Commun. Math. Phys. 137, 567–586 (1991)

Communications in
Mathematical
Physics

Conformal Embeddings, Rank-Level Duality and Exceptional Modular Invariants

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Received August 15, 1990

Abstract. We compute the branching rules of the conformal embeddings $SO(4nk)_1 \supset Sp(2n)_k \oplus Sp(2k)_n$ and $SO(rq)_1 \supset SO(r)_q \oplus SO(q)_r$ for rq even. Using this we prove that the affine algebras $Sp(2n)_k$ and $Sp(2k)_n$ have the same S matrix and modular invariants. As a second application, we show how the triality of $SO(8)$ leads to an exceptional modular invariant for $SU(2)$ at level 16 and for all $SO(q \geqq 4)$ at level 8.

1. Introduction

An important class of conformal field theories (CFT) [4] is provided by the Wess-Zumino-Witten (WZW) models [35]. They are characterized by the presence of a Kac-Moody (KM) symmetry [19, 17]. The WZW models lead to many other CFT's through the Goddard-Kent-Olive coset construction [16]. Examples are the minimal unitary Virasoro models [10] and the $N=2$ superconformal theories [23]. The latter have been used as building blocks for the internal sector of 4-dimensional heterotic strings [13, 23, 9]. The classification of modular invariant partition functions for Kac-Moody algebras is thus also important for string phenomenology.

Several authors have recently pointed out intriguing relations between a priori rather different KM algebras. Kac and Wakimoto [22] showed that $Sp(2n)_1$ and $SU(2)_n$ ($=Sp(2)_n$) have the same modular S matrix (the subscript is the level; all the algebras considered here are untwisted). Walton [34] computed the branching rules of the conformal embeddings $SU(n)_k \oplus SU(k)_n \subset SU(nk)_1$ and observed that the modular invariant partition functions of $SU(n)_k$ are naturally related to those of $SU(k)_n$. Considering the same conformal embeddings, Altschuler, Bauer, and Itzykson [1] expressed the S matrix of $SU(n)_k$ in terms of that of $SU(k)_n$. These embeddings were systematically used in [33] to obtain new exceptional modular

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invariants. Naculich and Schnitzer [26] derived relations between the holomorphic blocks of the four-point functions of $SU(n)_k$ and $SU(k)_n$. It was observed in [1, 25, 12] that there is also a duality for the ratios $S_{0,\lambda}/S_{0,0}$ for the following pairs of theories:

$$SU(n)_k \text{ and } SU(k)_n,$$
$$Sp(2n)_k \text{ and } Sp(2k)_n,$$
$$SO(r)_q \text{ and } SO(q)_r.$$

Here, we use conformal embeddings to investigate the relations between some of these KM algebras. Using the branching rules of

$$SO(4nk)_1 \supset Sp(2n)_k \oplus Sp(2k)_n, \tag{1.1}$$

we show that the S matrices of $Sp(2n)_k$ and $Sp(2k)_n$ are exactly equal, a stronger result than the equality of the ratios $S_{0,\lambda}/S_{0,0}$. The conformal dimensions are also closely related, and this allows us to prove that these two theories have the same partition functions. The isomorphism between representations is given by the transposition of the corresponding Young tableaux.

We also compute the branching rules of the conformal embeddings

$$SO(rq)_1 \supset SO(r)_q \oplus SO(q)_r \tag{1.2}$$

if rq is even. Here the number of fields of $SO(r)_q$ and $SO(q)_r$ is different, and the relation between the S matrices is not as simple as in the previous case. We will thus instead concentrate on the construction of modular invariant partition functions. A method due to Bouwknegt [6] indeed allows one to compute a modular invariant of $\hat{h}$ from modular invariants of $\hat{h}'$ and $\hat{g}$ if the branching rules of the conformal embedding $\hat{g} \supset \hat{h}' \oplus \hat{h}$ are known.

Many modular invariants have already been constructed for KM algebras [15, 5, 30, 8] but the only algebra for which a complete classification exists is $\widehat{SU(2)}$ [14, 7]. In [33], we developed a computer algorithm that finds all modular invariants of a Kac-Moody algebra provided the rank, the level and the coefficients are not too big. One of the results was that the $\widehat{SO(q)}$ $(4 \leqq q \leqq 11)$ have an exceptional invariant at level $k=8$. We conjectured that this is true for all $SO(q)$ and we also suggested that this is related, through (1.2), to the triality of $SO(8)$. We show here that this is indeed correct and we derive this exceptional invariant.

The paper is organized as follows. We derive the branching rules for (1.1) and prove the equality of the S matrices and modular invariants of $Sp(2n)_k$ and $Sp(2k)_n$ in Sect. 2. We compute the branching rules for (1.2) in Sect. 3 and in Sect. 4 we construct the exceptional modular invariant of $SU(2)_{16}$ and $SO(q \geqq 4)_8$. Section 5 is a short conclusion.

2. Duality Between $Sp(2n)_k$ and $Sp(2k)_n$

The Kac-Moody algebras $Sp(2n)_k$ and $Sp(2k)_n$ have the same number of primary fields: $(n+k)!/n!k!$. To relate the transformation properties under $\tau \to -1/\tau$ of the characters of these theories, we use the fact that they appear jointly in the conformal embedding (1.1). We denote by $\chi_\Lambda(\tau)$ and $\chi_\lambda(\tau)$ the characters of the integrable representations of $SO(4nk)_1$ and $Sp(2k)_n$, and by $S_{\Lambda\Lambda'}$ and $S_{\lambda\lambda'}$ the corresponding S matrices (to simplify the notation, we use the same symbols – but different indices – for the various characters and S matrices that appear). S is always symmetric and is real if the associated Lie algebra has only selfcontragradient representations (this includes the $\widehat{SO(2n+1)}$, $\widehat{Sp(2n)}$, and $\widehat{SO(4n)}$ algebras).

The integrable representations of $SO(4nk)_1$ reduce as follows:

$$\chi_\Lambda(\tau)=\sum_\lambda b^\lambda_\Lambda(\tau)\chi_\lambda(\tau), \tag{2.1}$$

where the branching functions $b^\lambda_\Lambda(\tau)$ are linear combinations (with constant coefficients) of the characters of $Sp(2n)_k$. Transforming $\tau\to-1/\tau$ in (2.1) and using the unitarity of S, one finds:

$$b^\lambda_\Lambda(-1/\tau)=\sum_{\Lambda',\lambda'} S_{\Lambda\Lambda'}S_{\lambda\lambda'}b^{\lambda'}_{\Lambda'}(\tau). \tag{2.2}$$

The rest of this section is as follows. We first compute the matrix $S_{\Lambda\Lambda'}$ of $SO(4nk)_1$. We then derive the decomposition formulas for the representations of this algebra (Eqs. (2.31–32) and (2.35–36)). Inserting these results in Eq. (2.2), we show that with the appropriate identification of representations, the S matrices of $Sp(2n)_k$ and $Sp(2k)_n$ are equal. We also prove that their partition functions are in one-to-one correspondence and finally we deduce a few new partition functions.

$SO(4nk)_1$ has four integrable representations: o, v, $+$, $-$ which have respectively the identity, the vector and the two spinor representations of $SO(4nk)$ as ground states. The S matrix is easy to compute since v, $+$, and $-$ are simple currents [11, 30]. A simple current is a primary field whose fusion product with any other primary field consists of only one field. Here,

$$(v)(v)=(+)(+)=(-)(-)=(o);\ (+)(-)=(v). \tag{2.3}$$

The other fusion rules follow from the commutativity and the associativity of the fusion product.

It has been shown in [31, 18] that

$$S_{Ja,b}=e^{2\pi i Q_J(b)}S_{a,b}, \tag{2.4}$$

where Ja is the fusion product of J and a, and where the "charge" $Q_J(b)$ (which characterizes the monodromy of the field b with respect to the simple current J) is defined by:

$$Q_J(b)=h(b)+h(J)-h(Jb)\ \mathrm{mod}\ 1. \tag{2.5}$$

The conformal dimensions are given by:

$$h(o)=0,\quad h(v)=1/2,\quad h(+)=h(-)=nk/4. \tag{2.6}$$

The S matrix of $SO(4nk)_1$ is thus:

$$S=S_{o,o}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & \varepsilon & -\varepsilon\\ 1 & -1 & -\varepsilon & \varepsilon\end{pmatrix} \tag{2.7}$$

with $\varepsilon=(-1)^{nk}$ (the ordering of the fields is: o, v, $+$, $-$). The reality and the unitarity of S imply $S_{o,o}=\pm\frac{1}{2}$. The sign is determined as follows. For any affine Lie algebra, $S_{a,b}$ can be expressed as a certain sum over the (finite) Weyl group [21, 15].

If a (or b) = identity, this sum can be rewritten using the Weyl denominator identity as a product over the (finite) positive roots:

$$S_{o,b}=\frac{\det^{-1/2}\{\bar{\beta}_i\cdot\bar{\beta}_j\}}{(k+h^\vee)^{r/2}}\prod_{\bar{\alpha}>0}2\sin\frac{\pi\bar{\alpha}\cdot(\bar{\varrho}+\bar{b})}{k+h^\vee}, \tag{2.8}$$

where bars distinguish finite roots and weights from the affine ones. The $\bar{\beta}_i$ span a fundamental cell of the coroot lattice, k is the level, $h^\vee$ the dual Coxeter number, r the rank of the associated Lie algebra and $\bar{\varrho}$ the finite Weyl vector. It follows from

$$0<\bar{\alpha}\cdot\bar{\varrho}\leqq h^\vee-1 \tag{2.9}$$

and

$$0<\bar{\alpha}\cdot\bar{b}\leqq k \tag{2.10}$$

that $S_{o,b}$ (and in particular $S_{o,o}$) is always strictly positive.

To compute the branching functions $b_\Lambda^\lambda(\tau)$, we use the "Parthasarathy-Kac-Peterson-Nahm theorem." This theorem is due to Kac and Peterson [20] with a correction by Nahm [27] and gives the decomposition of the spinor representations (denoted + and − here) of some orthogonal KM algebra. It is a generalization of the finite dimensional analog, which was proved by Parthasarathy [28]. A proof and an example of application can be found in [2]. The theorem reads:

Let g be a simple Lie algebra, $g_0\subset g$ a semi-simple subalgebra of the same rank, such that $g=g_0\oplus V$ defines a symmetric space V, i.e., $[V,V]\subset g_0$; let W, W_0 be the Weyl groups and ϱ, ϱ_0 be the Weyl vectors of the affine Lie algebras $\hat{g}$, $\hat{g}_0$; let W_1 be a set of coset representatives of W/W_0 such that $w(\varrho)-\varrho_0$ is a dominant weight of $\hat{g}_0$ for any $w\in W_1$. Then the spinor representations + and − of the affine algebra $\widehat{SO(\dim V)}$ associated to the isometry group of V reduce as follows in representations of $\hat{g}_0$:

$$\begin{cases}+\\-\end{cases}=\sum_{w\in W_1^\pm}(w(\varrho)-\varrho_0), \tag{2.11}$$

where

$$W_1^\pm=\{w\in W_1:\det(w)=\pm1\}, \tag{2.12}$$

and where we denote a representation of $\hat{g}_0$ by its highest weight $w(\varrho)-\varrho_0$.

The Weyl group W of an affine Lie algebra is the semi-direct product of the corresponding finite Weyl group $\bar{W}$ and of the group T of "translations" by the vectors of the coroot lattice $\left(\text{this is the lattice generated by the vectors } \bar{\alpha}^\vee=\frac{2\bar{\alpha}}{\bar{\alpha}^2}\right)$ [17]. Hence,

$$W_1=(\bar{W}\ltimes T)/(\bar{W}_0\ltimes T_0)=(\bar{W}/\bar{W}_0)\times(T/T_0). \tag{2.13}$$

Here $g_0=Sp(2n)\oplus Sp(2k)$ and $g=Sp(2n+2k)$ turns out to be the right choice: the symmetric space V defined by

$$Sp(2n+2k)=(Sp(2n)\oplus Sp(2k))\oplus V \tag{2.14}$$

has dimension:

$$\dim V=2(n+k)^2+(n+k)-(2n^2+n+2k^2+k)=4nk. \tag{2.15}$$

The roots of $Sp(2n)$ can be taken as

$$\{(\pm e_i \pm e_j)/\sqrt{2} \quad (i \neq j), \ \pm\sqrt{2}e_i\}, \tag{2.16}$$

where the e_i $(1 \leqq i \leqq n)$ are orthonormal vectors. The (finite) Weyl group is the semi-direct product $S_n \ltimes Z_2^n$, where S_n consists of all permutations of the e_i and Z_2^n of all sign changes: $e_i \to \pm e_i$. The regular embedding $Sp(2n+2k) \supset Sp(2n) \oplus Sp(2k)$ is taken such that the roots of these three algebras are given by (2.16), with $1 \leqq i$, $j \leqq n+k$, $1 \leqq i, j \leqq n$, $n+1 \leqq i, j \leqq n+k$ respectively. $\bar{W}/\bar{W}_0$ reduces to $S_{n+k}/S_n \otimes S_k$. T/T_0 is trivial because the coroot lattice of a symplectic algebra is generated by the vectors $\{\sqrt{2}e_i\}$.

We now describe the set $W_1 = S_{n+k}/S_n \otimes S_k$. Consider the following permutation of S_{n+k}:

$$\mathscr{P} = \begin{pmatrix} 1 & 2 \dots l_1+1 \dots l_2+2 \dots l_k+k \dots \\ 1 & 2 \dots n+1 \dots n+2 \dots n+k \dots \end{pmatrix}, \tag{2.17}$$

with

$$0 \leqq l_1 \leqq l_2 \dots \leqq l_k \leqq n. \tag{2.18}$$

The sets $\{1, 2, \dots, n\}$ and $\{n+1, n+2, \dots, n+k\}$ on the second line are both arranged in increasing order. With these ordering conventions, $\mathscr{P}$ is uniquely specified by the set of l_i's. $\mathscr{P}$ transforms the vectors e_i with indices in the first line in the vectors with indices in the second line. For example,

$$\mathscr{P} e_{l_1+1} = e_{n+1}. \tag{2.19}$$

The identical permutation corresponds to $l_1 = l_2 = \dots = l_k = n$. The number of such permutations is $(n+k)!/n!k!$. It is also easy to see that it is impossible to transform a permutation of this set into another one by multiplication with permutations of S_n or S_k. We will show below that $\mathscr{P}(\varrho) - \varrho_0$ is a dominant weight of $Sp(2n)_k \oplus Sp(2k)_n$ for all $\mathscr{P}$. The permutations $\mathscr{P}$ are thus coset representatives of W_1.

The Weyl vector ϱ of $\hat{g} = \widehat{Sp(2n+2k)}$ is given by

$$\varrho = \bar{\varrho} + h_g^\vee \Lambda_0, \tag{2.20}$$

where $\bar{\varrho}$ is the (finite) Weyl vector of g (i.e., half the sum of the positive roots), and where the dual Coxeter number of g is $n+k+1$; Λ_0 is the fundamental weight corresponding to α_0, the highest root of the affine algebra $\hat{g}$. Since the embedding $g_0 \subset g$ is regular, Λ_0 reduces as: $\Lambda_0 = \Lambda'_0 + \Lambda''_0$, where Λ'_0 and Λ''_0 are the analogs of Λ_0 for $\widehat{Sp(2n)}$ and $\widehat{Sp(2k)}$. ϱ_0 is the sum of the Weyl vectors of $\widehat{Sp(2n)}$ and $\widehat{Sp(2k)}$. Combining all this we find

$$\mathscr{P}(\varrho) - \varrho_0 = \mathscr{P}(\bar{\varrho}) - \bar{\varrho}_0 + k\Lambda'_0 + n\Lambda''_0. \tag{2.21}$$

As expected, the spinor representations of $SO(4nk)_1$ decompose in combinations of (products of) representations of level k of $\widehat{Sp(2n)}$ and representations of level n of $\widehat{Sp(2k)}$. In the following, we will concentrate on the finite part $\mathscr{P}(\bar{\varrho}) - \bar{\varrho}_0$.

The Weyl vector of $Sp(2n+2k)$ is given by:

$$\bar{\varrho} = \frac{1}{\sqrt{2}} \sum_{i=1}^{n+k} (n+k+1-i) e_i, \tag{2.22}$$

and similarly for $Sp(2n)$ and $Sp(2k)$. It is convenient to write

$$\mathscr{P}(\bar{\varrho})-\bar{\varrho}_0=(\mathscr{P}(\bar{\varrho})-\bar{\varrho})+(\bar{\varrho}-\bar{\varrho}_0), \tag{2.23}$$

with

$$\bar{\varrho}-\bar{\varrho}_0=\frac{k}{\sqrt{2}}\sum_{i=1}^{n} e_i. \tag{2.24}$$

We also need the expression of an arbitrary vector in weight space in terms of the fundamental weights $\bar{\Lambda}_i$:

$$\frac{1}{\sqrt{2}}\sum_{i=1}^{r} x_i e_i=\sum_{i=1}^{r-1}(x_i-x_{i+1})\bar{\Lambda}_i+x_r\bar{\Lambda}_r; \tag{2.25}$$

the coefficients of the $\bar{\Lambda}_i$ are the Dynkin labels. This gives a dominant weight of $Sp(2r)$ iff $x_1 \geqq x_2 \geqq \ldots \geqq x_r \geqq 0$. The key property of $\bar{\varrho}$ is that the coefficients of the e_i's decrease regularly as i increases. The coefficients of the e_i's in $\mathscr{P}(\bar{\varrho})-\bar{\varrho}$ are then simply related to the l_i's that characterize the permutation $\mathscr{P}$. We first consider the weight of $Sp(2k)$ contained in $\mathscr{P}(\bar{\varrho})-\bar{\varrho}_0$:

$$\mathscr{P}(\bar{\varrho})-\bar{\varrho}_0|_{Sp(2k)}=\frac{1}{\sqrt{2}}\sum_{i=1}^{k}(n-l_i)e_{n+i}. \tag{2.26}$$

In terms of Dynkin labels, this gives rise to the representation with highest weight

$$(l_2-l_1, l_3-l_2, \ldots, l_k-l_{k-1}, n-l_k) \tag{2.27}$$

which, because of (2.18), is a dominant weight of $Sp(2k)$. But it must also be a dominant weight of $Sp(2k)_n$. This means that if we express

$$n\Lambda''_0+(l_2-l_1)\bar{\Lambda}''_1+\ldots+(l_k-l_{k-1})\bar{\Lambda}''_{r-1}+(n-l_k)\bar{\Lambda}''_r \tag{2.28}$$

in terms of the affine fundamental weights Λ''_0 and $\bar{\Lambda}''_i+\Lambda''_0$ $(i=1,\ldots,k)$, all generalized Dynkin labels (=coefficients of Λ''_0 and $\Lambda''_0+\bar{\Lambda}''_i$) should be positive. This is the case, since the above representation corresponds to $(l_1; l_2-l_1, \ldots, l_k-l_{k-1}, n-l_k)$ and $l_1 \geqq 0$ (the 0^{th} Dynkin label will always be separated by a semicolon from the others).

We now turn to the terms involving $e_1, \ldots, e_n$ in $\mathscr{P}(\bar{\varrho})-\bar{\varrho}_0$, i.e., to the weights of $Sp(2n)$. It is easy to see that the coefficients of $e_1, \ldots, e_{l_1}$ are k, those of $e_{l_1+1}, \ldots, e_{l_2}$ are $k-1, \ldots$. Because of (2.25), this gives a contribution equal to 1 for the ${l_1}^{\text{th}}$, ${l_2}^{\text{th}}, \ldots, {l_k}^{\text{th}}$ Dynkin label. The weight of $Sp(2n)$ contained in $\mathscr{P}(\bar{\varrho})-\bar{\varrho}_0$ has thus the Dynkin labels

$$(N(1), N(2), \ldots, N(n)), \tag{2.29}$$

where $N(i)$ is the number of times i appears among the l_j's. Note that this notation smoothly extends from the finite to the affine case. In the former case, the l_j's equal to 0 are ignored, while in the latter, each of them contributes 1 to the coefficient of Λ'_0. In terms of generalized Dynkin labels, the above representation is $(N(0); N(1), \ldots, N(n))$.

We finally compute the sign of $\mathscr{P}$. Writing $\mathscr{P}$ as a product of elementary permutations on two indices, one verifies that this sign is equal to

$$(-1)^{\sum_{i=1}^{k}(n-l_i)}=(-1)^{nk+\sum_{i=1}^{k} l_i}. \tag{2.30}$$

Collecting these results, we find that the spinor representations of $SO(4nk)_1$ decompose as follows into representations of $Sp(2n)_k \oplus Sp(2k)_n$:

$$\left\{\begin{matrix}+\\-\end{matrix}\right. = \sum_{0 \leqq l_1 \ldots \leqq l_k \leqq n} (N(0); N(1), \ldots, N(n))(l_1; l_2 - l_1, \ldots, l_k - l_{k-1}, n - l_k), \quad (2.31)$$

with

$$nk + \sum_{i=1}^{k} l_i = \left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right. . \quad (2.32)$$

The decomposition of the two other representations, o and v, is obtained through automorphisms [2]. The nontrivial automorphism of $Sp(2k)_n$ takes the form:

$$\mu(m_0; m_1, \ldots, m_k) = (m_k; m_{k-1}, \ldots, m_0) \quad (2.33)$$

$\left(\text{here } \sum_{i=0}^{k} m_i = n\right)$. Its action is equivalent to the fusion product with the simple current $J = (0; 0, \ldots, 0, n)$:

$$\mu(m) = J \cdot m . \quad (2.34)$$

The automorphisms of $Sp(2n)_k \oplus Sp(2k)_n$ extend to automorphisms of $SO(4nk)_1$ and hence transform $+$ and $-$ into one of the integrable representations $+, -, o, v$. One finds which one by examining the terms of lowest conformal dimension. This yields

$$\left\{\begin{matrix}o\\v\end{matrix}\right. = \sum_{0 \leqq l_1 \ldots \leqq l_k \leqq n} (N(0); N(1), \ldots, N(n))(n - l_k; l_k - l_{k-1}, \ldots, l_2 - l_1, l_1) \quad (2.35)$$

with

$$\sum_{i=1}^{k} l_i = \left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right. . \quad (2.36)$$

The set $\{b_o^\lambda(\tau), b_v^\lambda(\tau)\}$ is thus the set of characters of $Sp(2n)_k$, since each such character appears exactly once, either in the decomposition of o or of v. One could also use the set $\{b_+^\lambda(\tau), b_-^\lambda(\tau)\}$ with

$$b_+^\lambda(\tau) = b_o^{J\lambda}(\tau), b_-^\lambda(\tau) = b_v^{J\lambda}(\tau) \quad (2.37)$$

if nk is even and

$$b_+^\lambda(\tau) = b_v^{J\lambda}(\tau), b_-^\lambda(\tau) = b_o^{J\lambda}(\tau) \quad (2.38)$$

if nk is odd. Using (2.2) and (2.7) (with $S_{o,o} = 1/2$), we find

$$\begin{aligned} b_o^\lambda(-1/\tau) &= \sum_{\Lambda', \lambda'} S_{o\Lambda'} S_{\lambda\lambda'} b_{\Lambda'}^{\lambda'}(\tau) \\ &= \sum_{\lambda'} \frac{1}{2} S_{\lambda\lambda'} (b_o^{\lambda'}(\tau) + b_v^{\lambda'}(\tau) + b_o^{J\lambda'}(\tau) + b_v^{J\lambda'}(\tau)) \\ &= \sum_{\lambda'} \frac{1}{2} (S_{\lambda\lambda'} + S_{\lambda, J\lambda'})(b_o^{\lambda'}(\tau) + b_v^{\lambda'}(\tau)) . \end{aligned} \quad (2.39)$$

Similarly,

$$b_v^\lambda(-1/\tau)=\sum_{\lambda'}\frac{1}{2}(S_{\lambda\lambda'}-S_{\lambda,J\lambda'})(b_o^{\lambda'}(\tau)+b_v^{\lambda'}(\tau)). \tag{2.40}$$

$S_{\lambda,J\lambda'}$ can be related to $S_{\lambda\lambda'}$ using (2.4) and (2.5). The charge $Q_J(\lambda)$ is simply half the congruence of λ (mod 1). Hence,

$$\begin{aligned}Q_J(n-l_k;\, l_k-l_{k-1},\dots,l_1)&=\frac{1}{2}(l_k-l_{k-1}+l_{k-2}-l_{k-3}+\dots)\\&=\frac{1}{2}\sum_{i=1}^{k} l_i(\mathrm{mod}\,1).\end{aligned}\tag{2.41}$$

Note that the charge of $(N(0);\dots,N(n))$ with respect to the simple current of $Sp(2n)_k$ is given by

$$\frac{1}{2}(N(1)+N(3)+\dots)=\frac{1}{2}\sum_{i=1}^{k} l_i(\mathrm{mod}\,1). \tag{2.42}$$

Thus o contains only representations of charge 0 and v only representations of charge 1/2 with respect to the simple currents of $Sp(2n)_k$ and $Sp(2k)_n$. The final result is:

$$\begin{aligned}b_0^\lambda(-1/\tau)&=\sum_{\lambda'}S_{\lambda\lambda'}(b_0^{\lambda'}(\tau)+b_v^{\lambda'}(\tau)),\\b_v^\lambda(-1/\tau)&=\sum_{\lambda'}S_{\lambda\lambda'}(b_0^{\lambda'}(\tau)+b_v^{\lambda'}(\tau)).\end{aligned}\tag{2.43}$$

The characters $(n-l_k;\, l_k-l_{k-1},\dots,l_2-l_1,l_1)$ of $Sp(2k)_n$ and $(N(0);N(1),\dots,N(n))$ of $Sp(2n)_k$ (with $N(i)$ the number of times i appears among the l_j's) transform thus in the same way under $\tau\to-1/\tau$. The fusion rules can be expressed in terms of the S matrix elements using Verlinde's formula [32] and are thus also the same.

This isomorphism has a simple interpretation in terms of Young tableaux. Consider the representation of $Sp(2r)$ with Dynkin labels $(a_1,a_2,\dots,a_r)$. In the orthonormal basis of the e_i's, its highest weight is

$$\frac{1}{\sqrt{2}}[(a_1+a_2+\dots+a_r)e_1+(a_2+\dots+a_r)e_2+\dots+a_re_r]. \tag{2.44}$$

One associates to it a Young tableau with rows of length $a_1+a_2+\dots+a_r$, $a_2+\dots+a_r,\dots,a_r$. It is not too difficult to convince oneself that the representations $(l_k-l_{k-1},\dots,l_2-l_1,l_1)$ of $Sp(2k)$ and $(N(1),\dots,N(n))$ of $Sp(2n)$ correspond to Young tableaux that are the transpose of each other. A similar identification of fields has been discussed in [25, 12, 29].

We conclude this section with a proof that the modular invariant partition functions of these theories are also the same. We first note that $\chi_a(\tau)\chi_b^*(\tau)$ can be invariant under T (i.e., $h(a)-h(b)=0$ mod 1) only if $Q_J(a)=Q_J(b)$. This follows from:

$$\begin{aligned}4(k+n+1)h(a)&=1\cdot a_1^2+2\cdot a_2^2+3\cdot a_3^2+\dots\ \mathrm{mod}\,2\\&=a_1+a_3+\dots\ \mathrm{mod}\,2\end{aligned}\tag{2.45}$$

for the representation of $Sp(2n)_k$ or $Sp(2k)_n$ with Dynkin labels $(a_1,a_2,\dots)$. Let a^T be the field of $Sp(2n)_k$ associated by the above isomorphism to the field a of $Sp(2k)_n$.

They satisfy

$$h(a)+h(a^T)=0 \bmod 1 \quad \text{if} \quad Q_J(a)=0 \tag{2.46}$$

and

$$h(a)+h(a^T)=1/2 \bmod 1 \quad \text{if} \quad Q_J(a)=1/2. \tag{2.47}$$

This implies

$$h(a)-h(b)=0 \bmod 1 \Leftrightarrow h(a^T)-h(b^T)=0 \bmod 1, \tag{2.48}$$

since $Q_J(a)=Q_J(b)$.

This one-to-one correspondence between the bilinears invariant under T (and with the same transformation properties under S) implies that there is a one-to-one correspondence between modular invariants of $Sp(2n)_k$ and $Sp(2k)_n$.

$SU(2)_{16}=Sp(2)_{16}$ and $SO(5)_8=Sp(4)_8$ (cf. Sect. 4) have an exceptional modular invariant. This implies the existence of an exceptional invariant for $Sp(32)_1$ and $Sp(16)_2$ (they were obtained using other methods in [33]). For example, to get that of $Sp(32)_1$, one has simply to replace the representation of $SU(2)$ with Dynkin label i in (4.9) by the representation of $Sp(32)$ whose i^{th} Dynkin label is 1 (and the others are 0). On the other hand, the exceptional invariant of $Sp(8)_4$ [33] cannot be obtained using the present method.

There are four "isolated" conformal embeddings [3] $Sp(2n)_k \subset \hat{l}$; by this we mean that there is no conformal embedding $Sp(2k)_n \subset \hat{l}'$. For example, $SU(2)_{10}=Sp(2)_{10} \subset SO(5)_1$ while there is no conformal embedding of $Sp(20)_1$ in some KM algebra. The above results imply that $Sp(20)_1$, $Sp(56)_1$, $Sp(10)_3$, and $Sp(14)_4$ have an exceptional integer spin modular invariant (sum of squares).

3. Branching Rules for $SO(rq)_1 \supset SO(r)_q \oplus SO(q)_r$ (rq Even)

These branching rules are an essential ingredient in the derivation of new modular invariants for the orthogonal algebras. We use the same theorem as in Sect. 2. Here $g_0=SO(r)\oplus SO(q)$ and we must find a simple group g with the same rank as g_0 and such that the symmetric space V defined by $g=g_0\oplus V$ has dimension rq. This is only possible if r and q are not both odd, and $g=SO(r+q)$. In the following, we will put $r=2n$ and $q=2k+1$ or $2k$.

In terms of r orthonormal vectors e_i's, the roots of $SO(2r+1)$ can be chosen as

$$\{\pm e_i, \pm e_i \pm e_j \ (i \neq j)\}, \tag{3.1}$$

and those of $SO(2r)$ as

$$\{\pm e_i \pm e_j \ (i \neq j)\}. \tag{3.2}$$

The Weyl group of $SO(2r+1)$ is then the semi-direct product of all permutations of the e_i's and of all sign changes (like in the symplectic case). The Weyl group of $SO(2r)$ has the same form, except that the number of sign changes must be even. We choose the roots of the three algebras appearing in the regular embedding $SO(2n+2k+1) \supset SO(2n)\oplus SO(2k+1)$ to be given by (3.1) with $1 \leqq i, j \leqq n+k$, by (3.2) with $1 \leqq i, j \leqq n$ and by (3.1) with $n+1 \leqq i, j \leqq n+k$ respectively; and similarly for $SO(2n+2k) \supset SO(2n)\oplus SO(2k)$.

The set of coset representatives is:

$$W_1=(S_{n+k}/S_n \otimes S_k)\otimes Z_2 \otimes (T/T_0). \tag{3.3}$$

We will make the same choice of coset representatives $\mathscr{P} \in S_{n+k}/S_n \otimes S_k$ as in Sect. 2. The non-trivial sign change of Z_2 can be taken as

$$\sigma: e_n \to -e_n \quad \text{if} \quad q=2k+1 \tag{3.4}$$

and

$$\sigma: e_n \to -e_n, \quad e_{n+k} \to -e_{n+k} \quad \text{if} \quad q=2k. \tag{3.5}$$

With these choices, $\mathscr{P}(\varrho)-\varrho_0$ and $\sigma\mathscr{P}(\varrho)-\varrho_0$ will be dominant weights of $SO(2n)_q \oplus SO(q)_{2n}$. T/T_0 is also non-trivial: the coroot lattice of $SO(2n+q)$ is generated by the long roots $\pm e_i \pm e_j$, and a translation t by $e_n + e_{n+k}$ for example cannot be undone using translations of the Weyl groups of $\widehat{SO(2n)}$ or $\widehat{SO(q)}$. T/T_0 has two elements; we will discuss below which element has to be taken as representative of the non-trivial class.

The computation of the highest weights $\mathscr{P}(\varrho)-\varrho_0$ appearing in the decomposition of the spinors + and − of $SO(2nq)_1$ closely parallels that of Sect. 2. The finite Weyl vectors are given by:

$$\bar{\varrho} = \sum_{i=1}^{r} \left(r + \frac{1}{2} - i\right) e_i \quad \text{for} \quad SO(2r+1),$$

$$\bar{\varrho} = \sum_{i=1}^{r} (r-i) e_i \quad \text{for} \quad SO(2r). \tag{3.6}$$

Note that the coefficients decrease in the same regular way as in (2.22). The relation between the $\{e_i\}$ basis and the Dynkin basis is provided by

$$\sum_{i=1}^{r} x_i e_i = \sum_{i=1}^{r-1} (x_i - x_{i+1}) \bar{\Lambda}_i + 2x_r \bar{\Lambda}_r \tag{3.7}$$

for $SO(2r+1)$, and

$$\sum_{i=1}^{r} x_i e_i = \sum_{i=1}^{r-1} (x_i - x_{i+1}) \bar{\Lambda}_i + (x_{r-1} + x_r) \bar{\Lambda}_r \tag{3.8}$$

for $SO(2r)$. These weights are dominant iff $x_1 \geqq x_2 \geqq \ldots \geqq x_r \geqq 0$ and $x_1 \geqq x_2 \geqq \ldots \geqq x_{r-1} \geqq |x_r|$ respectively. If $q=2k+1$, the representation of $SO(2n)_q \oplus SO(q)_{2n}$ corresponding to the permutation $\mathscr{P}$ characterized by $0 \leqq l_1 \leqq l_2 \ldots \leqq l_k \leqq n$ has the highest weight

$$\begin{aligned} \mathscr{P}(\varrho) - \varrho_0 = {} & (2N(0) + N(1); N(1), N(2), \ldots, N(n-1), N(n-1) \\ & + 2N(n) + 1)(l_2 + l_1; l_2 - l_1, l_3 - l_2, \ldots, l_k - l_{k-1}, 2n - 2l_k) \end{aligned} \tag{3.9}$$

with the same notations as in Sect. 2. If $q=2k$, one has instead

$$\begin{aligned} \mathscr{P}(\varrho) - \varrho_0 = {} & (2N(0) + N(1); N(1), \ldots, N(n-1), N(n-1) \\ & + 2N(n))(l_2 + l_1; l_2 - l_1, \ldots, l_k - l_{k-1}, 2n - l_{k-1} - l_k). \end{aligned} \tag{3.10}$$

We now consider the contribution of $\sigma\mathscr{P}(\varrho)-\varrho_0$ to the decomposition of the representations + and −. Since ϱ_0 does not contain e_n (nor e_{n+k} if $q=2k$) on which σ acts nontrivially,

$$\sigma\mathscr{P}(\varrho) - \varrho_0 = \sigma(\mathscr{P}(\varrho) - \varrho_0). \tag{3.11}$$

Changing the sign of e_n in $\mathscr{P}(\varrho)$ is thus equivalent to interchanging the last two Dynkin labels of the weight of $SO(2n)_q$ contained in $\mathscr{P}(\varrho)-\varrho_0$, and similarly for e_{n+k} and $SO(q)_{2n}$ if $q=2k$.

We now turn to the action of T/T_0. $\mathscr{P}(\varrho)$ is a weight of $\widehat{SO(2n)} \oplus \widehat{SO(q)}$ of level $h_g^\vee = 2n+q-2$. The translation t by $e_n + e_{n+k}$ does not modify the level of the affine weight $\mathscr{P}(\varrho)$, and the finite part of $t\mathscr{P}(\varrho)$ is given by [17]:

$$\overline{t\mathscr{P}(\varrho)} = \overline{\mathscr{P}(\varrho)} + h_g^\vee (e_n + e_{n+k}). \tag{3.12}$$

We consider first the restriction of $\mathscr{P}(\varrho)$ and $t\mathscr{P}(\varrho)$ to $\widehat{SO(2n)}$. If we write

$$\mathscr{P}(\varrho)|_{\widehat{SO(2n)}} = (a_0; a_1, \ldots, a_{n-1}, a_n) = a, \tag{3.13}$$

with $a_0 + a_1 + 2(a_2 + \ldots + a_{n-2}) + a_{n-1} + a_n = h_g^\vee$, then the restriction of $t\mathscr{P}(\varrho)$ to $\widehat{SO(2n)}$ will be

$$(a_0; a_1, \ldots, a_{n-1} - h_g^\vee, a_n + h_g^\vee), \tag{3.14}$$

which we denote (slightly abusively) $t(a)$.

The restriction of ϱ_0 is $(1; 1, \ldots, 1)$, and $t(a) - (1; 1, \ldots, 1)$ will not be a dominant weight of $\widehat{SO(2n)}_q$. We use reflections of the Weyl group of $\widehat{SO(2n)}$, level $h_g^\vee$, to bring $t(a)$ back in the strictly dominant affine Weyl chamber. The reflection w_i with respect to the simple root α_i $(0 \leqq i \leqq n)$ takes the form

$$w_i(a) = a - a_i \alpha_i. \tag{3.15}$$

The components of α_i in the generalized Dynkin basis are given by the i^{th} line of the generalized Cartan matrix, which can easily be read off the extended Dynkin diagram. We find

$$\begin{aligned} w_{n-1} w_{n-2} \ldots w_2 w_0 w_1 w_2 \ldots w_{n-1} t(a) &= (a_1; a_0, a_2, \ldots, a_{n-2}, a_n, a_{n-1}) \\ &= J_v \cdot a, \end{aligned} \tag{3.16}$$

with J_v one of the simple currents of $\widehat{SO(2n)}$.

Note that this operation commutes with the subtraction of ϱ_0:

$$J_v \cdot a - (1; 1, \ldots, 1, 1) = J_v \cdot (a - (1; 1, \ldots, 1, 1)). \tag{3.17}$$

The sign of the above transformation is $+1$, since the sign of t is $+1$, and each w_i contributes a factor -1.

If q is even, the same result applies to the restriction of $t\mathscr{P}(\varrho) - \varrho_0$ to $\widehat{SO(q)}$. Take now $q = 2k+1$. Let a be the restriction of $\mathscr{P}(\varrho)$ to $\widehat{SO(2k+1)}$, with

$$a_0 + a_1 + 2(a_2 + \ldots a_{k-1}) + a_k = h_g^\vee. \tag{3.18}$$

Then

$$t(a) = (a_0; a_1, \ldots, a_{k-1} - h_g^\vee, a_k + 2h_g^\vee) \tag{3.19}$$

and

$$\begin{aligned} w_k w_{k-1} \ldots w_2 w_0 w_1 \ldots w_{k-1} t(a) &= (a_1; a_0, a_2, \ldots, a_{k-1}, a_k) \\ &= J \cdot a \end{aligned} \tag{3.20}$$

is a strictly dominant affine weight. This transformation also commutes with the subtraction of ϱ_0; its sign is -1.

One can check that $t(a)$ is Weyl-equivalent to $J_v \cdot a$ or $J \cdot a$ also in the "degenerate" cases of $\widehat{SO(3)}, \ldots, \widehat{SO(6)}$.

Collecting these results, one finds the following branching rules for $SO(2n(2k+1))_1 \supset SO(2n)_{2k+1} \oplus SO(2k+1)_{2n}$:

$$\begin{cases} + \\ - \end{cases} = \sum_{\beta,\gamma=0}^{1} \sum_{\{l_i\}} C^\beta J_v^\gamma (2N(0)+N(1); N(1), \ldots, N(n-1), N(n-1) \\ + 2N(n)+1) J^\gamma (l_2+l_1; l_2-l_1, \ldots, l_k-l_{k-1}, 2n-2l_k), \tag{3.21}$$

with $0 \leqq l_1 \leqq \ldots \leqq l_k \leqq n$ (implicit hereafter) and

$$\sum_{i=1}^{k} l_i + nk + \beta + \gamma = \begin{cases} \text{even} \\ \text{odd} \end{cases}. \tag{3.22}$$

C interchanges the last two Dynkin labels; C, J_v, and J act only on the representation immediately to their right.

The decomposition of the two other representations can be obtained by acting with some appropriate automorphism on $+$ and $-$:

$$\begin{cases} o \\ v \end{cases} = \sum_{\beta,\gamma=0}^{1} \sum_{\{l_i\}} J_s C^\beta J_v^\gamma (2N(0)+N(1); N(1), \ldots, N(n-1), N(n-1) \\ + 2N(n)+1) J^\gamma (l_2+l_1; l_2-l_1, \ldots, l_k-l_{k-1}, 2n-2l_k), \tag{3.23}$$

with

$$\sum_{i=1}^{k} l_i + nk + \beta + \gamma + n = \begin{cases} \text{even} \\ \text{odd} \end{cases}. \tag{3.24}$$

The action of the simple current $J_s = (0; 0, \ldots, 0, q)$ of $SO(2n)_q$ is [30]:

$$J_s \cdot (a_0; a_1, \ldots, a_n) = \begin{cases} (a_n; a_{n-1}, \ldots, a_0) \text{ if } n \text{ is even}, \\ (a_{n-1}; a_n, a_{n-2}, \ldots, a_1, a_0) \text{ if } n \text{ is odd}. \end{cases} \tag{3.25}$$

$-$, o, and v can be expressed in terms of $+$:

$$\begin{aligned} - &= C(+), \\ o &= J_s C^n(+), \\ v &= J_s C^{n+1}(+), \end{aligned} \tag{3.26}$$

where C and J_s only act on the representation of $SO(2n)_{2k+1}$.

The components in the above decomposition formulas are a bit different if $n \leqq 2$ or $k=1$ but they can be easily worked out. In particular, if $k=1$ (and $n \geqq 3$), the embedding is such that the root of $SO(3)$ is e_{n+1}, with norm-squared $=1$. The finite Weyl vector and the fundamental weight are $\frac{1}{2}e_{n+1}$; the coroot is $2e_{n+1}$. In general the level of a representation is $2k'/\psi^2$, where k' is the central element and ψ is the highest root of the algebra. Usually one can arrange $\psi^2=2$, so that k' and the level coincide. Here $\psi^2=1$, hence the conformal embedding under study is actually

$$SO(6n)_1 \supset SO(2n)_3 \oplus SU(2)_{4n}. \tag{3.27}$$

In formulas (3.21) and (3.23), $J^\gamma(l_2+l_1; \ldots, 2n-2l_k)$ has to be replaced by $(2n+(-1)^\gamma 2l;\ 2n-(-1)^\gamma 2l)$.

For $SO(4nk)_1 \supset SO(2n)_{2k} \oplus SO(2k)_{2n}$, one finds

$$\begin{cases} + \\ - \end{cases} = \sum_{\beta,\gamma=0}^{1} \sum_{\{l_i\}} C^\beta J_v^\gamma (2N(0)+N(1); N(1), \ldots, N(n-1), N(n-1) \\ + 2N(n)) C^\beta J_v^\gamma (l_2+l_1; l_2-l_1, \ldots, l_k-l_{k-1}, 2n-l_{k-1}-l_k), \tag{3.28}$$

with

$$\sum_{i=1}^{k} l_i + nk = \begin{cases} \text{even} \\ \text{odd} \end{cases} \tag{3.29}$$

and

$$\begin{cases} o \\ v \end{cases} = \sum_{\beta,\gamma=0}^{1} \sum_{\{l_i\}} J_s C^\beta J_v^\gamma (2N(0)+N(1); N(1), \ldots, N(n-1), N(n-1) \\ + 2N(n)) C^\beta J_v^\gamma (l_2+l_1; l_2-l_1, \ldots, l_k - l_{k-1}, 2n - l_{k-1} - l_k)\,, \tag{3.30}$$

with

$$\sum_{i=1}^{k} l_i = \begin{cases} \text{even} \\ \text{odd} \end{cases}. \tag{3.31}$$

If $k=2$, $(l_2+l_1; \ldots, 2n-l_{k-1}-l_k)$ has to be replaced in (3.28) and (3.30) by the following representation of $SU(2)_{2n} \oplus SU(2)_{2n}$:

$$(2n+l_1-l_2; l_2-l_1)(l_1+l_2; 2n-l_1-l_2) \tag{3.32}$$

on which J_v acts as follows:

$$J_v(2n-a; a)(2n-b; b) = (a; 2n-a)(b; 2n-b)\,. \tag{3.33}$$

4. Exceptional Modular Invariant of $SU(2)_{16}$ and $SO(q)_8$

We will construct these exceptional invariants using a method due to Bouwknegt [6]. If the branching rules for $\hat{g} \supset \hat{h}' \oplus \hat{h}$ are known, any invariant of $\hat{g}$ can be written as follows:

$$Z'' = \sum_{\mu,\nu,m,n} Z''_{\mu m, \nu n} \chi'_\mu \chi_m \chi'^*_\nu \chi^*_n\,, \tag{4.1}$$

where χ'_μ, χ'_ν are characters of $\hat{h}'$ and χ_m, χ_n characters of $\hat{h}$. It follows from the unitarity of T and S that if

$$Z' = \sum_{\mu,\nu} Z'_{\mu,\nu} \chi'_\mu \chi'^*_\nu \tag{4.2}$$

is an invariant of $\hat{h}'$, then the contracted tensor

$$Z_{m,n} = \sum_{\mu,\nu} Z'_{\mu,\nu} Z''_{\mu m, \nu n} \tag{4.3}$$

will give rise to a modular invariant quantity with positive and integer coefficients. $Z_{0,0}$ is however generally not equal to 1. Sometimes all the $Z_{m,n}$ are multiples of $Z_{0,0}$ and it suffices to divide $Z_{m,n}$ by $Z_{0,0}$ to obtain a physical invariant. In other cases, one needs to take more involved combinations of modular invariant quantities to obtain a physical one.

We now try to derive invariants of $SU(2)_{16}$ using this method. The $+$ spinor of $SO(24)_1$ reduces as

$$+ = (0;0,0,0,3)(0+16) + (0;0,1,0,1)(4+12) + (1;1,0,1,0)(6+10) \\ + (0;0,0,2,1)(2+14) + [(0;2,0,0,1)+(2;0,0,0,1)](8) \tag{4.4}$$

and $-$, o, v are given by (3.26). We have used a hybrid notation: representations of $SO(8)_3$ are denoted by generalized Dynkin labels, and those of $SU(2)_{16}$ by the associated finite Dynkin labels.

If we contract the diagonal invariant tensors of $SO(24)_1$ and $SO(8)_3$ we find:

$$4(|0+16|^2+|2+14|^2+|4+12|+|6+10|^2+2|8|^2)\,. \tag{4.5}$$

Here 0 stands for the character χ_0. After dividing by 4, we thus obtain the regular integer spin invariant of $SU(2)_{16}$.

Consider now the following modular invariant tensor of $SO(8)_3$:

$$Z'_{\mu,\phi}=\delta_{\mu,\mathscr{T}\phi}\,, \tag{4.6}$$

where $\mathscr{T}$ is the transformation

$$\mathscr{T}(\phi_0;\,\phi_1,\phi_2,\phi_3,\phi_4)=(\phi_0;\,\phi_3,\phi_2,\phi_1,\phi_4)\,. \tag{4.7}$$

This modular invariant arises from one of the automorphisms of the *finite* Dynkin diagram of $SO(8)$ (triality). Contract $Z'_{\mu,\phi}$ with the invariant tensor of $SO(24)_1$ associated with

$$Z''=|+|^2+|o|^2+(-)(v)^*+(v)(-)^*\,. \tag{4.8}$$

After division by 4, this yields the exceptional modular invariant of $SU(2)_{16}$:

$$Z=|0+16|^2+|4+12|^2+|6+10|^2+(2+14)(8)^*+(8)(2+14)^*+|8|^2\,. \tag{4.9}$$

This result gives us a simple explanation for the existence of this invariant and it shows (see below) that it belongs to an infinite series. In other words, it is not that exceptional after all!

In the following we show that contracting the tensors of (4.6) and (4.8) gives an exceptional modular invariant for all $SO(2k+1)_8$. To establish this, we formulate a series of propositions. The proofs are easy and are omitted. The aim is first to characterize which representations of $SO(8)_{2k+1}$ and $SO(2k+1)_8$ occur in the decomposition of $+$, $-$, o, v and which products of representations are possible. We then describe which combinations of terms can appear in the exceptional invariant and finally we show that this invariant is physical (i.e., $Z_{0,0}=1$ and all coefficients are nonnegative integers).

1. Solving the constraint (3.22) we find

$$\begin{aligned}+ = \sum_{\gamma=0}^{1}\sum_{\{l_i\}} & C^{\Sigma l_i}(CJ_v)^{\gamma}(2N(0)+N(1);\,N(1),N(2),N(3),N(3)\\ & +2N(4)+1)J^{\gamma}(l_2+l_1;\,l_2-l_1,\ldots,l_k-l_{k-1},8-2l_k);\\ -=C(+);\quad & o=J_s(+);\quad v=J_sC(+)\,.\end{aligned} \tag{4.10}$$

2. All representations of $SO(2k+1)_8$ of charge ($=$congruence) 0 appear once in the decomposition of $+$, and twice if they are fixed points under the simple current J; and similarly for $-$, o, and v.

3. The representations $(\phi_0;\phi_1,\phi_2,\phi_3,\phi_4)$ of $SO(8)_{2k+1}$ present in $+$ satisfy:

$$\phi_0=\phi_1 \bmod 2=\phi_3 \bmod 2;\qquad \phi_4=\phi_0+1 \bmod 2\,. \tag{4.11}$$

All such representations appear once in $+$, and twice if they are fixed points under CJ_v (i.e., if $\phi_0=\phi_1$).

4. Denote by R the representations of $SO(2k+1)_8$ such that $JR \neq R$, and by f (and g, h) the fixed points: $Jf = f$. Then $+$ decomposes as a sum of two types of terms:

$$\phi(R+JR) \quad (\text{with } CJ_v\phi = \phi) \tag{4.12}$$

and

$$(\phi + CJ_v\phi)f \quad (\text{with } CJ_v\phi \neq \phi). \tag{4.13}$$

The crux of the proof is that $l_1 = 0$ (respectively $\neq 0$) is equivalent to $N(0) \neq 0$ (respectively $= 0$).

5. There are three possibilities for the components ϕ_0, ϕ_1, ϕ_3 of ϕ: they can all be equal, or two can be equal or they can all be different. If a representation with labels $(\phi_0;\ \phi_1, \phi_2, \phi_3, \phi_4)$ appears in the decomposition of $+$, all those obtained by permutations of ϕ_0, ϕ_1, and ϕ_3 also appear.

6. One can thus naturally group the terms in $+$ as follows:

$$\begin{aligned}
&\mathrm{I}: (a;\ a, \phi_2, a, \phi_4)(R+JR)\\
&\mathrm{II}: [(a;\ b, \phi_2, b, \phi_4) + (b;\ a, \phi_2, b, \phi_4)]f + (b;\ b, \phi_2, a, \phi_4)(R+JR)\\
&\quad \text{with } a \neq b\\
&\mathrm{III}: [(a;\ b, \phi_2, c, \phi_4) + (b;\ a, \phi_2, c, \phi_4)]f\\
&\quad + [(a;\ c, \phi_2, b, \phi_4) + (c;\ a, \phi_2, b, \phi_4)]g\\
&\quad + [(c;\ b, \phi_2, a, \phi_4) + (b;\ c, \phi_2, a, \phi_4)]h\\
&\quad \text{with } a > b > c\,.
\end{aligned} \tag{4.14}$$

$+$ is the sum of all these combinations, each appearing once (they must of course satisfy Eq. (4.11) and level $(\phi) = 2k+1$). The representations R, f, g, h of $SO(2k+1)_8$ are determined by the representations ϕ they multiply. Contracting (4.6) with the first term in (4.8) then gives three types of contributions to the invariant of $SO(2k+1)_8$:

$$\begin{aligned}
&\mathrm{I}: |R+JR|^2\\
&\mathrm{II}: |f|^2 + (R+JR)f^* + f(R+JR)^* \equiv \{f, R, JR\}\\
&\mathrm{III}: f(g+h)^* + g(f+h)^* + h(f+g)^* \equiv [f, g, h]
\end{aligned} \tag{4.15}$$

(we have used the same notation for the representations and for their character). Note that the identity is multiplied by $\phi = (0;\ 0, 0, 0, 2k+1)$ and is thus contained in a term of type I.

7. The four terms in (4.8) give exactly the same contribution after contraction; one can thus divide the total result by 4 to get a physical invariant.

The recipe for constructing the exceptional modular invariant of $SO(2k+1)_8$ is thus: group the terms appearing in the decomposition of $+$ as in (4.14); the contraction with $\delta_{\mu,\mathcal{T}\phi}$ gives rise to terms of type I, II or III, depending on ϕ. For example, for $SO(5)_8$, one has:

$$\begin{aligned}
Z = {} & |00+80|^2 + |20+60|^2 + |22+42|^2 + |04+44|^2 + |14+34|^2\\
& + \{40, 02, 62\} + \{32, 06, 26\} + \{24, 12, 52\} + \{16, 30, 50\}\\
& + \{08, 10, 70\}\,,
\end{aligned} \tag{4.16}$$

where 00 stands for the character $\chi_{(0,0)} \cdots$

The exceptional invariant of $SO(7)_8$ also displays a term of type III: $[f, g, h]$. These invariants had already been obtained numerically in [33]. They can be given a natural interpretation in terms of an automorphism of the fusion rules of the extension of $SO(2k+1)_8$ by the simple current J [24]. Consider the regular invariant generated by the integer spin simple current J:

$$Z_{A,A'} = \begin{cases} \delta_{A',A} + \delta_{A',JA} & \text{if } Q_J(A) = 0 \\ 0 & \text{otherwise} \end{cases} . \tag{4.17}$$

Instead of the combinations (4.15), (4.17) gives rise to

$$\begin{aligned} &\mathrm{I}: |R + JR|^2 \\ &\mathrm{II}: 2|f|^2 + |R + JR|^2 \\ &\mathrm{III}: 2|f|^2 + 2|g|^2 + 2|h|^2 . \end{aligned} \tag{4.18}$$

The theory which has this invariant as partition function has a larger symmetry than the original $\widehat{SO(2k+1)}$. Because of the factor 2, the fixed points f, g, h give rise to 2 primary fields (denoted by $\tilde{f}^+, \tilde{f}^- \ldots$) of the extended algebra [24]. On the other hand, the pair R, JR leads to one primary field (denoted by $\tilde{R}$). Consider the following transformation of the primary fields of the extended algebra:

$$\begin{aligned} &\mathrm{I}: \tilde{R} \leftrightarrow \tilde{R} \\ &\mathrm{II}: \tilde{f}^+ \leftrightarrow \tilde{f}^+ , \quad \tilde{f}^- \leftrightarrow \tilde{R} \\ &\mathrm{III}: \tilde{f}^+ \leftrightarrow \tilde{g}^- , \quad \tilde{g}^+ \leftrightarrow \tilde{h}^- , \quad \tilde{h}^+ \leftrightarrow \tilde{f}^- . \end{aligned} \tag{4.19}$$

Acting with this transformation on one chiral sector of the regular invariant gives the exceptional invariant discovered above. The existence of such a transformation (we did not prove that it constitutes an automorphism of the fusion rules, but this is likely) provides a test of the correctness of our computations.

We now turn to the case of $SO(2k)_8$. In [33], we found that $SO(4)_8 = SU(2)_8 \oplus SU(2)_8$ has an exceptional modular invariant given by

$$\begin{aligned} Z = &|00 + 88 + 08 + 80|^2 + |22 + 66 + 26 + 62|^2 \\ &+ (02 + 86 + 06 + 82 + 20 + 68 + 28 + 60)(44)^* + \text{c.c.} \\ &+ 2|44|^2 + |24 + 64 + 42 + 46|^2 + |04 + 84 + 40 + 48|^2 , \end{aligned} \tag{4.20}$$

where each pair ij stands for a representation of $SU(2)_8 \oplus SU(2)_8$ with (finite) Dynkin labels i and j (the 0^{th} labels are $8-i$ and $8-j$ respectively). These representations appear in the decomposition of $+$ and o, but not in $-$ nor v (cf. Eqs. (3.28–33) with $n=4$ and $k=2$). One can easily verify that contracting the tensor of $SO(16k)_1$ (with $k=2$) arising from $Z'' = |(o) + (+)|^2$ with the tensor $\delta_{\mu, \mathcal{T}\phi}$ of $SO(8)_{2k}$ (cf. Eq. (4.7)) and dividing by 4 reproduces this exceptional invariant.

In the following, we show through a series of propositions that this construction gives an exceptional invariant for all $SO(2k)_8$, $k \geqq 2$. We will have to analyze the possible patterns for the terms of this invariant and to prove that all the coefficients are integers (after division by 4).

1. The decomposition of the representation $+$ of $SO(16k)_1$ is given by Eq. (3.28) with $\Sigma l_i = \text{even}$. From now on we note J'_v, J'_s, $J'_c = J'_v J'_s$ the simple currents of $SO(8)_{2k}$, and J_v, J_s, J_c those of $SO(2k)_8$.

$$o = J'_s(+) = J_s(+) . \tag{4.21}$$

Note that $J_s^2 = J_v$ if k is odd and $J_s^2 = \mathbb{1}$ if k is even.

2. The representations of $SO(2k)_8$ present in $+$ and o are of charge (=congruence) 0 with respect to J_v and J_s and reciprocally, all representations of charge 0 appear in $+$ and o. Remember that a representation $(\Lambda_0; \ldots, \Lambda_k)$ has charge 0 with respect to J_v if $\Lambda_{k-1}+\Lambda_k=0 \bmod 2$ and has charge 0 with respect to J_s if

$$2\Lambda_1+4\Lambda_2+\ldots+2(k-2)\Lambda_{k-2}+(k-2)\Lambda_{k-1}+k\Lambda_k=0 \bmod 4\,. \tag{4.22}$$

The multiplicity of Λ is usually 1, but it is 2 if Λ is a fixed point under C or CJ_v and 4 if it is a fixed point under J_v (i.e., under C and CJ_v).

3. The same statement applies to the representations ϕ of $SO(8)_{2k}$; ϕ_0, ϕ_1, ϕ_3, and ϕ_4 are thus all equal mod 2.

4. All products $\phi\Lambda$ appearing in $+$ satisfy

$$C'\phi=\phi\,, \qquad C\Lambda\neq\Lambda \tag{4.23}$$

or

$$C'\phi\neq\phi\,, \qquad C\Lambda=\Lambda \tag{4.24}$$

and similarly with C', C replaced by $C'J'_v$, CJ_v.

5. There are 5 possibilities for the components ϕ_0, ϕ_1, ϕ_3, and ϕ_4 of ϕ: from all equal to all different (see below). If the representation ϕ appears in $+$, all those obtained by permutations of ϕ_0, ϕ_1, ϕ_3, ϕ_4 are also present.

6. $+$ decomposes as the sum of the following combinations of terms and each allowed combination appears once. The behaviour of the representations $\Lambda, \Lambda_1, \ldots$ under C and CJ_v follows from Proposition 4.

$$\begin{aligned}
\mathrm{I}:\,& (a;a,\phi_2,a,a)(1+C+CJ_v+J_v)\Lambda\\
\mathrm{II}:\,& [(a;a,\phi_2,a,b)+(a;a,\phi_2,b,a)](1+CJ_v)\Lambda_1\\
&+[(a;b,\phi_2,a,a)+(b;a,\phi_2,a,a)](1+C)\Lambda_2\\
&\text{with } a\neq b\\
\mathrm{III}:\,& (a;a,\phi_2,b,b)(1+C+CJ_v+J_v)\Lambda_1\\
&+(b;b,\phi_2,a,a)(1+C+CJ_v+J_v)\Lambda_2\\
&+[(a;b,\phi_2,a,b)+(b;a,\phi_2,a,b)+(a;b,\phi_2,b,a)+(b;a,\phi_2,b,a)]\Lambda_3\\
&\text{with } a>b\\
\mathrm{IV}:\,& [(a;a,\phi_2,b,c)+(a;a,\phi_2,c,b)](1+CJ_v)\Lambda_1\\
&+[(b;c,\phi_2,a,a)+(c;b,\phi_2,a,a)](1+C)\Lambda_2\\
&+[(a;b,\phi_2,a,c)+(b;a,\phi_2,a,c)+(a;b,\phi_2,c,a)+(b;a,\phi_2,c,a)]\Lambda_3\\
&+[(a;c,\phi_2,a,b)+(c;a,\phi_2,a,b)+(a;c,\phi_2,b,a)+(c;a,\phi_2,b,a)]\Lambda_4\\
&\text{with } a\neq b,c;\ b>c\\
\mathrm{V}:\,& [(a;b,\phi_2,c,d)+(b;a,\phi_2,c,d)+(a;b,\phi_2,d,c)+(b;a,\phi_2,d,c)]\Lambda_1\\
&+\ldots+[(b;c,\phi_2,a,d)+\ldots+(c;b,\phi_2,d,a)]\Lambda_6\\
&\text{with } a>b>c>d\,.
\end{aligned} \tag{4.25}$$

Combinations of type IV are absent for $k<3$ and those of type V for $k<6$.

7. Consider the product

$$\phi\Lambda = (2N(0)+N(1);\, N(1), N(2), N(3), N(3) + 2N(4))\,(l_2+l_1;\, l_2-l_1, \ldots, l_k-l_{k-1}, 8-l_k-l_{k-1}). \tag{4.26}$$

Then

$$N(0)=N(4),\, N(1)=N(3) \Leftrightarrow l_i = 4-l_{k+1-i} \quad (1 \leqq i \leqq k). \tag{4.27}$$

The behaviours of ϕ under J'_s and of Λ under J_s (cf. 3.25) are thus closely related.

8. Contracting the tensors and dividing by 4 yields the following types of terms:

$$\mathrm{I}: |(1+C+CJ_v+J_v)\Lambda|^2. \tag{4.28}$$

If k is even, Λ is a fixed point under J_s (because of Proposition 7) and we can rewrite this expression as

$$|(1+J_v)\Lambda+(1+J_v)C\Lambda|^2. \tag{4.29}$$

If k is odd, Λ is not a fixed point under J_s; using Proposition 7, we can rewrite expression (4.28) as

$$|(1+J_v+J_s+J_c)\Lambda|^2. \tag{4.30}$$

Using $o=J'_s(+)$, contracting and dividing, we find next:

$$\mathrm{II}: |(1+CJ_v)\Lambda_1+(1+C)\Lambda_2|^2. \tag{4.31}$$

Λ_1 and Λ_2 can be related using $J'_s(+)=J_s(+)$:

$$J_s(1+CJ_v)\Lambda_1=(1+C)\Lambda_2. \tag{4.32}$$

This yields the contribution

$$|(1+J_v+J_s+J_c)\Lambda_1|^2 \tag{4.33}$$

to the exceptional modular invariant.

We proceed in the same way for the terms of type III, IV, V. They organize themselves into orbits under J_s if k is odd and into orbits under J_v and J_s if k is even; incomplete orbits never occur. Whenever an expression like $(1+J_s+J_v+J_c)\Lambda$ appears, the representations Λ, $J_s\Lambda$, $J_v\Lambda$, and $J_c\Lambda$ are all different.

$$\begin{aligned}
&\mathrm{III}: [(1+J_s+J_v+J_c)(1+C)\Lambda_1]\Lambda_3^* + \text{c.c.} + 2|\Lambda_3|^2 \\
&\mathrm{IV}: [(1+J_s+J_v+J_c)\Lambda_1]\,[(1+J_s)\Lambda_3]^* + \text{c.c.} + |(1+J_s)\Lambda_3|^2 \\
&\mathrm{V}: [(1+J_s)\Lambda_1]\,[(1+J_s)(\Lambda_3+\Lambda_5)]^* + [(1+J_s)\Lambda_3]\,[(1+J_s)(\Lambda_1+\Lambda_5)]^* \\
&\qquad + [(1+J_s)\Lambda_5]\,[(1+J_s)(\Lambda_1+\Lambda_3)]^*.
\end{aligned} \tag{4.34}$$

This completes the proof of the existence of an exceptional modular invariant for $SO(2k)_8$.

The recipe for constructing this invariant is now clear. To get the terms of type I, it suffices to identify all nonnegative integers a, ϕ_2 satisfying $4a+2\phi_2=2k$. One finds the corresponding representations Λ using Eqs. (3.28–29); this gives rise to (4.29) or to (4.30) (depending on the parity of k). One then proceeds similarly for the terms of type II, ..., V.

Here the identity appears in the combination (of type II) $|\mathbb{1}+J_v+J_s+J_c|^2$. It is thus likely that this exceptional invariant is related by an automorphism of the extended algebra to the regular invariant:

$$Z_{\Lambda,\Lambda'}=\begin{cases}\delta_{\Lambda',\Lambda}+\delta_{\Lambda',J_v\Lambda}+\delta_{\Lambda',J_s\Lambda}+\delta_{\Lambda',J_c\Lambda} & \text{if } Q_v(\Lambda)=Q_s(\Lambda)=0\\ 0 & \text{otherwise}\end{cases} \tag{4.35}$$

This is the integer spin invariant tensor of $SO(2k)_8$ generated by J_s if k is odd and the product of the tensors generated by J_v and J_s if k is even. It gives rise to:

$$\begin{aligned}
\mathrm{I}:&\,2|(1+J_v)\Lambda|^2+2|(1+J_v)C\Lambda|^2 \text{ if } k \text{ is even}\\
&\,|(1+J_s+J_v+J_c)\Lambda|^2 \text{ if } k \text{ is odd}\\
\mathrm{II}:&\,|(1+J_v+J_s+J_c)\Lambda_1|^2\\
\mathrm{III}:&\,|(1+J_s+J_v+J_c)\Lambda_1|^2+|(1+J_s+J_v+J_c)C\Lambda_1|^2+4|\Lambda_3|^2\\
\mathrm{IV}:&\,|(1+J_s+J_v+J_c)\Lambda_1|^2+2|(1+J_s)\Lambda_3|^2\\
\mathrm{V}:&\,2|(1+J_s)\Lambda_1|^2+2|(1+J_s)\Lambda_3|^2+2|(1+J_s)\Lambda_5|^2\,.
\end{aligned} \tag{4.36}$$

Comparing expressions (4.29–30, 4.33–34), and (4.36), one sees that there is a simple transformation (the analog of (4.19)) which transforms the regular invariant into the exceptional one and which is probably an automorphism of the fusion rules of the extended algebra.

5. Conclusion

An elegant method for finding new modular invariants of a KM algebra $\hat{h}$ is to use the connection with another algebra $\hat{h}'$ provided by a conformal embedding $\hat{g} \supset \hat{h}' \oplus \hat{h}$. In this way, we have exhibited a one-to-one correspondence between the modular invariants of $Sp(2n)_k$ and those of $Sp(2k)_n$.

We have also discovered an exceptional modular invariant for all $SO(q \geqq 4)$ at level 8. The mechanism behind this – the triality of $SO(8)$ – also provides a simple explanation of the exceptional invariant of $SU(2)_{16}$. These invariants belong to a class that comprises all the modular invariants which are due to an automorphism of an extension of a KM algebra by integer spin simple currents. Many (but not all of them) can be obtained using the methods of this paper. On the other hand, *all* the known ones can be accounted for by the conjecture of [33]. This conjecture predicts (among others) modular invariants of the above type for all simple Kac-Moody algebras with fixed points under a simple current of order 2 and spin 4. An interesting problem would be to prove this conjecture directly, by constructing the regular extensions of KM algebras and by identifying their representations.

Acknowledgements. I am grateful to S. Schrans, W. Troost and especially to J. Figueroa-O'Farrill for many interesting discussions and for reading the manuscript.

Note added. The branching rules used here have also been examined by K. Hasegawa, Publ. RIMS **25**, 741 (1989)

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Communicated by N. Yu. Reshetikhin