ip_0\sqrt{\frac{2\pi}{h}}; \quad \hat{\varpi} := ip_0\sqrt{\frac{2\pi}{\hat{h}}}; \quad \hat{\varpi} = \varpi\frac{h}{\pi}; \quad \varpi = \hat{\varpi}\frac{\hat{h}}{\pi}; \tag{1.4}$$

so that for any function f

$$\psi_{m\hat{m}}^{(J\hat{J})} f(\varpi) = f(\varpi + 2m + 2\hat{m}\pi/h)\psi_{m\hat{m}}^{(J\hat{J})}. \tag{1.5}$$

The operators $\psi_{m\hat{m}}^{(J\hat{J})}$ thus live in Hilbert spaces[2] of the form

$$\mathscr{H}(\varpi_0) \equiv \bigoplus_{n,\hat{n}=-\infty}^{+\infty} \mathscr{F}(\varpi_0 + n + \hat{n}\pi/h). \tag{1.6}$$

ϖ^0 is to be determined, and $\mathscr{F}(\varpi)$ is the Fock space of states with quasi momentum ϖ. The $SL(2,C)$-invariant vacuum corresponds to $\varpi_0 = 1 + \pi/h$ [1], but this choice is not appropriate here.

The above ψ family is closed by fusion and braiding. However, the fusion coefficients and R-matrix are functions of ϖ, and according to (1.5) do not commute with the ψ fields. Thus the structure is unusual and its connection with the standard quantum group not very transparent. This was overcome in [1, 8] by changing basis to a new set of chiral operators noted $\xi_{M,\hat{M}}^{(J,\hat{J})}(\sigma)$ with $-J \leqq M \leqq J$, $-\hat{J} \leqq \hat{M} \leqq \hat{J}$. The change of basis was determined so that all ϖ dependence disappear from the R-matrix and from the fusion coefficients. For the ξ fields the quantum group structure is transparent. In particular, it was found that, for $\xi_{M,0}^{(J,0)} \equiv \xi_M^{(J)}$, the R-matrix coincides with the standard R-matrix of the quantum group $SL(2)_q$. Hence we may call the family of ξ fields the universal conformal family. We shall see that the (so-called quantum) mathematical deformation of the underlying group is dictated by the non-commutativity of the conformal field operators due to physical quantum effects, i.e. by the uncertainty principle of quantum mechanics, if the fields $\xi_M^{(J)}(\sigma) - J \leqq M \leqq J$ span a representation of spin J of $SL(2)_q$. The properties of the universal family are thus completely determined by the quantum group structure (more about this later on) and this is instrumental in the proof of the unitary decoupling theorem which is our present aim. The whole discussion is carried out assuming that q is not a root of unity, so that the representations of the quantum algebra are trivial deformations of the $q=1$ case.

Natural as they may be, group theoretically, the ξ operators are not always the most useful operators to use, nevertheless. Indeed the ψ fields, contrary to the ξ fields, have well-defined shift properties of the associated quasi momentum ϖ. Thus the former are more useful to discuss properties pertaining to the Hilbert space in which conformal theories are defined. Thus we shall use each basis in its turn.

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[2] Mathematically they are not really Hilbert spaces since their metrics are not positive definite

The plan of the article is as follows. At first (Sect. 2) we complete the analysis of [1] without specializing the value of C. After developing some useful machinery about q-deformed special functions, we re-express the entries of the matrix that relates ξ fields and ψ fields in terms of q-hypergeometric functions and determine its inverse. The connection is next made between the short-distance operator-product expansion of the universal family and the co-product associated with the quantum group structure unravelled in [1]. The fusion coefficients are thereby shown to coincide with the associated 3-j symbols. In Sect. 3, the mathematical properties of the special choices $C=7$, 13, and 19 are established. The two quantum modifications are then such that $h+\hat{h}=s\pi$, s integer, and we show that, as a result, the Clebsch-Gordan coefficients, and braiding matrices, are respectively related by very simple formulae. In addition, the physical Hilbert space $\mathscr{H}_{\mathrm{phys}}$ is introduced so that the change of basis between ψ and ξ fields and between $\hat{\psi}$ and $\hat{\xi}$ fields are suitably connected. The structure so obtained allows us next to prove the unitary truncation theorem, for the fields with negative weights (Sect. 4), and for the complete set of physical fields (Sect. 5). In the latter discussion, operators with negative spins are needed. This case is handled by continuation from the case of positive spins, making use of the connection made in Sect. 2 with q-deformed special functions. A useful symmetry between spins J and $-J-1$ is put forward. Some concluding remarks are made in Sect. 6.

2. More on the General Properties of the ξ Fields

At first we consider the fields $\xi_M^{(J)}$ separately – it goes without saying that each part of the following discussion has its "hatted" counterpart. For any positive integer $2J$, introduce group-theoretic state-vectors noted $|J,M\rangle$, with $-J \leqq M \leqq J$, and operators $J_\pm$, J_3, such that

$$J_\pm|J,M\rangle=\sqrt{\lfloor J\mp M\rfloor\lfloor J\pm M+1\rfloor}\,|J,M\pm 1\rangle\,,\qquad J_3|J,M\rangle=M|J,M\rangle\,,\tag{2.1}$$

where, following [1–3], we started to use the notation $\lfloor x\rfloor$, defined by

$$\lfloor x\rfloor:=\sin(hx)/\sin h\,.\tag{2.2}$$

These operators satisfy the $SL(2)_q$ commutation relations,

$$[J_3,J_\pm]=\pm J_\pm\,,\qquad [J_+,J_-]=\lfloor 2J_3\rfloor\,.\tag{2.3}$$

First recall two basic results of [1].

Theorem (2.1). *For* $2J$ *a positive integer let*

$$\xi_M^{(J)}(\sigma):=\sum_{-J\leqq m\leqq J}|J,\varpi)_M^m\psi_m^{(J)}(\sigma)\,,\tag{2.4}$$

$$|J,\varpi)_M^m:=\sqrt{\binom{2J}{J+M}}\,e^{ihm/2}\sum_t e^{iht(\varpi+m)}\binom{J-M}{(J-M+m-t)/2}\binom{J+M}{(J+M+m+t)/2}\tag{2.5}$$

$$\binom{P}{Q}:=\frac{\lfloor P\rfloor!}{\lfloor Q\rfloor!\lfloor P-Q\rfloor!}\,,\qquad \lfloor n\rfloor!:=\prod_{r=1}^{n}\lfloor r\rfloor\,,\tag{2.6}$$

where the variable t takes all values such that the entries of the binomial coefficients are non-negative integers. 1) *For $\pi>\sigma>\sigma'>0$, these operators obey the exchange algebra*

$$\xi_M^{(J)}(\sigma)\xi_{M'}^{(J')}(\sigma')=\sum_{-J\leqq N\leqq J;\,-J'\leqq N'\leqq J'}(J,J')_{MM'}^{N'N}\xi_{N'}^{(J')}(\sigma')\xi_N^{(J)}(\sigma), \tag{2.7}$$

$$(J,J')_{MM'}^{N'N}=(\langle J,M|\otimes\langle J',M'|)\mathbf{R}(|J,N\rangle\otimes|J',N'\rangle), \tag{2.8}$$

$$\mathbf{R}=e^{(-2ihJ_3\otimes J_3)}\left(1+\sum_{n=1}^{\infty}\frac{(1-e^{2ih})^n e^{ihn(n-1)/2}}{\lfloor n\rfloor!}e^{-ihnJ_3}(J_+)^n\otimes e^{ihnJ_3}(J_-)^n\right). \tag{2.9}$$

2) *For $0<\sigma<\sigma'<\pi$, the ξ fields obey the exchange algebra*

$$\xi_M^{(J)}(\sigma)\xi_{M'}^{(J')}(\sigma')=\sum_{-J\leqq N\leqq J;\,-J'\leqq N'\leqq J'}\overline{(J,J')}_{MM'}^{N'N}\xi_{N'}^{(J')}(\sigma')\xi_N^{(J)}(\sigma), \tag{2.10}$$

$$\overline{(J,J')}_{MM'}^{N'N}=(\langle J,M|\otimes\langle J',M'|)\bar{\mathbf{R}}(|J,N\rangle\otimes|J',N'\rangle), \tag{2.11}$$

$$\bar{\mathbf{R}}=e^{(2ihJ_3\otimes J_3)}\left(1+\sum_{n=1}^{\infty}\frac{(1-e^{-2ih})^n e^{-ihn(n-1)/2}}{\lfloor n\rfloor!}e^{-ihnJ_3}(J_-)^n\otimes e^{ihnJ_3}(J_+)^n\right) \tag{2.12}$$

3) *The two exchange formulae are related by the inverse relation*

$$\sum_{-J\leqq N\leqq J;\,-J'\leqq N'\leqq J'}(J,J')_{MM'}^{N'N}\overline{(J',J)}_{N'N}^{PP'}=\delta_{M,P}\,\delta_{M',P'}. \tag{2.13}$$

Proof. Based on the braiding properties of the ψ fields it is spelled out in [1]. The normalization constants of this article are chosen to be $\kappa_J=1$ and $a_2=a_1e^{ih}$. □

Theorem (2.2). *For $C>1$, and to leading order*[3] *in the short distance singularity at $\sigma\to\sigma'$, the product of ξ fields behaves as*

$$\xi_M^{(J)}(\sigma)\xi_{M'}^{(J')}(\sigma')\sim(1-e^{-i(\sigma-\sigma')})^{\frac{-2hJJ'}{\pi}}\lambda(J,M;J',M')\xi_{M+M'}^{(J+J')}(\sigma), \tag{2.14}$$

where

$$\lambda(J,M;J',M')=\sqrt{\frac{\binom{2J}{J+M}\binom{2J'}{J'+M'}}{\binom{2J+2J'}{J+J'+M+M'}}}\,e^{ih(M'J-MJ')}. \tag{2.15}$$

Proof. See [1] with $\kappa_J=1$. □

We now come to new results about the ξ family. First, in order to continue to negative J (see [3]) we relate the coefficients of (2.4) to q-deformed (so-called basic) hypergeometric functions. For this purpose we need to introduce *q-Gamma functions* [9, 10] which are such that

$$\Gamma(a+1)=\lfloor a\rfloor\Gamma(a),\qquad \Gamma(0)=1. \tag{2.16}$$

Proposition (2.3). *If $\operatorname{Im}h<0$, the solution of* (2.16) *is*

$$\Gamma(a)=e^{iha(a-1)/2}(2i\sin h)^{1-a}\prod_{\mu=1}^{\infty}\left(\frac{1-e^{-2ih\mu}}{1-e^{-2ih(\mu+a-1)}}\right). \tag{2.17}$$

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[3] If $C<1$ this term is no more leading, but still there. It corresponds to adding the spins J_1 and J_2 to obtain the maximum spin J_1+J_2. The theorem still holds

Proof. Easily verified explicitly. The result may be understood as follows. Choose an arbitrary integer A, and write, for a integer,

$$\Gamma(a)=\lfloor a-1\rfloor !=\prod_{\mu=1}^{A+a-1}\lfloor\mu\rfloor\Big/\prod_{\mu=1}^{A}\lfloor\mu+a-1\rfloor ,$$

or equivalently

$$\Gamma(a)=e^{iha(a-1)/2}(2i\sin h)^{1-a}\prod_{\mu=1}^{A+a-1}(1-e^{-2ih\mu})\Big/\prod_{\mu=1}^{A}(1-e^{-2ih(\mu+a-1)}).$$

Taking $A\to\infty$ gives (2.17) which, being convergent for $\mathrm{Im}\, h<0$, is the correct interpolation. □

We shall also need the following

Theorem (2.4). *The function*

$$S(x):=\frac{\pi}{\Gamma(x)\Gamma(1-x)} \tag{2.18}$$

is given by

$$S(x)=\frac{h}{\sin h}\,\frac{\Theta_1(x|\pi/h)}{\Theta_1'(0|\pi/h)}, \tag{2.19}$$

where Θ_1 denotes the standard Jacobi Theta function [11].

Proof. Making use of (2.17) one gets

$$S(x)=e^{-ihx^2}\lfloor x\rfloor\prod_{\mu=1}^{\infty}\frac{(1-2\cos(2hx)e^{-2ih\mu}+e^{-4ih\mu})}{(1-e^{-2ih\mu})^2}\equiv\frac{\pi}{\sin h}\,e^{i\pi v^2/\tau}\,\frac{\Theta_1(v|\tau)}{\Theta_1'(0|\tau)} \tag{2.20}$$

with $x=\pi v/h$, $\tau=-h/\pi$. Equation (2.19) follows from the Jacobi transformation from the period $\tau=-h/\pi$ to $\hat{\tau}=-1/\tau$. □

It is interesting to note that since $h\hat{h}=\pi^2$, $\hat{\tau}=\hat{h}/\pi$. Thus the Jacobi transformation exchanges the two quantum modifications h and $\hat{h}$ up to a sign, and one has

$$S(x)=\frac{\sin\pi x}{\pi}\,\frac{h}{\sin h}\prod_{\mu=0}^{\infty}\left(\frac{1-2e^{2i\mu\hat{h}}\cos(2\pi x)+e^{4i\mu\hat{h}}}{(1-e^{2i\mu\hat{h}})^2}\right). \tag{2.21}$$

We shall often use the fact that, for N integer,

$$S(x+N)=(-1)^N S(x). \tag{2.22}$$

Next, as in [3], it is convenient to define *basic or q-hypergeometric functions* as[4]

$$F(a,b;c;z):=\sum_{\nu=0}^{\infty}\frac{\lfloor a\rfloor_\nu\lfloor b\rfloor_\nu}{\lfloor c\rfloor_\nu\lfloor\nu\rfloor !}z^\nu;\qquad \lfloor a\rfloor_\nu:=\frac{\Gamma(a+\nu)}{\Gamma(a)}. \tag{2.23}$$

The relationship with the definition commonly used in mathematics [10, 12, 13] is displayed in Appendix B. Next we have the

[4] Since all the standard functions we shall use are q-deformed, the index q is omitted

Theorem (2.5). *Equation* (2.5) *is equivalent to*

$$|J,\varpi)_M^m=\sqrt{\binom{2J}{J+M}}\,e^{ih(m/2+(\varpi+m)(J+1-c))}\binom{2J+a}{c-1}F(a,b;c;e^{-2ih(\varpi+m)}) \tag{2.24}$$

$$a=M-J,\quad b=-m-J,\quad c=1+M-m,\quad \mathit{for}\quad M>m, \tag{2.25}$$

$$a=-M-J,\quad b=m-J,\quad c=1-M+m,\quad \mathit{for}\quad M<m. \tag{2.26}$$

Proof. Take $M>m$ for instance. Continue temporarily Eq. (2.5) to non-integer J by replacing factorials by Γ functions. Letting $t=-2v+J-M+m$, one gets

$$|J,\varpi)_M^m=\sqrt{\binom{2J}{J+M}}\,e^{ih[(J-M+m)(\varpi+m)+m/2]}\sum_{v>0}\frac{e^{ihv(\varpi+m)}\Gamma(J+M+1)\Gamma(1-a)}{\Gamma(1-a-v)\Gamma(1-b-v)\Gamma(c+v)} \tag{2.27}$$

Next, it is an easy consequence of Eq. (2.22) that, in general,

$$\Gamma(1-x)/\Gamma(1-x-v)=(-1)^v\Gamma(x+v)/\Gamma(x). \tag{2.28}$$

Equation (2.27) is transformed into the desired expression (2.24) by means of (2.28) taken with $x=a$ and with $x=b$. □

Next the inversion of the transformation (2.4, 5) is determined by the

Theorem (2.6). *If Eq.* (2.4) *holds, the* ψ *fields are given by the formulae*

$$\psi_m^{(J)}(\sigma)=\sum_{M=-J}^{J}\xi_M^{(J)}(\sigma)(J,\varpi|_m^M, \tag{2.29}$$

where the coefficients $(J,\varpi|_m^M$ *are such that*

$$(J,\varpi|_m^M:=(-1)^{J+M}e^{ih(J+M)}|J,\varpi)_{-M}^{-m}/C_{-m}^{(J)}(\varpi), \tag{2.30}$$

$$C_m^{(J)}(\varpi):=(-1)^{J-m}(2i\sin h)^{2J}e^{ihJ}\binom{2J}{J-m}\frac{\lfloor\varpi-J+m\rfloor_{2J+1}}{\lfloor\varpi+2m\rfloor}. \tag{2.31}$$

Proof. First, it is an easy algebra to check that the theorem holds for $J=1/2$, using the explicit expressions:

$$|\tfrac{1}{2},\varpi)_{\pm1/2}^{\pm1/2}=e^{ih(\varpi\pm1)/2}\qquad |\tfrac{1}{2},\varpi)_{\pm1/2}^{\mp1/2}=e^{-ih\varpi/2}; \tag{2.32}$$

that follow from (2.5). Next, one establishes a recurrence relation in J by using the leading-order fusion relations (we omit the divergent factors which are the same throughout the proof)

$$\psi_m^{(J)}(\sigma)\psi_\alpha^{(1/2)}(\sigma')\sim\psi_{m+\alpha}^{(J+1/2)}(\sigma)N(1/2,-\alpha;J,-m;\varpi), \tag{2.33}$$

$$\xi_M^{(J)}(\sigma)\xi_A^{(1/2)}(\sigma')\sim\lambda(J,M;1/2,A)\xi_{M+A}^{(J+1/2)}(\sigma). \tag{2.34}$$

The explicit expression of $N(1/2,-\alpha;J,-m;\varpi)$ is derived in [1]. Assume the theorem holds up to spin J. Then

$$\psi_m^{(J)}(\sigma)\psi_\alpha^{(1/2)}(\sigma')=\sum_M(-1)^{J+M}e^{ih(J+M)}\xi_M^{(J)}(\sigma)$$

$$\times\sum_{A=\pm1/2}\xi_A^{(1/2)}(\sigma')(-1)^{1/2+A}e^{ih(1/2+A)}\frac{|1/2,\varpi)_{-A}^{-\alpha}}{C_{-\alpha}^{(1/2)}(\varpi)}\frac{|J,\varpi-2\alpha)_{-M}^{-m}}{C_{-m}^{(J)}(\varpi-2\alpha)}. \tag{2.35}$$

Choose, for instance, $\alpha=1/2$. From the expression (2.5) of $|J,\varpi)_M^m$ one may verify that

$$\sum_{M+A=R} \lambda(J,M;1/2,A)|1/2,\varpi)_{-A}^{-1/2}|J,\varpi-1)_{-M}^{-m}=\frac{\lfloor J+m+1\rfloor}{\lfloor 2J+1\rfloor}|J+1/2,\varpi)_{-R}^{-m-1/2} \tag{2.36}$$

Since

$$C_{-1/2}^{(1/2)}=-(2i\sinh h)e^{ih/2}\lfloor\varpi\rfloor, \qquad N(1/2,-1/2;J,-m;\varpi)=\lfloor\varpi-J+m\rfloor/\lfloor\varpi\rfloor, \tag{2.37}$$

and making use of (2.33, 34), one sees that the theorem holds for spin $J+1/2$ provided

$$C_{-m-1/2}^{(J+1/2)}(\varpi)=-C_{-m}^{(J)}(\varpi-1)(2i\sinh h)e^{ih/2}\frac{\lfloor J+m+1\rfloor}{\lfloor 2J+1\rfloor}\lfloor\varpi-J+m\rfloor, \tag{2.38}$$

and formula (2.31) follows. □

Corollary (2.7). *The coefficients* $|J,\varpi)_M^m$ *satisfy the relation*

$$\sum_{M=-J}^{J}(-1)^{J-M}e^{ih(J-M)}|J,\varpi)_M^m|J,\varpi+2p)_{-M}^{-p}=\delta_{m,p}C_m^{(J)}(\varpi). \tag{2.39}$$

Proof. Substitute (2.29) into (2.4). □

Theorem (2.1) indicates that the braiding properties of the ξ fields are given by the universal R-matrix of $SL(2)_q$, and are thus completely determined group theoretically. This is also true for the short-distance operator-product expansion, as the following shows.

Theorem (2.8). *The short-distance operator-product expansion of the* ξ *fields is of the form:*

$$\xi_{M_1}^{(J_1)}(\sigma)\xi_{M_2}^{(J_2)}(\sigma')=\sum_{J=|J_1-J_2|}^{J_1+J_3}\{(d(\sigma-\sigma'))^{\Delta(J)-\Delta(J_1)-\Delta(J_2)}$$
$$\times g_{J_1J_2}^{J}(J_1,M_1;J_2,M_2|J_1,J_2;J,M_1+M_2)(\xi_{M_1+M_2}^{(J)}(\sigma)+\text{descendants})\}, \tag{2.40}$$

where $d(\sigma-\sigma')\equiv 1-e^{-i(\sigma-\sigma')}$, $(J_1,M_1;J_2,M_2|J_1,J_2;J,M_1+M_2)$ *denotes the Clebsch-Gordan coefficients of* $SL(2)_q$ *(see Appendix C),* $g_{J_1J_2}^{J}$ *are numerical constants, and* $\Delta(J):=-hJ(J+1)/\pi-J$ *is the Virasoro-weight of* $\xi_M^{(J)}(\sigma)$.

Proof. This was already proven in [1] (see Theorem (2.2)) to leading order in the singularity. Indeed, comparing the explicit expression given in [1] (Eqs. (2.14, 15)) with Eq. (C.3), one sees that

$$\lambda(J_1,M_1;J_2,M_2)\equiv(J_1,M_1;J_2,M_2|J_1,J_2;J_1+J_2,M_1+M_2). \tag{2.41}$$

On the other hand, $\Delta(J_1+J_2)-\Delta(J_1)-\Delta(J_2)=-2hJ_1J_2/\pi$. The complete proof works in the same way as in [1]. One proceeds by recursion in J_1. First one derives (2.40) in the case of $J_1=1/2$ for arbitrary J_2. For this, one performs the same calculations as in Appendix D of [1] except that both terms $\xi^{(J_2\pm 1/2)}$ are retained. Next, assuming that (2.40) holds up to J_1, one multiplies both sides of this relation by $\xi_\alpha^{(1/2)}(\sigma_1)$ and lets $\sigma_1\to\sigma$ first and $\sigma\to\sigma'$ last. This gives a recurrence relation between the fusion coefficients for spins J_1, J_2 and those for spins $J_1+1/2$, J_2 which is used, together with the explicit form of the Clebsch-Gordan coefficients (see Appendix C) to derive (2.40) for spins $J_1+1/2$, J_2 and this establishes the recursion. We shall not go into more details of this lengthy discussion. □

In the following $g_{J_1J_2}^{J}$ is omitted since it plays no role in the discussions.

Theorem (2.9). *Define the quantum group action on the ξ fields by*

$$J_3\xi_M^{(J)}=M\xi_M^{(J)},\quad J_\pm\xi_M^{(J)}=\sqrt{\lfloor J\mp M\rfloor\lfloor J\pm M+1\rfloor}\,\xi_{M\pm1}^{(J)}. \tag{2.42}$$

Then the operator-product $\xi_{M_1}^{(J_1)}(\sigma)\xi_{M_2}^{(J_2)}(\sigma')$ also gives a representation of the quantum group algebra (2.3) *with generators*

$$\mathbf{J}_\pm:=J_\pm\otimes e^{ihJ_3}+e^{-ihJ_3}\otimes J_\pm,\quad \mathbf{J}_3:=J_3\otimes 1+1\otimes J_3, \tag{2.43}$$

where the tensor product is defined so that

$$(A\otimes B)(\xi_{M_1}^{(J_1)}(\sigma)\xi_{M_2}^{(J_2)}(\sigma')):=(A\xi_{M_1}^{(J_1)}(\sigma))(B\xi_{M_2}^{(J_2)}(\sigma')), \tag{2.44}$$

and where each term in the expansion over J transforms according to a representation of spin J.

Proof. This is obvious for $\mathbf{J}_3$ since the M_i's add up. For $\mathbf{J}_\pm$, compute

$$\begin{aligned}\mathbf{J}_\pm(\xi_{M_1}^{(J_1)}(\sigma)\xi_{M_2}^{(J_2)}(\sigma'))&=e^{ihM'}\sqrt{\lfloor J_1\mp M_1\rfloor\lfloor J_1\pm M_1+1\rfloor}(\xi_{M_1\pm1}^{(J_1)}(\sigma)\xi_{M_2}^{(J_2)}(\sigma'))\\&+e^{-ihM}\sqrt{\lfloor J_2\mp M_2\rfloor\lfloor J_2\pm M_2+1\rfloor}(\xi_{M_1}^{(J_1)}(\sigma)\xi_{M_2\pm1}^{(J_2)}(\sigma')).\end{aligned} \tag{2.45}$$

Substitute (2.40) in both sides, and make use of the following recurrence relations for q-C.G. coefficients (see Appendix C):

$$\begin{aligned}&\sqrt{\lfloor J\mp M_1\mp M_2\rfloor\lfloor J\pm M_1\pm M_2+1\rfloor}(J_1,M_1;J_2,M_2|J_1,J_2;J,M_1+M_2)\\&=e^{ihM_2}\sqrt{\lfloor J_1\mp M_1\rfloor\lfloor J\pm M_1+1\rfloor}(J_1,M_1\pm1;J_2,M_2|J_1,J_2;J,M_1+M_2\pm1)\\&\quad+e^{-ihM_1}\sqrt{\lfloor J_2\mp M_2\rfloor\lfloor J_2\pm M_2+1\rfloor}(J_1,M_1;J_2,M_2\pm1|J_1,J_2;J,M_1+M_2\pm1).\end{aligned} \tag{2.46}$$

One gets

$$\begin{aligned}&\mathbf{J}_\pm(\xi_{M_1}^{(J_1)}(\sigma)\xi_{M_2}^{(J_2)}(\sigma'))\\&=\sum_{J=|J_1-J_2|}^{J_1+J_2}\{(d(\sigma-\sigma'))^{\Delta(J)-\Delta(J_1)-\Delta(J_2)}\sqrt{\lfloor J\mp M_1\mp M_2\rfloor\lfloor J\pm M_1\pm M_2+1\rfloor}\\&\quad\times(J_1,M_1;J_2,M_2|J_1,J_2;J,M_1+M_2)(\xi_{M_1+M_2\pm1}^{(J)}(\sigma)+\text{descendants})\},\end{aligned} \tag{2.47}$$

and the result follows by inspection. □

Equation (2.43) coincides with the definition of the co-product of $SL(2)_q$ [14]. One thus see that, in conformal theories, the physical origin of the co-product is the operator-product expansion.

Next we come to properties of the general fields $\xi_{M,\hat{M}}^{(J,\hat{J})}$ fields, with both J and $\hat{J}$ non-vanishing. The hatted quantities, which now appear together with the unhatted ones, are noted $\widehat{\lfloor x\rfloor}:=\sin\hat{h}x/\sin\hat{h}$, $\widehat{|J,\varpi)}_M^m$, $\hat{\lambda}(\hat{J}_1,\hat{M}_1;\hat{J}_2,\hat{M}_2)$, and so on. First the fusion to leading order is given by the

Theorem (2.10). *To leading order in the singularity at $\sigma\to\sigma'$, the O.P.E. of the fields $\xi_{M,\hat{M}}^{(J,\hat{J})}$ is*

$$\begin{aligned}\xi_{M_1,\hat{M}_1}^{(J_1,\hat{J}_1)}(\sigma)\xi_{M_2,\hat{M}_2}^{(J_2,\hat{J}_2)}(\sigma')&\sim(d(\sigma-\sigma'))^{p(J_1,J_2,\hat{J}_1\hat{J}_2;J_1+J_2,\hat{J}_1+\hat{J}_2)}\\&\times\lambda(J_1,M_1,\hat{J}_1,\hat{M}_1;J_2,M_2,\hat{J}_2,\hat{M}_2)\xi_{M_1+M_2,\hat{M}_1+\hat{M}_2}^{(J_1+J_2,\hat{J}_1+\hat{J}_2)}(\sigma),\end{aligned} \tag{2.48}$$

$$p(J_1,J_2,\hat{J}_1\hat{J}_2;J,J):=\Delta_{\text{Kac}}(J,\hat{J};C)-\Delta_{\text{Kac}}(J_1,\hat{J}_1;C)-\Delta_{\text{Kac}}(J_2,\hat{J}_2;C), \tag{2.49}$$

$$\begin{aligned}\lambda(J_1,M_1,\hat{J}_1,\hat{M}_1;J_2,M_2,\hat{J}_2,\hat{M}_2)&:=e^{i\pi[M_1\hat{J}_2-M_2\hat{J}_1+\hat{M}_1J_2-\hat{M}_2J_1]}\\&\times\lambda(J_1,M_1;J_2,M_2)\hat{\lambda}(\hat{J}_1,\hat{M}_1;\hat{J}_2,\hat{M}_2).\end{aligned} \tag{2.50}$$

Proof. Proceed by recursion from the relation (omitting the divergent factors)

$$\xi_M^{(J)}(\sigma)\xi_{\hat M}^{(\hat J)}(\sigma')\sim e^{i\pi(M\hat J-\hat M J)}\xi_{M,\hat M}^{(J,\hat J)}(\sigma)$$

derived in [1]. First compute $\lambda(1/2, \alpha, 0, 0; J, M, \hat J, \hat M)$, $\alpha=\pm 1/2$, from the associativity of the fusion between operators. Consider $\xi_\alpha^{(1/2)}(\sigma)\xi_M^{(J)}(\sigma')\xi_{\hat M}^{(\hat J)}(\sigma'')$. Let $\sigma\to\sigma'$ first followed by $\sigma\to\sigma''$. It must lead to the same result as $\sigma'\to\sigma''$ first followed by $\sigma\to\sigma''$. This gives

$$\lambda(1/2,\alpha,0,0;J,M,\hat J,\hat M)=e^{i\pi(\alpha\hat J-\hat M/2)}\lambda(1/2,\alpha;J,M).$$

Similarly

$$\lambda(0,0,1/2,\hat\alpha;J,M,\hat J,\hat M)=e^{i\pi(\hat\alpha J-M/2)}\hat\lambda(1/2,\hat\alpha;\hat J,\hat M).$$

Next impose the same associativity condition on the product

$$\xi_\alpha^{(1/2)}(\sigma)\xi_{M_1,\hat M_1)}^{(J_1,\hat J_1)}(\sigma')\xi_{M_2,\hat M_2}^{(J_2,\hat J_2)}(\sigma'').$$

One obtains

$$\lambda(J_1+1/2,M_1+\alpha,\hat J_1,\hat M_1;J_2,M_2,\hat J_2,\hat M_2)=\lambda(J_1,M_1,\hat J_1,\hat M_1;J_2,M_2,\hat J_2,\hat M_2)\times\frac{\lambda(1/2,\alpha,0,0;J_1+J_2,M_1+M_2,\hat J_1+\hat J_2,\hat M_1+\hat M_2)}{\lambda(1/2,\alpha,0,0;J_1,M_1,\hat J_1,\hat M_1)},$$

and using the above relation

$$\lambda(J_1+1/2,M_1+\alpha,\hat J_1,\hat M_1;J_2,M_2,\hat J_2,\hat M_2)=\lambda(J_1,M_1,\hat J_1,\hat M_1;J_2,M_2,\hat J_2,\hat M_2)\times e^{i\pi(\alpha\hat J_2-\hat M_2/2)}\frac{\lambda(1/2,\alpha;J_1+J_2,M_1+M_2)}{\lambda(1/2,\alpha;J_1,M_1)}.$$

This together with the similar hatted recurrence leads to (2.50). □

Next the complete quantum group action on the ξ family is given by the following generalisation of Theorem (2.9):

Theorem (2.11). *For the general $\xi_{M,\hat M}^{(J,\hat J)}$ fields, the natural quantum group action is*

$$J_\pm\xi_{M,\hat M}^{(J,\hat J)}=\sqrt{\lfloor J\mp M\rfloor\lfloor J\pm M+1\rfloor}\,\xi_{M\pm 1,\hat M}^{(J,\hat J)},\qquad J_3\xi_{M,\hat M}^{(J,\hat J)}=M\xi_{M,\hat M}^{(J,\hat J)},\tag{2.51a}$$

$$\hat J_\pm\xi_{M,\hat M}^{(J,\hat J)}=\sqrt{\hat{\lfloor}\hat J\mp\hat M\hat{\rfloor}\hat{\lfloor}\hat J\pm\hat M+1\hat{\rfloor}}\,\xi_{M,\hat M\pm 1}^{(J,\hat J)},\qquad \hat J_3\xi_{M,\hat M}^{(J,\hat J)}=\hat M\xi_{M,\hat M}^{(J,\hat J)}.\tag{2.51b}$$

These operators satisfy the $SL(2)_q\times SL(2)_{\hat q}$ commutation relations

$$[J_+,J_-]=\lfloor 2J_3\rfloor,\qquad [\hat J_+,\hat J_-]=\hat{\lfloor}2\hat J_3\hat{\rfloor},\qquad [\hat J_l,J_m]=0.\tag{2.52}$$

The associated co-product is

$$\begin{aligned}\mathbf{J}_\pm&=J_\pm\otimes e^{ihJ_3\mp i\pi\hat J}+e^{-ihJ_3\pm i\pi\hat J}\otimes J_\pm,\\ \hat{\mathbf{J}}_\pm&=\hat J_\pm\otimes e^{i\hat h\hat J_3\mp i\pi J}+e^{-i\hat h\hat J_3\pm i\pi J}\otimes\hat J_\pm.\end{aligned}\tag{2.53}$$

Proof. We shall skip details, since they become cumbersome. It is elementary to check that (2.51a, b) satisfies (2.52). Moreover it is straightforward to check that the co-product is indeed consistent with the leading-order fusion-coefficient (2.50) using the recurrence realisation for C.G. coefficients (2.46). □

Theorem (2.12). General Operator-Product Expansion. *The general fusion-coefficients of the* $\xi_{M,\hat{M}}^{(J,\hat{J})}$ *fields are given by*

$$\xi_{M_1,\hat{M}_1}^{(J_1,\hat{J}_1)}(\sigma)\xi_{M_2,\hat{M}_2}^{(J_2,\hat{J}_2)}(\sigma') \sim \sum_{J,\hat{J}=|J_1-J_2|}^{J_1+J_2} ((d(\sigma-\sigma'))^{p(J_1,J_2,\hat{J}_1,\hat{J}_2;J,\hat{J})}$$
$$\times e^{i\pi[M_1\hat{J}_2-M_2\hat{J}_1+\hat{M}_1J_2-\hat{M}_2J_1]}$$
$$\times (J_1,M_1;J_2,M_2|J)(\hat{J}_1,\hat{M}_1;\hat{J}_2,\hat{M}_2|\hat{J})(\xi_{M_1+M_2,\hat{M}_1+\hat{M}_2}^{(J,\hat{J})}(\sigma)+\ldots)). \quad (2.54)$$

Proof. A straightforward extension of the proof of Theorem (2.8). □

We need general braiding properties of $\xi_{M_1,\hat{M}_1}^{(J_1,\hat{J}_1)}(\sigma)$ and $\xi_{M_2,\hat{M}_2}^{(J_2,\hat{J}_2)}(\sigma')$. They are given by the

Theorem (2.13). *For* $\pi>\sigma>\sigma'>0$, *one has:*

$$\xi_{M_1,\hat{M}_1}^{(J_1,\hat{J}_1)}(\sigma)\xi_{M_2,\hat{M}_2}^{(J_2,\hat{J}_2)}(\sigma') = \sum_{N_1,N_2,\hat{N}_1,\hat{N}_2} \mathscr{R}_{M_1,\hat{M}_1;M_2,\hat{M}_2}^{N_2,\hat{N}_2;N_1,\hat{N}_1}\xi_{N_2,\hat{N}_2}^{(J_2,\hat{J}_2)}(\sigma')\xi_{N_1,\hat{N}_1}^{(J_1,\hat{J}_1)}(\sigma), \quad (2.55)$$

where

$$\mathscr{R}_{M_1,\hat{M}_1;M_2,\hat{M}_2}^{N_2,\hat{N}_2;N_1,\hat{N}_1} = (J_1,J_2)_{M_1M_2}^{N_2N_1}\widehat{(J_1,J_2)}_{\hat{M}_1\hat{M}_2}^{\hat{N}_2\hat{N}_1}$$
$$\times e^{i\pi[J_2(\hat{N}_1+\hat{M}_1)+\hat{J}_2(N_1+M_1)-J_1(\hat{N}_2+\hat{M}_2)-\hat{J}_1(N_2+M_2)]}. \quad (2.56)$$

Proof. Consider, for $\sigma>\sigma_1>\sigma'>\sigma'_1$,

$$\xi_{M_1}^{(J_1)}(\sigma)\xi_{\hat{M}_1}^{(\hat{J}_1)}(\sigma_1)\xi_{M_2}^{(J_2)}(\sigma')\xi_{\hat{M}_2}^{(\hat{J}_2)}(\sigma'_1).$$

First, letting $\sigma\to\sigma_1$ and $\sigma'\to\sigma'_1$, and working to leading order, gives (once again we omit the divergent factors)

$$e^{i\pi[M_1\hat{J}_1-\hat{M}_1J_1+M_2\hat{J}_2-\hat{M}_2J_2]}\xi_{M_1,\hat{M}_1}^{(J_1,\hat{J}_1)}(\sigma)\xi_{M_2,\hat{M}_2}^{(J_2,\hat{J}_2)}(\sigma').$$

Second, exchanging first the operators, one gets for the same product

$$\sum_{N_1,N_2,\hat{N}_1,\hat{N}_2}(J_1,J_2)_{M_1M_2}^{N_2N_1}\widehat{(J_1,J_2)}_{\hat{M}_1\hat{M}_2}^{\hat{N}_2\hat{N}_1}e^{2\pi i[\hat{M}_1J_2-M_2\hat{J}_1+N_1\hat{J}_2-\hat{N}_2J_1]}$$
$$\times e^{-2\pi i(J_1\hat{J}_1+J_2\hat{J}_2)}e^{i\pi[N_1\hat{J}_1-\hat{N}_1J_1+N_2\hat{J}_2-\hat{N}_2J_2]}\xi_{N_2,\hat{N}_2}^{(J_2,\hat{J}_2)}(\sigma')\xi_{N_1,\hat{N}_1}^{(J_1,\hat{J}_1)}(\sigma).$$

Equating the two expressions establishes the theorem. □

3. Mathematical Properties at the Special Values

From now we derive the properties which only hold if

$$h+\hat{h}=s\pi, \quad s=0,\pm1, \quad \text{or, equivalently}, \quad C=1+6(s+2), \quad (3.1)$$

$$h=\tfrac{\pi}{2}(s-i\sqrt{4-s^2}), \quad \hat{h}=\tfrac{\pi}{2}(s+i\sqrt{4-s^2})=h^*. \quad (3.2)$$

Our aim in this section is to show that, for these values, hatted and unhatted quantum-group quantities are very simply related. The simplest [2, 3] is the

Theorem (3.1). *If* (3.1) *holds, and for any positive integer N and arbitrary real number* α, *one has*

$$(\widehat{\lfloor N\rfloor})^\alpha = e^{-i\alpha(N-1)s\pi}(\lfloor N\rfloor)^\alpha. \quad (3.3)$$

Proof. Compute

$$\lfloor N \rfloor \equiv \frac{e^{ihN}-e^{-ihN}}{e^{ih}-e^{-ih}} = \frac{e^{iN\pi(s-i\sqrt{4-s^2})/2}-e^{-iN\pi(s-i\sqrt{4-s^2})/2}}{e^{i\pi(s-i\sqrt{4-s^2})/2}-e^{-i\pi(s-i\sqrt{4-s^2})/2}}$$
$$= e^{is\pi(N-1)/2}\,\frac{e^{N\pi\sqrt{4-s^2}/2}-(-1)^{Ns}e^{-N\pi\sqrt{4-s^2}/2}}{e^{\pi\sqrt{4-s^2}/2}-(-1)^{s}e^{-\pi\sqrt{4-s^2}/2}}\,. \tag{3.4}$$

The second term is real and positive. Thus the phase of $\lfloor N \rfloor$ is $-s\pi(N-1)/2$. Since $\widehat{\lfloor N \rceil}$ is the complex conjugate of $\lfloor N \rfloor$ the relation follows. □

Corollary (3.2). *The following relations hold (general q-hypergeometric functions are defined in Appendix B)*

$$\widehat{\lfloor N \rceil}! = e^{-is\pi N(N-1)/2}\lfloor N \rfloor!\,, \qquad \widehat{\lfloor a \rceil}_\nu = \lfloor a \rfloor_\nu\, e^{-is\pi\nu(\nu+2N-3)/2}\,, \tag{3.5}$$

$${}_{r+1}\hat{F}_r\begin{pmatrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{pmatrix}, z\Bigg) = {}_{r+1}F_r\begin{pmatrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{pmatrix}; z e^{-is\pi[\Sigma a_j - \Sigma b_j - 1]}\Bigg)\,. \tag{3.6}$$

Proof. A straightforward algebra from (3.3). □

Next we relate the Clebsch-Gordan coefficients by the

Theorem (3.3). *If* (3.1) *holds the hatted and unhatted Clebsch-Gordan coefficients are related by*

$$\widehat{(J_1,M_1;J_2,M_2|J_1,J_2;J,M)} = (J_1,-M_1;J_2,-M_2|J_1,J_2;J,-M)$$
$$\times(-1)^{s\{(J_1-M_1)(J_2+M_2)+(J_1+J_2-J)(J_1+M_1+J_2-M_2)\}}(-1)^{J_1+J_2-J}\,. \tag{3.7}$$

Proof. Take expression (C.3) of Appendix C for the hatted C.G. coefficients:

$$\widehat{(J_1,M_1;J_2,M_2|J_1,J_2;J,M_1+M_2)} = e^{i\hat{h}(J_1+J_2-J)(J_1+J_2+J+1)/2}$$
$$\times\sqrt{\widehat{\lfloor 2J+1 \rceil}}\sqrt{\frac{\widehat{\lfloor J_1+J_2-J \rceil}!\,\widehat{\lfloor -J_1+J_2+J \rceil}!\,\widehat{\lfloor J_1-J_2+J \rceil}!}{\widehat{\lfloor J_1+J_2+J+1 \rceil}!}}$$

$$\times\sqrt{\widehat{\lfloor J_1-M_1 \rceil}!\,\widehat{\lfloor J_1+M_1 \rceil}!\,\widehat{\lfloor J_2-M_2 \rceil}!\,\widehat{\lfloor J_2+M_2 \rceil}!\,\widehat{\lfloor J-M_1-M_2 \rceil}!\,\widehat{\lfloor J+M_1+M_2 \rceil}!}$$
$$\times e^{i\hat{h}(M_2J_1-M_1J_2)}\sum_{\mu=0}^{J_1+J_2-J}\Bigg\{\frac{e^{-i\hat{h}\mu(J+J_1+J_2+1)}(-1)^\mu}{\widehat{\lfloor \mu \rceil}!\,\widehat{\lfloor J_1+J_2-J-\mu \rceil}!}$$
$$\times\frac{1}{\widehat{\lfloor J_1-M_1-\mu \rceil}!\,\widehat{\lfloor J-J_2+M_1+\mu \rceil}!\,\widehat{\lfloor J_2+M_2-\mu \rceil}!\,\widehat{\lfloor J-J_1-M_2+\mu \rceil}!}\Bigg\}\,. \tag{3.8}$$

One relates this expression to unhatted quantities piece by piece, using (3.1, 3, 5), and recalling that the $2J$'s and $2M$'s are integers, as well as $J_i \pm M_i$. First, write

$$\sqrt{\frac{\widehat{\lfloor 2J+1 \rceil}\,\widehat{\lfloor J-M_1-M_2 \rceil}!\,\widehat{\lfloor J+M_1+M_2 \rceil}!}{\widehat{\lfloor J_1+J_2+J+1 \rceil}!}}$$
$$= e^{is\pi\varphi_1}\sqrt{\frac{\lfloor 2J+1 \rfloor\,\lfloor J-M_1-M_2 \rfloor!\,\lfloor J+M_1+M_2 \rfloor!}{\lfloor J_1+J_2+J+1 \rfloor!}}\,.$$

One finds

$$\varphi_1 = \tfrac{1}{2}[(J_1+J_2)^2-(M_1+M_2)^2] - \tfrac{1}{4}(J_1+J_2-J)(J_1+J_2-J-1)\,. \tag{3.9}$$

Second, let

$$\sqrt{\widehat{\lfloor J_1+J_2-J\rceil}!}=e^{is\pi\varphi_2}\sqrt{\lfloor J_1+J_2-J\rfloor!}\,,$$

where, according to Eq. (3.5),

$$\varphi_2=-\tfrac{1}{4}(J_1+J_2-J)(J_1+J_2-J-1)\,. \tag{3.10}$$

Third, write

$$\frac{\sqrt{\widehat{\lfloor J_1-J_2+J\rceil}\,\widehat{\lfloor J_1-M_1\rceil}!\,\widehat{\lfloor J_1+M_1\rceil}!}}{\widehat{\lfloor J_1-M_1-\mu\rceil}!\,\widehat{\lfloor J-J_2+M_1+\mu\rceil}!}$$

$$=e^{is\pi\{\mu^2-\mu[(J-J_1-J_2)+2M_1)]+\varphi_3\}}\,\frac{\sqrt{\lfloor J_1-J_2+J\rfloor\,\lfloor J_1-M_1\rfloor!\,\lfloor J_1+M_1\rfloor!}}{\lfloor J_1-M_1-\mu\rfloor!\,\lfloor J-J_2+M_1+\mu\rfloor!}\,,$$

obtaining

$$\varphi_3=\tfrac{1}{2}(M_1^2-J_1^2)+\tfrac{1}{4}(J_1+J_2-J)(1-4M_1+J-J_1-J_2)\,. \tag{3.11}$$

Fourth, in a similar way,

$$\frac{\sqrt{\widehat{\lfloor -J_1+J_2+J\rceil}!\,\widehat{\lfloor J_2+M_2\rceil}!\,\widehat{\lfloor J_2-M_2\rceil}!}}{\widehat{\lfloor J_2+M_2-\mu\rceil}!\,\widehat{\lfloor J-J_1-M_2+\mu\rceil}!}$$

$$=e^{is\pi\{\mu^2-\mu[(J-J_1-J_2)-2M_2)]+\varphi_4\}}\,\frac{\sqrt{\lfloor -J_1+J_2+J\rfloor!\,\lfloor J_2+M_2\rfloor!\,\lfloor J_2-M_2\rfloor!}}{\lfloor J_2+M_2-\mu\rfloor!\,\lfloor J-J_1-M_2+\mu\rfloor!}\,,$$

with

$$\varphi_4=\tfrac{1}{2}(M_2^2-J_2^2)+\tfrac{1}{4}(J_1+J_2-J)(1+4M_2+J-J_1-J_2)\,. \tag{3.12}$$

Fifth consider

$$\frac{1}{\widehat{\lfloor \mu\rceil}!\,\widehat{\lfloor J_1+J_2-J-\mu\rceil}!}=\frac{e^{is\pi(\mu^2-\mu(J_1+J_2-J)+\varphi_5)}}{\lfloor \mu\rfloor!\,\lfloor J_1+J_2-J-\mu\rfloor!}\,,$$

obtaining

$$\varphi_5=\tfrac{1}{2}(J_1+J_2-J)(J_1+J_2-J-1)\,. \tag{3.13}$$

Sixth let

$$e^{i\hat{h}(M_2J_1-M_1J_2)}=e^{is\pi\varphi_6}\,e^{-ih(M_2J_1-M_1J_2)}\,,$$

where

$$\varphi_6=M_2J_1-M_1J_2\,. \tag{3.14}$$

Next consider the sum over μ. The μ-dependent term of the phase factor is

$$\exp\{[is\pi]\,[3\mu^2-\mu[(2J-2J_1-2J_2)+2M_1-2M_2]-\mu(J_1+J_2-J)]\}$$
$$=\exp\{[is\pi]\,[2\mu^2+\mu(\mu+1)-\mu[2(M_1-J_1)-2(M_2+J_2)]-\mu(J_1+J_2+J+1)]\}$$

The first three terms in the second expression are even numbers, so that the μ-dependent terms simply gives $(-1)^{s\mu(J_1+J_2+J+1)}$. The summation over μ becomes

$$S=\sum_{\mu=0}^{J_1+J_2-J}\left\{\frac{e^{-i\hat{h}\mu(J+J_1+J_2+1)}(-1)^{\mu}(-1)^{s\mu(J_1+J_2+J+1)}}{\lfloor \mu\rfloor!\,\lfloor J_1+J_2-J-\mu\rfloor!}\right.$$
$$\left.\times\frac{1}{\lfloor J_1-M_1-\mu\rfloor!\,\lfloor J-J_2+M_1+\mu\rfloor!\,\lfloor J_2+M_2-\mu\rfloor!\,\lfloor J-J_1-M_2+\mu\rfloor!}\right\}.$$

Since $h+\hat{h}=s\pi$,

$$e^{-i\hat{h}\mu(J+J_1+J_2+1)}(-1)^{s\mu(J_1+J_2+J+1)}=e^{ih\mu(J+J_1+J_2+1)}.$$

One compensates for this change of sign by letting $\nu=J_1+J_2-J-\mu$:

$$S=e^{ih(J_1+J_2-J)(J_1+J_2+J+1)}(-1)^{J_1+J_2-J}\sum_{\nu=0}^{J_1+J_2-J}\left\{\frac{e^{ih\nu(J+J_1+J_2+1)}(-1)^{\nu}}{\lfloor\nu\rfloor!\lfloor J_1+J_2-J-\nu\rfloor!}\right.$$
$$\left.\times\frac{1}{\lfloor J-J_2-M_1+\nu\rfloor!\lfloor J_1+M_1-\nu\rfloor!\lfloor J-J_1+M_2+\nu\rfloor!\lfloor J_2-M_2-\nu\rfloor!}\right\}.$$

The sum coincides with the similar expression for the unhatted case if one replaces M_1 and M_2 by their opposite. Finally the factor in front combines with the first term on the right-hand side of the starting point (3.8) leading to:

$$e^{i(h+\hat{h}/2)(J_1+J_2-J)(J_1+J_2+J+1)}=e^{ih(J_1+J_2-J)(J_1+J_2+J+1)/2}\,e^{is\pi\varphi_7},$$
$$\varphi_7=(J_1+J_2-J)(J_1+J_2+J+1)/2. \tag{3.15}$$

Altogether one finds:

$$\widehat{(J_1,M_1;J_2,M_2|J_1,J_2;J,M)}=(J_1,-M_1;J_2,-M_2|J_1,J_2;J,-M)(-1)^{s\sum_{j=1}^{7}\varphi_j} \tag{3.16}$$

and the result is derived by collecting Eqs. (3.9–15). □

Closure under fusion of the physical family is ensured by the following

Corollary (3.4). *If* (3.1) *holds, the C.G. coefficients satisfy the orthogonality relation*

$$\sum_{M_1,M_2}\widehat{(J_1,-M_1;J_2,-M_2|J_1,J_2;\hat{J},-M)}(J_1,M_1;J_2,M_2|J_1,J_2;J,M)$$
$$\times(-1)^{s\{(J_1+M_1)(J_2-M_2)+(J_1+J_2-J)(J_1-M_1+J_2+M_2)\}}=(-1)^{J_1+J_2-J}\delta_{J\hat{J}}. \tag{3.17}$$

Proof. This is an easy consequence of the last theorem combined with the orthogonality properties of the C.G. coefficients. The latter are discussed in Appendix C. □

Finally the braiding properties are related by the

Theorem (3.5). *If* (3.1) *holds, the hatted and unhatted R-matrices are related by the relation*

$$\widehat{(J_1,J_2)}_{M_1M_2}^{N_2N_1}=\overline{(J_2,J_1)}_{-N_2-N_1}^{-M_1-M_2}\,e^{is\pi\{(J_1-M_1)(J_2+M_2)-J_1(J_2+N_2)+N_1(J_2-N_2)\}}. \tag{3.18}$$

Proof. Write formulae (2.8, 9) explicitly for the hatted braiding matrix:

$$\widehat{(J_1,J_2)}_{M_1M_2}^{N_2N_1}=\delta(M_1+M_2-N_1-N_2)e^{-2i\hat{h}M_1M_2}(1-e^{2i\hat{h}})^n$$
$$\times\frac{e^{i\hat{h}n(n-1)/2}}{\widehat{\lfloor n\rfloor}!}e^{-i\hat{h}n(M_1-M_2)}\sqrt{\frac{\widehat{\lfloor J_1+M_1\rfloor}!\widehat{\lfloor J_1-N_1\rfloor}!\widehat{\lfloor J_2-M_2\rfloor}!\widehat{\lfloor J_2+N_2\rfloor}!}{\widehat{\lfloor J_1-M_1\rfloor}!\widehat{\lfloor J_1+N_1\rfloor}!\widehat{\lfloor J_2+M_2\rfloor}!\widehat{\lfloor J_2-N_2\rfloor}!}}, \tag{3.19}$$

where $n=M_1-N_1=N_2-M_2$. Making use of (3.1, 3, 5) repeatedly, one gets

$$\widehat{(J_1,J_2)}_{M_1M_2}^{N_2N_1}=e^{is\pi A}\delta(M_1+M_2-N_1-N_2)e^{2ihM_1M_2}$$
$$\times(1-e^{-2ih})^n\frac{e^{-ihn(n-1)/2}}{\lfloor n\rfloor!}e^{ihn(M_1-M_2)}$$
$$\times\sqrt{\frac{\lfloor J_1+M_1\rfloor!\lfloor J_1-N_1\rfloor!\lfloor J_2-M_2\rfloor!\lfloor J_2+N_2\rfloor!}{\lfloor J_1-M_1\rfloor!\lfloor J_1+N_1\rfloor!\lfloor J_2+M_2\rfloor!\lfloor J_2-N_2\rfloor!}},$$

where, (using $M_1+M_2=N_1+N_2$),

$$\Lambda=(J_1-M_1)(J_2+M_2)-J_1(J_2+N_2)+N_1(J_2-N_2).$$

On the other hand, formulae (2.11, 12) give

$$\overline{(J_1,J_2)}_{M_1M_2}^{N_2N_1}=\delta(M_1+M_2-N_1-N_2)e^{2ihM_1M_2}$$
$$\times(1-e^{-2ih})^m\frac{e^{-ihm(m-1)/2}}{\lfloor m\rfloor!}e^{ihm(M_2-M_1)}$$
$$\times\sqrt{\frac{\lfloor J_1-M_1\rfloor!\lfloor J_1+N_1\rfloor!\lfloor J_2+M_2\rfloor!\lfloor J_2-N_2\rfloor!}{\lfloor J_1+M_1\rfloor!\lfloor J_1-N_1\rfloor!\lfloor J_2-M_2\rfloor!\lfloor J_2+N_2\rfloor!}},$$

where $m=M_2-N_2=N_1-M_1$. Comparing the above expressions and using the property $\overline{(J_1,J_2)}_{M_1M_2}^{N_2N_1}=\overline{(J_2,J_1)}_{N_2N_1}^{M_1M_3}$, one arrives at the desired relation. □

Finally, we shall need a relationship between $|J,\varpi)_m^M$ and its hatted counterpart $\widehat{|J,\hat{\varpi})}_m^M$. This will not hold for arbitrary values of ϖ, but only for the special ones that occur in the physical Hilbert space introduced in [2, 3, 6]:

Definition (3.6). The physical Hilbert space is defined by

$$\mathscr{H}_{\text{phys}}:=\bigoplus_{r=0}^{1-s}\bigoplus_{n=-\infty}^{\infty}\mathscr{F}(\varpi_{r,n}),\tag{3.20}$$

where $\mathscr{F}(\varpi_{r,n})$ is the Fock space with quasi momentum

$$\varpi_{r,n}\equiv\varpi_0^r+n\left(1-\frac{\pi}{h}\right)\equiv\left(\frac{r}{2-s}+n\right)\left(1-\frac{\pi}{h}\right).\tag{3.21}$$

The importance of this choice for our coming discussion is coming from the

Proposition (3.7). *In $\mathscr{H}_{\text{phys}}$ ϖ and $\hat{\varpi}$ are related by (N is an arbitrary integer)*

$$\hat{h}\hat{\varpi}_{r,n}-h\varpi_{r,n}=\pi(r+n(2-s)),\tag{3.22}$$
$$\widehat{\lfloor\hat{\varpi}_{r,n}+N\rfloor}=(-1)^{r+s(N-n-1)}\lfloor\varpi_{r,n}-N\rfloor.\tag{3.23}$$

Proof. Simple calculations [2, 3] using (1.2), (1.4), and (3.21). □

In the unitary truncation theorem, one will show that one may consistently restrict the operator algebra to $\mathscr{H}_{\text{phys}}$. This is satisfactory since one has the

Theorem (3.8). *In the space $\mathscr{H}_{\text{phys}}$, the Virasoro-highest-weights are real and positive, and the representation of the Virasoro algebra is unitary.*

Proof. As recalled in Appendix A, the Fock space $\mathscr{F}(\varpi)$ corresponds to the highest weight

$$\Delta(\varpi)\equiv\frac{1}{8\gamma}+\frac{(p_0)^2}{2}=\frac{h}{4\pi}\left(1+\frac{\pi}{h}\right)^2-\frac{h}{4\pi}\varpi^2.$$

An easy computation [2] shows that

$$\Delta(\varpi_{r,n})=\frac{s+2}{4}+\frac{1}{2}\left(\frac{r}{\sqrt{2(2-s)}}+\frac{n}{2}\sqrt{2(2-s)}\right)^2,\tag{3.24}$$

which is indeed real and positive. Since $C>1$ the corresponding representations of the Virasoro algebra are unitary. □

Finally in $\mathscr{H}_{\text{phys}}$, $|J,\varpi)_m^M$ and $\widehat{|J,\hat{\varpi})}_m^M$ are related by the

Theorem (3.9). *Introduce the operator*

$$\kappa := -e^{i(\hat{h}\hat{\varpi}-h\varpi)}e^{i(h-\hat{h})/2}. \tag{3.25}$$

In $\mathscr{H}_{\text{phys}}$ *one has*

$$\kappa^{J+M}\widehat{|J,\hat{\varpi})}_M^m = (-1)^{s(J+M)(J+M-1)/2+J+M} \times e^{ih(J+M-m)}e^{is\pi(-J^2+m/2)}e^{-im(\hat{h}\hat{\varpi}-h\varpi)}|J,\varpi-2m)_M^m. \tag{3.26}$$

Proof. Start from expression (2.5) for $\widehat{|J,\hat{\varpi})}_m^M$:

$$\widehat{|J,\hat{\varpi})}_M^m := \sqrt{\widehat{\begin{pmatrix}2J\\J+M\end{pmatrix}}}\, e^{i\hat{h}m/2}\sum_t e^{i\hat{h}t(\hat{\varpi}+m)}\widehat{\begin{pmatrix}J-M\\J-M+m-t)/2\end{pmatrix}}\widehat{\begin{pmatrix}J+M\\(J+M+m+t)/2\end{pmatrix}}. \tag{3.27}$$

t runs over values of the form $t=2\mu+J-M+m$, with μ integer. Thus, according to Eqs. (3.1, 22),

$$e^{i\hat{h}t(\hat{\varpi}+m)} = e^{iht(\varpi-m)}e^{is\pi mt}e^{ih(J-M+m)(\hat{h}\hat{\varpi}-h\varpi)}.$$

Next one has, according to Eq. (3.2),

$$\widehat{\begin{pmatrix}J-M\\(J-M+m-t)/2\end{pmatrix}}\widehat{\begin{pmatrix}J+M\\(J+M+m+t)/2\end{pmatrix}} = e^{-is\pi(J^2+M^2-m^2-t^2)/2}\begin{pmatrix}J-M\\(J-M+m-t)/2\end{pmatrix}\begin{pmatrix}J+M\\(J+M+m+t)/2\end{pmatrix}.$$

Collecting the t-dependent terms of the last two relations gives

$$e^{is\pi(t^2/2+mt)} = e^{is\pi(J-M+m)(J-M+3m)/2}.$$

The dependence in t disappears. One does find the same sum as in $|J,\varpi-2m)_M^m$ up to a factor. Applying (3.1) repeatedly completes the derivation. □

4. The Physical Fields with Negative Weights

At this point we have the mathematical machinery at our disposal. Our next task is to derive the unitary decoupling for the fields $\chi_-^{(J)}(\sigma)$ which take the general form

$$\chi_-^{(J)}(\sigma) = \sum_{M=-J}^{J}\kappa^{J-M}\varepsilon_M^{(J)}\xi_{M,-M}^{(J,J)}(\sigma), \tag{4.1}$$

where the $\varepsilon_M^{(J)}$'s are ± 1 factors to be determined below. For pedagogical reasons, we shall build them up, step by step, using the fusion to leading order. The decoupling theorem will be fully checked later on. In the expansion (4.1) the coefficients do not depend upon ϖ apart from the κ^{J-M} factor. We shall show later on that

$$\chi_-^{(J)}\mathscr{F}(\varpi_{r,n}) \in \bigoplus_{m=-J}^{J}\mathscr{F}(\varpi_{r,n+2m}), \tag{4.2}$$

so that, for any integer N,

$$\chi_-^{(J)}\kappa^N=(-1)^{s2JN}\kappa^N\chi_-^{(J)}. \tag{4.3}$$

Previous computations [2] show that

$$\chi_-^{(1/2)}(\sigma)=\xi_{1/2,-1/2}^{(1/2,1/2)}(\sigma)+\kappa\xi_{-1/2,1/2}^{(1/2,1/2)}(\sigma), \tag{4.4}$$

which is of the form (4.1). The field $\chi_-^{(J)}$ may be determined from the

Theorem (4.1). *If* (4.3) *holds, if to leading order in the singularity at* $\sigma\to\sigma'$,

$$\chi_-^{(1/2)}(\sigma)\chi_-^{(J)}(\sigma')\sim(d(\sigma-\sigma'))^{-J(s+2)}\chi_-^{(J+1/2)}(\sigma), \tag{4.5}$$

and if $\chi_-^{(1/2)}(\sigma)$ *is given by* (4.4); *then one has for any non-negative integer* $2J$,

$$\chi_-^{(J)}(\sigma)=\sum_{M=-J}^{J}\kappa^{J-M}(-1)^{s(J-M)(J-M-1)/2}\xi_{M,-M}^{(J,J)}(\sigma). \tag{4.6}$$

Proof. Introduce

$$\lambda^{\pm}(J,M)\equiv(1/2,\pm 1/2;J,M|1/2,J;J+1/2,M\pm 1/2),$$

and the corresponding $\hat{\lambda}^{\pm}(J,M)$. Assume that $\chi_-^{(J)}$ is of the form (4.1). The fusion to leading order gives (omitting the divergent factor as usual)

$$\begin{aligned}\chi_-^{(1/2)}(\sigma)\chi_-^{(J)}(\sigma')&\sim\varepsilon_J^{(J)}\xi_{J+1/2,-J-1/2}^{(J+1/2,J+1/2)}+\kappa^{2J+1}(-1)^{2sJ}\varepsilon_{-J}^{(J)}\xi_{-J-1/2,J+1/2}^{(J+1/2,J+1/2)}\\&+\sum_{M=-J}^{J-1}\kappa^{J-M}(-1)^{s(J-M)}\{\varepsilon_M^{(J)}\hat{\lambda}^-(J,-M)\lambda^+(J,M)\\&+(-1)^s\varepsilon_{M+1}^{(J)}\hat{\lambda}^+(J,-M-1)\lambda^-(J,M+1)\}\,\xi_{M+1/2,-M-1/2}^{(J+1/2,J+1/2)}(\sigma)\}.\end{aligned}$$

It follows from Corollary (3.4) that

$$\hat{\lambda}^-(J,-M)\lambda^+(J,M)+(-1)^{s(J-M)}\hat{\lambda}^-(J,M-1)\lambda^+(J,M+1)=(-1)^{s(J-M)}.$$

One verifies that $\chi_-^{(J)}$ is also of the form (4.1) if $\varepsilon_M^{(J)}$ obey the relations:

$$\varepsilon_{M+1}^{(J)}=\varepsilon_M^{(J)}(-1)^{s(J-M+1)};\qquad \varepsilon_{M+1/2}^{(J+1/2)}=\varepsilon_M^{(J)}.$$

The solution is indeed

$$\varepsilon_M^{(J)}=(-1)^{s(J-M)(J-M-1)/2}.\quad\square \tag{4.7}$$

Equation (4.6) is very simple since the coefficients do not explicitly depend upon J:

$$\chi_-^{(J)}(\sigma)=\xi_{J,-J}^{(J,J)}+\kappa\xi_{J-1,-J+1}^{(J,J)}+(-1)^s\kappa^2\xi_{J-2,-J+2}^{(J,J)}+\kappa^3\xi_{J-3,-J+3}^{(J,J)}+\dots. \tag{4.8}$$

The set of $\chi_-^{(J)}$ fields is closed under operator-product expansion as shown by the

Theorem (4.2). *If Eq.* (4.3) *holds, the operator-product expansion of the* $\chi_-^{(J)}$ *fields is given by*

$$\begin{aligned}\chi_-^{(J_1)}(\sigma)\chi_-^{(J_2)}(\sigma')&\sim\sum_{J=|J_1-J_2|}^{J_1+J_2}(-1)^{s(J_1+J_2-J)(J_1+J_2-J-1)/2+s(J_1+J_2-J)(J_2-J_1+J)}\\&\times(-1)^{J_1+J_2-J}\kappa^{J_1+J_2-J}((d(\sigma-\sigma'))^{p(J_1,J_1,J_2,J_2;J,J)}(\chi_-^{(J)}(\sigma)+\dots)).\end{aligned} \tag{4.9}$$

Proof. It follows from Theorem (2.12) that

$$\chi_-^{(J_1)}(\sigma)\chi_-^{(J_2)}(\sigma')\sim \sum_{M_1,M_2}\left\{\sum_{J,\hat{J}=|J_1-J_2|}^{J_1+J_2}((d(\sigma-\sigma'))^{p(J_1,J_1,J_2,J_2;J,\hat{J})}\kappa^{J_1+J_2-J}\right.$$
$$\left.\times(-1)^{sN}(J_1,M_1;J_2,M_2|\hat{J})(J_1,M_1;J_2,M_2|J)(\xi^{(J,\hat{J})}_{M_1+M_2,-M_1-M_2}(\sigma)+\ldots))\right\},$$

where

$$N=2J_1(J_2-M_2)+\tfrac{1}{2}(J_1-M_1)(J_1-M_1-1)+\tfrac{1}{2}(J_2-M_2)(J_2-M_2-1)$$
$$+(J_1+M_1)(J_2-M_2)+(J_1+J_2-J)(J_1-M_1+J_2+M_2).$$

This may be rewritten as

$$N=\tfrac{1}{2}(J-M)(J-M-1)+(J_1+J_2-J)(J_1+J_2-J-1)\tfrac{1}{2}$$
$$+(J_1+J_2-J)(J_2-J_1+J)\quad(\mathrm{mod}\,2),\tag{4.10}$$

where terms are arranged so that the integer-character of N is explicit. N (mod 2) only depends upon M and the summation for fixed M coincides with that of Corollary (3.14). Only fields with $J=\hat{J}$ appear on the right-hand side. The first term of (4.10) reproduces $\varepsilon_M^{(J)}$ so that the summation over M gives back the physical field $\chi_-^{(J)}$ and (4.9) follows. □

Next the braiding properties of the $\chi_-^{(J)}$ fields are particularly simple:

Theorem (4.3). *If Eq.* (4.3) *holds, the exchange properties of the* $\chi_-^{(J)}$ *fields are given by*

$$\chi_-^{(J_1)}(\sigma)\chi_-^{(J_2)}(\sigma')=e^{-2\pi i\varepsilon(2+s)J_1J_2}\chi_-^{(J_2)}(\sigma')\chi_-^{(J_1)}(\sigma),$$

where ε *is the sign of* $\sigma-\sigma'$.

Proof. Choose $\varepsilon=1$ for instance. Theorem (2.13) and Eq. (4.3) give

$$\chi_-^{(J_1)}(\sigma)\chi_-^{(J_2)}(\sigma')=\sum_{M_1,M_2}\kappa^{J_1-M_1}\varepsilon^{(J_1)}_{M_1}\xi^{(J_1,\hat{J}_1)}_{M_1,-M_1}(\sigma)\kappa^{J_2-M_2}\varepsilon^{(J_2)}_{M_2}\xi^{(J_2,\hat{J}_2)}_{M_2,-M_2}(\sigma')$$
$$=\sum_{M_1,M_2;N_1,N_2;P_1,P_2}\kappa^{J_1+J_2-M_1-M_2}\varepsilon^{(J_1)}_{M_1}\varepsilon^{(J_2)}_{M_2}(-1)^{2sJ_1(J_2-M_2)}$$
$$\mathscr{R}^{N_2,-P_2;N_1,-P_1}_{M_1,-M_1;M_2,-M_2}\,\xi^{(J_2,\hat{J}_2)}_{N_2,-P_2}(\sigma')\xi^{(J_1,\hat{J}_1)}_{N_1,-P_1}(\sigma).\tag{4.11}$$

Next apply Theorem (3.5) and make use of the identity

$$\varepsilon^{(J_1)}_{M_1}\varepsilon^{(J_2)}_{M_2}(-1)^{2sJ_1(J_2-M_2)}=\varepsilon^{(J_1+J_2)}_{M_1+M_2}(-1)^{s(J_1+M_1)(J_2-M_2)},$$

which comes out in the derivation of Theorem (4.2). One finds altogether

$$\chi_-^{(J_1)}(\sigma)\chi_-^{(J_2)}(\sigma')=e^{-4i\pi J_1J_2}\sum_{M_1,M_2;N_1,N_2;P_1,P_2}\kappa^{J_1+J_2-N_1-N_2}\varepsilon^{(J_1+J_2)}_{N_1+N_2}$$
$$\times e^{i\pi[(N_1-P_1)J_2-(N_2-P_2)J_1]}(J_1,J_2)^{N_2N_1}_{M_1M_2}\overline{(J_2,J_1)}^{M_1M_2}_{P_2P_1}$$
$$\times e^{is\pi[N_1(J_2-N_2)-J_1(J_2+N_2)]}\xi^{(J_2,\hat{J}_2)}_{N_2,-P_2}(\sigma')\xi^{(J_1,\hat{J}_1)}_{N_1,-P_1}(\sigma).$$

The sum over M_1 and M_2 reduces to the inverse relation (2.13) of Theorem (2.1), and only $P_1=N_1$ and $P_2=N_2$ contribute as required by the closure. Pulling $\kappa^{J_1-N_1}$ to the right of $\xi^{(J_2,\hat{J}_2)}_{N_2,-P_2}$ to reconstruct $\chi_-^{(J_1)}$ on the right-hand side, one gets the overall phase factor in the remaining summation over N_1 and N_2. It is given by

$$\varepsilon_{N_1+N_2}^{(J_1+J_2)} e^{is\pi 2J_2(J_1-N_1)} e^{is\pi[N_1(J_2-N_2)-J_1(J_2+N_2)]} = \varepsilon_{N_1}^{(J_1)} \varepsilon_{N_2}^{(J_2)} e^{-2\pi s J_1 J_2},$$

which completes the proof. □

The last part of this section is concerned with the fact that the physical family may be consistently restricted to $\mathscr{H}_{\text{phys}}$. This is seen by going back to the ψ-fields, making use of the

Theorem (4.4). *The fields* $\chi_-^{(J)}$ *defined by* (4.6) *are equivalently given by*

$$\chi_-^{(J)}(\sigma) = (-1)^{4J^2} e^{-is\pi J^2} \sum_{m=-J}^{J} C_m^{(J)}(\varpi) e^{im[\hat{h}\hat{\varpi} - h\varpi + (h-\hat{h})/2]} \psi_{m,-m}^{(J,J)}. \tag{4.12}$$

Proof. Equation (4.9) of [1] gives

$$\xi_{M,-M}^{(J,J)}(\sigma) = \sum_{-J \le m \le J;\, -J \le \hat{m} \le J} (-1)^{4J^2} |J,\varpi)_M^m \widehat{|J,\hat{\varpi})}_{-M}^{-\hat{m}} \psi_{m,-\hat{m}}^{(JJ)}(\sigma), \tag{4.13}$$

which is to be substituted into (4.6). Making use of Theorem (3.9) one gets, on the other hand,

$$\begin{aligned}
&\sum_M \kappa^{J-M}(-1)^{s(J-M)(J-M-1)/2} |J,\varpi)_M^m \widehat{|J,\hat{\varpi})}_{-M}^{-\hat{m}} \\
&= \sum_M e^{ih(J-M+\hat{m})}(-1)^{J-M} e^{-is\pi(J^2+\hat{m}/2)} e^{i\hat{m}(\hat{h}\hat{\varpi}-h\varpi)} |J,\varpi)_M^m |J,\varpi+2m)_{-M}^{-\hat{m}} \\
&= e^{-is\pi(J^2+\hat{m}/2)} e^{im[(\hat{h}\hat{\varpi}-h\varpi)+h]} C_m^{(J)}(\varpi) \delta_{m,\hat{m}},
\end{aligned} \tag{4.14}$$

where the last equality is a consequence of Corollary (2.7). Substituting into (4.6) completes the proof. □

Corollary (4.5). *Equations* (4.2, 3) *hold, namely, for any integer N,*

$$\chi_-^{(J)} \mathscr{F}(\varpi_{r,n}) \in \bigoplus_{m=-J}^{J} \mathscr{F}(\varpi_{r,n+2m}); \qquad \chi_-^{(J)} \kappa^N = (-1)^{s2JN} \chi_-^{(J)} \kappa^N.$$

Proof. Easy consequence of Theorem (4.4) together with the shift properties of the ψ-fields:

$$\psi_{m,-m}^{(J,J)} \mathscr{F}(\varpi_{r,n}) \in \mathscr{F}(\varpi_{r,n-2m}), \tag{4.15}$$

that follow from (1.5). □

Finally collecting the results of the section we arrive at the

Theorem (4.6). Unitary Truncation Theorem for the $\chi_-^{(J)}$ Fields. *For* $C = 1 + 6(s+2)$, $s = 0, \pm 1$, *and when it acts on* $\mathscr{H}_{\text{phys}}$; *the set* $\mathscr{A}_{\text{phys}}^-$ *of operators* $\chi_-^{(J)}$, *with positive integer 2J, is closed by fusion and braiding, and only gives states that belong to* $\mathscr{H}_{\text{phys}}$.

Proof. Closure by fusion and braiding was derived above (Theorems (4.1, 2)) assuming that (4.3) holds, which is confirmed by Corollary (4.5). The fact that $\mathscr{H}_{\text{phys}}$ is left invariant by the $\chi_-^{(J)}$ fields is an immediate consequence of Corollary (4.5). □

Proposition (4.7). *The set of operators $\chi_-^{(J)}(\sigma)$ has Virasoro weights*

$$\Delta^-(J) := \Delta_{\mathrm{Kac}}(J, J; C) = -\frac{C-1}{6} J(J+1), \tag{4.16}$$

which are real and negative.

Proof. Simple computations using Eq. (1.3). □

5. The Physical Fields with Positive Weights

There remains to study $\mathscr{A}^+_{\mathrm{phys}}$ which is the part of physical family $\mathscr{A}_{\mathrm{phys}}$ made up with fields of positive weights. As shown in [3], one has to combine unhatted fields of spin $-J-1$ with hatted fields of spin J, and operators with negative spins must be discussed. First we have the

Theorem (5.1). *In $\mathscr{H}_{\mathrm{phys}}$ there exist operators $\psi_m^{(J)}$ with negative J which satisfy*

$$\psi_\alpha^{(1/2)}(\sigma)\psi_m^{(J)}(\sigma') = \sum_{\beta=\pm 1/2;\, m'} S_{\alpha m}^{(J)m'\beta} \psi_{m'}^{(J)}(\sigma')\psi_\beta^{(1/2)}(\sigma); \tag{5.1}$$

$$S_{-1/2\; m}^{(J)m,\,-1/2}(\varpi) = S_{1/2\; -m}^{(J)-m\; 1/2}(-\varpi) = \frac{\lfloor \varpi + J + m \rfloor}{\lfloor \varpi \rfloor} e^{ihm\varepsilon},$$

$$S_{-1/2\; m}^{(J)m-1\, 1/2}(\varpi) = \frac{\lfloor J+m \rfloor}{\lfloor \varpi \rfloor} e^{ih\varepsilon(1-m-\varpi)} = S_{1/2,\, -m}^{(J)-m+1,\, -1/2}(-\varpi); \tag{5.2}$$

$$\psi_m^{(J)}(\sigma)\hat{\psi}_{\hat{m}}^{(\hat{J})}(\sigma') = e^{-2i\pi J\hat{J}\varepsilon}\hat{\psi}_{\hat{m}}^{(\hat{J})}(\sigma')\psi_m^{(J)}(\sigma); \tag{5.3}$$

$$\psi_m^{(J)}\hat{\psi}_{\hat{m}}^{(\hat{J})} \sim \hat{\psi}_{\hat{m}}^{(\hat{J})}\psi_m^{(J)} \sim (-1)^{2(J\hat{J}-m\hat{m})}\psi_{m\hat{m}}^{(J\hat{J})}. \tag{5.4}$$

Proof. Since $C>1$, $\mathscr{H}_{\mathrm{phys}}$ is a direct sum of trivial Verma modules. In each such space, the states generated by powers of the Virasoro generators applied to the highest-weight vector are linearly independent and form a basis. Thus, any matrix element of a primary field may be computed from its matrix elements between highest-weight states, by only making use of its transformation law under conformal transformation. It follows that for positive J the fields $\psi_m^{(J)}$ are uniquely characterized, up to normalization, by their conformal weights $\Delta(J) = -hJ(J+1)/\pi - h$ and their shift properties $\psi_m^{(J)} f(\varpi) = f(\varpi + 2m)\psi_m^{(J)}$ recalled in Appendix A (formula (A.20)). For negative J, we define the $\psi_m^{(J)}$ operators by the same two conditions. Then going back to [1], one sees that the starting point – that is the fusion and braiding properties of $\psi_{\pm 1/2}^{(1/2)}$ with $\psi_m^{(J)}$ – has a natural and well defined continuation to negative J. Indeed, the discussion was solely based on the following facts. First, the fields $\psi_{\pm 1/2}^{(1/2)}$ satisfy a quantum Schrödinger equation; second, the fields $\psi_m^{(J)}$ are primary with conformal weight $\Delta(J)$; third, the total shift of the product $\psi_{\pm 1/2}^{(1/2)}(\sigma)\psi_m^{(J)}(\sigma')$ is $2m \pm 1$. By definition, these properties hold true for negative J. Moreover, the solution of the recursions that determine the properties of the ψ fields (Appendix A of [1]) remains basically the same except for trivial changes. Thus the exchange properties of $\psi_{\pm 1/2}^{(1/2)}(\sigma)$ and $\psi_m^{(J)}(\sigma')$ takes the same form for negative and positive J. We shall skip details in order to avoid another lengthy discussion. □

After this continuation is made the key question is: which values of m should one consider? We shall see that, for negative J, there exists a consistent family of fields $\psi_m^{(J)}$ with $J+1 \leqq m \leqq -J-1$. This is first indicated by the

Theorem (5.2). *The S-matrix of Eq.* (5.1, 2) *satisfies*

$$S_{\alpha m}^{(-J-1)m'\beta} = S_{\alpha m}^{(J)m'\beta} \lambda_m^{(J)}(\varpi + 2\alpha)/\lambda_{m'}^{(J)}(\varpi), \tag{5.5}$$

$$\lambda_m^{(J)}(\varpi) := (-1)^{J+m} \binom{2J}{J+m} \lfloor \varpi + m - J \rfloor_{2J+1}. \tag{5.6}$$

Proof. Simple computations. □

Corollary (5.3). *For $J < -1$, $S_{\alpha m}^{(J)m'\beta}$ satisfies Yang and Baxter's equations with m and m' going from $J+1$ to $-J-1$, and it is consistent to restrict* (5.1) *to this interval.*

Proof. Equation (5.5) has the form of a "gauge transformation" of the R-matrix that preserves Yang and Baxter's equations as is well known. □

Next the ξ fields with negative J are defined by the

Definition (5.4). For any negative integer $2J < -2$, the fields $\xi_M^{(J)}$, $J+1 \leqq M \leqq -J-1$, are given by

$$\xi_M^{(J)}(\sigma) := \sum_{J+1 \leqq m \leqq -J-1} |J, \varpi)_M^m \psi_m^{(J)}(\sigma), \tag{5.7}$$

$$|J,\varpi)_M^m = (-1)^{J+m+1} \binom{-2(J+1)}{a-c}^{-1} \sqrt{\binom{-2J-2}{-J-1+M}} \binom{-J-1+M}{c-1}$$
$$\times e^{ih(m/2 + (\varpi+m)(J+1-c))} F(a,b;c;e^{-2ih(\varpi+m)}), \tag{5.8a}$$

$$\begin{aligned} &a = M-J, \quad b = -m-J, \quad c = 1+M-m, \quad \text{when } M > m, \\ &a = -M-J, \quad b = m-J, \quad c = 1-M+m, \quad \text{when } M < m. \end{aligned} \tag{5.8b}$$

This definition is motivated by the

Theorem (5.5). *Up to a factor that only depends on J, Eqs.* (5.8) *are the continuation of* (2.24–26) *to negative integer values of J.*

Proof. The hypergeometric functions are identical. Concerning the binomial coefficients in front of (2.24), write (say for $M > m$)

$$\sqrt{\binom{2J}{J+M}} \binom{J+M}{M-m} \equiv \sqrt{\frac{\Gamma(1+2J)\Gamma(1+J+M)}{\Gamma(1+J-M)}} \frac{1}{\lfloor M-m \rfloor! \, \Gamma(1+J+m)}$$
$$= \frac{S(-J-m)}{\sqrt{S(-2J)}} \sqrt{\frac{S(M-J)}{S(-M-J)}} \sqrt{\frac{\Gamma(M-J)}{\Gamma(-2J)\Gamma(-M-J)}} \frac{\Gamma(-J-m)}{\lfloor M-m \rfloor!}.$$

According to Eq. (2.22), $S(-J-m) = -S(-J-m-1)$ so that $S(-J-m) \propto (-1)^{J+m}$, and

$$S(M-J)/S(-M-J) = S(M+1-J)/S(-M-1-J)$$

is independent of M. Thus

$$\frac{S(-J-m)}{\sqrt{S(-2J)}} \sqrt{\frac{S(M-J)}{S(-M-J)}} = (-1)^{J+m} \alpha_J,$$

where α_J only depends on J. The binomial factor of (2.24) finally becomes

$$(-1)^{J+m}\alpha_J \begin{pmatrix} -2(J+1) \\ -J-1+m \end{pmatrix}^{-1} \sqrt{\begin{pmatrix} -2(J+1) \\ -J-1+M \end{pmatrix}} \begin{pmatrix} -J-1+M \\ M-m \end{pmatrix},$$

and this established the theorem. □

The $\xi_M^{(J)}$ and $\xi_M^{(-J-1)}$ operators are closely related. This is first apparent in the

Theorem (5.6). *One has*

$$|J, \varpi)_M^m = (2i\sin(h))^{1+2J} \lambda_m^{(J)}(\varpi) |-J-1, \varpi)_M^m. \tag{5.12}$$

Proof. An immediate consequence of the following identity due to Rodgers [10] (see Appendix B)

$$F(a,b;c;e^{-2ihu})$$
$$= (2i\sin h)^{c-a-b} e^{ihu(a+b-c)} \frac{\Gamma(u-(a+b-c-1)/2)}{\Gamma(u+(a+b-c+1)/2)} F(c-a, c-b; c; e^{-2ihu}), \tag{5.13}$$

together with Theorem (2.5). □

This symmetry between J and $-J-1$ is also present at the purely group theoretical level:

Proposition (5.7). *The explicit formulae for the universal R-matrices* (2.8, 9), *and* (2.11, 12), *and for the q-C.G. coefficients* (C.3) *are invariant under the change* $J \to -J-1$.

Proof. The explicit expressions of the R-matrices follow from computing the matrix elements of powers of $J_\pm$ obtaining products of terms of the type $\sqrt{\lfloor J\mp M\rfloor \lfloor J\pm M+1\rfloor}$ that are left invariant. Moreover, the q-C.G. coefficients are computed by solving equations (C.2) where the explicit dependence in J is only through similar factors and which is thus invariant. They are thus themselves unchanged if J is replaced by $-J-1$. □

With this proposition, together with Theorem (5.6), one sees that the only important change in going from J to $-J-1$ is the factor $\lambda_m^{(J)}$ in the transformation from the ψ fields to the ξ fields. This explains why the S-matrix of Eq. (5.1, 2) transforms as proven in Theorem (5.2). The above considerations put together lead to the

Conjecture (5.8). *For* $J>0$, *the fields* $\xi_M^{(-J-1)}$ *with* $-J<M<J$, *introduced by Definition* (5.4), *span a representation of spin J of* $SL(2)_q$, *and enjoy the same fusion and braiding properties as the fields* $\xi_M^{(J)}$.

Discussion. A mathematically rigourous proof is not available at present, although the following consistency arguments are very convincing. The basic point is that, since Eq. (2.1) are invariant by $J\to -J-1$, the operator-algebra of the ξ-fields has a consistent extension which is symmetric with respect to $J=-1/2$. Indeed, it follows from Theorem (5.5) that, for $J_1>0$, $J_2>0$, $-J_1\leqq M_1\leqq J_1$, and $-J_2\leqq M_2 \leqq J_2$, the exchange and braiding properties of $\xi_{M_1}^{(-J_1-1)}$ with $\xi_{M_2}^{(J_2)}$ are the continuation of those of $\xi^{(J_1)}_{M_1}$ with $\xi^{(J_2)}_{M_2}$ and are thus given by the quantum-group R matrix and C.G. coefficients. Then a consistent solution of the conformal algebra is obtained by taking the exchange and braiding of $\xi^{(-J_1-1)}_{M_1}$ with $\xi^{(-J_2-1)}_{M_2}$ to be also deduced by symmetry from the positive spin case. According to Proposition (5.7), this means that they are also given by quantum-group R matrix and C.G. coefficients. Due to (5.12), the coefficients $|-J-1)^m_M$, with $J>0$, $-J\leqq M\leqq J$, satisfy a pseudo-orthogonality relation similar to (2.39). After inverting the relation between ξ- and ψ-fields, one deduces that the $\psi^{(-J-1)}_m$ form a closed algebra. It is related with the algebra of the $\psi^{(J)}_m$-fields by transformations similar to (5.5). By construction one arrives at a consistent solution of the conformal bootstrap for any sign of J. The corresponding operator-algebra of two operators with negative spins is not yet mathematically established, but we shall assume that it holds in order to proceed. □

The symmetry between J and $-J-1$ motivates the

Definition (5.9). The set $\mathscr{A}^+_{\text{phys}}$ is made up with the operators

$$\chi^{(J)}_+(\sigma):=\sum_{M=-J}^{J}\kappa^{J-M}(-1)^{s(J-M)(J-M-1)/2}\,\xi^{(-J-1,J)}_{M,-M}(\sigma), \tag{5.14}$$

with $2J$ a positive integer. The associated set of physical weights is given by the

Proposition (5.10). *The set of operators $\chi^{(J)}_+(\sigma)$ has Virasoro weights*

$$\Delta^+(J):=\Delta_{\text{Kac}}(-J-1,J;C)=1+\frac{25-C}{6}J(J+1), \tag{5.15}$$

which are real and positive.

Proof. Simple computations. □

Finally we arrive at the

Theorem (5.11). Unitary Truncation Theorem for the $\chi^{(J)}_+$ Fields. *For $C=1+6(s+2)$, $s=0,\pm1$, and when it acts on $\mathscr{H}_{\text{phys}}$; the set $\mathscr{A}^+_{\text{phys}}$ of operators $\chi^{(J)}_+$ is closed by fusion and braiding, and only gives states that belong to $\mathscr{H}_{\text{phys}}$.*

Proof. Thanks to Theorems (5.7, 8), closure by fusion and braiding is verified by computations identical to the ones performed for the $\mathscr{A}^-_{\text{phys}}$ family in Sect. 4. (Theorems (4.2, 3)). Moreover, Theorem (5.6) shows that the proof of Theorem (4.4) may be repeated for the $\chi^{(J)}_+$'s obtaining a formula similar to (4.12). Up to the factor $\lambda^{(J)}_m(\varpi)$, they are thereby expressed as a linear combination of the operators $\psi^{(-J-1,J)}_{m,-m}$, similar to the expression of $\chi^{(J)}_-$, in terms of $\psi^{(J,J)}_{m,-m}$. Thus $\mathscr{A}^+_{\text{phys}}$ leaves $\mathscr{H}_{\text{phys}}$ invariant. □

Finally, one has the complete unitary truncation theorem:

Theorem (5.12). Truncation Theorem for the Fields with Real Virasoro-Weights. *For $C=1+6(s+2)$, $s=0$, ±1, and when it acts on $\mathscr{H}_{\text{phys}}$; the set $\mathscr{A}_{\text{phys}}:=\mathscr{A}^-_{\text{phys}}\cup\mathscr{A}^+_{\text{phys}}$ of operators $\chi^{(J)}_\pm$ is closed by fusion and braiding, and only gives states that belong to $\mathscr{H}_{\text{phys}}$.*

Proof. There only remains to check closure under fusion and braiding. The previous argument based on the symmetry between J and $-J-1$ immediately leads to the conclusion. □

Concluding Remarks

Let us turn, at last, to physics. First, the present discussion only considered each chiral component separately. It is straightforward to combine the two chiralities in order to achieve modular invariance [2]. Second, taking D free fields as worldsheet matter [5, 6], one sees that one may construct consistent string emission vertices if $D=26-C_{\mathrm{grav}}=19$, 13, 7. The mass squared of the emitted string-ground-state is $m^2=2(\Delta-1)$, where Δ is the conformal weight of the 2D-gravity-dressing operator [5]. Since an infinite number of tachyons is unacceptable, this selects the $\mathscr{A}^+_{\mathrm{phys}}$ family with positive weights Δ^+. Bilal and I have already unravelled striking properties of the associated Liouville strings [17] at space-time dimensions 7 and 13, as well as, of the Liouville superstrings at 3 and 5 dimensions. Remarkably, one finds that the target-spaces which are selected by the consistency of the Liouville dynamics, as we just discussed, have very special properties that do ensure the consistency of the string dynamics. In particular, the above-mentioned Liouville superstring theories are space-time supersymmetric. Another striking point is that no tachyon remains, even for the bosonic Liouville-strings. Third, clearly, $\mathscr{A}^+_{\mathrm{phys}}$ is also selected if we consider the associated conformal theories by themselves, in order to avoid correlation functions that grow at very large distance. In this connection, and although bona fide target-spaces are suitable, one may "play the game" of fractal gravity, since Eqs. (4.6), (5.15) show that $\Delta^-(J,C)+\Delta^+(J,26-C)=1$, and since the set of values 7, 13, 19 is left invariant by $C\to 26-C$. Comparing with the string case, one sees that the $\mathscr{A}^-_{\mathrm{phys}}$ describes the matter. The calculation of the associated critical exponents is in progress. It is a challenge to derive these models from the matrix-approach to 2D gravity.

From the technical side, the next step is the determination of the Green functions. There is no problem for $\mathscr{A}^-_{\mathrm{phys}}$ since the Coulomb gas representation is applicable [18]. For $\mathscr{A}^+_{\mathrm{phys}}$, on the contrary, negative spins appear and the situation is more involved. It is likely that the symmetry $J\to -J-1$ put forward in Sect. 5 will be of help.

Appendix A

First recall some basic points about the weak coupling regime following [1–3, 15]. In the conformal gauge, the classical dynamics is governed by the action:

$$S=\frac{1}{2\pi\gamma}\int d\sigma d\tau\left(\frac{1}{2}\left(\frac{\partial\Phi}{\partial\sigma}\right)^2+\frac{1}{2}\left(\frac{\partial\Phi}{\partial\tau}\right)^2+e^{2\Phi}\right). \tag{A.1}$$

Classically, it is a conformal theory such that $\exp(2\Phi)$ is conformal with weights (1, 1). The canonical Poisson brackets (P.B.) give one P.B. realization of the Virasoro algebra with $C_{\mathrm{Liou}}=3/\gamma$ for each chiral component. The chiral modes may be separated very simply using the

Theorem (A.1). *The function $\Phi(\sigma,\tau)$ satisfies the equation*

$$\frac{\partial^2\Phi}{\partial\sigma^2}+\frac{\partial^2\Phi}{\partial\tau^2}=2e^{2\Phi}, \tag{A.2}$$

if and only if

$$e^{-\Phi}=\frac{i}{\sqrt{2}}\sum_{j=1,2}f_j(x_+)g_j(x_-),\qquad x_\pm=\sigma\mp i\tau, \tag{A.3}$$

where f_j (respectively g_j), which are functions of a single variable, are solutions of the same Schrödinger equation

$$-f_j''+T(x_+)f_j=0, \quad (\text{respectively } -g_j''+\bar{T}(x_-)g_j=0). \tag{A.4}$$

The solutions are normalized such that their Wronskians $f_1'f_2-f_1f_2'$ and $g_1'g_2-g_1g_2'$ are equal to one.

Proof. 1) First, check that (A.3) is indeed a solution. Taking the Laplacian of the logarithm of the right-hand side gives

$$\frac{\partial^2\Phi}{\partial\sigma^2}+\frac{\partial^2\Phi}{\partial\tau^2}\equiv 4\partial_+\partial_-\Phi=-4\Big/\Big(\sum_{i=1,2}f_ig_i\Big)^2,$$

where $\partial_\pm=(\partial/\partial\sigma\pm i\partial/\partial\tau)/2$. The numerator has been simplified by means of the Wronskian condition. This is equivalent to (A.2).

2) Next, check that any solution of (A.2) may be put under the form (A.3). If (A.2) holds, one deduces

$$\partial_\mp T^{(\pm)}=0; \quad \text{with} \quad T^{(\pm)}:=e^{\Phi}\partial_\mp^2 e^{-\Phi}. \tag{A.5}$$

$T^{(\pm)}$ are thus functions of a single variable. The equation involving $T^{(+)}$ may be rewritten as

$$(-\partial_+^2+T^{(+)})e^{-\Phi}=0,$$

with solution

$$e^{-\Phi}=\frac{i}{\sqrt{2}}\sum_{j=1,2}f_j(x_+)g_j(x_-); \quad \text{with} \quad -f_j''+T^{(+)}f_j=0,$$

where the g_j are arbitrary functions of x_-. Using Eq. (A.5) that involves $T^{(-)}$, one finally derives the Schrödinger equation $-g_j''+T^{(-)}g_j=0$. Thus the theorem holds with $T=T^{(+)}$ and $\bar{T}=T^{(-)}$. □

Equation (A.4) shows that the potentials of the two Schrödinger equations coincide with the two chiral components of the stress-energy tensor. *Thus these equations are the classical equivalent of the Ward identities that ensure the decoupling of Virasoro-null-vectors.* From the canonical Poisson brackets (P.B.) one finds two P.B. realizations of the Virasoro algebra such that the f_i (respectively g_i) are primary fields with weights $(-1/2,0)$ (respectively $(0,-1/2)$). At the classical level it is trivial to compute powers of $e^{-\Phi}$:

$$e^{-N\Phi}=\left(\frac{i}{2}\right)^N\sum_{p=0}^{N}\frac{N!}{p!(N-p)!}(f_1g_1)^p(f_2g_2)^{N-p} \tag{A.6}$$

which is primary with weight $(-N/2,-N/2)$. $e^{-N\Phi}$ is thus built up from powers of the solutions of the basic fields f_j and g_j. For positive N one has a finite number of terms but the weights are negative. Operators with positive weights have N negative so that (A.6) involves an infinite number of terms. Setting $N=-2$ gives weights $(1,1)$ in agreement with the fact that the potential term of (A.1) is equal to $e^{2\Phi}$ which must be a marginal operator. It is natural that f_j^{-2} has weight one since it is the classical equivalent of the screening operators.

Consider for instance the + chiral component. One may work at $\tau=0$ without loss of generality. The potential $T(\sigma)$ is periodic with period, say, 2π and we are working on the unit circle. Any two independent solutions of the Schrödinger equation are suitable. It seems natural at first sight to diagonalize the monodromy matrix, that is to choose two solutions noted $\psi_j, j=1,2$, that are periodic up to a constant[5]. It is convenient to introduce

$$\phi_j(\sigma) := \ln(\psi_j)/\sqrt{\gamma} - \ln d_j,$$

d_j are suitable normalization constants. The fields ϕ_j are periodic up to additive constants and have the expansion

$$\phi_j(\sigma) = q_0^{(j)} + p_0^{(j)}\sigma + i \sum_{n \neq 0} e^{-in\sigma} p_n^{(j)}/n, \quad j=1,2. \tag{A.7}$$

The canonical P.B. structure of the action (A.1) leads to the

Theorem (A.2). *The chiral fields ϕ_j, are such that*

$$\{\phi_1'(\sigma_1), \phi_1'(\sigma_2)\}_{\text{P.B.}} = \{\phi_2'(\sigma_1), \phi_2'(\sigma_2)\}_{\text{P.B.}} = 2\pi\delta'(\sigma_1 - \sigma_2), \tag{A.8}$$

$$\{q_0^{(j)}, p_0^{(j)}\}_{\text{P.B.}} = 1, \tag{A.9}$$

$$T/\gamma = (\phi_1')^2 + \phi_1''/\sqrt{\gamma} = (\phi_2')^2 + \phi_2''/\sqrt{\gamma}, \tag{A.10}$$

$$p_0^{(1)} = -p_0^{(2)}. \tag{A.11}$$

Proof. Equation (A.10) is trivial to derive from the Schrödinger Eq. (A.4). It is the associated Riccati equation. Equation (A.11) follows from the fact that the product of the two eigenvalues of the monodromy matrix is equal to one, as a standard Wronskian argument shows. For the P.B. relations see [4]. □

In the language of field theory, the ϕ's are two equivalent free fields such that (A.10) takes the form of a U_1-Sugawara stress-tensor with a linear term. The latter is responsible for the classical Virasoro central charge $C_{\text{Liou}} = 3/\gamma$. Clearly the two free fields play a symmetric role and one could as well build $e^{-\Phi}$ from different sets of Schrödinger solutions. Such a possibility is at the origin of the quantum group action as [1–3] show.

Let us now come to the quantum case. The basic point of the method is to quantize the above classical structure in such a way that the conformal structure is maintained. In particular, the quantum version of $e^{-\Phi}$ must be a primary field. This is ensured by the following

Theorem (A.3). *On the unit circle, $z=e^{i\sigma}$, and for generic γ, there exist two equivalent free fields:*

$$\phi_j(\sigma) = q_0^{(j)} + p_0^{(j)}\sigma + i \sum_{n \neq 0} e^{-in\sigma} p_n^{(j)}/n, \quad j=1,2, \tag{A.12}$$

such that

$$[\phi_1'(\sigma_1), \phi_1'(\sigma_2)] = [\phi_2'(\sigma_1), \phi_2'(\sigma_2)] = 2\pi i\delta'(\sigma_1 - \sigma_2), \quad p_0^{(1)} = -p_0^{(2)}, \tag{A.13}$$

$$N^{(1)}(\phi_1')^2 + \phi_1''/\sqrt{\gamma} = N^{(2)}(\phi_2')^2 + \phi_2''/\sqrt{\gamma}. \tag{A.14}$$

$N^{(1)}$ (respectively $N^{(2)}$) denote the normal orderings with respect to the modes of ϕ_1 (respectively of ϕ_2). γ is an arbitrary coupling constant.

Proof. See [4]. □

Equation (A.14) defines the quantum Virasoro-generators. The corresponding central charge is $C = 1 + 3/\gamma$. It is noted C instead of C_{Liou} since, clearly, the

[5] We only deal with the generic case where the monodromy matrix is diagonalizable

structure is intrinsically defined. The chiral family is built up [4] from the following four solutions of an operator Schrödinger equation equivalent to the decoupling of Virasoro null vectors

$$\psi_j = d_j N^{(j)}(e^{\sqrt{h/2\pi}\phi_j}), \quad \hat{\psi}_j = \hat{d}_j N^{(j)}(e^{\sqrt{\hat{h}/2\pi}\phi_j}), \quad j=1,2, \tag{A.15}$$

$$h = \frac{\pi}{12}(C-13-\sqrt{(C-25)(C-1)}), \quad \hat{h} = \frac{\pi}{12}(C-13+\sqrt{(C-25)(C-1)}), \tag{A.16}$$

where d_j and $\hat{d}_j$ are normalization constants. Since there are two possible quantum modifications h and $\hat{h}$, there are four solutions. By operator-product, ψ_j, $j=1,2$, and $\hat{\psi}_j$, $j=1,2$, generate two infinite families of chiral fields which are denoted $\psi_m^{(J)}$, $-J \leqq m \leqq J$, and $\hat{\psi}_{\hat{m}}^{(\hat{J})}$, $-\hat{J} \leqq \hat{m} \leqq \hat{J}$; respectively, with $\psi_{-1/2}^{(1/2)} = \psi_1$, $\psi_{1/2}^{(1/2)} = \psi_2$, and $\hat{\psi}_{-1/2}^{(1/2)} = \hat{\psi}_1$, $\hat{\psi}_{1/2}^{(1/2)} = \hat{\psi}_2$. An easy computation shows that

$$\sqrt{\frac{h}{2\pi}} = \frac{1}{4}\left(\sqrt{\frac{C-1}{3}} - \sqrt{\frac{C-25}{3}}\right), \quad \sqrt{\frac{\hat{h}}{2\pi}} = \frac{1}{4}\left(\sqrt{\frac{C-1}{3}} + \sqrt{\frac{C-25}{3}}\right). \tag{A.17}$$

$\psi_m^{(J)}$, $\hat{\psi}_{\hat{m}}^{(\hat{J})}$, are of the type $(1, 2J)$ and $(2\hat{J}, 1)$, respectively, in the BPZ classification. For the zero-modes, it is simpler [1–3] to define the rescaled variables

$$\varpi = i p_0^{(1)} \sqrt{\frac{2\pi}{h}}; \quad \hat{\varpi} = i p_0^{(1)} \sqrt{\frac{2\pi}{\hat{h}}}; \quad \hat{\varpi} = \varpi \frac{h}{\pi}; \quad \varpi = \hat{\varpi} \frac{\hat{h}}{\pi}. \tag{A.18}$$

From now on $p_0^{(1)}$ is simply denoted by p_0. At this point a pedagogical parenthesis may be in order: the hatted and unhatted ψ fields have the same chirality. If we go to $\tau \neq 0$ they are both functions of x_+. There are two counterparts $\bar{\psi}_m^{(J)}(x_-)$ and $\bar{\hat{\psi}}_m^{(J)}(x_-)$ with opposite chirality, which may be discussed in exactly the same way. Returning to our main line we recall that the Hilbert space in which the operators ψ and $\hat{\psi}$ live, is a direct sum [1–3, 15] of Fock spaces $\mathscr{F}(\varpi)$ spanned by the harmonic excitations of highest-weight Virasoro states noted $|\varpi, 0\rangle$. They are eigenstates of the quasi momentum ϖ, and satisfy $L_n|\varpi, 0\rangle = 0, n>0$; $(L_0 - \Delta(\varpi))|\varpi, 0\rangle = 0$. The corresponding highest weights $\Delta(\varpi)$ may be rewritten as

$$\Delta(\varpi) \equiv \frac{1}{8\gamma} + \frac{(p_0^{(1)})^2}{2} = \frac{h}{4\pi}\left(1 + \frac{\pi}{h}\right)^2 - \frac{h}{4\pi}\varpi^2. \tag{A.19}$$

The commutation relations (A.13) are to be supplemented by the zero-mode ones:

$$[q_0^{(1)}, p_0^{(1)}] = [q_0^{(2)}, p_0^{(2)}] = i.$$

The fields ψ and $\hat{\psi}$ shift the quasi momentum $p_0^{(1)} = -p_0^{(2)}$ by a fixed amount. For an arbitrary c-number function f one has

$$\psi_m^{(J)} f(\varpi) = f(\varpi + 2m)\psi_m^{(J)}, \quad \hat{\psi}_{\hat{m}}^{(\hat{J})} f(\varpi) = f(\varpi + 2\hat{m}\pi/h)\hat{\psi}_{\hat{m}}^{(\hat{J})}. \tag{A.20}$$

The fields ψ and $\hat{\psi}$ together with their products live in Hilbert spaces[6] of the form

$$\mathscr{H}(\varpi_0) \equiv \bigoplus_{n,\hat{n}=-\infty}^{+\infty} \mathscr{F}(\varpi_0 + n + \hat{n}\pi/h). \tag{A.21}$$

ϖ^0 is a constant which is arbitrary so far. The $SL(2, C)$-invariant vacuum corresponds to $\varpi_0 = 1 + \pi/h$ [1], but this choice is not appropriate for our purpose.

6 Mathematically they are not really Hilbert spaces since their metrics are not positive definite

At the quantum level, one makes use of the above chiral conformal family, since the quantum field equation is likely to imply that the quantum Schrödinger equation holds for each chiral component. Associated with each quantum modification one finds a quantum version of (A.3). Since (A.15) involves h and $\hat{h}$ instead of γ, these should be considered as defining the quantum operators $\exp(-\eta\Phi)$ and $\exp(-\hat{\eta}\Phi)$ with $\eta=\sqrt{h/(2\pi\gamma)}$ and $\hat{\eta}=\sqrt{\hat{h}/(2\pi\gamma)}$. Indeed, they have the same conformal weights as $\psi_\alpha^{(1/2)}$ and $\hat{\psi}_\alpha^{(1/2)}$, respectively, for each chiral component. By short-distance operator-product expansion, one generates the fields $\exp[-(2J\eta+2\hat{J}\hat{\eta})\Phi]$. The basic point of introducing the two normal orderings $N^{(j)}$, $j=1,2$, was to obtain a conformal regularization of the metric tensor operators such as $e^{-\eta\Phi}$ and $e^{-\hat{\eta}\Phi}$. In terms of the Liouville field Φ, it is rather involved and field-dependent. Which η should one choose? For γ going to zero, η has a finite limit while $\hat{\eta}$ blows up. Thus if one wants to keep a smooth classical limit, only η should appear. This is possible with open boundary condition [5]. With closed boundary conditions, both η's should be kept in order to couple rational theories with gravity [15].

In any case, the quantum modifications are real only if $C>25$ or $C<1$. Thus the construction of the metric tensor operator just recalled fails for $1<C<25$, which is the case considered here. The chiral families may be continued, however, and this is taken to be the way to deal with 2D gravity in the strong coupling regime, if a consistent truncation may be found as we show in the main body of the paper fully generalising the partial results of [2, 6].

Appendix B

The purpose of the present section is to make connection between the conventions used in the present series of articles, and the notations of the mathematical literature concerning q-deformed special functions. The basic difference is that, in quantum group discussions, one makes use of q-symbols of the type (2.2), that is $\lfloor a \rfloor=\sin(ha)/\sin h$, that are symmetric in $h\to -h$ while q-deformed special functions are formulated [10, 12, 13] in terms of q-factors of the type

$$(y;\varrho)_\nu := \prod_{\tau=0}^{\nu-1}(1-y\varrho^\tau). \tag{B.1}$$

The quantum parameter is temporarily noted ϱ since it does not coincide with the parameter q. Indeed an easy computation shows that

$$\lfloor a \rfloor_\nu \equiv \frac{\Gamma(a+\nu)}{\Gamma(a)} = (2i\sin h)^{-\nu} e^{ih[\nu(\nu-1)/2+\nu a]}(e^{-2iha};e^{-2ih})_\nu, \tag{B.2}$$

so that for unhatted quantities $\varrho=e^{-2ih}$. This choice of sign, which is the same as in Eq. (2.17), is such that the infinite product $(y;\varrho)_\infty$ is convergent for $\operatorname{Im} h<0$ which is the case of interest (see (3.2)).

Concerning q-hypergeometric functions, it is convenient to write the standard definition [10, 12, 13] under the form

$$ {}_s\varphi_r\begin{pmatrix} e^{-2iha_1},\ldots,e^{-2iha_s} \\ e^{-2ihc_1},\ldots,e^{-2ihc_r} \end{pmatrix}; e^{-2ih}; z\Bigg): $$

$$= \sum_{\nu=0}^{\infty} \frac{(e^{-2iha_1};e^{-2ih})_\nu \ldots (e^{-2iha_s};e^{-2ih})_\nu}{(e^{-2ihc_1};e^{-2ih})_\nu \ldots (e^{-2ihc_r};e^{-2ih})_\nu (e^{-2ih};e^{-2ih})_\nu} z^\nu . \tag{B.3}$$

We shall only need hypergeometric functions with $s=r+1$. The generalization of (2.23) is

$$ {}_{r+1}F_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ c_1,\dots,c_r \end{matrix};z\right) \equiv \sum_\nu \frac{\lfloor a_1 \rfloor_\nu \dots \lfloor a_{r+1}\rfloor_\nu}{\lfloor c_1\rfloor_\nu \dots \lfloor c_r \rfloor_\nu \lfloor \nu \rfloor !} z^\nu . \tag{B.4} $$

Its relation with (B.3) is

$$ {}_{r+1}F_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ c_1,\dots,c_r \end{matrix};z\right) = {}_{r+1}\varphi_r\left(\begin{matrix} e^{-2iha_1},\dots,e^{-2iha_{r+1}} \\ e^{-2ihc_1},\dots,e^{-2ihc_r} \end{matrix}; e^{-2ih}; ze^{ih(\Sigma a_i - \Sigma c_i - 1)}\right), \tag{B.5} $$

and, in particular,

$$ F(a,b;c;z) \equiv {}_2\varphi_1\left(\begin{matrix} e^{-2iha}, e^{-2ihb} \\ e^{-2ihc} \end{matrix}; e^{-2ih}; ze^{ih(a+b-c-1)}\right). \tag{B.6} $$

The definition of ${}_{r+1}F_r$ is symmetric in h contrary to ${}_{r+1}\varphi_r$, where the quantum parameter is identified with e^{-2ih}. This choice is made on account of the negative sign of the imaginary part of h (see (3.2)) which ensures convergence if needed. In most cases, we are dealing with finite series so that this point is not essential, however, and the other sign could be used. This will be the case in Appendix C.

The continuation to negative J is based on an identity due to Rodgers [10],

$$ \begin{aligned} &{}_2\varphi_1\left(\begin{matrix} e^{-2iha}, e^{-2ihb} \\ e^{-2ihc} \end{matrix}; e^{-2ih}; x\right) \\ &= \frac{(e^{-2ih(a+b-c)}x; e^{-2ih})_\infty}{(x; e^{-2ih})_\infty} \, {}_2\varphi_1\left(\begin{matrix} e^{-2ih(c-a)}, e^{-2ih(c-b)} \\ e^{-2ihc} \end{matrix}; e^{-2ih}; xe^{-2ih(a+b-c)}\right), \end{aligned} \tag{B.8} $$

which is the deformation of a standard identity on hypergeometric functions. This may be retransformed using (B.2) and (B.6) and the relation

$$ \Gamma(a) = e^{iha(a-1)/2}(2i\sin h)^{1-a}(e^{-2ih}; e^{-2ih})_\infty/(e^{-2iha}; e^{-2ih})_\infty , \tag{B.9} $$

obtaining

$$ \begin{aligned} F(a,b;c;e^{-2ihu}) = (2i\sin h)^{c-a-b} e^{ihu(a+b-c)} \frac{\Gamma(u-(a+b-c-1)/2)}{\Gamma(u+(a+b-c+1)/2)} \\ \times F(c-a, c-b; c; e^{-2ihu}) . \end{aligned} \tag{B.10} $$

Similarly, the hatted hypergeometric functions are defined by

$$ {}_{r+1}\hat{F}_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix};z\right) \equiv \sum_\nu \frac{\hat{\lfloor} a_1 \hat{\rfloor}_\nu \dots \hat{\lfloor} a_{r+1} \hat{\rfloor}_\nu}{\hat{\lfloor} b_1 \hat{\rfloor}_\nu \dots \hat{\lfloor} b_r \hat{\rfloor}_\nu \hat{\lfloor} \nu \hat{\rfloor}!} z^\nu , \tag{B.11} $$

and their relation with the standard q-hypergeometric functions is best written as

$$ {}_{r+1}\hat{F}_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ c_1,\dots,c_r \end{matrix};z\right) = {}_{r+1}\varphi_r\left(\begin{matrix} e^{2i\hat{h}a_1},\dots,e^{2i\hat{h}a_{r+1}} \\ e^{2i\hat{h}c_1},\dots,e^{2i\hat{h}c_r} \end{matrix}; e^{2i\hat{h}}; ze^{-i\hat{h}(\Sigma a_i - \Sigma c_i - 1)}\right). \tag{B.12} $$

The change of sign with respect to (B.5) is for convergence purposes, since the imaginary part of $\hat{h}$ is positive.

For $h+\hat{h}=s\pi$, one has

$$ \hat{\lfloor} a \hat{\rfloor}_\nu = \lfloor a \rfloor_\nu e^{-is\pi\nu(\nu+2a-3)/2} . \tag{B.13} $$

Thus

$$ {}_{r+1}\hat{F}_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix};z\right) \equiv {}_{r+1}F_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix}; ze^{-is\pi[\Sigma a_i - \Sigma b_i - 1]}\right). \tag{B.14} $$

Appendix C

In this section we recall for completeness some basic material concerning q-Clebsch Gordan (q-C.G.) coefficients. We shall make frequent use of the following identity

$$\sum_{Q_1+Q_2=Q, Q_1>0, Q_2>0} \binom{P_1}{Q_1}\binom{P_2}{Q_2} e^{\pm ih(P_1Q_2-P_2Q_1)} = \binom{P_1+P_2}{Q_1+Q_2}, \tag{C.1}$$

which is straightforwardly proven by recursion.

The expression of the q-C.G. coefficients $(J_1, M_1; J_2, M_2|J_1, J_2; J, M)$ follows from the

Theorem (C.1). *The solution of the recurrence relation*

$$\begin{aligned}&\sqrt{\lfloor J\mp M_1\mp M_2\rfloor\lfloor J\pm M_1\pm M_2+1\rfloor}(J_1, M_1; J_2, M_2|J_1, J_2; J, M_1+M_2)\\ &\quad= e^{ihM_2}\sqrt{\lfloor J_1\mp M_1\rfloor\lfloor J\pm M_1+1\rfloor}(J_1, M_1\pm 1; J_2, M_2|J_1, J_2; J, M_1+M_2\pm 1)\\ &\qquad+e^{-ihM_1}\sqrt{\lfloor J_2\mp M_2\rfloor\lfloor J_2\pm M_2+1\rfloor}(J_1, M_1; J_2, M_2\pm 1|J_1, J_2; J, M_1+M_2\pm 1),\end{aligned} \tag{C.2}$$

is given by

$$\begin{aligned}&(J_1, M_1; J_2, M_2|J_1, J_2; J, M) = \delta_{M, M_1+M_2} e^{ih(J_1+J_2-J)(J_1+J_2+J+1)/2}\\ &\quad\times\sqrt{\lfloor 2J+1\rfloor}\sqrt{\frac{\lfloor J_1+J_2-J\rfloor!\lfloor -J_1+J_2+J\rfloor!\lfloor J_1-J_2+J\rfloor!}{\lfloor J_1+J_2+J+1\rfloor!}}\\ &\times\sqrt{\lfloor J_1-M_1\rfloor!\lfloor J_1+M_1\rfloor!\lfloor J_2-M_2\rfloor!\lfloor J_2+M_2\rfloor!\lfloor J-M_1-M_2\rfloor!\lfloor J+M_1+M_2\rfloor!}\\ &\quad\times e^{ih(M_2J_1-M_1J_2)}\sum_{\mu=0}^{J_1+J_2-J}\left\{\frac{e^{-ih\mu(J+J_1+J_2+1)}(-1)^\mu}{\lfloor\mu\rfloor!\lfloor J_1+J_2-J-\mu\rfloor!}\right.\\ &\quad\times\left.\frac{1}{\lfloor J_1-M_1-\mu\rfloor!\lfloor J-J_2+M_1+\mu\rfloor!\lfloor J_2+M_2-\mu\rfloor!\lfloor J-J_1-M_2+\mu\rfloor!}\right\}.\end{aligned} \tag{C.3}$$

Proof. It goes in strict parallel with the standard case of $SU(2)$ [16]. One proceeds by recursion [14]. First let

$$f(M_1)\equiv(J_1, M_1; J_2, J-M_1|J_1, J_2; J, J),$$

and choose the upper sign in (C.2). Its left-hand side vanishes. This gives the recursion

$$f(M_1) = -f(M_1-1)e^{ih(J+1)}\sqrt{\frac{\lfloor J_2-M_1+J+1\rfloor\lfloor J_2+M_1-J-1\rfloor}{\lfloor J_1+M_1\rfloor\lfloor J_1-M_1+1\rfloor}},$$

with solution

$$\begin{aligned}f(M_1) &= f(J_1)e^{ih(J+1)(M_1-J_1)}(-1)^{J_1-M_1}\\ &\quad\times\sqrt{\frac{\lfloor J_2+J-M_1\rfloor!\lfloor J_1+J_2-J\rfloor!\lfloor J_1+M_1\rfloor!}{\lfloor 2J_1\rfloor!\lfloor -J_1+J_2+J\rfloor!\lfloor J_2-J+M_1\rfloor!\lfloor J_1-M_1\rfloor!}}.\end{aligned}$$

Equation (C.1) leads to the relation

$$\sum_{M_1} \frac{\lfloor J_1+M_1\rfloor!\lfloor J_2+J-M_1\rfloor!}{\lfloor J_1-M_1\rfloor!\lfloor J_2-J+M_1\rfloor!} e^{ih(J_1+J_2-J)(J_2-J_1+J+1)+2ih(J+1)(M_1-J_1)}$$
$$=\frac{\lfloor J_1+J_2+J+1\rfloor!\lfloor J_1-J_2+J\rfloor!\lfloor -J_1+J_2+J\rfloor!}{\lfloor J_1+J_2-J\rfloor!\lfloor 2J+1\rfloor!}.$$

So that we may impose the condition $\sum_{M_1} (f(M_1))^2=1$. This gives

$$f(J_1)=e^{ih(J_1+J_2-J)(J_2-J_1+J+1)/2}\sqrt{\frac{\lfloor 2J_1\rfloor!\lfloor 2J+1\rfloor!}{\lfloor J_1+J_2+J+1\rfloor!\lfloor J_1-J_2+J\rfloor!}},$$

and the partial result

$$(J_1,M_1;J_2,J-M_1|J_1,J_2;J,J)$$
$$=e^{ih[(J+1)(M_1-J_1)+(J_1+J_2-J)(J_2-J_1+J+1)/2]}(-1)^{J_1-M_1}$$
$$\times\sqrt{\frac{\lfloor 2J+1\rfloor!\lfloor J_1+J_2-J\rfloor!\lfloor J_2+J-M_1\rfloor!\lfloor J_1+M_1\rfloor!}{\lfloor J_1+J_2+J+1\rfloor!\lfloor J_1-J_2+J\rfloor!\lfloor -J_1+J_2+J\rfloor!\lfloor J_2-J+M_1\rfloor!\lfloor J_1-M_1\rfloor!}}.$$

Next define

$$g(M_1,M_2):=(J_1,M_1;J_2,M_2|J_1,J_2;J,M_1+M_2)$$
$$\times\sqrt{\frac{\lfloor J_1+M_1\rfloor!\lfloor J_2+M_2\rfloor!\lfloor J-M\rfloor!}{\lfloor J_1-M_1\rfloor!\lfloor J_2-M_2\rfloor!\lfloor J+M\rfloor!}}.$$

Equation (C.2) with the plus sign leads to

$$g(M_1,M_2)=e^{ihM_2}g(M_1+1,M_2)+e^{-ihM_1}g(M_1,M_2+1),$$

and, after iteration,

$$g(M_1,M_2)=\sum_{r=0}^{n} e^{ih[M_2r-M_1(n-r)]}\binom{n}{r}g(M_1+r,M_2+n-r).$$

If we choose $n=J-M_1-M_2$ the summation involves

$$g(M_1+r,J-M_1-r)=f(M_1+r)$$
$$\times\sqrt{\frac{\lfloor J_1+M_1+r\rfloor!\lfloor J_2+J-M_1-r\rfloor!\lfloor 0\rfloor!}{\lfloor J_1-M_1-r\rfloor!\lfloor J_2-J+M_1+r\rfloor!\lfloor 2J\rfloor!}}.$$

Since by definition $\lfloor N+1\rfloor!=\lfloor N+1\rfloor\,\lfloor N\rfloor!$ one has $\lfloor 1\rfloor!=\lfloor 0\rfloor!$ and, by consistency, $\lfloor 0\rfloor!=1$. Collecting everything, one obtains a first expression

$$(J_1,M_1;J_2,M_2|J_1,J_2;J,M)=\delta_{M,M_1+M_2}e^{ih[J_2(J_2+1)-J_1(J_1+1)-J(J+1)]/2}$$
$$\times\sqrt{\lfloor 2J+1\rfloor}\sqrt{\frac{\lfloor J_1+J_2-J\rfloor!}{\lfloor -J_1+J_2+J\rfloor!\lfloor J_1-J_2+J\rfloor!\lfloor J_1+J_2+J+1\rfloor!}}$$
$$\times\sqrt{\frac{\lfloor J_1-M_1\rfloor!\lfloor J_2-M_2\rfloor!\lfloor J-M\rfloor!\lfloor J+M\rfloor!}{\lfloor J_1+M_1\rfloor!\lfloor J_2+M_2\rfloor!}}$$
$$\times(-1)^{J_1-M_1}e^{ihM_1(M+1)}\sum_{\nu=0}^{J-M}\left\{\frac{e^{ih\nu(M+J+1)}(-1)^\nu}{\lfloor \nu\rfloor!\lfloor J-M-\nu\rfloor!}\right.$$
$$\left.\times\frac{\lfloor J_1+M_1+\nu\rfloor!\lfloor J_2+J-M_1-\nu\rfloor!}{\lfloor J_1-M_1-\nu\rfloor!\lfloor J_2-J+M_1+\nu\rfloor!}\right\}. \quad \text{(C.4)}$$

Next we transform the result into a more symmetric expression. Apply (C.1) to the term

$$\frac{\lfloor J_2+J-M_1-\nu\rfloor!}{\lfloor J_1-M_1-\nu\rfloor!\lfloor J_2+J-J_1\rfloor!}\equiv\begin{pmatrix} J-M-\nu+J_2+M_2 \\ J_1-J_2-M-\nu+J_2+M_2\end{pmatrix}.$$

This gives

$$\frac{\lfloor J_2+J-M_1-\nu\rfloor!}{\lfloor J_1-M_1-\nu\rfloor!\lfloor J_2+J-J_1\rfloor!}=e^{-ih(J-J_1+J_2)(J_2+M_2)}$$
$$\times\sum_{\mu} e^{ih\mu(J+J_2-M_1-\nu)}\begin{pmatrix} J-M-\nu \\ J_1-J_2-M-\nu+\mu\end{pmatrix}\begin{pmatrix} J_2+M_2 \\ J_2+M_2-\mu\end{pmatrix}.$$

Substitute into Eq. (C.4) and sum over ν. One finally makes use of the identity

$$\sum_{\varrho}(-1)^{\varrho}\frac{\lfloor p+\varrho\rfloor!}{\lfloor \varrho\rfloor!\lfloor q+\varrho\rfloor!\lfloor r-\varrho\rfloor!}e^{ih\varrho(p-q-r+1)}=\frac{e^{ihr(p+1)}\lfloor p\rfloor!\lfloor p-q\rfloor!(-1)^r}{\lfloor r\rfloor!\lfloor q+r\rfloor!\lfloor p-q+r\rfloor!},$$

which is a consequence of Eq. (C.1), and the result follows. □

Next we establish the connection between q-C.G. coefficients and orthogonal polynomials. First we need the

Theorem (C.2). *The Hahn polynomials defined by*

$$\mathscr{P}_n(x;a,b,N):={}_3\varphi_2\begin{pmatrix} e^{-2ihn}, e^{2ih(a+b+n+1)}, e^{-2ihx} \\ e^{2ih(a+1)}, e^{-2ihN} \end{pmatrix}; e^{2ih}; e^{2ih}\Bigg), \tag{C.5}$$

or equivalently by

$$\mathscr{P}_n(x;a,b,N)\equiv{}_3F_2\left(\begin{matrix} -n, a+b+n+1, -x \\ a+1, -N\end{matrix}; e^{ih(b+N-x+1)}\right), \tag{C.6}$$

satisfy the orthogonality relations

$$\sum_{x=0,1,\dots,N}\mathscr{P}_n(x;a,b,N)\mathscr{P}_m(x;a,b,N)\varrho(x)=(d_n)^2\delta_{n,m}, \tag{C.7}$$

$$\varrho(x)=e^{-ih(a+b+2)x}\frac{\lfloor a+1\rfloor_x\lfloor N+1-x\rfloor_x}{\lfloor N+b+1\rfloor_x\lfloor x\rfloor!}, \tag{C.8}$$

$$d_n=e^{ih[n(a+b+n+1)-N(a+1)]/2}\sqrt{\frac{\lfloor n\rfloor!\lfloor N-n\rfloor!}{\lfloor N\rfloor!}}$$
$$\times\sqrt{\frac{\lfloor a+b+N+n+1\rfloor!\lfloor b+n\rfloor!\lfloor a\rfloor!}{\lfloor a+b+n\rfloor!\lfloor b+N\rfloor!\lfloor a+n\rfloor!\lfloor a+b+2n+1\rfloor}}. \tag{C.9}$$

Proof. See e.g. [13]. □

The connection is established by the

Theorem (C.3). *Equation* (*C*.3) *is equivalent to* ($M=M_1+M_2$),

$$(J_1,M_1;J_2,M_2|J_1,J_2;J,M)=(-1)^{J_1-M_1}\frac{\sqrt{\varrho(x)}}{d_n}\mathscr{P}_n(x;a,b,N),$$

$$\begin{aligned} x&:=J_1-M_1, \quad n:=J-M, \quad N:=J_1+J_2-M \\ a&:=M+J_2-J_1, \quad b:=M+J_1-J_2. \end{aligned} \tag{C.10}$$

Proof. Using the obvious relations

$$\lfloor m+\nu\rfloor! = \lfloor m\rfloor!\lfloor m+1\rfloor_\nu; \qquad \lfloor m-\nu\rfloor! = (-1)^\nu \frac{\lfloor m\rfloor!}{\lfloor m-1\rfloor_\nu},$$

one first rewrites (C.3) as

$$\begin{aligned}(J_1, M_1; J_2, M_2|J_1, J_2; J, M) &= e^{ih[J_2(J_2+1)-J_1(J_1+1)-J(J+1)]/2} \\ &\times \sqrt{\lfloor 2J+1\rfloor}\sqrt{\frac{\lfloor J_1+J_2-J\rfloor!}{\lfloor -J_1+J_2+J\rfloor!\lfloor J_1-J_2+J\rfloor!\lfloor J_1+J_2+J+1\rfloor!}} \\ &\times \frac{\lfloor J_2+J-M_1\rfloor!}{\lfloor J_2-J-M_1\rfloor!}\sqrt{\frac{\lfloor J_1+M_1\rfloor!\lfloor J_2-M_2\rfloor!\lfloor J+M\rfloor!}{\lfloor J_1-M_1\rfloor!\lfloor J_2+M_2\rfloor!\lfloor J-M\rfloor!}}(-1)^{J_1-M_1} \\ &\times e^{ihM_1(M+1)}{}_3F_2\left(\begin{matrix}-J_1+M_1, J_1+M_1+1, M-J \\ J_2-J+M_1+1, M_1-J-J_2\end{matrix}; e^{ih(M+J+1)}\right). \end{aligned} \tag{C.11}$$

Next the desired hypergeometric function is obtained by making use of the general relation[7] $(s=\gamma+\delta-\alpha-\beta+n)$

$$\begin{aligned}{}_3\varphi_2\left(\begin{matrix}e^{2ih\alpha}, e^{2ih\beta}, e^{-2ihn} \\ e^{2ih\gamma}, e^{2ih\delta}\end{matrix}; e^{2ih}; e^{2ih}\right) &= e^{2ihn\alpha}\frac{(e^{2ih(\gamma-\alpha)}; e^{2ih})_n}{(e^{2ih\gamma}; e^{2ih})_n} \\ \times \frac{(e^{2ih(\delta-\alpha)}; e^{2ih})_n}{(e^{2ih\delta}; e^{2ih})_n}\, {}_3\varphi_2 &\left(\begin{matrix}e^{2ih\alpha}, e^{2ih(1-s)}, e^{-2ihn} \\ e^{2ih(1+\alpha-\gamma-n)}, e^{2ih(1+\alpha-\delta+n)}\end{matrix}; e^{2ih}; e^{2ih}\right),\end{aligned}$$

or equivalently

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}\alpha, \beta, -n \\ \gamma, \delta\end{matrix}; e^{-ih(s-1)}\right) \\ &= \frac{\lfloor\gamma-\alpha\rfloor_n\lfloor\delta-\alpha\rfloor_n}{\lfloor\gamma\rfloor_n\lfloor\delta\rfloor_n}\, {}_3F_2\left(\begin{matrix}\alpha, 1-s, -n \\ 1+\alpha-\gamma-n, 1+\alpha-\delta-n\end{matrix}; e^{ih\beta}\right).\end{aligned} \tag{C.12}$$

Choosing $n=J-M$, $\alpha=M_1-J_1$, $\beta=J_1+M_1+1$, $\gamma=J_2-J+M_1+1$, and $\delta=-J_2-J+M_1$ gives

$$\begin{aligned}{}_3F_2\left(\begin{matrix}M_1-J_1, J_1+M_1+1, M-J \\ J_2-J+M_1+1, M_1-J-J_2\end{matrix}; e^{ih(J+M+1)}\right) &= \frac{\lfloor J_1-J_2-J\rfloor_n}{\lfloor J_2-J+M_1+1\rfloor_n} \\ \times \frac{\lfloor J_1+J_2-J+1\rfloor_n}{\lfloor J_2-J+M_1\rfloor_n}\, {}_3F_2\left(\begin{matrix}M_1-J_1, 1+J+M, -n \\ 1+M_1+M_2+J_2-J_1, M-J_1-J_2\end{matrix}; e^{ih(J_1+M_1+1)}\right).\end{aligned}$$

Finally substituting this last relation in Eq. (C.11) and comparing with the general expression (C.6) of the Hahn polynomials, one completes the derivation. □

The above discussion essentially follows [14], where the transformation (C.12) is not applied, however. This change of parameter in the Hahn polynomial is instrumental for deriving the orthogonality properties of the C.-C.G. coefficients which are expressed by the

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[7] The $h=0$ limit of this relation is in Ref. [12]

Corollary (C.4). *q-C.G. coefficients satisfy the relations*

$$\sum_{M_1} (J_1, M_1; J_2, M_2|J_1, J_2; J, M)(J_1, M_1; J_2, M_2|J_1, J_2; J', M) = \delta_{J,J'}. \quad \text{(C.13)}$$

Proof. Immediate consequence of the above two theorems. □

Appendix D

In this appendix we rederive some of the properties of the $\chi_-^{(J)}$ fields by only making use of the ψ fields, in order to show the full equivalence of the two formulations. The braiding properties follow from the

Theorem (D.1). *Acting on $\mathscr{F}(\varpi_{r,n})$ with $\varpi_{r,n}$ given by Eq. (3.21), the fields*

$$\phi_-^{(J)} := \sum_{m=-J}^{J} a_m^{(J)}(\varpi)\psi_{m,-m}^{(J,J)} \quad \text{(D.1)}$$

satisfy the braiding equation

$$\phi_-^{(1/2)}(\sigma)\phi_-^{(J)}(\sigma') = e^{-i\pi(2+s)J\varepsilon}\phi_-^{(J)}(\sigma')\phi_-^{(1/2)}(\sigma) \quad \text{(D.2)}$$

(ε is the sign of $\sigma - \sigma'$), if $a_m^{(J)}$ is given by

$$a_m^{(J)} = \binom{2J}{J-m}(-1)^{(r+1)(J-m)+2nJ} e^{i[-s\pi mn + (J-m)(\hat{h}-h)/2]} \frac{\lfloor \varpi_{r,n} - J + m \rfloor_{2J+1}}{\lfloor \varpi_{r,n+2m} \rfloor}. \quad \text{(D.3)}$$

Proof. It is a generalisation of the particular case worked out in [2, 6]. Recall some more results of [1]. The fields $\psi_m^{(J)}$ satisfy

$$\psi_\alpha^{(1/2)}(\sigma)\psi_m^{(J)}(\sigma') = \sum_{\beta=\pm 1/2;\, m'=-J\ldots J} S_{\alpha m}^{(J)m'\beta}\psi_{m'}^{(J)}(\sigma')\psi_\beta^{(1/2)}(\sigma), \quad \text{(D.4)}$$

where the non-vanishing $S_{\alpha m}^{(J)m'\beta}$ are

$$S_{-1/2\,m}^{(J)m,-1/2}(\varpi) = S_{1/2\,-m}^{(J)-m\,1/2}(-\varpi) = \frac{\lfloor \varpi + J + m \rfloor}{\lfloor \varpi \rfloor} e^{ihm\varepsilon},$$

$$S_{-1/2\,m}^{(J)m-1\,1/2}(\varpi) = \frac{\lfloor J + m \rfloor}{\lfloor \varpi \rfloor} e^{ih\varepsilon(1-m-\varpi)} = S_{1/2,-m}^{(J)-m+1,-1/2}(-\varpi), \quad \text{(D.5)}$$

with similar formulae for the hatted fields. The braiding of the ψ and $\hat{\psi}$ fields is trivial:

$$\psi_m^{(J)}(\sigma)\psi_{\hat{m}}^{(\hat{J})}(\sigma') = e^{-2i\pi J\hat{J}\varepsilon}\psi_{\hat{m}}^{(\hat{J})}(\sigma')\psi_m^{(J)}(\sigma). \quad \text{(D.6)}$$

The fusion to leading order is of the form

$$\psi_{m0}^{(J0)}\psi_{0\hat{m}}^{(0\hat{J})} \sim \psi_{0\hat{m}}^{(0\hat{J})}\psi_{m0}^{(J0)} \sim (-1)^{2(J\hat{J}-m\hat{m})}\psi_{m\hat{m}}^{(J\hat{J})}. \quad \text{(D.7)}$$

The beginning of the derivation is to start from the ansatz (D.1), assume that $a_{\pm 1/2}^{(1/2)}$ is given by (D.3), and make use the relations just recalled to derive that, in $\mathscr{F}(\varpi_{r,n})$,

$$\phi_-^{(1/2)}(\sigma)\phi_-^{(J)}(\sigma') = e^{-2i\pi J\varepsilon}\sum_{t=1}^{4} W_t, \quad \text{(D.8)}$$

where

$$W_1 \equiv \sum_m [a_{1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n+1}) \hat{S}_{-1/2\,-m}^{-m\,-1/2} S_{1/2\,m}^{m\,1/2}$$
$$+ a_{-1/2}^{(1/2)}(\varpi_n) a_{m+1}^{(J)}(\varpi_{n-1}) \hat{S}_{1/2\,-m-1}^{-m\,-1/2} S_{-1/2\,m+1}^{m\,1/2}] \psi_{m,\,-m}^{(J,J)}(\sigma') \psi_{1/2,\,-1/2}^{(1/2,\,1/2)}(\sigma), \tag{D.9a}$$

$$W_2 \equiv \sum_m [a_{1/2}^{(1/2)}(\varpi_n) a_{m-1}^{(J)}(\varpi_{n+1}) \hat{S}_{-1/2\,-m+1}^{-m\,1/2} S_{1/2\,m-1}^{m\,-1/2}$$
$$+ a_{-1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n-1}) \hat{S}_{1/2\,-m}^{-m\,1/2} S_{-1/2\,m}^{m\,-1/2}] \psi_{m,\,-m}^{(J,J)}(\sigma') \psi_{-1/2,\,1/2}^{(1/2,\,1/2)}(\sigma), \tag{D.9b}$$

$$W_3 \equiv \sum_m [a_{1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n+1}) \hat{S}_{-1/2\,-m}^{-m\,-1/2} S_{1/2 m}^{m+1\,-1/2}$$
$$+ a_{-1/2}^{(1/2)}(\varpi_n) a_{m+1}^{(J)}(\varpi_{n-1}) \hat{S}_{1/2\,-m-1}^{-m\,-1/2} S_{-1/2\,m+1}^{m+1\,-1/2}] \psi_{m+1,\,-m}^{(J,J)}(\sigma') \psi_{-1/2,\,-1/2}^{(1/2,\,1/2)}(\sigma), \tag{D.9c}$$

$$W_4 \equiv \sum_m [a_{1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n+1}) \hat{S}_{-1/2\,-m}^{-m-1\,1/2} S_{1/2\,m}^{m 1/2}$$
$$+ a_{-1/2}^{(1/2)}(\varpi_n) a_{m+1}^{(J)}(\varpi_{n-1}) \hat{S}_{1/2\,-m-1}^{-m-1\,1/2} S_{-1/2\,m+1}^{m\,1/2}] \psi_{m,\,-m-1}^{(J,J)}(\sigma') \psi_{1/2,\,1/2}^{(1/2,\,1/2)}(\sigma). \tag{D.9d}$$

Since it is the same throughout, the index r of $\varpi_{r,n}$ is not explicitly written any more. For the braiding matrices, one makes use of the fact that, for any integer v,

$$S_{\alpha m}^{(J) m'\beta}\left(\varpi + v\frac{\pi}{h}\right) = S_{\alpha m}^{(J) m'\beta}(\varpi), \qquad \hat{S}_{\alpha m}^{(J) m'\beta}\left(\hat{\varpi} + v\frac{h}{\pi}\right) = \hat{S}_{\alpha m}^{(J) m'\beta}(\hat{\varpi}),$$

to let their arguments equal to ϖ_n. The symbols $S_{\alpha m}^{(J) n\beta}(\varpi_n)$, and $\hat{S}_{\hat{\alpha}\hat{m}}^{(J)\hat{n}\hat{\beta}}(\hat{\varpi}_n)$ have been replaced by $S_{\alpha m}^{n\beta}$, and $\hat{S}_{\hat{\alpha}\hat{m}}^{\hat{n}\hat{\beta}}$ in order to avoid clumsy notations as much as possible. Since they involve ψ operators that do not appear in Eq. (D.2), W_3 and W_4 must disappear. The determinant of the corresponding linear system with unknowns $a_{1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n+1})$ and $a_{-1/2}^{(1/2)}(\varpi_n) a_{m+1}^{(J)}(\varpi_{n-1})$ should vanish. This gives

$$\hat{S}_{-1/2\,-m}^{-m\,-1/2} S_{1/2 m}^{m+1\,-1/2} \hat{S}_{1/2\,-m-1}^{-m-1\,1/2} S_{-1/2\,m+1}^{m 1/2} = -\hat{S}_{1/2\,-m-1}^{-m\,-1/2} S_{-1/2\,m+1}^{m+1\,-1/2} \hat{S}_{-1/2\,-m}^{-m-1\,1/2} S_{1/2\,m}^{m 1/2}$$

or equivalently

$$\frac{\widehat{\lfloor \hat{\varpi}_n + J - m \rfloor}\,\widehat{\lfloor \hat{\varpi}_n - J - m - 1 \rfloor}}{\widehat{\lfloor J - m \rfloor}\,\widehat{\lfloor J + m + 1 \rfloor}} = \frac{\lfloor \varpi_n - J + m \rfloor \lfloor \varpi_n + J + m + 1 \rfloor}{\lfloor J - m \rfloor \lfloor J + m + 1 \rfloor}.$$

This equation is easily seen to follow from Theorem (3.1) and Proposition (3.7). The vanishing of D_3 next gives the recurrence relation

$$\frac{a_m^{(J)}(\varpi_{n+1})}{a_{m+1}^{(J)}(\varpi_{n-1})} = -e^{i(\hat{h}-h)/2}(-1)^{r+s(2J+1+n)} \frac{\lfloor \varpi_n + J + m + 1 \rfloor \lfloor J + m + 1 \rfloor}{\lfloor \varpi_n - J + m \rfloor \lfloor J - m \rfloor}. \tag{D.10}$$

Considering now the first two contributions (W_1, and W_2), one sees that the theorem will be fulfilled if

$$a_{1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n+1}) \hat{S}_{-1/2\,-m}^{-m\,-1/2} S_{1/2\,m}^{m 1/2} + a_{-1/2}^{(1/2)}(\varpi_n) a_{m+1}^{(J)}(\varpi_{n-1}) \hat{S}_{1/2\,-m-1}^{-m\,-1/2} S_{-1/2\,m+1}^{m 1/2}$$
$$= e^{-i\pi s J \varepsilon} a_{1/2}^{(1/2)}(\varpi_{n+2m}) a_m^{(J)}(\varpi_n), \tag{D.11a}$$

$$a_{1/2}^{(1/2)}(\varpi_n) a_{m-1}^{(J)}(\varpi_{n+1}) \hat{S}_{-1/2\,-m+1}^{-m\,1/2} S_{1/2\,m-1}^{m\,-1/2} + a_{-1/2}^{(1/2)}(\varpi_n) a_m^{(J)}(\varpi_{n-1}) \hat{S}_{1/2\,-m}^{-m\,1/2} S_{-1/2\,m}^{m\,-1/2}$$
$$= e^{-i\pi s J \varepsilon} a_{-1/2}^{(1/2)}(\varpi_{n+2m}) a_m^{(J)}(\varpi_n). \tag{D.11b}$$

Combine Eqs. (D.10) and (D.11a). After some computations one gets

$$a_m^{(J)}(\varpi_{n+1}) = -a_m^{(J)}(\varpi_n) \frac{\lfloor \varpi_{n+2m} \rfloor}{\lfloor \varpi_{n+1+2m} \rfloor} \frac{\lfloor \varpi_n + J + m + 1 \rfloor}{\lfloor \varpi_n - J + m \rfloor} e^{-i\pi s m}. \tag{D.12}$$

This recursion has the solution

$$a_m^{(J)}(\varpi_n)=c_m^{(J)}(-1)^{2Jn}e^{-is\pi mn}\lfloor\varpi_n-J+m\rfloor_{2J+1}/\lfloor\varpi_{n+2m}\rfloor, \tag{D.13}$$

where $c_m^{(J)}$ is still arbitrary. Determine it from the other recursion relation, which is satisfied if:

$$c_{m+1}^{(J)}=c_m^{(J)}(-1)^{r+1}e^{-i(\hat{h}-h)/2}\lfloor J-m\rfloor/\lfloor J+m+1\rfloor\,,$$

so that we find

$$c_m^{(J)}=\begin{pmatrix}2J\\ J-m\end{pmatrix}(-1)^{(r+1)(J-m)}e^{i(J-m)(\hat{h}-h)/2}\,.$$

The normalization is fixed by letting $c_J^{(J)}=1$, and Eq. (D.3) follows. □

Next, the fusion properties to leading order of the fields $\phi_-^{(1/2)}$ with $\phi_-^{(J)}$ are given by the

Theorem (D.2). *In $\mathscr{F}(\varpi_{r,n})$ and to leading order, the fusion of the fields $\phi_-^{(J)}$ is given by*

$$\phi_-^{(1/2)}(\sigma)\phi_-^{(J)}(\sigma')\sim(d(\sigma-\sigma'))^{-J(s+2)}e^{-is\pi J}\phi_-^{(J+1/2)}(\sigma)+\dots. \tag{D.14}$$

Proof. Recall the relevant fusion properties to leading order, from App. E of [1]:

$$\begin{aligned}&\psi_{\alpha,-\alpha}^{(1/2,1/2)}\psi_{m,-m}^{(J,J)}\sim(-1)^{2[J-2\alpha m]}\\ &\quad\times N(1/2,\alpha;J,m;\varpi)\hat{N}(1/2,-\alpha;J,-m;\hat{\varpi})\psi_{\alpha+m,-\alpha-m}^{(J+1/2,J+1/2)}\,,\\ &\qquad N(1/2,\pm1/2;J,m;\varpi)=\frac{\lfloor\varpi\mp J+m\rfloor}{\lfloor\varpi\rfloor}\,,\end{aligned} \tag{D.15}$$

(omitting the σ and σ' dependence as usual). Substitute the ansatz (D.1) into the left-hand side of (D.14) and make use of the last equalities. This gives

$$\begin{aligned}\phi_-^{(1/2)}\phi_-^{(J)}&\sim a_{1/2}^{(1/2)}(\varpi_n)a_J^{(J)}(\varpi_{n+1})\psi_{J+1/2,-J-1/2}^{(J+1/2,J+1/2)}\\ &+a_{-1/2}^{(1/2)}(\varpi_n)a_{-J}^{(J)}(\varpi_{n-1})\psi_{-J-1/2,J+1/2}^{(J+1/2,J+1/2)}+\sum_{m=-J}^{J-1}\alpha_m^{(J)}(\varpi_n)\psi_{m+1/2,-m-1/2}^{(J+1/2,J+1/2)}\,,\end{aligned}$$

where

$$\begin{aligned}\alpha_m^{(J)}(\varpi_n)\equiv{}&a_{1/2}^{(1/2)}(\varpi_n)a_m^{(J)}(\varpi_{n+1})(-1)^{s(J-m)}\frac{\lfloor\varpi_n-J+m\rfloor^2}{\lfloor\varpi_n\rfloor^2}\\ &+a_{-1/2}^{(1/2)}(\varpi_n)a_{m+1}^{(J)}(\varpi_{n-1})(-1)^{s(J+m+1)}\frac{\lfloor\varpi_n+J+m+1\rfloor^2}{\lfloor\varpi_n\rfloor^2}\,.\end{aligned}$$

Using the recurrence relation (D.10) one finds

$$\alpha_m^{(J)}(\varpi_n)=-a_m^{(J)}(\varpi_{n+1})\lfloor 2J+1\rfloor\frac{\lfloor\varpi_n-J+m\rfloor}{\lfloor J+m+1\rfloor}(-1)^{s(J-m)}e^{i\pi n(2-s)/2}\,.$$

Finally, from the above expression of $a_m^{(J)}(\varpi_n)$ one verifies that

$$\begin{aligned}\alpha_m^{(J)}(\varpi_n)&=e^{-is\pi J}a_{m+1/2}^{(J+1/2)}(\varpi_n)\,,\\ a_{\pm1/2}^{(1/2)}(\varpi_n)a_{\pm J}^{(J)}(\varpi_{n\pm1})&=e^{-is\pi J}a_{\pm J\pm1/2}^{(J+1/2)}(\varpi_n)\,.\end{aligned}$$

Thus (D.3) follows. □

Corollary (D.3). *In $\mathscr{F}(\varpi_{r,n})$ the fields $\phi_{-}^{(J)}$ and $\chi_{-}^{(J)}$ are related by*

$$\chi_{-}^{(J)} = d^{(J)}\phi_{-}^{(J)}, \qquad d^{(J)} = e^{-is\pi(J^2 - J/2)}(2i\sin h)^{2J} e^{i(h-\hat{h}+r\pi)J}. \tag{D.16}$$

Proof. Consider first Eq. (4.4):

$$\chi_{-}^{(1/2)}(\sigma) = \xi_{1/2,-1/2}^{(1/2,1/2)}(\sigma) + \kappa\xi_{-1/2,1/2}^{(1/2,1/2)}(\sigma).$$

The expression $\chi_{-}^{(1/2)}$ in terms of ψ fields was the starting point of [2]. Recalling Eq. (3.25), that is $\kappa = -e^{i(\hat{h}\varpi - h\varpi)}e^{i(h-\hat{h})/2}$, one does find, after some computations,

$$\chi_{-}^{(1/2)}(\sigma) = d^{(1/2)}[a_{1/2}^{(1/2)}(\varpi_n)\psi_{1/2,-1/2}^{(1/2,1/2)} + a_{-1/2}^{(1/2)}(\varpi_n)\psi_{-1/2,1/2}^{(1/2,1/2)}],$$
$$d^{(1/2)} := 2i\sin h\, e^{i(h-\hat{h}+r\pi)/2}, \tag{D.17}$$

$$a_{1/2}^{(1/2)}(\varpi_n) = (-1)^{n+1}e^{-i\pi sn/2}\lfloor\varpi_n\rfloor, \qquad a_{-1/2}^{(1/2)}(\varpi_n) = (-1)^{r+n}e^{i\pi sn/2}\lfloor\varpi_n\rfloor e^{i(\hat{h}-h)/2}, \tag{D.18}$$

which proves the theorem for $J=1/2$. Next, comparing (D.14) with (4.5), one sees that the theorem will be verified by recursion if the $d^{(J)}$'s satisfy

$$d^{(J+1/2)} = d^{(J)}d^{(1/2)}e^{-is\pi J},$$

and formulae (D.16) follow. □

As an overall crosscheck, one may of course directly verify that (D.16) is equivalent to (4.12) in $\mathscr{H}_{\text{phys}}$. This appendix thus shows that one may equivalently work with the ψ fields. However calculations become messy and this should be avoided unless one is specifically interested into the shifts of the quasi momentum ϖ.

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