Commun. Math. Phys. 137, 149–160 (1991)

Communications in
Mathematical
Physics

q-Oscillator Realizations of the Quantum Superalgebras $sl_q(m, n)$ and $osp_q(m, 2n)$

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Received May 7, 1990

Abstract. Realizations of the quantum superalgebras corresponding to the $A(m, n)$, $B(m, n)$, $C(n+1)$, and $D(m, n)$ series are given in terms of the creation and annihilation operators of q-deformed Bose and Fermi oscillators.

1. Introduction

Let $\mathscr{G}$ be a (simple) Lie algebra. The quantum Lie algebra [1–5] $\mathscr{G}_q$ is a deformation of the universal envelopping algebra of $\mathscr{G}$ which is endowed with a Hopf algebra structure [6]. This mathematical object is currently drawing a lot of attention, in part because of its connections with integrable systems and conformal field theories. The quantum algebra $\mathscr{G}_q$ can be characterized by giving its generators together with defining relations based on the Cartan matrix of $\mathscr{G}$.

The Weyl and Clifford algebras also admit quantum deformations [7] with q-analogues of the Bose, and respectively, Fermi oscillator operators as generators [7–10]. These quantized algebras have been used to construct oscillator realizations of the quantum algebras that correspond to all classical Lie algebras [7]. Here, we provide similar representations of the quantum Lie superalgebras associated to the unitary and the orthosymplectic series. Algebra homomorphisms from the quantized envelopping algebras of type $A(m, n)$, $B(m, n)$, $C(n+1)$, and $D(m, n)$ into the quantum Weyl superalgebra will be presented by expressing the generators of the quantum superalgebras as linears and bilinears in the creation and annihilation operators of q-bosons and q-fermions.

In Sect. 2 we review some results on the classification of contragredient Lie superalgebras. A general description of the quantum Lie superalgebras is given in

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Sect. 3. We introduce in Sect. 4 the q-analogue of the Bose and Fermi oscillators and present the quantized Weyl superalgebra. Section 5 comprises our main results, that is the q-oscillator realization of the quantum Lie superalgebras $sl_q(m,n)$ and $osp_q(m,n)$. Unless stated otherwise, we shall stick to the conventions of Kac regarding superalgebras [11–13], this means in particular, that we shall use non-symmetric Cartan matrices. We discuss in the Appendix the modifications that arise if one adopts instead, symmetric Cartan matrices.

2. Unitary and Orthosymplectic Lie Superalgebras

The Lie superalgebras $sl(m,n)$ and $osp(m,n)$ that respectively form the unitary and orthosymplectic series are in many ways similar to the classical Lie algebras. A superalgebra $\mathcal{G}$ of rank r belonging to either series can be characterized [11–13] by a Cartan matrix (a_{ij}) and a subset $\tau \subset I \equiv \{1, \ldots, r\}$ that identifies the odd generators. Unless $\mathcal{G}$ is an ordinary Lie algebra, in which case $\tau = \emptyset$, the set τ can actually be taken to consist of only one element [11, 12]. Let $[\,,\,]$ stand for the graded product defined by

$$[x,y] = -(-)^{\deg x \deg y}[y,x] \quad \text{and} \quad [x,[y,z]] = [[x,y],z] + (-)^{\deg x \deg y}[y,[z,x]],$$

and denote as usual by $\mathrm{ad}\,x$ the adjoint operation $(\mathrm{ad}\,x)y = [x,y]$. The algebra $\mathcal{G}$ can be constructed from the $3r$ generators $\hat{e}_i$, $\hat{f}_i$, and $\hat{h}_i$, $i \in I$, which satisfy the relations [13]

$$\begin{aligned} &[\hat{e}_i, \hat{f}_j] = \delta_{ij}\hat{h}_i, \quad [\hat{h}_i, \hat{h}_j] = 0, \\ &[\hat{h}_i, \hat{e}_j] = a_{ij}\hat{e}_j, \quad [\hat{h}_i, \hat{f}_j] = -a_{ij}\hat{f}_j, \end{aligned} \tag{2.1}$$

and

$$(\mathrm{ad}\,\hat{e}_i)^{1-\tilde{a}_{ij}}\hat{e}_j = 0, \quad (\mathrm{ad}\,\hat{f}_i)^{1-\tilde{a}_{ij}}\hat{f}_j = 0, \quad i \neq j, \tag{2.2}$$

with

$$\deg \hat{h}_i = 0; \quad \deg \hat{e}_i = \deg \hat{f}_i = 0, \quad i \notin \tau; \quad \deg \hat{e}_i = \deg \hat{f}_i = 1, \quad i \in \tau,$$

and $(\tilde{a}_{ij})$ the matrix which is obtained from the non-symmetric Cartan matrix (a_{ij}) by substituting -1 for the strictly positive elements in the rows with 0 on the diagonal entry. In the case of Lie algebras the matrices (a_{ij}) and $(\tilde{a}_{ij})$ coincide and Eq. (2.2) reduce to the standard Serre relations [14].

Following the established notation [11, 12], we put

$$\begin{aligned} &A(m,n) = sl(m+1, n+1), \quad m,n \geqq 0,\ m \neq n, \\ &A(m,m) = sl(m+1, m+1)/\{\lambda \mathbf{1}_{2m+2}\}, \quad m > 0,\ \lambda \in \mathbf{C}, \\ &B(m,n) = osp(2m+1, 2n), \quad m \geqq 0,\ n > 0, \\ &C(n+1) = osp(2, 2n), \quad n > 0, \\ &D(m,n) = osp(2m, 2n), \quad m \geqq 2,\ n > 0. \end{aligned}$$

We give below the Cartan matrix (a_{ij}), the set τ and the rank r, which are associated to the superalgebras belonging to these series [12, 13]. In each case, we also specify a set of rational numbers d_i, $i = 1, \ldots, r$, such that: $d_i a_{ij} = d_j a_{ji}$. These numbers d_i will enter in the defining relations of the quantum superalgebras (see next section). In what follows

$$\mathscr{A}_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{2.3}$$

stands for the $n \times n$ Cartan matrix of the rank n ordinary Lie algebra A_n.

- $A(m, n)$

$$(a_{ij}) = \begin{pmatrix} \mathscr{A}_m & & & & \\ & & -1 & & \\ & -1 & 0 & 1 & \\ & & -1 & & \\ & & & & \mathscr{A}_n \end{pmatrix}, \tag{2.4}$$

$$\tau = \{m+1\}, \qquad r = m+n+1, \tag{2.5}$$

$$d_i = (\underbrace{1, \ldots, 1}_{m+1}, \underbrace{-1, \ldots, -1}_{n}). \tag{2.6}$$

When $m=n$, the algebra generated by the elements $\hat{e}_i$, $\hat{f}_i$, and $\hat{h}_i$, $i=1, \ldots, 2m+1$, has a one-dimensional center [12] which consists of the element

$$\hat{c} \equiv (\hat{h}_1 - \hat{h}_{2m+1}) + 2(\hat{h}_2 - \hat{h}_{2m}) + \ldots + m(\hat{h}_m - \hat{h}_{m+2}) + (m+1)\hat{h}_{m+1}.$$

The identification with $A(m, m)$ is achieved once this center has been factored out. This is the only case where such a situation occurs [11].

- $B(m, n)$

$$(a_{ij}) = \begin{pmatrix} \mathscr{A}_{n-1} & & & & & & \\ & & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & -1 & & & & \\ & & & & \mathscr{A}_{m-1} & & \\ & & & & & & -1 \\ & & & & & -2 & 2 \end{pmatrix}, \tag{2.7}$$

$$\tau = \{n\}, \qquad r = m+n, \tag{2.8}$$

$$d_i = (\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m-1}, -\tfrac{1}{2}). \tag{2.9}$$

- $B(0, n)$

$$(a_{ij}) = \begin{pmatrix} \mathscr{A}_{n-1} & & \\ & & -1 \\ & -2 & 2 \end{pmatrix}, \tag{2.10}$$

$$\tau = \{n\}, \qquad r = n, \tag{2.11}$$

$$d_i = (\underbrace{1, \ldots, 1}_{n-1}, \tfrac{1}{2}). \tag{2.12}$$

- $C(n+1)$

$$(a_{ij}) = \begin{pmatrix} 0 & 1 & & & \\ -1 & & & & \\ & & \mathscr{A}_{n-1} & & \\ & & & & -2 \\ & & & -1 & 2 \end{pmatrix}, \tag{2.13}$$

$$\tau = \{1\}, \quad r = n+1, \tag{2.14}$$

$$d_i = (1, \underbrace{-1, \ldots, -1}_{n-1}, -2). \tag{2.15}$$

- $D(m, n)$

$$(a_{ij}) = \begin{pmatrix} \mathscr{A}_{n-1} & & & & & & \\ & & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & -1 & & & & \\ & & & & \mathscr{A}_{m-1} & & -1 \\ & & & & & & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}, \tag{2.16}$$

$$\tau = \{n\}, \quad r = m+n, \tag{2.17}$$

$$d_i = (\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m}). \tag{2.18}$$

3. Quantum Lie Superalgebras

Let $\mathscr{G}$ be a rank r superalgebra belonging to the unitary or the orthosymplectic series, described in the previous section. Let $q \in \mathbb{C} \setminus \{0\}$ be the deformation parameter which we shall sometimes write $q = e^{\eta/2}$. We shall also use $q_i = q^{d_i}$, with d_i the numbers, given in the previous section, that symmetrize the Cartan matrix (a_{ij}), and shall assume $q_i^4 \neq 1$. The quantum superalgebras $\mathscr{G}_q$ of the universal envelopping algebra of $\mathscr{G}$ is again generated by $3r$ elements e_i, f_i, and h_i, $i \in I$, which satisfy [10]

$$\begin{aligned} &[e_i, f_j] = \delta_{ij} \frac{\sinh(\eta d_i h_i)}{\sinh(\eta d_i)}, \quad &&[h_i, h_j] = 0, \\ &[h_i, e_j] = a_{ij} e_j, &&[h_i, f_j] = -a_{ij} f_j, \end{aligned} \tag{3.1}$$

with

$$\deg h_i = 0; \quad \deg e_i = \deg f_i = 0, \quad i \notin \tau; \quad \deg e_i = \deg f_i = 1, \quad i \in \tau,$$

and further obey certain generalized Serre relations which will be specified. It is convenient to introduce the quantities $k_i = q_i^{h_i}$ in terms of which the defining relations (3.1) become:

$$\begin{aligned} &k_i k_i^{-1}=k_i^{-1}k_i=1\,, \qquad k_i k_j=k_j k_i\,,\\ &k_i e_j k_i^{-1}=q_i^{a_{ij}}e_j\,, \qquad k_i f_j k_i^{-1}=q_i^{-a_{ij}}f_j\,,\\ &[e_j,f_i]=\delta_{ij}\frac{k_i^2-k_i^{-2}}{q_i^2-q_i^{-2}}\,. \end{aligned} \tag{3.2}$$

The quantum superalgebra $\mathscr{G}_q$ is endowed with a Hopf algebra structure [6]. The action of the coproduct $\Delta:\mathscr{G}_q\to\mathscr{G}_q\otimes\mathscr{G}_q$, antipode $S:\mathscr{G}_q\to\mathscr{G}_q$ and counit $\varepsilon:\mathscr{G}_q\to\mathbf{C}$ on the generators is as follows [10]:

$$\begin{aligned} &\Delta(h_i)=h_i\otimes 1+1\otimes h_i\,, && \Delta(k_i)=k_i\otimes k_i\,,\\ &\Delta(e_i)=e_i\otimes k_i+k_i^{-1}\otimes e_i\,, && \Delta(f_i)=f_i\otimes k_i+k_i^{-1}\otimes f_i\,,\\ &S(h_i)=-h_i\,, && S(k_i)=k_i^{-1}\,,\\ &S(e_i)=-q_i^{a_{ii}}e_i\,, && S(f_i)=-q_i^{a_{ii}}f_i\,,\\ &\varepsilon(h_i)=\varepsilon(e_i)=\varepsilon(f_i)=0\,, && \varepsilon(1)=1\,. \end{aligned} \tag{3.3}$$

One can define the q-analogue ad_q of the adjoint operation by [15, 10]

$$\mathrm{ad}_q=(\mu_L\otimes\mu_R)(\mathrm{id}\otimes S)\Delta\,, \tag{3.4}$$

with id the identity operator and μ_L, μ_R the left and right (graded) multiplications: $\mu_L(x)y=xy$, $\mu_R(x)y=(-)^{\deg x\deg y}yx$. The quantum Serre relations are most simply expressed in terms of the following rescaled generators [15],

$$\mathscr{E}_i=e_ik_i^{-1}\,, \qquad \mathscr{F}_i=f_ik_i^{-1}\,. \tag{3.5}$$

They then take a form similar to (2.2) and read

$$(\mathrm{ad}_q\mathscr{E}_i)^{1-\tilde{a}_{ij}}\mathscr{E}_j=0\,, \qquad (\mathrm{ad}_q\mathscr{F}_i)^{1-\tilde{a}_{ij}}\mathscr{F}_j=0\,, \qquad i\neq j\,. \tag{3.6}$$

The defining system for the generators of $\mathscr{G}_q$ is thus completed by adding these generalized Serre relations to Eq. (3.1) or Eq. (3.2).

Let us record for reference, the explicit forms that conditions (3.6) take for $sl_q(m,n)$ and $osp_q(m,2n)$. One has, always with $i\neq j$,

$\tilde{a}_{ij}=0$:

$$e_ie_j-(-1)^{\deg e_i\deg e_j}e_je_i=0\,; \tag{3.7}$$

$\tilde{a}_{ij}=-1$:
for $\deg e_i=0$,

$$e_i^2e_j-2\cosh(\eta d_i)e_ie_je_i+e_je_i^2=0\,, \tag{3.8a}$$

for $\deg e_i=1$,

$$e_i^2e_j-(\cosh(2\eta d_i)-\sinh(2\eta d_i))e_je_i^2=0\,; \tag{3.8b}$$

$\tilde{a}_{ij}=-2$:
for $\deg e_i=0$,

$$e_i^3e_j-(1+2\cosh(2\eta d_i))(e_i^2e_je_i-e_ie_je_i^2)-e_je_i^3=0\,, \tag{3.9a}$$

for $\deg e_i=1$,

$$e_i^3e_j+(1-2\cosh(2\eta d_i))((-1)^{\deg e_j}e_i^2e_je_i+e_ie_je_i^2)+(-1)^{\deg e_j}e_je_i^3=0\,. \tag{3.9b}$$

In deriving these equations one should recall that $q_i^{a_{ij}} = q_j^{a_{ji}}$. Substituting $e_k \to f_k$ and $\eta \to -\eta$ in the above relations, one obtains the corresponding conditions on the generators f_k.

4. q-Analogues of the Bose and Fermi Oscillators

Let s and t be two positive integers. The Weyl superalgebra, here denoted by $W(s,t)$, is generated by the annihilation and creation operators of s Bose and t Fermi oscillators. The q-deformation of $W(s,t)$ is obtained by introducing the quantum analogues of these oscillators [7].

The annihilation, creation, and number operators b_i, $b_i^\dagger$, and N_i, $i=1,\dots,s$, of bosonic q-oscillators are taken to satisfy,

$$b_i b_i^\dagger - q^2 b_i^\dagger b_i = q^{-2N_i}, \qquad b_i b_i^\dagger - q^{-2} b_i^\dagger b_i = q^{2N_i}, \tag{4.1}$$

$$[N_i, b_j] = -\delta_{ij} b_i, \qquad [N_i, b_j^\dagger] = \delta_{ij} b_i^\dagger, \tag{4.2}$$

and for $i \neq j$

$$[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = [b_i, b_j^\dagger] = 0, \qquad [N_i, N_j] = 0, \tag{4.3}$$

with $\deg b_i = \deg b_i^\dagger = \deg N_i = 0$.

Similarly, the annihilation, creation and number operators, ψ_i, $\psi_i^\dagger$, and M_i, $i=1,\dots,t$, of fermionic q-oscillators are defined through,

$$\psi_i \psi_i^\dagger + q^2 \psi_i^\dagger \psi_i = q^{2M_i}, \qquad \psi_i \psi_i^\dagger + q^{-2} \psi_i^\dagger \psi_i = q^{-2M_i}, \tag{4.4}$$

$$[M_i, \psi_j] = -\delta_{ij} \psi_j, \qquad [M_i, \psi_j^\dagger] = \delta_{ij} \psi_j^\dagger, \tag{4.5}$$

$$\{\psi_i, \psi_j\} = 0, \qquad \{\psi_i^\dagger, \psi_j^\dagger\} = 0, \tag{4.6a}$$

and for $i \neq j$,

$$\{\psi_i, \psi_j^\dagger\} = 0, \qquad [M_i, M_j] = 0, \tag{4.6b}$$

with $\deg \psi_i = \deg \psi_i^\dagger = 1$, $\deg M_i = 0$, and $\{x, y\} = xy + yx$. It is further assumed that bosonic and fermionic operators commute,

$$[b_i, \psi_j] = [b_i, \psi_j^\dagger] = [b_i^\dagger, \psi_j] = [b_i^\dagger, \psi_j^\dagger] = 0, \tag{4.7a}$$

$$[N_i, \psi_j] = [N_i, \psi_j^\dagger] = [M_i, b_j] = [M_i, b_j^\dagger] = [N_i, M_j] = 0. \tag{4.7b}$$

The algebra $W_q(s,t)$ generated by the operators b_i, $b_i^\dagger$, N_i, $i=1,\dots,s$, and ψ_j, $\psi_j^\dagger$, M_j, $j=1,\dots,t$, subjected to Eqs. (4.1)–(4.7), will be referred to as q-analogue of the Weyl superalgebra $W(s,t)$. The second conditions in (4.1) and (4.4) are sometimes omitted [8–10], their presence amounts to requiring the invariance [16] of the defining system under $q \to q^{-1}$. Note that Eqs. (4.1) are equivalent to

$$b_i b_i^\dagger = \frac{q^{2(N_i+1)} - q^{-2(N_i+1)}}{q^2 - q^{-2}}, \qquad b_i^\dagger b_i = \frac{q^{2N_i} - q^{-2N_i}}{q^2 - q^{-2}}, \tag{4.8}$$

and (4.4) to

$$\psi_i \psi_i^\dagger = \frac{q^{2(1-M_i)} - q^{-2(1-M_i)}}{q^2 - q^{-2}}, \qquad \psi_i^\dagger \psi_i = \frac{q^{2M_i} - q^{-2M_i}}{q^2 - q^{-2}}. \tag{4.9}$$

When $q=1$, Eqs. (4.1)–(4.7) reduce to the canonical commutation and anticommutation relations of ordinary bosonic and fermionic annihilation and creation operators. We shall denote by $\hat{b}_i$, $\hat{b}_i^\dagger$, $\hat{\psi}_i$, and $\hat{\psi}_i^\dagger$ the classical relatives of b_i, $b_i^\dagger$, ψ_i, and $\psi_i^\dagger$; note that $N_i \to \hat{N}_i = \hat{b}_i^\dagger \hat{b}_i$, and $M_i \to \hat{M}_i = \hat{\psi}_i^\dagger \hat{\psi}_i$ as $q \to 1$.

The defining relations of the q-Weyl superalgebra can be realized by expressing the q-oscillator operators in terms of their classical analogues. For the bosonic operators take [9]

$$b_i = \sqrt{\frac{f(\hat{N}_i+1)}{\hat{N}_i+1}}\,\hat{b}_i, \qquad b_i^\dagger = \sqrt{\frac{f(\hat{N}_i)}{\hat{N}_i}}\,\hat{b}_i^\dagger, \qquad N_i = \hat{N}_i, \tag{4.10}$$

with

$$f(\hat{N}_i) = \frac{q^{2\hat{N}_i} - q^{-2\hat{N}_i}}{q^2 - q^{-2}} = \frac{\sinh(\eta \hat{N}_i)}{\sinh \eta}. \tag{4.11}$$

[Notice that q has to be real or a pure phase, i.e. η has to be real or purely imaginary, for b_i and $b_i^\dagger$ in (4.10) to be hermitian conjugates.] For the fermionic operators set

$$\psi_i = \hat{\psi}_i, \qquad \psi_i^\dagger = \hat{\psi}_i^\dagger, \qquad M_i = \hat{M}_i. \tag{4.12}$$

It is easy to check that Eqs. (4.1)–(4.7) are verified under such identifications. For instance, since $\hat{M}_i^2 = \hat{M}_i$, one has $q^{2\hat{M}_i} = (1-\hat{M}_i) + q^2 \hat{M}_i = \hat{\psi}_i \hat{\psi}_i^\dagger + q^2 \hat{\psi}_i^\dagger \hat{\psi}_i$.

5. q-Oscillator Representations of Quantum Superalgebras

We shall now construct q-oscillator representations of the quantum superalgebras $sl_q(m,n)$ and $osp_q(m,2n)$. We shall provide explicit expressions for the corresponding generators as linears and bilinears in q-deformed bosonic and fermionic oscillator operators. We shall successively consider the quantum superalgebras $A_q(m,n)$, $B_q(m,n)$, $C_q(n+1)$, and $D_q(m,n)$ associated to the $A(m,n)$, $B(m,n)$, $C(n+1)$, and $D(m,n)$ Lie superalgebra series described in Sect. 2.

Let us observe first that the quantum algebra corresponding to the classical Lie algebra A_n admits the following four representations [7, 17]:

$\pi_{A_n}^{(1)}$:

$$e_k = b_k^\dagger b_{k+1}, \qquad f_k = b_{k+1}^\dagger b_k, \qquad h_k = N_k - N_{k+1}, \qquad k = 1, \ldots, n. \tag{5.1}$$

$\pi_{A_n}^{(2)}$:

$$\begin{aligned} &e_{n-2k+1} = i b_{n-2k+1}^\dagger b_{n-2k+2}^\dagger, \\ &f_{n-2k+1} = i b_{n-2k+1} b_{n-2k+2}, \qquad k = 1, \ldots, [\![n/2]\!], \\ &h_{n-2k+1} = N_{n-2k+1} + N_{n-2k+2} + 1, \\ &e_{n-2k} = i b_{n-2k} b_{n-2k+1}, \\ &f_{n-2k} = i b_{n-2k}^\dagger b_{n-2k+1}^\dagger, \qquad k = 0, \ldots, [\![(n-1)/2]\!]. \\ &h_{n-2k} = -(N_{n-2k} + N_{n-2k+1} + 1), \end{aligned} \tag{5.2}$$

$\pi^{(3)}_{A_n}$:

$$e_k=\psi_k^\dagger\psi_{k+1}\,, \quad f_k=\psi_{k+1}^\dagger\psi_k\,, \quad h_k=M_k-M_{k+1}\,, \quad k=1,\ldots,n. \tag{5.3}$$

$\pi^{(4)}_{A_n}$:

$$\begin{aligned}
&e_{n-2k+1}=i\psi_{n-2k+1}^\dagger\psi_{n-2k+2}^\dagger\,, \\
&f_{n-2k+1}=i\psi_{n-2k+1}\psi_{n-2k+2}\,, \qquad k=1,\ldots,[\![n/2]\!]\,, \\
&h_{n-2k+1}=M_{n-2k+1}+M_{n-2k+2}-1\,, \\
&e_{n-2k}=i\psi_{n-2k}\psi_{n-2k+1}\,, \\
&f_{n-2k}=i\psi_{n-2k}^\dagger\psi_{n-2k+1}^\dagger\,, \qquad k=0,\ldots,[\![(n-1)/2]\!]\,. \\
&h_{n-2k}=-(M_{n-2k}+M_{n-2k+1}-1)\,,
\end{aligned} \tag{5.4}$$

The symbol $[\![x]\!]$ stands for the integer part of x. Equivalent representations are obtained upon exchanging e_i and f_i, and letting $h_i \to -h_i$.

Under the standard inner product on the Hilbert space of oscillator states, $\pi^{(1)}_{A_n}$, $\pi^{(3)}_{A_n}$, and $\pi^{(4)}_{A_n}$ are unitary, while $\pi^{(2)}_{A_n}$ is antiunitary. Upon suitably combining these representations, realizations of the quantum superalgebras $sl_q(m,n)$ and $osp_q(m,2n)$ will be obtained.

- $A_q(m,n)$

In this case we can form four algebra homomorphisms of $A_q(m,n)$ into

$$W_q(n+1,m+1)\,.$$

For instance, we can take the first m generators (e_i, f_i, h_i) to be realized as in $\pi^{(3)}_{A_m}$ and the last n ones given as in $\pi^{(1)}_{A_n}$. Explicitly, this provides the following unitary representation of $A_q(m,n)$,

$$\begin{aligned}
&e_k=\psi_k^\dagger\psi_{k+1}\,, && f_k=\psi_{k+1}^\dagger\psi_k\,, && h_k=M_k-M_{k+1}\,, && k=1,\ldots,m\,, \\
&e_{m+1}=\psi_{m+1}^\dagger b_2\,, && f_{m+1}=\psi_{m+1}b_2^\dagger\,, && h_{m+1}=M_{m+1}+N_2\,, && \\
&e_{m+l}=b_l^\dagger b_{l+1}\,, && f_{m+l}=b_{l+1}^\dagger b_l\,, && h_{m+l}=N_l-N_{l+1}\,, && l=2,\ldots,n+1\,.
\end{aligned} \tag{5.5}$$

This construction has been sketched in [10].

One can also join the representation $\pi^{(3)}_{A_m}$ with the representation $\pi^{(2)}_{A_n}$ (or its equivalent under $e_i \leftrightarrow f_i$, $h_i \to -h_i$) using for e_{m+1}, f_{m+1}, and h_{m+1} the expressions given in (5.5). One has then,

$$\begin{aligned}
&e_k=\psi_k^\dagger\psi_{k+1}\,, && f_k=\psi_{k+1}^\dagger\psi_k\,, && h_k=M_k-M_{k+1}\,, && k=1,\ldots,m\,, \\
&e_{m+1}=\psi_{m+1}^\dagger b_1\,, && f_{m+1}=\psi_{m+1}b_1^\dagger\,, && h_{m+1}=M_{m+1}+N_1\,, && \\
&e_{m+2}=ib_1^\dagger b_2^\dagger\,, && f_{m+2}=ib_1b_2\,, && h_{m+2}=N_1+N_2+1\,, && \\
&e_{m+3}=ib_2b_3\,, && f_{m+3}=ib_2^\dagger b_3^\dagger\,, && h_{m+3}=-(N_2+N_3+1)\,, &&
\end{aligned} \tag{5.6}$$

and so until the index $n+m+1$ is reached. The representation of $A_q(m,n)$ thus obtained is not unitary anymore. However, it becomes unitary when the symmetric Cartan matrix $(a_{ij}^s)=(d_ia_{ij})$ is adopted (see Appendix).

The representation $\pi^{(4)}_{A_m}$ can similarly be attached to either representation $\pi^{(1)}_{A_m}$ or representation $\pi^{(2)}_{A_n}$ to form two additional representations of $A_q(m,n)$. The first one is unitary, while the second one becomes unitary once the rescalings associated to the use of the symmetric Cartan matrix have been performed. When $m=n$, the center

$$c=(h_1-h_{2m+1})+2(h_2-h_{2m})+\ldots+m(h_m-h_{m+2})+(m+1)h_{m+1}$$

should be factored out.

From the four representations that we have just described one can obtain four additional homomorphisms of $A_q(m,n)$ in $W_q(m+1,n+1)$ by exchanging in an obvious fashion the bosonic and fermionic operators.

- $B_q(m,n)$, $m>0$

Four algebra homomorphisms of $B_q(m,n)$ into $W_q(n,m)$ are obtained by combining $\pi^{(1)}_{A_{n-1}}$ or $\pi^{(2)}_{A_{n-1}}$ with $\pi^{(3)}_{A_{m-1}}$ or $\pi^{(4)}_{A_{m-1}}$. A unitary representation follows from using $\pi^{(1)}_{A_{n-1}}$ and $\pi^{(3)}_{A_{m-1}}$. This is the only one that we shall describe explicitly; the others are similarly constructed. Set

$$\begin{aligned}
&e_k=b_k^\dagger b_{k+1}, && f_k=b_{k+1}^\dagger b_k, && h_k=N_k-N_{k+1}, \quad k=1,\ldots,n-1,\\
&e_n=\psi_1 b_n^\dagger, && f_n=\psi_1^\dagger b_n, && h_n=M_1+N_n,\\
&e_{n+l}=\psi_l^\dagger\psi_{l+1}, && f_{n+l}=\psi_{l+1}^\dagger\psi_l, && h_{n+l}=M_l-M_{l+1}, \quad l=1,\ldots,m-1,\\
&e_{m+n}=(-1)^M\psi_m^\dagger, && f_{m+n}=\psi_m(-1)^M, && h_{n+m}=2M_m-1.
\end{aligned} \tag{5.7}$$

where $M=\sum_{i=1}^{m} M_i$. It is not difficult to check that the defining relations of $B_q(m,n)$ are then satisfied. Note that a Klein operator enters in the expression of e_{m+n} and f_{m+n}.

Let us point out that different homomorphisms of $B_q(m,n)$ into $W(n,m)$ can be obtained by exchanging the b's and the ψ's. However, one then needs to use a set τ with more than one element. For $\tau=\{n,m+n\}$ in particular, a representation of $B_q(m,n)$ is obtained through combining $\pi^{(3)}_{A_{n-1}}$ and $\pi^{(1)}_{A_{m-1}}$ as follows:

$$\begin{aligned}
&e_k=\psi_k^\dagger\psi_{k+1}, && f_k=\psi_{k+1}^\dagger\psi_k, && h_k=M_k-M_{k+1}, \quad k=1,\ldots,n-1,\\
&e_n=b_1\psi_n^\dagger, && f_n=b_1^\dagger\psi_n, && h_n=N_1+M_n,\\
&e_{n+l}=b_l^\dagger b_{l+1}, && f_{n+l}=b_{l+1}^\dagger b_l, && h_{n+l}=N_l-N_{l+1}, \quad l=1,\ldots,m-1,\\
&e_{m+n}=(-1)^N b_m^\dagger, && f_{m+n}=b_m(-1)^N, && h_{n+m}=2N_m+1.
\end{aligned} \tag{5.8}$$

where $N=\sum_{i=1}^{m} N_i$.

- $B_q(0,n)$

The representations of $B_q(0,n)$ only require q-bosons. Homomorphisms of $B_q(0,n)$ into $W_q(n,0)$ can be constructed from either $\pi^{(1)}_{A_{n-1}}$ or $\pi^{(2)}_{A_{n-1}}$. In the first case one has

$$\begin{aligned}
&e_k=b_k^\dagger b_{k+1}, \quad f_k=b_{k+1}^\dagger b_k, \quad h_k=N_k-N_{k+1}, \quad \text{for} \quad k=1,\ldots,n-1,\\
&e_n=b_n^\dagger, \qquad f_n=b_n, \qquad h_n=2N_n+1.
\end{aligned} \tag{5.9}$$

This representation is unitary. The other one has (e_k, f_k, h_k), $k=1,\ldots,n-1$, as in $\pi^{(2)}_{A_{n-1}}$, with (e_n, f_n, h_n) as in (5.9). These bosonic realizations of $osp_q(1,2n)$ were given in [17].

• $C_q(n+1)$

We have two homomorphisms of $C_q(n+1)$ in $W_q(n,1)$. There is one which is constructed out of the representation $\pi^{(2)}_{A_{n-1}}$ given in (5.2) when n is odd, or, when n is even, out of the equivalent representation obtained from the substitution $e_i \leftrightarrow f_i$ and $h_i \to -h_i$. It is explicitly defined by

$$\begin{aligned} &e_1=\psi_1 b_1, \quad f_1=\psi_1^\dagger b_1^\dagger, \quad n_1=N_1-M_1+1,\\ &e_2=ib_1^\dagger b_2^\dagger, \quad f_2=ib_1 b_2, \quad h_2=N_1+N_2+1,\\ &e_3=ib_2 b_3, \quad f_3=ib_2^\dagger b_3^\dagger, \quad h_3=-(N_2+N_3+1), \end{aligned} \tag{5.10}$$

and so on, till:

$$e_{n+1}=\frac{i}{2\cosh\eta}b_n^2, \quad f_{n+1}=\frac{i}{2\cosh\eta}(b_n^\dagger)^2, \quad h_{n+1}=-\left(N_n+\frac{1}{2}\right), \quad \text{for } n \text{ even},$$

$$e_{n+1}=\frac{i}{2\cosh\eta}(b_n^\dagger)^2, \quad f_{n+1}=\frac{i}{2\cosh\eta}b_n^2, \quad h_{n+1}=N_n+\frac{1}{2}, \quad \text{for } n \text{ odd}.$$

This representation becomes unitary when referred to the symmetric Cartan matrix $(a^s_{ij})=(d_i a_{ij})$ (see Appendix).

The other representation of $C_q(n+1)$ in $W_q(n,1)$, uses $\pi^{(1)}_{A_{n-1}}$ and is defined as follows:

$$\begin{aligned} &e_1=\psi_1^\dagger b_1, \quad f_1=\psi_1 b_1^\dagger, \quad h_1=M_1+N_1,\\ &e_{k+1}=b_k^\dagger b_{k+1}, \quad f_{k+1}=b_{k+1}^\dagger b_k, \quad h_{k+1}=N_k-N_{k+1}, \quad k=1,\dots,n-1,\\ &e_{n+1}=\frac{i}{2\cosh\eta}(b_n^\dagger)^2, \quad f_{n+1}=\frac{i}{2\cosh\eta}b_n^2, \quad h_{n+1}=N_n+\frac{1}{2}. \end{aligned} \tag{5.11}$$

• $D_q(m,n)$

Two homomorphisms of $D_q(m,n)$ into $W_q(m,n)$ are obtained upon combining $\pi^{(1)}_{A_{n-1}}$ or $\pi^{(2)}_{A_{n-1}}$ with $\pi^{(3)}_{A_{m-1}}$. The first produces the following unitary realization:

$$\begin{aligned} &e_k=b_k^\dagger b_{k+1}, \quad f_k=b_{k+1}^\dagger b_k, \quad h_k=N_k-N_{k+1}, \quad k=1,\dots,n-1,\\ &e_n=\psi_1 b_n^\dagger, \quad f_n=\psi_1^\dagger b_n, \quad h_n=N_n+M_1,\\ &e_{n+l}=\psi_l^\dagger \psi_{l+1}, \quad f_{n+l}=\psi_{l+1}^\dagger \psi_l, \quad h_{n+l}=M_l-M_{l+1}, \quad l=1,\dots,m-1,\\ &e_{m+n}=\psi_m^\dagger \psi_{m-1}^\dagger, \quad f_{m+n}=\psi_{m-1}\psi_m, \quad h_{m+n}=M_{m-1}+M_m-1. \end{aligned} \tag{5.12}$$

For $D_q(m,1)$, the form of this q-oscillator representation had been conjectured in [16]. A second realization is formed by taking the first $n-1$ generators (e_k, f_k, h_k) as in representation $\pi^{(2)}_{A_{n-1}}$ keeping the remaining generators as in (5.12). Finally, two new homomorphisms of $D_q(m,n)$ into $W_q(m,n)$ can be obtained from the representations just described by letting $b_i \leftrightarrow \psi_i$, $b_i^\dagger \leftrightarrow \psi_i^\dagger$, $N_i \to M_i - 1$, and $M_i \to N_i + 1$ in all the generators, except for e_{m+n} and f_{m+n}, which are realized as $e_{m+n}=ib_{m-1}^\dagger b_m^\dagger$, $f_{m+n}=ib_{m-1}b_m$.

Appendix. Conversion to Symmetric Cartan Matrices

Two Cartan matrices $A=(a_{ij})$ and $A'=(a'_{ij})$ are equivalent [13] if there exists a matrix D such that $\det D \neq 0$ and $A'=DA$. Using this freedom, we can symmetrize the Cartan matrices of the basic Lie superalgebras. In fact, let $D_i = d_i\delta_{ij}$ with d_i the components of the vector given in Sect. 2; the symmetric Cartan matrices $A^s = (a^s_{ij})$ of [18] are related to those listed in Sect. 2 by $A^s = DA$.

We here indicate how various formulas translate when one chooses to describe quantum superalgebras with (a^s_{ij}) instead of (a_{ij}). Let E_i, F_i, and H_i, $i=1, \ldots, r$, be the elements that generate the quantum superalgebra characterized by (a^s_{ij}) and τ. They satisfy the defining relations [10]

$$\begin{aligned} &[E_i, F_j] = \delta_{ij}\frac{\sinh(\eta h_i)}{\sinh\eta}, \qquad [H_i, H_j] = 0, \\ &[H_i, E_j] = a^s_{ij}E_j, \qquad [H_i, F_j] = -a^s_{ij}F_j. \end{aligned} \tag{A.1}$$

$$\deg H_i = 0; \quad \deg E_i = \deg F_i = 0, \quad i \notin \tau; \quad \deg E_i = \deg F_i = 1, \quad i \in \tau,$$

together with the Serre relations (3.6), still involving the Cartan matrix (a_{ij}) and the rescaled generators

$$\mathscr{E}_i = E_i e^{-\frac{\eta}{2}H_i}, \qquad \mathscr{F}_i = F_i e^{-\frac{\eta}{2}H_i}.$$

This set of generators is straightforwardly related to the set e_i, f_i, and h_i, $i=1, \ldots, r$, that satisfy (3.1) and (3.6). One has

$$E_i = \sqrt{\frac{\sinh(\eta d_i)}{\sinh\eta}}\, e_i, \qquad F_i = \sqrt{\frac{\sinh(\eta d_i)}{\sinh\eta}}\, f_i, \qquad H_i = d_i h_i. \tag{A.2}$$

When d_i is negative, E_i and F_i will no longer be hermitian conjugate, if e_i and f_i were. Conversely, as indicated in Sect. 5, there might be cases where one needs to use (a^s_{ij}) and the generators E_i, F_i, and H_i for certain representations to be unitary.

Acknowledgements. We thank E. D'Hoker and N. Yu. Reshetikhin for discussions and comments. V.P.S. gratefully acknowledges the hospitality and support extended to him at Caltech during the course of this work. L.V. is supported in part by the UCLA Department of Physics, the National Sciences and Engineering Research Council (NSERC) of Canada and the Fonds FCAR of the Quebec Ministry of Education.

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Communicated by N. Yu. Reshetikhin