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Bosonic and Fermionic Realizations of the Affine Algebra $\hat{gl}_n$

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Abstract. We give an explicit description of all inequivalent Heisenberg subalgebras of the affine Lie algebra $\hat{gl}_n(\mathbb{C})$ and the associated vertex operator constructions of the level one integrable highest weight representations of this algebra. The construction uses multicomponent fermionic fields and yields a correspondence between bosons (elements of the Heisenberg subalgebra) and fermions.

1. Introduction

In 1978 Lepowsky and Wilson [1] gave the first explicit realization of the basic representation $L(\Lambda_0)$ of the simplest affine Lie algebra $\hat{sl}_2$. Their construction was soon generalized to the case of arbitrary simply-laced affine Lie algebras and to the case of twisted affine Lie algebras (see [2]). In this so-called principal realization of the basic representation an important role was played by a set of formal generating operators, which resembled closely the vertex operators, known at that time from the theories of dual models and strings in physics. Inspired by this discovery, Frenkel and Kac [3], and independently Segal [4] gave a different construction of the same module. The formal generating operators used in their construction are precisely the vertex operators from physics.

So in the early 80's there already existed more than one realization of a level one highest weight representation of a simply-laced affine Lie algebra. At that time these constructions seemed totally disconnected. The link between distinguished constructions was first made in [5] and, from a somewhat different point of view in [6]. It turned out that each realization depends on the choice of a so-called Heisenberg subalgebra, i.e., a subalgebra of the affine algebra with basis $\{p_k, q_k\}_{k\in\mathbb{N}}$ and the canonical central element c, whose elements satisfy the well known Heisenberg commutation relations:

$$[p_k, q_j] = \delta_{kj} c. \tag{1.1}$$

One also encounters this algebra in a slightly different disguise, namely as a collection of bosonic oscillators $\alpha(k)$, $k \in \mathbf{Z}$ with commutations relations:

$$[\alpha(k), \alpha(j)] = k\delta_{k+j,0}c. \tag{1.2}$$

The connection is made by leaving out the zero mode $\alpha(0)$ and defining for $k > 0$: $p_k := \alpha(k)$, $q_k := 1/k\alpha(-k)$.

It is easy to construct a representation of a Heisenberg algebra; take the polynomial ring $\mathbf{C}[x_k; k \in \mathbf{N}]$ and represent the p_k's and q_k's by:

$$p_k \mapsto \frac{\partial}{\partial x_k} \quad q_k \mapsto x_k \quad c \mapsto I. \tag{1.3}$$

It is well known that any irreducible representation V in which c acts as the identity and which contains a vacuum vector, i.e., a vector which is annihilated by all p_k's, is isomorphic to this module. Under certain mild restrictions on the Heisenberg subalgebra (see, e.g., [7]) it is possible to show that the integrable highest weight representations of an affine Lie algebra are completely reducible with respect to the action of the Heisenberg subalgebra. Restricting ourselves to the basic representation $L(\Lambda_0)$, we conclude that it can be written as a tensor product of an irreducible module over the Heisenberg subalgebra and a so-called vacuum space $\Omega(\Lambda_0)$, consisting of all vectors in $L(\Lambda_0)$ which are killed by the p_k's;

$$L(\Lambda_0) \cong \Omega(\Lambda_0) \otimes \mathbf{C}[x_k],$$
$$\Omega(\Lambda_0) := \{v \in L(\Lambda_0) \mid p_k(v) = 0 \ \forall k\}. \tag{1.4}$$

The choice of the Heisenberg subalgebra will be reflected in the structure of the vacuum space $\Omega(\Lambda_0)$. This can be illustrated with the simple example of the affine algebra $\hat{sl}_2$. In this case there are essentially two inequivalent Heisenberg subalgebras, namely the principal one used by Lepowsky and Wilson:

$$p_k := e^{i(k-1)\theta}e + e^{ik\theta}f \quad q_k := \frac{1}{2k-1}(e^{-i(k-1)\theta}e + e^{-ik\theta}f) \quad \forall k > 0, \tag{1.5}$$

and the homogeneous one used by Frenkel–Kac and Segal:

$$p_k := e^{ik\theta}h \quad q_k := \frac{1}{2k}e^{-ik\theta}h \quad \forall k > 0, \tag{1.6}$$

where we have used the standard basis $\{e, f, h\}$ of $sl_2(\mathbf{C})$. The difference between the vacuum spaces for these two Heisenberg subalgebras is tremendous; one can show by a character theoretical argument that in the principal case the vacuum space contains only multiples of the highest weight vector v_{Λ_0}, while in the homogeneous case it has a basis $\{T^k \cdot v_{\Lambda_0}\}_{k \in \mathbf{Z}}$, where T is some lift of the matrix

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \widetilde{SL}_2 \tag{1.7}$$

to the associated Kac–Moody group $\widehat{SL}_2$. In particular, it is infinite dimensional!

The importance of the non-homogeneous realizations for physics was soon realized. It turns out that the non-homogeneous Heisenberg subalgebras become homogeneous if one works in a twisted realization of the affine algebra. Such a twisted realization is constructed by means of a finite order automorphism of the underlying finite dimensional Lie algebra. These automorphisms can also be used to construct so-called orbifold models in string theory (see, e.g., [8]).

Apart from their applications in string theory the different realizations of representations also play an important role in the theory of integrable partial differential equations (soliton equations). In such equations one often encounters an infinite dimensional symmetry group, showing that the solutions of the equations lie on (an) orbit(s) in some representation space of this group. Conversely, it is well known (see [9]), that the orbit through the highest weight vector in a level one highest weight representation of an affine Kac–Moody group can be described by a family of integrable partial differential equations. The point is that these describing equations can look different in different realizations of this representation, whence one can associate different integrable systems to the same object, namely the orbit of a Kac–Moody group. This was illustrated in [7, 10] for the simple case of $\hat{SL}_2$; the principal realization yields the KdV-family of p.d.e.'s, the homogeneous one the so-called Toda-AKNS system.

These applications provide a strong motivation to consider more general cases than the principal and homogeneous Heisenberg subalgebras and more general affine algebras than $\hat{sl}_2$. To do this, one first has to classify all inequivalent Heisenberg subalgebras of an affine Lie algebra $\hat{g}$. In this context inequivalent means of course: nonconjugate under the adjoint action of the associated Kac–Moody group. One can show (see [5, 11, 12]) that the inequivalent Heisenberg subalgebras, which come from a compact form $\hat{g}_0$ of $\hat{g}$ are parameterized by the conjugacy classes in the Weyl group of the underlying finite dimensional Lie algebra g. For the simply-laced Lie algebras the associated vertex operator constructions were given in [5] and in a somewhat different manner in [6]. Kac and Wakimoto [13] have used these constructions to calculate the hierarchies of p.d.e.'s describing the group orbit through the highest weight vector in the principal and homogeneous realizations.

In this paper we readdress the case $\hat{sl}_n(\mathbb{C})$ or rather $\hat{gl}_n(\mathbb{C})$. Because of the particular simple structure of the Weyl group of $sl_n(\mathbb{C})$ it is easy to describe all inequivalent Heisenberg subalgebras. The associated vertex operator constructions can be given in a very simple and explicit manner by using the language of multicomponent free fermions from two dimensional quantum field theory. In a forthcoming paper we will generalize this description to the algebras $D_n^{(1)}$. We hope that these fermionic constructions can be used to calculate other hierarchies of soliton equations then the ones found in [13] (see Sect. 8.3).

The Weyl group of $sl_n(\mathbb{C})$ is the symmetric group S_n, consisting of all permutations of n elements. Every permutation is conjugate to a product of disjoint cycles, say $c_1 c_2 \cdots c_s$. Denoting the length of the cycle c_i by n_i, and choosing the ordering such that $n_1 \leqq n_2 \leqq \cdots \leqq n_s$, we see that the conjugacy classes in S_n and hence the inequivalent Heisenberg subalgebras of $\hat{sl}_n$ are parametrized by partitions of n;

$$\underline{n} = \{n_1, n_2, \ldots, n_s\}; \quad n_1 \leqq n_2 \leqq \cdots n_s; \quad \sum_{i=1}^{s} n_i = n. \tag{1.8}$$

The conjugacy class of Heisenberg subalgebras corresponding to one cycle of length n is in some way generic. The principal Heisenberg subalgebra is a representative from this conjugacy class. In this case a fermionic construction of the level one highest weight representations was given by the Kyoto school [see, e.g. [14]). In order to motivate what is coming, we will give a slightly adjusted version here in the spirit of the paper [15].

Let Cl be the Clifford algebra on generators $\psi(k)$, $\psi^*(k)$, $k \in \mathbf{Z}$ with relations

$$\{\psi(k), \psi^*(j)\} = \delta_{kj} \quad \{\psi(k), \psi(j)\} = 0 = \{\psi^*(k), \psi^*(j)\}. \tag{1.9}$$

Define the so-called spin module V as the unique irreducible Cl-module, which admits a vacuum vector $|0\rangle$, such that

$$\begin{aligned} \psi(k)|0\rangle &= 0 \quad \forall k \leqq 0, \\ \psi^*(k)|0\rangle &= 0 \quad \forall k > 0. \end{aligned} \tag{1.10}$$

This motivates the following normal ordering prescription

$$:\psi(k)\psi^*(j): \stackrel{\text{def}}{=} \begin{cases} \psi(k)\psi^*(j) & \text{if } j > 0 \\ -\psi^*(j)\psi(k) & \text{if } j \leqq 0 \end{cases}. \tag{1.11}$$

Next one introduces formal fermionic fields by

$$\begin{aligned} \psi(z) &:= \sum_{k \in \mathbf{Z}} \psi(k) z^k, \\ \psi^*(z) &:= \sum_{k \in \mathbf{Z}} \psi^*(k) z^{-k}. \end{aligned} \tag{1.12}$$

It can then be shown that the identity operator together with the homogeneous components of

$$\begin{aligned} &:\psi(\omega^p z)\psi^*(\omega^q z): - \frac{\omega^{p-q}}{1-\omega^{p-q}} \quad 1 \leqq p,\, q \leqq n,\, p \neq q \\ &:\psi(z)\psi^*(z): \end{aligned} \tag{1.13}$$

where $\omega = e^{2\pi i/n}$ is a primitive root of unity, span a Lie algebra of operators on V isomorphic to $\hat{gl}_n(\mathbf{C})$.

Next one introduces the charge decomposition of the module V; this is done by setting the charge of the vacuum $|0\rangle$ to be zero and by agreeing that the fermions $\psi(k)(\psi^*(k))$ raise (lower) the charge by one. One can then write

$$V = \bigoplus_{m \in \mathbf{Z}} V_m, \tag{1.14}$$

V_m being the subspace of all vectors of charge m. The V_m's are irreducible level one highest weight modules over $\hat{gl}_n(\mathbf{C})$.

Notice that this is a pure fermionic approach to the representations of $\hat{gl}_n(\mathbf{C})$. There is also a well known bosonization procedure, which can be described as follows; in (1.13) we have introduced the bosonic field $\alpha(z) := :\psi(z)\psi^*(z):$. Expanding this field as a Laurent series, $\alpha(z) = \sum_{k \in \mathbf{Z}} \alpha(k) z^{-k}$, one readily finds

$$\alpha(k) = \sum_{i \in \mathbf{Z}} :\psi(i)\psi^*(i+k):. \tag{1.15}$$

With this expression one easily verifies the oscillator commutation relations (1.2). One can then prove the following fundamental theorem.

Theorem 1.1.

$$\psi(z) = Qz^{\alpha(0)+1}\exp\left(-\sum_{k<0}\frac{1}{k}z^{-k}\alpha(k)\right)\exp\left(-\sum_{k>0}\frac{1}{k}z^{-k}\alpha(k)\right)$$

$$\psi^*(z) = Q^{-1}z^{-\alpha(0)}\exp\left(\sum_{k<0}\frac{1}{k}z^{-k}\alpha(k)\right)\exp\left(\sum_{k>0}\frac{1}{k}z^{-k}\alpha(k)\right) \tag{1.16}$$

where the operator $Q: V \to V$ is defined by

$$\begin{aligned} Q|0\rangle &= \psi(1)|0\rangle, \\ Q\psi(k) &= \psi(k+1)Q, \\ Q\psi^*(k) &= \psi^*(k+1)Q. \end{aligned} \tag{1.17}$$

In particular, this operator raises the fermionic charge;

$$Q: V_m \to V_{m+1}. \tag{1.18}$$

Inserting this result in the expression (1.13), one finds after some calculations for $p \neq q$:

$$:\psi(\omega^p z)\psi^*(\omega^q z): - \frac{\omega^{p-q}}{1-\omega^{p-q}}$$
$$= \frac{\omega^{(p-q)\alpha(0)}}{1-\omega^{q-p}}\exp\left(-\sum_{k<0}\frac{1}{k}(\omega^{-pk}-\omega^{-qk})\alpha(k)\right)\exp\left(-\sum_{k>0}\frac{1}{k}(\omega^{-pk}-\omega^{-qk})\alpha(k)\right). \tag{1.19}$$

Note that Q and Q^{-1} have cancelled in this expression, whence everything can be expressed in terms of oscillators. This is the reason that the modules V_m remain irreducible when restricted to the action of the principal Heisenberg subalgebra.

How to generalize this construction to a more general partition $\underline{n} = \{n_1, n_2, \ldots, n_s\}$? To answer this, we divide an $n \times n$ matrix in s^2 blocks of size $n_i \times n_j$. The diagonal blocks correspond to Lie algebras $gl_{n_i}(\mathbf{C})$ and the principal construction above tells us how to make vertex operators describing the action of the affine algebra $\hat{gl}_{n_i}$. So we just should take s copies of the construction above or, what is the same thing, we should work with s-component fermions $\psi_i(k)$, $\psi_i^*(k)$, $1 \leqq i \leqq s$, $k \in \mathbf{Z}$. The problem is of course how to find vertex operators associated to the off diagonal blocks.

In the case that every cycle has the same length, everything is rather straightforward. As an example we sketch the homogeneous construction, corresponding to the partition $\underline{n} = \{1, 1, \ldots, 1\}$. In this case we take the Clifford algebra generated by the fermions $\psi_i(k)$, $\psi_i^*(k)$, $1 \leqq i \leqq n$, $k \in \mathbf{Z}$ with relations

$$\{\psi_i(k), \psi_j^*(l)\} = \delta_{ij}\delta_{kl} \quad \{\psi_i(k), \psi_j(l)\} = 0 = \{\psi_i^*(k), \psi_j^*(l)\}. \tag{1.20}$$

In fact we will see that these n-component fermions can be obtained by relabeling the 1-component fermions, so that this Clifford algebra is really the same as the one considered before. We again consider the irreducible Cl-module generated by a vacuum $|0\rangle$ satisfying

$$\begin{aligned} \psi_i(k)|0\rangle &= 0 \quad \forall k \leqq 0\, \forall i, \\ \psi_i^*(k)|0\rangle &= 0 \quad \forall k > 0\, \forall i. \end{aligned} \tag{1.21}$$

Normal ordering and fermionic fields are defined as in (1.11) and (1.12). The vertex operators associated to the off diagonal blocks are given by $:\psi_i(z)\psi_j^*(z):$, $1 \leqq i,j \leqq n$, $i \neq j$.

The bosonization procedure is also analogous to the principal case; introduce $\alpha_i(z) = \sum_{k\in\mathbf{Z}} \alpha_i(k)z^{-k} := :\psi_i(z)\psi_i^*(z):$. One can then prove the following theorem.

Theorem 1.2.

$$\psi_i(z) = Q_i z^{\alpha_i(0)+1} \exp\left(-\sum_{k<0}\frac{1}{k}z^{-k}\alpha_i(k)\right)\exp\left(-\sum_{k>0}\frac{1}{k}z^{-k}\alpha_i(k)\right),$$

$$\psi_i^*(z) = Q_i^{-1} z^{-\alpha_i(0)} \exp\left(\sum_{k<0}\frac{1}{k}z^{-k}\alpha_i(k)\right)\exp\left(\sum_{k>0}\frac{1}{k}z^{-k}\alpha_i(k)\right), \tag{1.22}$$

where the operators $Q_i: V \to V$ *are defined by*

$$\begin{aligned} &Q_i|0\rangle = \psi_i(1)|0\rangle,\\ &Q_i\psi_i(k) = \psi_i(k+1)Q_i,\\ &Q_i\psi_i^*(k) = \psi_i^*(k+1)Q_i,\\ &Q_i\psi_j(k) = -\psi_j(k)Q_i \quad \text{if } i \neq j,\\ &Q_i\psi_j^*(k) = -\psi_j^*(k)Q_i \quad \text{if } i \neq j. \end{aligned} \tag{1.23}$$

These operators satisfy:

$$\{Q_i, Q_j\} = 0 \quad \text{if } i \neq j. \tag{1.24}$$

With this result one can find an expression for the vertex operators associated to the off diagonal elements; for $i \neq j$ we get:

$$\begin{aligned} :\psi_i(z)\psi_j^*(z): = Q_iQ_j^{-1} z^{\alpha_i(0)-\alpha_j(0)+1} &\exp\left(\sum_{k<0}\frac{1}{k}z^{-k}(\alpha_j(k)-\alpha_i(k))\right)\\ &\cdot\exp\left(\sum_{k>0}\frac{1}{k}z^{-k}(\alpha_j(k)-\alpha_i(k))\right). \end{aligned} \tag{1.25}$$

It is important to notice that the Q's do not cancel in this formula, simply because there are n different types of them. It is easy to see that the vacuum space $\Omega(\Lambda_k)$ of the $\hat{sl}_n$-module $L(\Lambda_k)$ is spanned by the vectors $T_1^{m_1}T_2^{m_2}\cdots T_{n-1}^{m_{n-1}}\cdot v_{\Lambda_k}$, $m_i \in \mathbf{Z}$; $T_i := Q_iQ_{i+1}^{-1}$, $1 \leqq i \leqq n-1$.

We will complete this introduction by formulating the general result of this paper.

Theorem 1.3. *Let* $\underline{n} = \{n_1, n_2, \ldots, n_s\}$ *be a partition of* n, N' *the least common multiple of the numbers* $n_1, n_2, \ldots, n_s$ *and define*

$$N := \begin{cases} N' & \text{if } N'\left(\frac{1}{n_i}+\frac{1}{n_j}\right) \in 2\mathbf{Z} \quad \forall ij \\ 2N' & \text{if } N'\left(\frac{1}{n_i}+\frac{1}{n_j}\right) \notin 2\mathbf{Z} \quad \text{for a pair } (i,j) \end{cases} \tag{1.26}$$

Let Cl be the Clifford algebra generated by the fermions $\psi(k)$, $\psi^*(k)$, $k \in \mathbf{Z}$ *with* *relations (1.9) and define multicomponent fermions $\psi_i(k)$, $\psi_i^*(k)$, $1 \leqq i \leqq s$, $k \in \mathbf{Z}$ by:*

$$\begin{aligned}\psi_i(l+mn_i) &:= \psi(n_1+n_2+\cdots+n_{i-1}+l+nm),\\ \psi_i^*(l+mn_i) &:= \psi^*(n_1+n_2+\cdots+n_{i-1}+l+nm) \quad 1 \leqq l \leqq n_i,\ m \in \mathbf{Z}.\end{aligned} \tag{1.27}$$

These fermions satisfy the relations:

$$\{\psi_i(k), \psi_j^*(l)\} = \delta_{ij}\delta_{kl} \quad \{\psi_i(k), \psi_j(l)\} = 0 = \{\psi_i^*(k), \psi_j^*(l)\}. \tag{1.28}$$

In terms of these fermions the spin module V can also be defined as the unique irreducible Cl-module generated by a vacuum $|0\rangle$ satisfying

$$\begin{aligned}\psi_i(k)|0\rangle &= 0 \quad \forall k \leqq 0\, \forall i,\\ \psi_i^*(k)|0\rangle &= 0 \quad \forall k > 0\, \forall i.\end{aligned} \tag{1.29}$$

Introduce formal fermionic and bosonic fields by

$$\begin{aligned}\psi_i(z) &:= \sum_{k \in \mathbf{Z}} \psi_i(k) z^{(N/n_i)k},\\ \psi_i^*(z) &:= \sum_{k \in \mathbf{Z}} \psi_i^*(k) z^{-(N/n_i)k},\\ \alpha_i(z) &= \sum_{k \in \mathbf{Z}} \alpha_i(k) z^{-(N/n_i)k} := :\psi_i(z)\psi_i^*(z):\end{aligned} \tag{1.30}$$

where normal ordering is defined as in (1.11). Then we have

$$\begin{aligned}\psi_i(z) &= Q_i z^{(N/n_i)(\alpha_i(0)+1)} \exp\left(-\sum_{k<0} \frac{1}{k} z^{-(N/n_i)k}\alpha_i(k)\right) \exp\left(-\sum_{k>0} \frac{1}{k} z^{-(N/n_i)k}\alpha_i(k)\right),\\ \psi_i^*(z) &= Q_i^{-1} z^{-(N/n_i)\alpha_i(0)} \exp\left(\sum_{k<0} \frac{1}{k} z^{-(N/n_i)k}\alpha_i(k)\right) \exp\left(\sum_{k>0} \frac{1}{k} z^{-(N/n_i)k}\alpha_i(k)\right),\end{aligned} \tag{1.31}$$

where the operators $Q_i: V \to V$ are defined by (1.23).

The homogeneous components of the normal ordered products

$$:\psi_i(u)\psi_j^*(v): \quad 1 \leqq i, j \leqq s \tag{1.32}$$

together with the identity operator provide an irreducible level one representation of the affine Lie algebra of infinite rank, A_∞, on each charge sector V_m. Similarly, by setting $\omega := e^{2\pi i/N}$, one obtains an irreducible level one representation of the subalgebra $\widehat{gl}_n(\mathbf{C}) \subset A_\infty$ on V_m by the homogeneous components of

$$\begin{aligned}&:\psi_i(\omega^p z)\psi_j^*(\omega^q z): - \delta_{ij}\frac{\omega^{(N/n_i)(p-q)}}{1-\omega^{(N/n_i)(p-q)}} \quad i \neq j,\ 1 \leqq p \leqq n_i,\ 1 \leqq q \leqq n_j \quad \text{or} \quad i = j,\ 1 \leqq p \neq q \leqq n_i,\\ &:\psi_i(z)\psi_i^*(z): \quad 1 \leqq i \leqq s\end{aligned} \tag{1.33}$$

and the identity operator.

The definition (1.30) of the bosonic and fermionic fields will be motivated in Sect. 5 after a detailed study of a finite order automorphism associated to the partition $\underline{n}$ and the corresponding twisted realization of the affine algebra $\widehat{gl}_n(\mathbf{C})$. The expression (1.31) for the fermionic fields in terms of the bosonic oscillators $\alpha_i(k)$ and the operators Q_i will be derived by exploiting the conformal symmetry of these fields, i.e., their commutation relations with the Virasoro algebra.

2. The Lie Algebra $gl_n(\mathbf{C})$

2.1. Introduction. Let $gl_n(\mathbf{C})$ be the Lie algebra of all $n \times n$ matrices with complex entries. The usual basis for this algebra is the collection $\{E_{ij}\}_{1 \leqq i,j \leqq n}$, where E_{ij} is the matrix with a 1 on the $(i,j)^{\text{th}}$ entry and zeros elsewhere. The commutation relations for these basis elements are:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}. \tag{2.1.1}$$

In this paper we will work with partitions of an $n \times n$ matrix in blocks. To be more precise: let $\underline{n} = \{n_1, n_2, \dots, n_s\}$ be a partition of n, i.e., $n_1 \leqq n_2 \leqq \cdots \leqq n_s$ and $\sum_{i=1}^{s} n_i = n$. The associated partition of $n \times n$ matrices is then given schematically by:

$$\begin{array}{c c} & \begin{array}{cccc} \overset{n_1}{\leftrightarrow} & \overset{n_2}{\leftrightarrow} & & \overset{n_s}{\leftrightarrow} \end{array} \\ \begin{array}{c} n_1 \updownarrow \\ n_2 \updownarrow \\ \\ n_s \updownarrow \end{array} & \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ B_{21} & B_{22} & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{ss} \end{pmatrix} \end{array}, \tag{2.12}$$

where B_{ij} is a block of size $n_i \times n_j$. With this blockform in mind we rephrase the commutation relations (2.1.1); the standard basis for the $(i,j)^{\text{th}}$ block is the set of matrices $\{E_{pq}^{ij}\}_{1 \leqq p \leqq n_i, 1 \leqq q \leqq n_j}$ defined by:

$$E_{pq}^{ij} := E_{n_1 + \cdots n_{i-1} + p, n_1 + \cdots n_{j-1} + q}, \tag{2.1.3}$$

and in terms of these elements we may write:

$$[E_{pq}^{ij}, E_{rs}^{kl}] = \delta_{jk} \delta_{qr} E_{ps}^{il} - \delta_{il} \delta_{ps} E_{rq}^{kj}. \tag{2.1.4}$$

Associated to each partition $\underline{n}$ we will define a Cartan subalgebra $\underline{h}_{\underline{n}}$ of $gl_n(\mathbf{C})$. For the case $\underline{n} = \{n\}$ this is the well known "principal" Cartan subalgebra, while for $\underline{n} = \{1, 1, \dots, 1\}$ $\underline{h}_n$ is the standard Cartan subalgebra $\underline{h}$ of $gl_n(\mathbf{C})$, i.e., the set of all diagonal matrices. We will also study the root space decomposition of $gl_n(\mathbf{C})$ with respect to these Cartan subalgebras. For each partition this will lead to the introduction of a basis $\{A_{pq}^{ij}\}_{1 \leqq i,j \leqq s, 1 \leqq p \leqq n_i, 1 \leqq q \leqq n_j}$ of $gl_n(\mathbf{C})$ consisting of eigenvectors for the adjoint action of $\underline{h}_{\underline{n}}$. It will turn out that the mapping $\psi_{\underline{n}}: gl_n(\mathbf{C}) \to gl_n(\mathbf{C})$ given by $\psi_{\underline{n}}(E_{pq}^{ij}) := A_{pq}^{ij}$ is an isomorphism of $gl_n(\mathbf{C})$, and hence the A_{pq}^{ij}'s satisfy the same commutation relations as the E_{pq}^{ij}'s:

$$[A_{pq}^{ij}, A_{rs}^{kl}] = \delta_{jk} \delta_{qr} A_{ps}^{il} - \delta_{il} \delta_{ps} A_{rq}^{ij}. \tag{2.1.5}$$

We will see that this isomorphism maps the standard Cartan subalgebra into the Cartan subalgebra $\underline{h}_{\underline{n}}$.

Before considering the general case, we will first review the well known principal case in Sect. 2.2. It is rather easy to generalize the results from this case to the general case, which will be treated in Sect. 2.3.

2.2. The principal case. The following construction goes back to Kostant [16], who introduced the so-called principal cyclic element in the context of an arbitrary semisimple Lie algebra $\underline{g}$. For the case $\underline{g} = sl_n(\mathbf{C})$ or rather the case $gl_n(\mathbf{C}) = \underline{g} \oplus \mathbf{C}I$, which we are interested in, all formulas become particularly transparent (see also [2, 7]); let E be the principal cyclic element given by

$$E = E_{n1} + \sum_{i=1}^{n-1} E_{i,i+1}. \tag{2.2.1}$$

This element is clearly associated to the cyclic permutation $\sigma:\{1,2,\dots n\}\to\{1,2,\dots n\}$ given by:

$$\sigma(i) := \begin{cases} i+1 & \text{if} \quad 1 \leqq i \leqq n-1 \\ 1 & \text{if} \quad i = n \end{cases}. \tag{2.2.2}$$

We have:

$$E = \sum_{i=1}^{n} E_{i,\sigma(i)} \tag{2.2.3}$$

and:

$$[E, E_{ij}] = E_{\sigma^{-1}(i),j} - E_{i,\sigma(j)}. \tag{2.2.4}$$

From this last formula it is easy to guess what the eigenvectors of ad E should be;

Lemma 2.2.1. *Let $\omega := e^{2\pi i/n}$ be an n^{th} root of unity and define*

$$A_{pq} := \frac{1}{n} \sum_{k,l=1}^{n} \omega^{pk}\omega^{-ql} E_{kl}, \tag{2.2.5}$$

then

$$[E, A_{pq}] = (\omega^p - \omega^q) A_{pq}. \tag{2.2.6}$$

More generally, we have

$$[E^k, A_{pq}] = (\omega^{kp} - \omega^{kq}) A_{pq}. \tag{2.2.7}$$

Proof.

$$\begin{aligned}
\left[E, \sum_{k,l=1}^{n} \omega^{pk}\omega^{-ql}E_{kl} \right] &= \sum_{i,k,l=1}^{n} \omega^{pk}\omega^{-ql}\{\delta_{k,\sigma(i)}E_{il} - \delta_{il}E_{k,\sigma(i)}\} \\
&= \sum_{i,l=1}^{n} \omega^{p\sigma(i)}\omega^{-ql}E_{il} - \sum_{i,k=1}^{n} \omega^{pk}\omega^{-qi}E_{k,\sigma(i)} \\
&= \omega^p \sum_{i,l=1}^{n} \omega^{pi}\omega^{-ql}E_{il} - \omega^q \sum_{i,k=1}^{n} \omega^{pk}\omega^{-q\sigma(i)}E_{k,\sigma(i)} \\
&= (\omega^p - \omega^q) \sum_{k,l=1}^{n} \omega^{pk}\omega^{-ql}E_{kl}.
\end{aligned}$$

Here we have used that $\sigma(i) = i + 1$ plus a multiple of n. The second formula is proved by replacing σ by $\tau \equiv \sigma^k$ in the calculation above. □

Of course the factor $1/n$ in the definition of the A_{pq}'s seems a bit arbitrary at first sight. In fact it is not, as we will explain now. Define the matrix $S = (S_{ij})$ by:

$$S_{ij} := \frac{1}{\sqrt{n}} \omega^{ij}. \tag{2.2.8}$$

Then one immediately verifies:

$$A_{pq} = SE_{pq}S^{\dagger}. \tag{2.2.9}$$

The important point is that the matrix S is unitary –

$$(SS^{\dagger})_{ij} = \frac{1}{n}\sum_{k=1}^{n} \omega^{ik}\omega^{-kj} = \delta_{ij} \tag{2.2.10}$$

– meaning that the mapping $\psi: gl_n(\mathbf{C}) \to gl_n(\mathbf{C})$ defined by $\psi: E_{pq} \mapsto A_{pq}$ is a similarity transformation and hence an isomorphism of $gl_n(\mathbf{C})$. In particular this implies that the set $\{A_{pq}\}_{1 \leq p,q \leq n}$ is a basis of $gl_n(\mathbf{C})$ whose elements satisfy the same commutation relations as the E_{ij}'s;

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{il}A_{kj}. \tag{2.2.11}$$

One easily expresses the E_{ij}'s in terms of the A_{ij}'s:

$$E_{ij} = \frac{1}{n}\sum_{k,l=1}^{n} \omega^{-ik}\omega^{jl}A_{kl}. \tag{2.2.12}$$

The lemma shows that the linear span of all powers of the principal cyclic element is a maximal commuting family of ad-diagonalizable elements. In other words: the linear span of all powers of E is a Cartan subalgebra of $gl_n(\mathbf{C})$; it is called the principal Cartan subalgebra of $gl_n(\mathbf{C})$:

$$\underline{h}_{\text{princ}} := \bigoplus_{i=1}^{n} \mathbf{C}E^{i}. \tag{2.2.13}$$

Moreover, formula (2.2.7) shows that the A_{pq}'s are root vectors with respect to this principal Cartan subalgebra. Note that the A_{pp}'s commute with all elements of $\underline{h}_{\text{princ}}$, and hence they must lie in $\underline{h}_{\text{princ}}$. To be more precise, we have:

Lemma 2.2.2.

$$A_{pp} = \frac{1}{n}\sum_{i=1}^{n} \omega^{-ip}E^{i}.$$

Proof.

$$\begin{aligned} A_{pp} &= \frac{1}{n}\sum_{k,l=1}^{n} \omega^{p(k-l)}E_{kl} \\ &= \frac{1}{n}\sum_{k,l=1}^{n} \omega^{-p(l-k)}E_{k,k+(l-k)} \\ &= \frac{1}{n}\sum_{i=1}^{n} \omega^{-ip}E^{i}. \quad \square \end{aligned}$$

With this lemma it is easy to show that the isomorphism ψ maps the standard Cartan subalgebra $\underline{h} := \bigoplus_{i=1}^{n} \mathbf{C}E_{ii}$ to the principal Cartan subalgebra $\underline{h}_{\text{princ}}$;

$$\psi\left(\bigoplus_{i=1}^{n} \mathbf{C}E_{ii}\right) = \bigoplus_{i=1}^{n} \mathbf{C}A_{ii} = \bigoplus_{i=1}^{n} \mathbf{C}E^{i} = \underline{h}_{\text{princ}}. \tag{2.2.14}$$

2.3. The General Case. Now let $\underline{n} = \{n_1, n_2, \ldots, n_s\}$ be an arbitrary partition of n

and recall the associated partition of an $n \times n$ matrix in s^2 blocks of size $n_i \times n_j$, $1 \leqq i, j \leqq s$ (see (2.12)). Isolating the $(i, i)^{\text{th}}$ diagonal block of $gl_n(\mathbf{C})$, we are led to define the "i^{th} cyclic element" $E_{(i)}$ by:

$$E_{(i)} := E^{ii}_{n_i,1} + \sum_{k=1}^{n_i-1} E^{ii}_{k,k+1}. \tag{2.3.1}$$

Analogous to the principal Cartan subalgebra $\underline{h}_{\text{princ}}$ we define the Cartan subalgebra $\underline{h}_{\underline{n}}$ associated to the partition $\underline{n}$ by:

$$\underline{h}_{\underline{n}} := \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_i} \mathbf{C} E_{(i)}{}^j. \tag{2.3.2}$$

It is easy to generalize the results of the previous section to this case; we will do this in the following lemma.

Lemma 2.3.1. *Let $\omega_j := e^{2\pi i/n_j}$ be an n_j^{th} root of unity and define*

$$A^{ij}_{pq} := \frac{1}{\sqrt{n_i n_j}} \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \omega_i^{pk} \omega_j^{-ql} E^{ij}_{kl}, \tag{2.3.3}$$

then we have

$$A^{ij}_{pq} = S_{\underline{n}} E^{ij}_{pq} S_{\underline{n}}^{-1}, \tag{2.3.4}$$

where $S_{\underline{n}} = \prod_{i=1}^{s} S_{(i)}$ and

$$S_{(i)} := \sum_{j \neq i} I_{n_j} + \sum_{k,l=1}^{n_i} \frac{1}{\sqrt{n_i}} \omega_i^{kl} E^{ii}_{kl}. \tag{2.3.5}$$

Consequently, the set $\{A^{ij}_{pq}\}_{1 \leqq i,j \leqq s, 1 \leqq p \leqq n_i, 1 \leqq q \leqq n_j}$ is a basis for $gl_n(\mathbf{C})$, whose elements satisfy the same commutation relations as the E^{ij}_{pq}'s;

$$[A^{ij}_{pq}, A^{kl}_{rs}] = \delta_{jk}\delta_{qr} A^{il}_{ps} - \delta_{il}\delta_{ps} A^{kj}_{rq}. \tag{2.3.6}$$

The elements E^{ij}_{pq} of the standard basis for $gl_n(\mathbf{C})$ can be expressed in terms of the A^{ij}_{pq}'s by

$$E^{ij}_{pq} := \frac{1}{\sqrt{n_i n_j}} \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \omega_i^{-pk} \omega_j^{ql} A^{ij}_{kl}. \tag{2.3.7}$$

Moreover, the A^{ij}_{pq}'s are eigenvectors for the adjoint action of $\underline{h}_{\underline{n}}$:

$$[E_{(l)}{}^k, A^{ij}_{pq}] = (\delta_{il}\omega_i^{kp} - \delta_{jl}\omega_j^{kq}) A^{ij}_{pq}. \tag{2.3.8}$$

Finally, the isomorphism $\psi_{\underline{n}} = \operatorname{Ad} S_{\underline{n}}$ maps $\underline{h}$ into $\underline{h}_{\underline{n}}$.

Proof. A straightforward generalization of proofs from the previous section. □

3. Automorphisms of Finite Order

3.1. Introduction. In this section we will explain how to associate an automorphism of finite order of $gl_n(\mathbf{C})$ to a partition $\underline{n}$ of n. The restriction of such an automorphism to the standard Cartan subalgebra $\underline{h}$ of $gl_n(\mathbf{C})$ coincides with an element $w_{\underline{n}}$ of the

Weyl group W of $gl_n(\mathbb{C})$. In the introduction we have explained that the conjugacy classes in $W(gl_n(\mathbb{C}))$ are parametrized by partitions of $\underline{n}$ and indeed, the assignment $\underline{n} \to w_{\underline{n}}$ induces a bijection between partitions of n and conjugacy classes in W.

Just as in Sect. 2 we will start with the case $\underline{n} = \{n\}$. The corresponding automorphism is known as the principal automorphism, while the restriction of this automorphism to the Cartan subalgebra $\underline{h}$ is called a Coxeter element of W. The general case will be treated in Sect. 3.3. Finally, in Sect. 3.4, we will argue that it is easier to conjugate these automorphisms by the automorphism $\psi_{\underline{n}}$.

3.2. The Principal Automorphism. In Sect. 2.2 we have noted that the principal cyclic element is related to the cyclic permutation σ: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ (see (2.2.2)). Let us define the mapping $\hat{\sigma}: gl_n(\mathbb{C}) \to gl_n(\mathbb{C})$ by:

$$\hat{\sigma}(E_{ij}) := E_{\sigma(i),\sigma(j)}. \tag{3.2.1}$$

With the commutation relations (2.1.1) it is immediate that this $\hat{\sigma}$ is an automorphism of $gl_n(\mathbb{C})$. Moreover, it has finite order: $\hat{\sigma}^n = I_{gl_n(\mathbb{C})}$. Note that the orders of σ and $\hat{\sigma}$ are the same. This $\hat{\sigma}$ is called the principal automorphism by Kostant [16]. The restriction $w = \hat{\sigma}_{|\underline{h}}$ is a Coxeter element of the Weyl group W of $sl_n(\mathbb{C})$ (see also [17]).

It is well known that $\hat{\sigma}$ can be written as $\exp\{2\pi i \operatorname{ad} X\}$, where $X = \psi(H)$ and H is the following element of the Cartan subalgebra $\underline{h} \cap sl_n(\mathbb{C})$:

$$H = \frac{1}{n} \sum_{i=1}^{n-1} t_i. \tag{3.2.2}$$

In this formula we have introduced the so-called "dual fundamental weights" t_i of $sl_n(\mathbb{C})$ by

$$[t_j, E_{i,i+1}] = \langle \alpha_i, t_j \rangle E_{i,i+1} = \delta_{ij} E_{i,i+1}. \tag{3.2.3}$$

Note that $X \in \underline{h}_{\text{princ}}$.

Lemma 3.2.1.

$$H = \frac{1}{2n} \sum_{i=1}^{n} (n - 2i + 1) E_{ii}. \tag{3.2.4}$$

Proof. Suppose that $H = \frac{1}{n} \sum_{i=1}^{n} \lambda_i E_{ii}$. The λ_i's are determined by the conditions $[H, E_{i,i+1}] = \frac{1}{n} E_{i,i+1}$ for $1 \leqq i \leqq n-1$ and $\sum_{i=1}^{n} \lambda_i = 0$. The first condition means: $\lambda_i - \lambda_{i+1} = 1$ for $1 \leqq i \leqq n-1$ or equivalently: $\lambda_i = \lambda_1 - (i-1)$ for $1 \leqq i \leqq n$. Imposing the second condition, one finds $\lambda_1 = \frac{1}{2}(n-1)$ and hence $\lambda_i = \frac{1}{2}(n - 2i + 1)$ as desired. □

With this lemma, the fact that $\psi(E_{ii}) = A_{ii}$ and Lemma 2.2.2 we can now compute X;

$$X = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{\omega^i - 1} E^i. \tag{3.2.5}$$

3.3. The General Case. Consider again an arbitrary partition $\underline{n} = \{n_1, n_2, \ldots, n_s\}$ of n. Analogous to (2.2.2) we define the cycle $\sigma_i: \{1, 2, \ldots, n_i\} \to \{1, 2, \ldots, n_i\}$ by:

$$\sigma_i(k) := \begin{cases} k+1 & \text{if} \quad 1 \leqq k \leqq n_i - 1 \\ 1 & \text{if} \quad k = n_i \end{cases}, \tag{3.3.1}$$

and the permutation $\sigma_{\underline{n}}:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ by

$$\sigma_{\underline{n}}(n_1+n_2+\cdots+n_{i-1}+k):=n_1+n_2+\cdots+n_{i-1}+\sigma_i(k) \quad 1\leqq k\leqq n_i. \tag{3.3.2}$$

To this permutation we associate the automorphism $\hat{\sigma}'_{\underline{n}}$ defined by

$$\hat{\sigma}'_{\underline{n}}(E^{ij}_{kl}):=E^{ij}_{\sigma_i(k),\sigma_j(l)}. \tag{3.3.3}$$

Note that $(\hat{\sigma}'_{\underline{n}})^{N'}=I_{gl_n(\mathbb{C})}$, where N' is the least common multiple of the numbers $n_1,n_2,\ldots,n_s$.

Let us try again to write $\hat{\sigma}'_{\underline{n}}$ as $e^{2\pi i adX_{\underline{n}}}$. To do this we set

$$H_{(i)}:=\frac{1}{2n_i}\sum_{j=1}^{n_i}(n_i-2j+1)E^{ii}_{jj}, \tag{3.3.4}$$

the analog of (3.2.4) placed on the $(i,i)^{\text{th}}$ diagonal block, and $X_{(i)}:=\psi_{\underline{n}}(H_{(i)})$. Our first guess for $X_{\underline{n}}$ would be of course: $X_{\underline{n}}:=\sum_{i=1}^{s}X_{(i)}=:\psi_{\underline{n}}(H_{\underline{n}})$. This turns out to be almost right;

Lemma 3.3.2. *Let $X_{\underline{n}}=\sum_{i=1}^{s}X_{(i)}$, then*

$$e^{2\pi i adX_{\underline{n}}}(E^{ij}_{kl})=\exp\left\{2\pi i\left(\frac{1}{2n_i}-\frac{1}{2n_j}\right)\right\}E^{ij}_{\sigma_i(k)\sigma_j(l)}. \tag{3.3.5}$$

Proof.

$$\begin{aligned}
&e^{2\pi i adX_{\underline{n}}}(E^{ij}_{kl})\\
&=\frac{1}{\sqrt{n_in_j}}e^{2\pi i adX_{\underline{n}}}\sum_{p=1}^{n_i}\sum_{q=1}^{n_j}\omega_i^{-kp}\omega_j^{lq}A^{ij}_{pq}\\
&=\frac{1}{\sqrt{n_in_j}}\psi_{\underline{n}}\left\{e^{2\pi i adH_{\underline{n}}}\sum_{p=1}^{n_i}\sum_{q=1}^{n_j}\omega_i^{-kp}\omega_j^{lq}E^{ij}_{pq}\right\}\\
&=\frac{1}{\sqrt{n_in_j}}\psi_{\underline{n}}\left\{\sum_{p=1}^{n_i}\sum_{q=1}^{n_j}\omega_i^{-kp}\omega_j^{lq}\exp\left\{2\pi i\left(\frac{1}{2n_i}(n_i-2p+1)-\frac{1}{2n_j}(n_j-2q+1)\right)\right\}E^{ij}_{pq}\right\}\\
&=\exp\left\{2\pi i\left(\frac{1}{2n_i}-\frac{1}{2n_j}\right)\right\}\frac{1}{\sqrt{n_in_j}}\psi_{\underline{n}}\left\{\sum_{p=1}^{n_i}\sum_{q=1}^{n_j}\omega_i^{-(k+1)p}\omega_j^{(l+1)q}E^{ij}_{pq}\right\}\\
&=\exp\left\{2\pi i\left(\frac{1}{2n_i}-\frac{1}{2n_j}\right)\right\}E^{ij}_{\sigma_i(k),\sigma_j(l)}. \quad \square
\end{aligned}$$

We see that $\hat{\sigma}'_{\underline{n}}$ and $e^{2\pi i adX_{\underline{n}}}$ coincide up to a "phase factor". It is not too difficult to remove this factor; set $Y_{\underline{n}}:=\frac{s}{2n}I_n-\sum_{i=1}^{s}\frac{1}{2n_i}I_{n_i}$ (the first term is meant to make $Y_{\underline{n}}$ traceless), then $[X_{\underline{n}},Y_{\underline{n}}]=0$ and if we define $X'_{\underline{n}}:=X_{\underline{n}}+Y_{\underline{n}}$ we can write:

$$\hat{\sigma}'_{\underline{n}}=e^{2\pi i adX'_{\underline{n}}} \tag{3.3.6}$$

However, for technical reasons one often prefers to exclude the diagonal part $Y_{\underline{n}}$ and works with the automorphism

$$\hat{\sigma}_{\underline{n}}:=e^{2\pi i adX_{\underline{n}}}. \tag{3.3.7}$$

Note that

$$\hat{\sigma}_{\underline{n}}^{N'}(E_{kl}^{ij}) = (-)^{N'((1/n_i)-(1/n_j))} E_{kl}^{ij}, \tag{3.3.8}$$

so that it may happen that the order N of $\hat{\sigma}_{\underline{n}}$ is $2N'$.

Lemma 3.3.2. *Let $\underline{n} = \{n_1, n_2, \ldots, n_s\}$ be a partition of n, N' the least common multiple of $n_1, n_2, \ldots, n_s$ and N the order of the automorphism $\hat{\sigma}_{\underline{n}}$ defined in (3.3.7), then:*

$$N = \begin{cases} N' & \text{if} \quad N'\left(\dfrac{1}{n_i} + \dfrac{1}{n_j}\right) \in 2\mathbf{Z} \quad \forall ij \\ 2N' & \text{if} \quad N'\left(\dfrac{1}{n_i} + \dfrac{1}{n_j}\right) \notin 2\mathbf{Z} \quad \text{for a pair } (i, j) \end{cases} \tag{3.3.9}$$

3.4. Rotation to $\underline{h}$. Let $\tau: gl_n(\mathbf{C}) \to gl_n(\mathbf{C})$ be an arbitrary automorphism of order N. It is well known that such a τ induces a $\mathbf{Z}_N \equiv \mathbf{Z}/N\mathbf{Z}$ gradation of the Lie algebra $gl_n(\mathbf{C})$; we have:

$$gl_n(\mathbf{C}) = \bigoplus_{\bar{j} \in \mathbf{Z}_N} gl_n(\mathbf{C})_{\bar{j}}, \tag{3.4.1}$$

where the eigenspace $gl_n(\mathbf{C})_{\bar{j}}$ is defined by

$$gl_n(\mathbf{C})_{\bar{j}} := \{ y \in gl_n(\mathbf{C}) \mid \tau(y) = \omega^j y \}. \tag{3.4.2}$$

Here we have introduced the N^{th} root of unity $\omega := e^{2\pi i/N}$ and the notation $\bar{j} := j \bmod N$. Notice that $[gl_n(\mathbf{C})_{\bar{k}}, gl_n(\mathbf{C})_{\bar{j}}] \subset gl_n(\mathbf{C})_{\overline{k+j}}$.

In order to compute the eigenspaces $gl_n(\mathbf{C})_{\bar{j}}$ it would be very convenient if τ were of the form $e^{2\pi i \operatorname{ad} h}$ for some h in the standard Cartan subalgebra $\underline{h}$; in that case one would find:

$$gl_n(\mathbf{C})_{\bar{j}} = \left\{ y \in gl_n(\mathbf{C}) \mid [h, y] = \frac{j + rN}{N} y \text{ for some } r \in \mathbf{Z} \right\}, \tag{3.4.3}$$

and the eigenspaces can be constructed from the root space decomposition of $gl_n(\mathbf{C})$ with respect to $\underline{h}$.

The automorphism $\hat{\sigma}_{\underline{n}}$, which we have constructed in the previous section, is of the form $e^{2\pi i \operatorname{ad} X_{\underline{n}}}$, where $X_{\underline{n}}$ is an element of the principal Cartan subalgebra but we can of course "rotate" everything to the standard Cartan subalgebra $\underline{h}$ by means of conjugation with the automorphism $\psi_{\underline{n}}$. In this way we obtain the automorphism $\hat{\tau}_{\underline{n}}$ defined by:

$$\hat{\tau}_{\underline{n}} := \psi_{\underline{n}}^{-1} \circ \hat{\sigma}_{\underline{n}} \circ \psi_{\underline{n}} = e^{2\pi i \operatorname{ad} \psi_{\underline{n}}^{-1}(X_{\underline{n}})} = e^{2\pi i \operatorname{ad} H_{\underline{n}}}. \tag{3.4.4}$$

Now the commutation relation

$$[H_{\underline{n}}, E_{kl}^{ij}] = \left(\frac{l}{n_j} - \frac{k}{n_i} + \frac{1}{2n_i} - \frac{1}{2n_j} \right) E_{kl}^{ij} \tag{3.4.5}$$

expresses that E_{kl}^{ij} is homogeneous in the gradation induced by $\hat{\tau}_{\underline{n}}$; its degree is given by

$$\deg E_{kl}^{ij} = N\left(\frac{l}{n_j} - \frac{k}{n_i} + \frac{1}{2n_i} - \frac{1}{2n_j} \right) \bmod N. \tag{3.4.6}$$

4. The Affine Algebra $\hat{gl}_n(\mathbf{C})$

4.1. Introduction. In this section we will briefly review the realizations of the affine algebra as algebras of loops twisted by the automorphisms $\hat{\tau}_{\underline{n}}$ (see [7]). In any such realization a Heisenberg subalgebra $\hat{\underline{s}}_{\underline{n}}$ naturally emerges as the collection of twisted loops in the Cartan subalgebra $\underline{h}_{\underline{n}}$. We will also introduce so-called vertex operators as generating series in a formal indeterminate z, whose coefficients describe the action of the affine algebra in an integrable highest weight representation. In Sect. 7 we will see that these operators are precisely the vertex operators one encounters in physics.

4.2. Twisted Realizations of $\tilde{gl}_n(\mathbf{C})$ [7, 18]. Let $\tilde{gl}_n(\mathbf{C}) := \bigoplus_{k\in\mathbf{Z}} e^{ik\theta} gl_n(\mathbf{C})$ be the loop algebra associated to $gl_n(\mathbf{C})$. Using the automorphism $\hat{\tau}_{\underline{n}}: gl_n(\mathbf{C}) \to gl_n(\mathbf{C})$, one defines the twisted realization $L(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}})$ of $\tilde{gl}_n(\mathbf{C})$ as the space

$$L(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}}) := \bigoplus_{k\in\mathbf{Z}} \lambda^k gl_n(\mathbf{C})_{\bar{k}} \tag{4.2.1}$$

with commutation relations

$$[\lambda^k x, \lambda^j y] := \lambda^{k+j}[x, y] \quad \forall x \in gl_n(\mathbf{C})_{\bar{k}}, y \in gl_n(\mathbf{C})_{\bar{j}}. \tag{4.2.2}$$

It is well known [7, 18] that $\tilde{gl}_n(\mathbf{C})$ and $L(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}})$ are isomorphic; an isomorphism is given by:

$$\Phi_{\underline{n}}: \lambda^k y \mapsto e^{-i\theta \operatorname{ad} H_{\underline{n}}} e^{i(k/N)\theta} y \quad \forall y \in gl_n(\mathbf{C})_{\bar{k}}. \tag{4.2.3}$$

This automorphism suggests to introduce the following spanning set of $\tilde{gl}_n(\mathbf{C})$ (see [5]):

$$y(k) := e^{-i\theta \operatorname{ad} H_{\underline{n}}} e^{i(k/N)\theta} y_{\bar{k}} \quad y \in gl_n(\mathbf{C}), \quad k \in \mathbf{Z}, \tag{4.2.4}$$

where $y_{\bar{k}}$ denotes the component of y, which is homogeneous of degree $\bar{k}$ in the $\mathbf{Z}_N$ gradation of $gl_n(\mathbf{C})$ induced by $\hat{\tau}_{\underline{n}}$ (see (3.4.1) and (3.4.2)).

The affine algebra $\hat{gl}_n(\mathbf{C}) := \tilde{gl}_n(\mathbf{C}) \oplus \mathbf{C}c$ is defined as the one dimensional central extension of $\tilde{gl}_n(\mathbf{C})$ with commutation relations

$$[\tilde{x} + \alpha c, \tilde{y} + \beta c] := [\tilde{x}, \tilde{y}] + \mu(\tilde{x}, \tilde{y})c, \tag{4.2.5}$$

where the two cocycle $\mu: \tilde{gl}_n(\mathbf{C}) \times \tilde{gl}_n(\mathbf{C}) \to \mathbf{C}$ is given by:

$$\mu(\tilde{x}, \tilde{y}) := \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left(\frac{d\tilde{x}(\theta)}{d\theta} \middle| \tilde{y}(\theta) \right). \tag{4.2.6}$$

Here we have introduced the notation $(A|B) = \operatorname{trace}(AB), \forall A, B \in gl_n(\mathbf{C})$.

The corresponding twisted realization $\hat{L}(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}})$ is defined as the vector space $L(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}}) \oplus \mathbf{C}c$ with commutation relations

$$[\lambda^k y_{\bar{k}} + \alpha c, \lambda^l z_{\bar{l}} + \beta c] = \lambda^{k+l}[y_{\bar{k}}, z_{\bar{l}}] + \frac{k}{N}\delta_{k+l,0}(y_{\bar{k}}|z_{\bar{l}})c. \tag{4.2.7}$$

The isomorphism $\Phi_{\underline{n}}: L(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}}) \to \tilde{gl}_n(\mathbf{C})$ can be extended to an isomorphism $\hat{\Phi}_{\underline{n}}$ between $\hat{L}(gl_n(\mathbf{C}), \hat{\tau}_{\underline{n}})$ and $\hat{gl}_n(\mathbf{C})$:

$$\hat{\Phi}_{\underline{n}}(\lambda^k y_{\bar{k}} + \alpha c) := \hat{y}(k) + \alpha c, \tag{4.2.8}$$

where the elements $\hat{y}(k)\in\hat{gl}_n(\mathbf{C})$ are defined by

$$\hat{y}(k):=y(k)-\delta_{k,0}(H_{\underline{n}}|y)c. \tag{4.2.9}$$

4.3. Heisenberg Subalgebras of $\hat{gl}_n(\mathbf{C})$. In this subsection we will consider Heisenberg subalgebras of the affine algebra $\hat{gl}_n(\mathbf{C})$. Let us start by recalling the definition of such subalgebras;

Definition 4.3.1. A Heisenberg subalgebra (HSA) $\underline{\hat{s}}$ of $\hat{gl}_n(\mathbf{C})$ is a subalgebra of $\hat{gl}_n(\mathbf{C})$ with basis $\{p_i, q_i\}_{i\in\mathbf{N}}$ and the central element c, whose elements satisfy the commutation relations

$$[p_i, q_j]=\delta_{ij}c. \tag{4.3.1}$$

For any partition $\underline{n}$ of n we will construct an associated HSA $\underline{\hat{s}}_{\underline{n}}$, thus obtaining a correspondence between conjugacy classes in the Weyl group $W(gl_n(\mathbf{C}))$ and HSA's.

Let α_i be the matrix consisting of 1's on the $(i,i)^{\text{th}}$ diagonal block B_{ii} (see (2.1.2)) and zeros elsewhere;

$$\alpha_i:=\sum_{k,l=1}^{n_i} E_{kl}^{ii}=\sum_{k=1}^{n_i} {E_{(i)}}^k. \tag{4.3.2}$$

According to (3.4.6), the homogeneous components of this matrix are precisely the powers of the i^{th} cyclic element $E_{(i)}$; their degree is given by

$$\deg {E_{(i)}}^k=\frac{N}{n_i}k \mod N. \tag{4.3.3}$$

Therefore, the loops $\alpha_i(j)$ introduced in Sect. 4.2 are only nonzero if $j=\dfrac{N}{n_i}k$ for some $k\in\mathbf{Z}$. In that case we can write:

$$\alpha_i\left(\frac{N}{n_i}k\right)=e^{-i\theta \operatorname{ad} H_{\underline{n}}}e^{i(k/n_i)\theta}{E_{(i)}}^k. \tag{4.3.4}$$

If we write $k=l+mn_i$, $1\leqq l\leqq n_i$, $m\in\mathbf{Z}$ this becomes

$$\alpha_i\left(\frac{N}{n_i}(l+mn_i)\right)=e^{im\theta}e^{-i\theta \operatorname{ad} H_{\underline{n}}}e^{i(l/n_i)\theta}{E_{(i)}}^l \tag{4.3.5}$$

corresponding to a matrix with zero entries, except on its $(i,i)^{\text{th}}$ diagonal block. For $1\leqq l\leqq n_i-1$ this block becomes:

$$e^{im\theta}\begin{array}{cc} & \overset{n_i-l}{\longleftrightarrow} \\ n_i-l\updownarrow & \begin{bmatrix} 0 & \cdots & 0 & 1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ 0 & \cdots & 0 & & & 1 \\ e^{i\theta} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & e^{i\theta} & 0 & \cdots & 0 \end{bmatrix}\end{array} \tag{4.3.6}$$

while the case $l=n_i$ yields $e^{im\theta}I_{n_i}$.

With the definition (4.2.6) on easily computes the value of the two cocycle μ on loops $\alpha_i\left(\frac{N}{n_i}k\right)$ and $\alpha_j\left(\frac{N}{n_j}l\right)$;

$$\mu\left(\alpha_i\left(\frac{N}{n_i}k\right),\alpha_j\left(\frac{N}{n_j}l\right)\right)=\frac{k}{n_i}\delta_{k+l,0}(E_{(i)}{}^k|E_{(j)}{}^l)$$
$$=k\delta_{ij}\delta_{k+l,0}. \tag{4.3.7}$$

Using this formula, one easily proves the following lemma.

Lemma 4.3.2. *Set* $\underline{\hat{s}}_n := \bigoplus_{k\in\mathbf{Z}-\{0\}} \bigoplus_{i=1}^{s} \mathbf{C}\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\oplus\mathbf{C}c$, *then* $\underline{\hat{s}}_n$ *is a* HSA.

Proof. Define for $1\leqq i\leqq s, 1\leqq j\leqq n_i, l\in\mathbf{Z}_+$ the elements

$$p_{n_1+\dots+n_{i-1}+j+nl}:=\hat{\alpha}_i\left(\frac{N}{n_i}(j+n_il)\right),$$

$$q_{n_1+\dots+n_{i-1}+j+nl}:=\frac{1}{j+n_il}\hat{\alpha}_i\left(-\frac{N}{n_i}(j+n_il)\right).$$

Together with the central element c these elements form a basis for $\underline{\hat{s}}_n$. The Heisenberg commutation relations (4.3.1) are immediately verified.

Remark 4.3.3. Note that $(H_{\underline{n}}|\alpha_i)=\operatorname{tr}H_{(i)}=0$ and therefore

$$\hat{\alpha}_i\left(\frac{N}{n_i}k\right)=\alpha_i\left(\frac{N}{n_i}k\right). \tag{4.3.8}$$

This is the reason that we have chosen to work with $\hat{\sigma}_{\underline{n}}$ and not with $\hat{\sigma}'_{\underline{n}}$ (cf. Sect. 3.3).

4.4. Vertex Operators. Recall from Sect. 2 the Cartan subalgebra $\underline{h}_{\underline{n}}:=\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{n_i}\mathbf{C}E_{(i)}{}^j$ and the root vectors $A_{pq}^{ij}, i\neq j\vee(i=j\wedge p\neq q)$. From the discussion in Sect. 4.3 it is clear now that the affine algebra $\hat{gl}_n(\mathbf{C})$ is spanned by the loops

$$\hat{\alpha}_i\left(\frac{N}{n_i}k\right)i=1,2,\dots,s,\quad k\in\mathbf{Z}\quad \hat{A}_{pq}^{ij}(k)i\neq j\vee(i=j\wedge p\neq q),\quad k\in\mathbf{Z}. \tag{4.4.1}$$

In the sequel we will be interested in representations $\rho:\hat{gl}_n(\mathbf{C})\to\operatorname{End}V$ of the affine algebra on a vector space V. In the construction of representations one uses the following formal generating series:

$$\hat{\alpha}_i(z):=\sum_{k\in\mathbf{Z}}z^{-(N/n_i)k}\rho\left(\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\right),$$
$$\hat{A}_{pq}^{ij}(z):=\sum_{k\in\mathbf{Z}}z^{-k}\rho(\hat{A}_{pq}^{ij}(k)). \tag{4.4.2}$$

The parameter z in these series is just a formal parameter. If one can somehow find explicit expressions for the operators $\hat{\alpha}_i(z)$ and $\hat{A}_{pq}^{ij}(z)$, one can – in principle – extract the coefficients of z^{-k} and find the action of the affine algebra on the representation space V.

Notice that the formal power series $\hat{\alpha}_i(z)$ and $\hat{A}^{ij}_{pq}(z)$ are not unrelated; recalling (4.3.2) and Lemma 2.2.2, we write $\alpha_i = \sum_{j=1}^{n_i} E_{(i)}{}^j = n_i A^{ii}_{n_i,n_i}$, and therefore we have

$$\hat{\alpha}_i(z) = n_i \hat{A}^{ii}_{n_i n_i}(z). \tag{4.4.3}$$

In fact we will only be interested in integrable highest weight representations of the affine algebra, in which the central element c is represented by the identity operator. So from now on we will take $\rho(c) = I$. We will see that in such representations the representation space V can be realized as a (direct sum of) polynomial ring(s) while the elements p_i and q_i of the HSA are represented by elementary differentiation and multiplication operators;

$$\rho(p_i) = \frac{\partial}{\partial x_i}; \quad \rho(q_i) = x_i. \tag{4.4.4}$$

It then remains to construct the so-called vertex operators $\hat{A}^{ij}_{pq}(z)$; inserting the definitions (4.2.4) and (4.2.9), we write for these operators:

$$\hat{A}^{ij}_{pq}(z) = \sum_{k \in \mathbf{Z}} z^{-k} \rho(e^{i(k/N)\theta} e^{-i\theta \operatorname{ad} H_{\underline{n}}}(A^{ij}_{pq})_{\bar{k}}) - (H_{\underline{n}} | A^{ij}_{pq}) I. \tag{4.4.5}$$

The first term can be worked out as:

$$\begin{aligned}
&\sum_{k \in \mathbf{Z}} z^{-k} \rho(e^{i(k/N)\theta} e^{-i\theta \operatorname{ad} H_{\underline{n}}}(A^{ij}_{pq})_{\bar{k}}) \\
&= \frac{1}{\sqrt{n_i n_j}} \sum_{k \in \mathbf{Z}} \sum_{m=1}^{n_i} \sum_{l=1}^{n_j} z^{-k} \omega_i^{pm} \omega_j^{-ql} \rho(e^{i(k/N)\theta} e^{-i\theta \operatorname{ad} H_{\underline{n}}} E^{ij}_{ml}) \\
&\quad \cdot \delta_{k, N((l/n_j) - (m/n_i) + (1/2n_i) - (1/2n_j)) + rN} \\
&= \frac{1}{\sqrt{n_i n_j}} z^{N((1/2n_j) - (1/2n_i))} \sum_{r \in \mathbf{Z}} \sum_{m=1}^{n_i} \sum_{l=1}^{n_j} z^{-N((l/n_j) - (m/n_i)) - rN} \omega_i^{pm} \omega_j^{-ql} \rho(e^{ir\theta} E^{ij}_{ml}),
\end{aligned} \tag{4.4.6}$$

while the central term is calculated as follows:

$$\begin{aligned}
(H_{\underline{n}} | A^{ij}_{pq}) &= \frac{1}{2n_i^2} \delta_{ij} \sum_{k=1}^{n_i} (n_i - 2k + 1) \omega_i^{k(p-q)} \\
&= \begin{cases} \dfrac{1}{n_i} \delta_{ij} \dfrac{\omega_i^{p-q}}{1 - \omega_i^{p-q}} & \text{if} \quad p \neq q \\ 0 & \text{if} \quad p = q \end{cases}.
\end{aligned} \tag{4.4.7}$$

In the following section we will construct a much simpler expression for the operators $\hat{A}^{ij}_{pq}(z)$ in terms of "fermionic fields."

5. Fermionic Fields

5.1. Introduction. In this section we will briefly review the Lie algebras $\overline{gl(\infty)}$ and its central extension $A_\infty = \overline{gl(\infty)} \oplus \mathbf{C}c$ (see, e.g., [13]). We will also discuss the embeddings $\tilde{gl}_n(\mathbf{C}) \hookrightarrow \overline{gl(\infty)}$ and $\hat{gl}_n(\mathbf{C}) \hookrightarrow A_\infty$, which, together with the wedge representation of A_∞, will enable us to find expressions for the vertex operators

$\hat{A}^{ij}_{pq}(z)$ in terms of normal ordered products of fermionic fields. Sections 5.2 and 5.3 have been taken from the paper [15].

5.2. Lie Algebras of Infinite Matrices. Let $gl(\infty)$ be the Lie algebra of matrices $(g_{ij})_{i,j\in\mathbf{Z}}$ such that all but a finite number of entries g_{ij} are zero. This Lie algebra has a basis of matrices $\mathscr{E}_{ij}$, $i,j\in\mathbf{Z}$, where $\mathscr{E}_{ij}$ is the matrix with a one on the $(i,j)^{\text{th}}$ entry and zeros elsewhere; any matrix $G\in gl(\infty)$ can be written as a finite linear combination of the $\mathscr{E}_{ij}$'s.

We also introduce the vector space $\mathbf{C}^\infty$ of infinite column vectors $(v_i)_{i\in\mathbf{Z}}$ such that all but a finite number of v_i's are zero. As a basis for this vector space we take the collection $\{\varepsilon_i\}_{i\in\mathbf{Z}}$, where ε_i is the vector with a one on the i^{th} entry and zeros elsewhere. The Lie algebra $gl(\infty)$ clearly acts on the vector space $\mathbf{C}^\infty$ by:

$$\mathscr{E}_{ij}\varepsilon_k = \delta_{jk}\varepsilon_i. \tag{5.2.1}$$

We will also need a completion $\overline{gl(\infty)}$ of the algebra $gl(\infty)$ consisting of all matrices of finite width around the main diagonal;

$$\overline{gl(\infty)} := \left\{ \sum_{ij\in\mathbf{Z}} g_{ij}\mathscr{E}_{ij} \,|\, g_{ij} = 0 \text{ if } |i-j| \gg 0 \right\}. \tag{5.2.2}$$

Note that the $gl(\infty)$-action on $\mathbf{C}^\infty$ can be extended to this completion. With this completion we can formulate the following well known result.

Lemma 5.2.1. *Let $\imath{:}\tilde{gl}_n(\mathbf{C}) \to \overline{gl(\infty)}$ be the mapping defined by*

$$\imath{:}\, e^{ik\theta}E_{ij} \mapsto \sum_{l\in\mathbf{Z}} \mathscr{E}_{i+n(l-k),\,j+nl}, \tag{5.2.3}$$

then $\imath$ is a Lie algebra homomorphism mapping $\tilde{gl}_n(\mathbf{C})$ injectively into $\overline{gl(\infty)}$. Its image is the collection of matrices $(g_{ij})\in\overline{gl(\infty)}$ satisfying the periodicity condition

$$g_{i+nr,\,j+nr} = g_{ij} \quad \forall i,j,r\in\mathbf{Z}. \tag{5.2.4}$$

5.3. The Semi-Infinite Wedge Space. In this subsection we consider highest weight representations for $gl(\infty)$ and $\overline{gl(\infty)}$. For this purpose we introduce the semi-infinite wedge space $\wedge^\infty\mathbf{C}^\infty$ as the vector space with a basis consisting of all semi-infinite exterior products of the basis elements ε_i of $\mathbf{C}^\infty$ of the form:

$$\varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \tag{5.3.1}$$

such that $i_0 > i_1 > i_2 > \cdots$ and such that $i_{l+1} = i_l - 1$ for $l \gg 0$. On this space $gl(\infty)$ acts as usual; denoting the action by τ, we can write:

$$\begin{aligned}\tau(A)(\varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots) &= (A\varepsilon_{i_0}) \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \\ &\quad + \varepsilon_{i_0} \wedge (A\varepsilon_{i_1}) \wedge \varepsilon_{i_2} \wedge \cdots \\ &\quad + \varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge (A\varepsilon_{i_2}) \wedge \cdots + \cdots \quad \forall A\in gl(\infty).\end{aligned} \tag{5.3.2}$$

We can distinguish the basis elements (5.3.1) by their behaviour at large l; we will say that an element of the form (5.3.1) has charge k if $i_l = k - l$ for all $l \gg 0$. For instance the vector

$$v_k := \varepsilon_k \wedge \varepsilon_{k-1} \wedge \varepsilon_{k-2} \wedge \cdots \tag{5.3.3}$$

has charge k. We will refer to v_k as the k^{th} vacuum. The vector space of all vectors of charge k is denoted by $\wedge_k^\infty \mathbf{C}^\infty$ and we clearly have a decomposition of the full semi-infinite wedge space in sectors of fixed charge;

$$\wedge^\infty \mathbf{C}^\infty = \bigoplus_{k\in\mathbf{Z}} \wedge_k^\infty \mathbf{C}^\infty. \tag{5.3.4}$$

The submodule $\wedge_k^\infty \mathbf{C}^\infty$ is an irreducible highest weight module for the algebra $gl(\infty)$. We have for $j>i$:

$$\tau(\mathscr{E}_{ij})(v_k) = \delta_{jk}\varepsilon_i \wedge \varepsilon_{k-1} \wedge \varepsilon_{k-2} \wedge \cdots + \delta_{j,k-1}\varepsilon_k \wedge \varepsilon_i \wedge \varepsilon_{k-2} \wedge \cdots + \cdots = 0 \tag{5.3.5}$$

and

$$\tau(\mathscr{E}_{ii})(v_k) = \langle \theta_k, \mathscr{E}_{ii}\rangle v_k, \tag{5.3.6}$$

where the linear mapping $\theta_k : \bigoplus_{i\in\mathbf{Z}} \mathbf{C}\mathscr{E}_{ii} \to \mathbf{C}$ is defined by

$$\langle \theta_k, \mathscr{E}_{ii}\rangle := \begin{cases} 0 & \text{if } i>k \\ 1 & \text{if } i\leqq k \end{cases}. \tag{5.3.7}$$

In the sequel we will denote the restriction of the representation τ to this module by τ_k.

For every $i\in\mathbf{Z}$ we define linear operators $\psi(i)$ and $\psi^*(i)$ on the semi-infinite wedge space by their action on basis vectors:

$$\begin{aligned} \psi(i)(\varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots) &:= \varepsilon_i \wedge \varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots, \\ \psi^*(i)(\varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots) &:= \sum_{k=0}^{\infty} (-)^k \delta_{i,i_k} \varepsilon_{i_0} \wedge \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \wedge \hat{\varepsilon}_{i_k} \wedge \cdots, \end{aligned} \tag{5.3.8}$$

where the notation $\hat{\varepsilon}_{i_k}$ means that the vector ε_{i_k} is deleted. It is easy to verify the anticommutation relations:

$$\{\psi(i), \psi(j)\} = 0 = \{\psi^*(i), \psi^*(j)\}, \quad \{\psi(i), \psi^*(j)\} = \delta_{ij}. \tag{5.3.9}$$

The importance of these operators lies in the fact that the action of the elements $\mathscr{E}_{ij}$ can be written as a product of these operators; one clearly has

$$\tau(\mathscr{E}_{ij}) = \psi(i)\psi^*(j). \tag{5.3.10}$$

It is not possible to extend the representation τ to the completion $\overline{gl(\infty)}$ by linearity. Instead one replaces τ by the assignment π defined by

$$\pi(\mathscr{E}_{ij}) := \tau(\mathscr{E}_{ij}) - \delta_{ij}\langle \theta_0, \mathscr{E}_{ii}\rangle I. \tag{5.3.11}$$

It can be proved that π can be extended to $\overline{gl(\infty)}$ by linearity; for the action of the identity matrix in $\overline{gl(\infty)}$ on the k^{th} vacuum we find:

$$\pi\left(\sum_{i\in\mathbf{Z}} \mathscr{E}_{ii}\right)(v_k) = \sum_{i\in\mathbf{Z}} \langle \theta_k - \theta_0, \varepsilon_{ii}\rangle v_k = kv_k. \tag{5.3.12}$$

Of course π is not a representation of $\overline{gl(\infty)}$ any more;

$$\begin{aligned} [\pi(\mathscr{E}_{ij}), \pi(\mathscr{E}_{kl})] &= [\tau(\mathscr{E}_{ij}), \tau(\mathscr{E}_{kl})] \\ &= \delta_{jk}\tau(\mathscr{E}_{il}) - \delta_{il}\tau(\mathscr{E}_{kj}) \\ &= \delta_{jk}\pi(\mathscr{E}_{il}) - \delta_{il}\pi(\mathscr{E}_{kj}) + \delta_{il}\delta_{jk}\langle \theta_0, \mathscr{E}_{ii} - \mathscr{E}_{jj}\rangle \\ &= \pi([\mathscr{E}_{ij}, \mathscr{E}_{kl}]) + \delta_{il}\delta_{jk}\langle \theta_0, \mathscr{E}_{ii} - \mathscr{E}_{jj}\rangle. \end{aligned} \tag{5.3.13}$$

Because of the extra term $\delta_{il}\delta_{jk}\langle\theta_0, \mathscr{E}_{ii}-\mathscr{E}_{jj}\rangle$ in the right-hand side of (5.3.13), π is called a projective representation of $\overline{gl(\infty)}$, or, equivalently, a $c=1$ representation of the central extension $A_\infty := \overline{gl(\infty)} \oplus \mathbf{C}c$ of $\overline{gl(\infty)}$ defined by the two cocycle

$$\mu: \overline{gl(\infty)} \times \overline{gl(\infty)} \to \mathbf{C},$$
$$\mu(\mathscr{E}_{ij}, \mathscr{E}_{kl}) := \delta_{il}\delta_{jk}\langle\theta_0, \mathscr{E}_{ii}-\mathscr{E}_{jj}\rangle. \tag{5.3.14}$$

One can of course restrict the central extension of $\overline{gl(\infty)}$ to the subalgebras $\tilde{gl}_n(\mathbf{C})$;

Lemma 5.3.1.

$$\mu(\imath(e^{ik\theta}A), \imath(e^{il\theta}B)) = k\delta_{k+l,0}(A|B) \quad \forall k, l \in \mathbf{Z}, A, B \in gl_n(\mathbf{C}). \tag{5.3.15}$$

The lemma means that the homomorphism $\imath: \tilde{gl}_n(\mathbf{C}) \hookrightarrow \overline{gl(\infty)}$ can be extended to a homomorphism $\hat{\imath}: \hat{gl}_n(\mathbf{C}) \hookrightarrow A_\infty$.

In physics the procedure above is usually formulated in terms of a normal ordering prescription on the fermionic creation and annihilation operators; one writes:

$$\pi(\mathscr{E}_{ij}) = :\psi(i)\psi^*(j): \tag{5.3.16}$$

where the normal ordered product $:\psi(i)\psi^*(j):$ is defined by:

$$:\psi(i)\psi^*(j): := \psi(i)\psi^*(j) - \delta_{ij}\langle\theta_0, \mathscr{E}_{ii}\rangle = \begin{cases} \psi(i)\psi^*(j) & \text{if } i>0 \\ -\psi^*(j)\psi(i) & \text{if } i \leqq 0 \end{cases}. \tag{5.3.17}$$

5.4. Multicomponent Fermions. In Sects. 5.2 and 5.3 we have explained how to embed the affine algebra $\hat{gl}_n(\mathbf{C})$ in A_∞ and how to obtain representations of this algebra in the semi-infinite wedge space. Up to now the block structure of $gl_n(\mathbf{C})$ (see (2.1.2)) has played no role in our discussion. In this section we will incorporate this block structure in the theory.

Let $\underline{n} = \{n_1, n_2, \ldots, n_s\}$ be again a partition of n. Associated to the block structure of an $n \times n$ matrix in $gl_n(\mathbf{C})$ one has of course a direct sum decomposition of $\mathbf{C}^n$:

$$\mathbf{C}^n = \mathbf{C}^{n_1} \oplus \mathbf{C}^{n_2} \oplus \cdots \mathbf{C}^{n_s}. \tag{5.4.1}$$

In view of this decomposition we relabel the basis vectors e_i of $\mathbf{C}^n$; we define:

$$e_i(l) := e_{n_1+n_2+\cdots n_{i-1}+l} \quad 1 \leqq i \leqq s,\ 1 \leqq l \leqq n_i. \tag{5.4.2}$$

Let $\tilde{\mathbf{C}}^n := \bigoplus_{k\in\mathbf{Z}} e^{ik\theta}\mathbf{C}^n$. The mapping $e^{-ik\theta}e_j \mapsto \varepsilon_{j+nk}$ gives an isomorphism between $\tilde{\mathbf{C}}^n$ and $\mathbf{C}^\infty$. The image of the vector $e^{-ik\theta}e_j(l)$ is $\varepsilon_{n_1+n_2+\cdots n_{j-1}+l+nk}$ so it is natural to relabel the basis vectors of $\mathbf{C}^\infty$ as follows:

$$\varepsilon_j(l+n_jk) := \varepsilon_{n_1+n_2+\cdots n_{j-1}+l+nk}. \tag{5.4.3}$$

With this definition it is clear that we can write:

$$\mathbf{C}^\infty = \bigoplus_{i=1}^{s} \bigoplus_{j\in\mathbf{Z}} \mathbf{C}\varepsilon_i(j). \tag{5.4.4}$$

Similarly, the semi-infinite wedge space is spanned by wedges of the form:

$$\varepsilon_{i_1}(k) \wedge \varepsilon_{i_2}(k_2) \wedge \cdots. \tag{5.4.5}$$

As an example we consider the expression for the k^{th} vacuum $v_k = \varepsilon_k \wedge \varepsilon_{k-1} \wedge \cdots$ in terms of the relabeled basis vectors; for $k = n_1 + n_2 + \cdots + n_{i-1} + l + nj$ we find:

$$\begin{aligned} v_k = {} & \varepsilon_i(l + n_i j) \wedge \varepsilon_i(l - 1 + n_i j) \wedge \cdots \varepsilon_i(1 + n_i j) \wedge \\ & \varepsilon_{i-1}(n_{i-1} + n_{i-1} j) \wedge \cdots \quad \varepsilon_{i-1}(1 + n_{i-1} j) \wedge \\ & \vdots \\ & \varepsilon_1(n_1 + n_1 j) \wedge \cdots \qquad\qquad \varepsilon_1(1 + n_1 j) \wedge \\ & \varepsilon_s(n_s + n_s(j-1)) \wedge \cdots \quad \varepsilon_s(1 + n_s(j-1)) \wedge \cdots . \end{aligned} \tag{5.4.6}$$

The next step is to introduce fermionic creation and annihilation operators for the vectors $\varepsilon_i(k)$ analogous to (5.3.8);

$$\begin{aligned} & \psi_i(k)(\varepsilon_{i_0}(k_0) \wedge \varepsilon_{i_1}(k_1) \wedge \varepsilon_{i_2}(k_2) \wedge \cdots) \\ & \quad := \varepsilon_i(k) \wedge \varepsilon_{i_0}(k_0) \wedge \varepsilon_{i_1}(k_1) \wedge \varepsilon_{i_2}(k_2) \wedge \cdots, \\ & \psi_i^*(k)(\varepsilon_{i_0}(k_0) \wedge \varepsilon_{i_1}(k_1) \wedge \varepsilon_{i_2}(k_2) \wedge \cdots) \\ & \quad := \sum_{j=0}^{\infty} (-)^j \delta_{i,i_j} \delta_{k,k_j} \varepsilon_{i_0}(k_0) \wedge \varepsilon_{i_1}(k_1) \wedge \varepsilon_{i_2}(k_2) \wedge \cdots\cdots \wedge \hat{\varepsilon}_{i_j}(k_j) \wedge \cdots . \end{aligned} \tag{5.4.7}$$

Clearly we have

$$\begin{aligned} \psi_i(k + n_i l) &= \psi(n_1 + n_2 + \cdots + n_{i-1} + k + nl), \\ \psi^*_i(k + n_i l) &= \psi^*(n_1 + n_2 + \cdots + n_{i-1} + k + nl). \end{aligned} \tag{5.4.8}$$

With this identification one easily checks the anti-commutation relations of these multicomponent fermions

$$\{\psi_i(k), \psi_j(l)\} = 0, \quad \{\psi_i^*(k), \psi_j^*(l)\} = 0, \quad \{\psi_i(k), \psi_j^*(l)\} = \delta_{ij}\delta_{kl}. \tag{5.4.9}$$

One also verifies

$$\begin{aligned} \psi_i(k)(v_0) &= 0 \quad \text{if} \quad k \leqq 0, \\ \psi_i^*(k)(v_0) &= 0 \quad \text{if} \quad k > 0, \end{aligned} \tag{5.4.10}$$

whence

$$:\psi_i(k)\psi_j^*(l): = \begin{cases} \psi_i(k)\psi_j^*(l) & \text{if } k > 0 \\ -\psi_j^*(l)\psi_i(k) & \text{if } k \leqq 0 \end{cases}. \tag{5.4.11}$$

5.5. Vertex Operators as Normal Ordered Products of Fermionic Fields. Here we return to the expression (4.4.5) for the vertex operator $\hat{A}_{pq}^{ij}(z)$. For the representation ρ occurring in this formula we take $\pi \circ \hat{\imath}$. So we first embed the algebra $\hat{gl}_n(\mathbf{C})$ in A_∞ by the homomorphism $\hat{\imath}$ and then we represent it on the semi-infinite wedge space by the representation π. Using (5.2.3) and (4.4.5–7), we write:

$$\begin{aligned} \hat{A}_{pq}^{ij}(z) + (H_{\underline{n}} | A_{pq}^{ij}) = {} & \frac{z^{N((1/2n_j) - (1/2n_i))}}{\sqrt{n_i n_j}} \sum_{r,s \in Z} \sum_{m=1}^{n_i} \sum_{l=1}^{n_j} z^{-N((l/n_j) - (m/n_i)) - rN} \omega_i^{pm} \omega_j^{-ql} \\ & \cdot \pi(\mathscr{E}_{n_1 + n_2 \cdots n_{i-1} + m + n(s-r), n_1 + n_2 \cdots n_{j-1} + l + ns}). \end{aligned} \tag{5.5.1}$$

Next we use the expression (5.3.16) and the definition (5.4.8) to rewrite this formula in terms of fermions;

$$\hat{A}^{ij}_{pq}(z) + (H_{\underline{n}} | A^{ij}_{pq})$$
$$= \frac{z^{N((1/2n_j)-(1/2n_i))}}{\sqrt{n_i n_j}} \sum_{r,s\in\mathbf{Z}} \sum_{m=1}^{n_i} \sum_{l=1}^{n_j} z^{-N((l/n_j)-(m/n_i))-rN} \omega_i^{pm} \omega_j^{-ql}$$
$$:\psi(n_1+n_2\cdots n_{i-1}+m+n(s-r))\psi^*(n_1+n_2\cdots n_{j-1}+l+ns):$$
$$= \frac{z^{N((1/2n_j)-(1/2n_i))}}{\sqrt{n_i n_j}} \sum_{r,s\in\mathbf{Z}} \sum_{m=1}^{n_i} \sum_{l=1}^{n_j} z^{-N((l/n_j)-(m/n_i))-rN} \omega_i^{pm} \omega_j^{-ql}$$
$$:\psi_i(m+n_i(s-r))\psi_j^*(l+n_j s):$$
$$= \frac{z^{N((1/2n_j)-(1/2n_i))}}{\sqrt{n_i n_j}} \sum_{r,s\in\mathbf{Z}} \sum_{m=1}^{n_i} \sum_{l=1}^{n_j} (z\omega^p)^{(N/n_i)(m+n_i(s-r))} (z\omega^q)^{-(N/n_j)(l+n_j s)}$$
$$:\psi_i(m+n_i(s-r))\psi_j^*(l+n_j s):, \tag{5.5.2}$$

where $\omega := e^{2\pi i/N}$. This formula shows that it is useful to introduce the following formal "fermionic fields:"

$$\psi_i(z) := \sum_{k\in\mathbf{Z}} z^{(N/n_i)(k-(1/2))} \psi_i(k),$$
$$\psi_i^*(z) := \sum_{k\in\mathbf{Z}} z^{-(N/n_i)(k-(1/2))} \psi_i^*(k). \tag{5.5.3}$$

One then has the following expression for the vertex operators in terms of normal ordered products of these fields:

$$\hat{A}^{ij}_{pq}(z) = \frac{e^{i\pi((p/n_i)-(q/n_j))}}{\sqrt{n_i n_j}} :\psi_i(\omega^p z)\psi_j^*(\omega^q z): - \frac{1}{n_i}\delta_{ij} \frac{\omega^{(N/n_i)(p-q)}}{1-\omega^{(N/n_i)(p-q)}} \quad \text{if } p \neq q,$$
$$\hat{A}^{ij}_{pp}(z) = \frac{e^{i\pi p((1/n_i)-(1/n_j))}}{\sqrt{n_i n_j}} :\psi_i(\omega^p z)\psi_j^*(\omega^p z):. \tag{5.5.4}$$

Note that, because $\hat{\alpha}_i(z) = n_i \hat{A}^{ii}_{n_i n_i}(z)$, we have:

$$\hat{\alpha}_i(z) = :\psi_i(z)\psi_i^*(z):. \tag{5.5.5}$$

Here we have absorbed factors $z^{(N/2n_i)}$ in the definition of the fermionic fields. This is slightly different from our definition in the introduction. The only reason for this is that the fields defined by (5.5.3) have nicer conformal transformation properties. We will come back to this in Sect. 6.

5.6. Hermitian Structure and Normal Ordering. It is well known that the semi-infinite wedge space can be equipped with an inner product, giving it the structure of a pre-Hilbert space;

Theorem 5.6.1. *There exists a unique positive definite Hermitian form* $(\,,\,): \Lambda^\infty \mathbf{C}^\infty \times \Lambda^\infty \mathbf{C}^\infty \to \mathbf{C}$ *such that*

$$\text{a)} \quad \psi(i)^\dagger = \psi^*(i),$$
$$\text{b)} \quad (v_0, v_0) = 1. \tag{5.6.1}$$

Instead of (,) we will also use Dirac's 'bra-ket' notation;

$$(v, \omega) \equiv \langle v | w \rangle. \tag{5.6.2}$$

Moreover, we will also write $|k\rangle$ for the k^{th} vacuum v_k in the sequel. Finally, we will leave out the symbols π and $\pi \circ \iota$ when a representation is understood.

With the inner product one formulates the following lemma:

Lemma 5.6.2.

$$\begin{aligned} &\text{a)} \quad \mathscr{E}_{ij}^{\dagger} = \mathscr{E}_{ji}, \\ &\text{b)} \quad \hat{\alpha}_i\left(\frac{N}{n_i}k\right)^{\dagger} = \hat{\alpha}_i\left(-\frac{N}{n_i}k\right), \\ &\text{c)} \quad (e^{ik\theta}X)^{\dagger} = e^{-ik\theta}\bar{X}^t \quad \forall X \in gl_n(\mathbf{C}). \end{aligned} \tag{5.6.3}$$

Proof. Relation a) is immediate from the expression $\pi(\mathscr{E}_{ij}) = :\psi(i)\psi^*(j):$ and (5.6.1). For b) one uses the following expression of $\hat{\alpha}_i\left(\frac{N}{n_i}k\right)$ in terms of fermions:

$$\hat{\alpha}_i\left(\frac{N}{n_i}k\right) = \sum_{r \in \mathbf{Z}} :\psi_i(r)\psi_i^*(r+k):, \tag{5.6.4}$$

which can itself easily be derived from (5.5.5). Finally, c) following from a) and (5.2.3). □

Next we use the Hermitian structure to define normal ordering for arbitrary operators on the semi-infinite wedge space. For this we note that the definition (5.3.17) can be written as:

$$:\psi(i)\psi^*(j):= \psi(i)\psi^*(j) - \langle 0|\psi(i)\psi^*(j)|0\rangle. \tag{5.6.5}$$

The notation $\langle 0|A|0\rangle := (v_0, Av_0)$ in this formula is called the vacuum expectation value of the operator A. Next we extend the normal ordering prescription to arbitrary operators on the semi-infinite wedge space.

Definition 5.6.3. *Let A and B be operators on $\wedge^{\infty}\mathbf{C}^{\infty}$, then the normal ordered product of these operators is defined by*

$$:AB: \stackrel{\text{def}}{=} AB - \langle 0|AB|0\rangle. \tag{5.6.6}$$

Of course this definition is different from the usual one, because we subtract only a c-number from the ordinary product and no operator structures. Our motivation for this definition is that it produces the right ordering prescription for a pair of bosonic oscillators as we will show below.

Here and in the sequel we will need the following useful lemma:

Lemma 5.6.4. *The operators $\hat{\alpha}_i\left(\frac{N}{n_i}k\right): \wedge^{\infty}\mathbf{C}^{\infty} \to \wedge^{\infty}\mathbf{C}^{\infty}$ are for $k > 0$ (bosonic) annihilation operators, i.e.:*

$$\hat{\alpha}_i\left(\frac{N}{n_i}k\right)|m\rangle = 0 \quad \forall k > 0, \quad m \in \mathbf{Z}. \tag{5.6.7}$$

Moreover, we have

$$\hat{\alpha}_i(0)|0\rangle = 0. \tag{5.6.8}$$

Proof. For $k>0$ the matrix $\alpha_i\left(\frac{N}{n_i}k\right)$ is upper triangular, for $k=0$ it is diagonal. □

It is now easy to compute the vacuum expectation value of the product of two oscillators;

$$\langle 0|\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}l\right)|0\rangle = \begin{cases} k\delta_{k+l,0} & \text{if } k>0 \\ 0 & \text{if } k\leqq 0 \end{cases}. \tag{5.6.9}$$

Hence we find for the normal ordered product:

$$:\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}l\right):= \begin{cases} \hat{\alpha}_i\left(\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}l\right) & \text{if } k\leqq 0 \\ \hat{\alpha}_i\left(\frac{N}{n_i}l\right)\hat{\alpha}_i\left(\frac{N}{n_i}k\right) & \text{if } k>0 \end{cases}. \tag{5.6.10}$$

One immediately checks that this is equivalent to (cf. [19]):

$$:\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}l\right):= \begin{cases} \hat{\alpha}_i\left(\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}l\right) & \text{if } k\leqq l \\ \hat{\alpha}_i\left(\frac{N}{n_i}l\right)\hat{\alpha}_i\left(\frac{N}{n_i}k\right) & \text{if } k>l \end{cases}. \tag{5.6.11}$$

6. Conformal Symmetry

6.1. Introduction. In this section we will review some well known constructions of the Virasoro algebra. First of all we will consider oscillator representations of this algebra, i.e., representations in terms of sums of normal ordered products of elements of the Heisenberg subalgebras $\hat{\underline{s}}_n$. Then we will briefly describe the so-called "Sugawara construction" of the Virasoro algebra, i.e., representations in terms of sums of normal ordered products of elements of the full affine algebra.

Using the rather well known fact (see, e.g. [5]) that these two constructions of the Virasoro algebra coincide in our case (which is a level one construction of a highest weight representation of $\hat{gl}_n(\mathbb{C})$), we will be able to describe the conformal transformation properties of the fermionic fields entirely in terms of oscillators. Together with the commutation relations of these fields with oscillators, which will be derived in Sect. 6.2, these properties will allow us to express the fields in terms of oscillators and fermionic "translation operators;" this will be done in Sect. 7.

6.2. Oscillators on $\wedge^\infty C^\infty$. The purpose of this subsection is to derive the commutation relations of the oscillators $\hat{\alpha}_i\left(\frac{N}{n_i}k\right)$ with the fermionic fields.

Lemma 6.2.1.

$$\left[\hat{\alpha}_i\left(\frac{N}{n_i}k\right), \psi_j(z)\right] = \delta_{ij} z^{(N/n_i)k} \psi_j(z),$$

$$\left[\hat{\alpha}_i\left(\frac{N}{n_i}k\right), \psi_j^*(z)\right] = -\delta_{ij} z^{(N/n_i)k} \psi_j^*(z). \tag{6.2.1}$$

Proof. We use again the expression (5.6.4) for $\hat{\alpha}_i\left(\frac{N}{n_i}k\right)$ in terms of fermions;

$$\begin{aligned}\left[\hat{\alpha}_i\left(\frac{N}{n_i}k\right), \psi_j(l)\right] &= -\sum_{r+k\leqq 0} [\psi_i^*(r+k)\psi_i(r), \psi_j(l)] + \sum_{r+k>0} [\psi_i(r)\psi_i^*(r+k), \psi_j(l)] \\ &= \sum_{r\in\mathbf{Z}} \delta_{ij}\delta_{r+k,l}\psi_i(r) \\ &= \delta_{ij}\psi_i(l-k).\end{aligned} \tag{6.2.2}$$

With the definition (5.5.3) of the field $\psi_i(z)$ the first relation of (6.2.1) follows. The second relation is proved by Hermitian conjugacy of the first. □

6.3. Oscillator Representations of the Virasoro Algebra. Recall that the Virasoro algebra is the universal one dimensional central extension of the conformal algebra. It has a basis consisting of elements d_k, $k\in\mathbf{Z}$ and a central element c_{vir}, which satisfy the commutation relations

$$[d_k, d_l] = (k-l)d_{k+l} + \frac{1}{12}\delta_{k+l,0}(k^3-k)c_{\text{vir}}. \tag{6.3.1}$$

The following construction of the Virasoro algebra is standard in physics.

Lemma 6.3.1. *Let $\{a_i\}_{i\in\mathbf{Z}}$ be a collection of operators on a vector space V, such that $[a_i, a_j] = i\delta_{i+j,0}I$ and $a_i(v) = 0, \forall v\in V$ and $i \gg 0$. Define normal ordering by: $:a_ia_j: = a_ia_j$ if $i \leqq j$ and $:a_ia_j: = a_ja_i$ if $i > j$, then the assignment $d_k \mapsto L_k$, where the operators L_k are defined by*

$$L_k := \frac{1}{2}\sum_{j\in\mathbf{Z}} :a_{-j}a_{j+k}: \tag{6.3.2}$$

is a representation of the Virasoro algebra with $c_{\text{vir}} \mapsto I$. Moreover, we have:

$$[L_k, a_i] = -ia_{i+k} \tag{6.3.3}$$

Proof. See, e.g., [19]. □

Translated to our situation, we get a $c_{\text{vir}} = 1$ representation of the Virasoro algebra on the semi-infinite wedge space by:

$$L_k^{(i)} := \frac{1}{2}\sum_{j\in\mathbf{Z}} :\hat{\alpha}_i\left(-\frac{N}{n_i}j\right)\hat{\alpha}_i\left(\frac{N}{n_i}(j+k)\right): \quad 1 \leqq i \leqq s. \tag{6.3.4}$$

Starting from these operators, we construct $c_{\text{vir}} = n_i$ representations of the Virasoro algebra in the following manner; set $T_k^{(i)} := 1/n_i L_{n_ik}^{(i)}$, then these $T_k^{(i)}$'s satisfy:

$$[T_k^{(i)}, T_l^{(i)}] = (k-l)T_{k+l}^{(i)} + \frac{1}{12}n_i\delta_{k+l,0}(k^3-k) + \frac{1}{12}\delta_{k+l,0}k\frac{n_i^2-1}{n_i}I. \tag{6.3.5}$$

The extra term in the right-hand side of (6.3.5) is a two coboundary and hence we define operators $D_k^{(i)} := T_k^{(i)} + \frac{1}{24}\delta_{k,0}\frac{n_i^2-1}{n_i}I$ to obtain the commutation relations in their standard form:

$$[D_k^{(i)}, D_l^{(i)}] = (k-l)D_{k+l}^{(i)} + \frac{1}{12}\delta_{k+l,0}n_i(k^3-k). \tag{6.3.6}$$

It is easily verified that $\frac{1}{24}\frac{n_i^2-1}{n_i} = \frac{1}{2}|H_{(i)}|^2$, where $H_{(i)}$ is defined in (3.3.4). Hence,

$$D_k^{(i)} = T_k^{(i)} + \tfrac{1}{2}\delta_{k0}|H_{(i)}|^2 I. \tag{6.3.7}$$

Since the operators $D_k^{(i)}$ and $D_l^{(j)}$ commute for $i \neq j$, we can also construct a $c_{\text{vir}} = n$ representation by summation over i. Let us summarize the result in the following theorem.

Theorem 6.3.2. *Let the operators D_k be defined by*

$$D_k := \frac{1}{2}\sum_{j\in\mathbf{Z}}\sum_{i=1}^{s}\frac{1}{n_i} : \hat{\alpha}_i\left(-\frac{N}{n_i}j\right)\hat{\alpha}_i\left(\frac{N}{n_i}(j+n_ik)\right) : + \tfrac{1}{2}\delta_{k0}|H_{\underline{n}}|^2 I, \tag{6.3.8}$$

then the assignment $d_k \mapsto D_k$ is a representation of the Virasoro algebra with $c_{\text{vir}} \mapsto nI$. Moreover, we have:

$$\left[D_k, \hat{\alpha}_i\left(\frac{N}{n_i}j\right)\right] = -\frac{j}{n_i}\hat{\alpha}_i\left(\frac{N}{n_i}(j+n_ik)\right). \tag{6.3.9}$$

6.4. The Sugawara Construction of the Virasoro Algebra. Let $\{u_m\}_{1\leqq m\leqq n^2-1}$ be a basis of $sl_n(\mathbf{C})$ and let $\{u^m\}_{1\leqq m\leqq n^2-1}$ be the dual basis with respect to the trace form on $sl_n(\mathbf{C})$, and define $u_{n^2} = u^{n^2} = \sqrt{\frac{n+1}{n}}I_n$. The following lemma is known as the Sugawara construction of the Virasoro algebra.

Lemma 6.4.1. *Define the operators $\hat{L}_k: \Lambda^\infty\mathbf{C}^\infty \to \Lambda^\infty\mathbf{C}^\infty$ by:*

$$\hat{L}_k := \frac{1}{2n+2}\sum_{j\in\mathbf{Z}}\sum_{m=1}^{n^2} :(e^{-ij\theta}u_m)(e^{i(j+k)\theta}u^m): \tag{6.4.1}$$

The assignment $d_k \mapsto \hat{L}_k$ defines a representation of the Virasoro algebra with $c_{\text{vir}} \mapsto nI$. Moreover, we have:

$$[\hat{L}_k, e^{il\theta}x] = -le^{i(k+l)\theta}x \quad \forall k,l\in\mathbf{Z}, \quad x\in gl_n(\mathbf{C}). \tag{6.4.2}$$

Proof. We have $\sum_{m=1}^{n^2}[u_m, u^m] = 0$, whence:

$$\sum_{m=1}^{n^2}[e^{ik\theta}u_m, e^{ij\theta}u^m] = k\delta_{k+j,0}\sum_{m=1}^{n^2}(u_m|u^m). \tag{6.4.3}$$

It is also clear that the operators $e^{ik\theta}u_m, k\geqq 0$ annihilate the vacuum $|0\rangle$. With these remarks and the normal ordering definition 5.6.3 one easily finds:

$$\sum_{m=1}^{n^2} :(e^{ik\theta}u_m)(e^{ij\theta}u^m): = \begin{cases} \sum_{m=1}^{n^2}(e^{ik\theta}u_m)(e^{ij\theta}u^m) & \text{if } k\leqq j \\ \sum_{m=1}^{n^2}(e^{ij\theta}u_m)(e^{ik\theta}u^m) & \text{if } k>j \end{cases} \tag{6.4.4}$$

The rest of the proof is standard; see, e.g., [19]. □

The lemma tells us that the Virasoro operators $\hat{L}_k$ act as $ie^{ik\theta}\frac{d}{d\theta}$ on the affine algebra. Now let us take an element $\hat{x}(l)$ of the affine algebra (see (4.2.9)). It is clear then that:

$$[\hat{L}_k, \hat{x}(l)] = e^{ik\theta} ad H_{\underline{n}}(x(l)) - \frac{l}{N} x(l+Nk). \tag{6.4.5}$$

In view of the right-hand side of this formula it is natural to consider the operator $\hat{L}_k - e^{ik\theta} H_{\underline{n}}$. Since $e^{ik\theta} H_{\underline{n}} = H_{\underline{n}}(Nk)$, we have:

$$[\hat{L}_k - e^{ik\theta} H_{\underline{n}}, \hat{x}(l)] = -\frac{l}{N} x(l+Nk) - \mu(H_{\underline{n}}(Nk), x(l))I. \tag{6.4.6}$$

The value of the two cocycle μ in the right-hand side of this formula is easily computed with the help of (4.2.6); using the ad-invariance of the trace form, we get:

$$\mu(H_{\underline{n}}(Nk), x(l)) = -\frac{l}{N} \delta_{l+Nk,0}(H_{\underline{n}} | x_{\bar{l}}). \tag{6.4.7}$$

Recalling the definition (4.2.9) of $\hat{x}(l+Nk)$, we find:

$$[\hat{L}_k - e^{ik\theta} H_{\underline{n}}, \hat{x}(l)] = -\frac{l}{N} \hat{x}(l+Nk). \tag{6.4.8}$$

We are now ready for the following lemma;

Lemma 6.4.2. *Define the operator* $\hat{D}_k : \wedge^\infty \mathbf{C}^\infty \to \wedge^\infty \mathbf{C}^\infty$ *by*

$$\hat{D}_k := \hat{L}_k - e^{ik\theta} H_{\underline{n}} + \tfrac{1}{2}\delta_{k0} |H_{\underline{n}}|^2 I, \tag{6.4.9}$$

then the assignment $d_k \mapsto \hat{D}_k$ *is a representation of the Virasoro algebra with* $c_{\mathrm{vir}} \mapsto nI$. *Moreover, we have*:

$$[\hat{D}_k, \hat{x}(l)] = -\frac{l}{N} \hat{x}(l+Nk). \tag{6.4.10}$$

Proof. An easy exercise. □

6.5. The Difference $\hat{D}_k - D_k$. In the representation theory of the Virasoro algebra one often considers the difference $\Delta_k := \hat{D}_k - D_k$. It is well known that these Δ_k's satisfy again the commutation relations of the Virasoro algebra. The algebra spanned by the Δ_k's is called a "coset Virasoro algebra" and one speaks of a coset construction. This coset construction can be found in many papers; we mention again our standard reference [19]. The key ingredient for the proof is the fact that the operators Δ_k commute with the elements of the HSA $\hat{\underline{s}}_{\underline{n}}$ –

$$\begin{aligned}\left[\Delta_k, \hat{\alpha}_i\left(\frac{N}{n_i} l\right)\right] &= \left[\hat{D}_k, \hat{\alpha}_i\left(\frac{N}{n_i} l\right)\right] - \left[D_k, \hat{\alpha}_i\left(\frac{N}{n_i} l\right)\right] \\ &= 0 \quad \text{see (6.3.9) and (6.4.10)}\end{aligned} \tag{6.5.1}$$

– and hence also with the D_j's;

$$[\Delta_k, D_j] = 0. \tag{6.5.2}$$

The value of the central charge of the coset algebra is simply the difference between the central charges of the Virasoro algebras spanned by the $\hat{D}_k$'s and the D_k's respectively. As a matter of fact these central charges coincide, so that the Δ_k's span a Virasoro algebra with $c_{\mathrm{vir}} = 0$, i.e., a conformal algebra. This numerical coincidence occurs for any level one highest weight representation of an affine algebra $\hat{\underline{g}}$, where $\underline{g}$ is a direct sum of simply laced simple Lie algebras and abelian Lie algebras.

Apart from the fact that any component $\wedge_k^\infty \mathbf{C}^\infty$ of the semi-infinite wedge space is a $c_{\mathrm{vir}} = 0$ representation of the Virasoro algebra, these components are also unitary representations for this algebra containing a highest weight vector;

Lemma 6.5.1.

$$\begin{aligned} &\text{a) } \Delta_l^\dagger = \Delta_{-l}, \\ &\text{b) } \Delta_l|k\rangle = 0 \quad \forall l > 0, \\ &\text{c) } \Delta_0|k\rangle = h(k)|k\rangle. \end{aligned} \tag{6.5.3}$$

Proof. With the help of Lemma 5.6.2 one immediately proves $D_k^\dagger = D_{-k}$ and $\hat{D}_k^\dagger = \hat{D}_{-k}$ and a) follows. Relation b) is clear from the corresponding property of $\hat{D}_l$ and $\hat{D}_l$. To prove c), we use the definitions (6.3.8) and (6.4.9) to write:

$$\begin{aligned} \Delta_0|k\rangle &= (\hat{D}_0 - D_0)|k\rangle \\ &= \left(\frac{1}{2n+2}\sum_{m=1}^{n^2} u_m u^m - \frac{1}{2}\sum_{i=1}^{s}\frac{1}{n_i}\hat{\alpha}_i(0)^2 - H_{\underline{n}}\right)|k\rangle. \end{aligned} \tag{6.5.4}$$

The terms between parentheses clearly act diagonally on the k^{th} vacuum. □

It is well known (see [20, 21]) that the only representation of the Virasoro algebra with these properties is the trivial one, i.e., $d_k \mapsto 0\ \forall k$, meaning that $\Delta_k \equiv 0$.

Theorem 6.5.2. *The oscillator and Sugawara constructions* (*see* (*6.3.8*) *and* (*6.4.9*)) *of the Virasoro algebra coincide,*

$$D_k = \hat{D}_k \quad \forall k. \tag{6.5.5}$$

6.6. Conformal Transformation Properties of the Fermionic Fields. Here we will derive useful expressions for the commutators $[D_k, \psi_i(z)]$ and $[D_k, \psi_i^*(z)]$. These expressions can be considered as the conformal transformation properties of the fermionic fields.

Recall that the operators $\hat{D}_k = D_k$ act on the affine algebra by: $[D_k, \hat{x}(l)] = -\frac{l}{N}\hat{x}(l + Nk)$. Using this relation and the definition (4.4.2) of the vertex operators, one immediately derives the following relation:

$$[D_k, \hat{A}_{pq}^{ij}(z)] = \frac{1}{N} z^{Nk}\left(z\frac{d}{dz} + Nk\right)\hat{A}_{pq}^{ij}(z). \tag{6.6.1}$$

It turns out that the commutators $[D_k, \psi_i(z)]$ and $[D_k, \psi_i^*(z)]$ are given by the same relation with Nk replaced by $\frac{1}{2}Nk$;

Theorem 6.6.1.

$$
\begin{aligned}
[D_k, \psi_i(z)] &= \frac{1}{N} z^{Nk}\left(z\frac{d}{dz} + \frac{1}{2}Nk\right)\psi_i(z),\\
[D_k, \psi_i^*(z)] &= \frac{1}{N} z^{Nk}\left(z\frac{d}{dz} + \frac{1}{2}Nk\right)\psi_i^*(z).
\end{aligned} \tag{6.6.2}
$$

Proof. The theorem is proved in a somewhat indirect manner; first one introduces operators $H'_k: \wedge^\infty \mathbf{C}^\infty \to \wedge^\infty \mathbf{C}^\infty$ such that the relations (6.6.2) hold for H'_k instead of D_k. In order to motivate the definition of these operators, we remark that the relations (6.6.2) can also be written in terms of their Fourier components;

$$
\begin{aligned}
[D_k, \psi_i(l)] &= \frac{1}{n_i}\left(l - \frac{n_ik+1}{2}\right)\psi_i(l - n_ik),\\
[D_k, \psi_i^*(l)] &= -\frac{1}{n_i}\left(l + \frac{n_ik-1}{2}\right)\psi_i^*(l + n_ik).
\end{aligned} \tag{6.6.3}
$$

It is easily verified that these relations hold, if we substitute for D_k the operator $H'_k: \wedge^\infty \mathbf{C}^\infty \to \wedge^\infty \mathbf{C}^\infty$ defined by

$$
H'_k := \sum_{i=1}^{s} \sum_{p\in\mathbf{Z}} \frac{1}{n_i}\left(p + \frac{n_ik-1}{2}\right):\psi_i(p)\psi_i^*(p+n_ik):. \tag{6.6.4}
$$

Let us compute the commutator $[H'_k, H'_l]$;

$$
\begin{aligned}
[H'_k, H'_l] = \sum_{i,j=1}^{s} \sum_{p,q\in\mathbf{Z}} &\frac{1}{n_in_j}\left(p + \frac{n_ik-1}{2}\right)\left(q + \frac{n_jl-1}{2}\right)\\
&\cdot[:\psi_i(p)\psi_i^*(p+n_ik):, :\psi_j(q)\psi_j^*(q+n_jl):].
\end{aligned} \tag{6.6.5}
$$

Inside the commutator we can leave out the normal ordering sign and hence we find for the right hand side of (6.6.5):

$$
\begin{aligned}
&\sum_{i=1}^{s} \sum_{p,q\in\mathbf{Z}} \frac{1}{n_i^2}\left(p + \frac{n_ik-1}{2}\right)\left(q + \frac{n_il-1}{2}\right)\\
&\quad\cdot(\delta_{q,p+n_ik}\psi_i(p)\psi_i^*(p+n_i(k+l)) - \delta_{p,q+n_il}\psi_i(q)\psi_i^*(q+n_i(k+l)))\\
&= \sum_{i=1}^{s} \sum_{p,q\in\mathbf{Z}} \frac{1}{n_i^2}\left(p + \frac{n_ik-1}{2}\right)\left(q + \frac{n_il-1}{2}\right)(\delta_{q,p+n_ik}:\psi_i(p)\psi_i^*(p+n_i(k+l)):\\
&\quad - \delta_{p,q+n_il}:\psi_i(q)\psi_i^*(q+n_i(k+l)): + \delta_{q,p+n_ik}\delta_{k+l,0}\bar{\theta}(p) + \delta_{p,q+n_il}\delta_{k+l,0}\bar{\theta}(q)).
\end{aligned} \tag{6.6.6}
$$

Here we have introduced the step function $\bar{\theta}: \mathbf{Z} \to \{0, 1\}$ as follows; $\bar{\theta}(k) := 1$ if $k \leqq 0$ and $\bar{\theta}(k) := 0$ if $k > 0$. The first two terms in the right-hand side of this formula yield $(k-l)H'_{k+l}$, while the central terms become:

$$
\begin{aligned}
&\delta_{k+l,0} \sum_{i=1}^{s} \frac{1}{n_i^2} \sum_{p,q\in\mathbf{Z}} \left(p + \frac{n_ik-1}{2}\right)\left(q + \frac{n_il-1}{2}\right)\delta_{q,p+n_ik}(\bar{\theta}(p) - \bar{\theta}(q))\\
&= \delta_{k+l,0} \sum_{i=1}^{s} \frac{1}{n_i^2} \sum_{p\in\mathbf{Z}} (\bar{\theta}(p) - \bar{\theta}(p+n_ik))\left(p + \frac{n_ik-1}{2}\right)^2
\end{aligned}
$$

$$= \frac{1}{12}\delta_{k+l,0}\sum_{i=1}^{s}\frac{n_i^3k^3 - n_ik}{n_i^2}$$

$$= \frac{1}{12}\delta_{k+l,0}n(k^3 - k) + \delta_{k+l,0}k|H_{\underline{n}}|^2. \tag{6.6.7}$$

This means that the operators

$$H_k := \sum_{i=1}^{s}\sum_{p\in Z}\frac{1}{n_i}\left(p + \frac{n_ik - 1}{2}\right):\psi_i(p)\psi_i^*(p + n_ik): + \tfrac{1}{2}\delta_{k0}|H_{\underline{n}}|^2 I \tag{6.6.8}$$

define a $c_{\text{vir}} = n$ representation of the Virasoro algebra.

The next step is to compute the commutation relations of the H_k's with the oscillators. Using the expression (5.6.4) of the oscillators in terms of fermions and the definition (6.6.8) of the H_k's, one readily finds:

$$\left[H_k, \hat{\alpha}_i\left(\frac{N}{n_i}l\right)\right] = -\frac{l}{n_i}\hat{\alpha}_i\left(\frac{N}{n_i}(l + n_ik)\right). \tag{6.6.9}$$

This implies again that the operators $H_k - D_k$ define a $c_{\text{vir}} = 0$ representation of the Virasoro algebra. Continuing in the same manner as in the previous section, one proves that $H_k = D_k$, which proves the theorem. □

The proof of the theorem yields the following corollary:

Corollary 6.6.2. *The operators*

$$H_k := \sum_{i=1}^{s}\sum_{p\in Z}\frac{1}{n_i}\left(p + \frac{n_ik - 1}{2}\right):\psi_i(p)\psi_i^*(p + n_ik): + \tfrac{1}{2}\delta_{k0}|H_{\underline{n}}|^2 I \tag{6.6.10}$$

define a $c_{\text{vir}} = n$ representation of the Virasoro algebra. This representation coincides with the oscillator and Sugawara constructions (see (6.3.8) and (6.4.9)).

7. Vertex Operators

7.1. Introduction. In this section we will show that the fermionic fields $\psi_i(z)$ and $\psi_i^*(z)$ can be expressed in terms of so-called fermionic translation operators $Q_i\colon \wedge_k^\infty \mathbf{C}^\infty \to \wedge_{k+1}^\infty \mathbf{C}^\infty$ and the bosonic oscillators $\hat{\alpha}_i\left(\frac{N}{n_i}k\right)$. As a consequence the vertex operators $\hat{A}_{pq}^{ij}(z)$ can be written as a product

$$Q_iQ_j^{-1} \times \Gamma_{pq}^{ij}, \tag{7.1.1}$$

where Γ_{pq}^{ij} is a complicated expression in the bosonic oscillators.

From this formula one easily reads off an alternative construction of the irreducible $\hat{gl}_n(\mathbf{C})$-module $\wedge_k^\infty \mathbf{C}^\infty$; it is the tensor product of the group algebra of the group generated by the operators $T_i := Q_iQ_{i+1}^{-1}$ $1 \leq i \leq s - 1$ and an irreducible representation of the Heisenberg algebra (a polynomial ring).

7.2. Expressions for $\psi_i(z)$ and $\psi_i^*(z)$. Recall the commutation relations (6.2.1):

$$\left[\hat{\alpha}_i\left(\frac{N}{n_i}k\right), \psi_j(z)\right] = \delta_{ij} z^{(N/n_i)k}\psi_i(z),$$
$$\left[\hat{\alpha}_i\left(\frac{N}{n_i}k\right), \psi_j^*(z)\right] = -\delta_{ij} z^{(N/n_i)k}\psi_i^*(z). \tag{7.2.1}$$

These relations can be seen as formal eigenvalue equations for the adjoint action of the HSA $\hat{\underline{s}}_{\underline{n}}$. They determine the formal eigenvectors $\psi_i(z)$ and $\psi_i^*(z)$ up to operators that commute with the action of the HSA. It is easy to find solutions for (7.2.1); define

$$E_i^{(+)}(z) := \exp\left(-\sum_{k>0}\frac{1}{k}z^{-(N/n_i)k}\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\right),$$
$$E_i^{(-)}(z) := \exp\left(-\sum_{k<0}\frac{1}{k}z^{-(N/n_i)k}\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\right). \tag{7.2.2}$$

Then one easily checks that the product $E_i(z) := E_i^{(-)}(z)E_i^{(+)}(z)$ satisfies the first relation of (7.2.1) for all $k \neq 0$. From this it is clear that we can write:

$$\psi_i(z) = Q_i(z)E_i(z), \tag{7.2.3}$$

where $Q_i(z) := E_i^{(-)}(z)^{-1}\psi_i(z)E_i^{(+)}(z)^{-1}$ is a formal operator valued Laurent series, which commutes with all oscillators except with the zero modes;

$$\left[\hat{\alpha}_j\left(\frac{N}{n_j}k\right), Q_i(z)\right] = \delta_{ij}\delta_{k0}Q_i(z). \tag{7.2.4}$$

Of course one has a similar expression for the Hermitian conjugate field;

$$\psi_i^*(z) = Q_i^*(z)E_i^{(-)}(z)^{-1}E_i^{(+)}(z)^{-1}. \tag{7.2.5}$$

Next we consider the formal operator valued Laurent series $Q_i(z)$ and $Q_i^*(z)$. In the lemma below we will show that the z-dependence of these operators is determined by the conformal transformation properties (6.6.2) of the fermionic fields.

Lemma 7.2.1. *We have*:

$$\begin{aligned} Q_i(z) &= z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_i = Q_i z^{(N/n_i)(\hat{\alpha}_i(0)+1/2)}, \\ Q_i^*(z) &= z^{-(N/n_i)(\hat{\alpha}_i(0)+1/2)}Q_i^* = Q_i^* z^{-(N/n_i)(\hat{\alpha}_i(0)-1/2)}, \end{aligned} \tag{7.2.6}$$

where Q_i and Q_i^* are Hermitian conjugate operators on $\wedge^\infty \mathbf{C}^\infty$ independent of z.

Proof. We only prove the first relation; the second follows by Hermitian conjugacy from the first. Consider the commutator $[D_0, \psi_i(z)]$. Using (6.6.2), the definition (6.3.8) for D_0 and the expression (7.2.3) for $\psi_i(z)$, we find:

$$\left[\frac{1}{2}\sum_{k\in\mathbf{Z}}\frac{N}{n_i}:\hat{\alpha}_i\left(-\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}k\right):, Q_i(z)E_i(z)\right] = z\frac{d}{dz}Q_i(z)E_i(z). \tag{7.2.7}$$

Using the Heisenberg commutation relations one easily derives:

$$\left[\frac{1}{2}\sum_{k\in\mathbf{Z}}\frac{N}{n_i}:\hat{\alpha}_i\left(-\frac{N}{n_i}k\right)\hat{\alpha}_i\left(\frac{N}{n_i}k\right):, E_i(z)\right] = z\frac{d}{dz}E_i(z). \tag{7.2.8}$$

Substituting this relation in (7.2.7) and recalling that $Q_i(z)$ commutes with all oscillators except with the zero modes, we find:

$$\left[\frac{1}{2}\frac{N}{n_i}\hat{\alpha}_i(0)^2, Q_i(z)\right] = z\frac{d}{dz}Q_i(z). \tag{7.2.9}$$

Using (7.2.4), this can be rewritten as:

$$\frac{N}{n_i}(\hat{\alpha}_i(0) - \tfrac{1}{2})Q_i(z) = z\frac{d}{dz}Q_i(z). \tag{7.2.10}$$

This differential equation is solved by the first relation of (7.2.6). □

With this lemma we can write:

$$\begin{aligned} \psi_i(z) &= z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_iE_i^{(-)}(z)E_i^{(+)}(z), \\ \psi_i^*(z) &= z^{-(N/n_i)(\hat{\alpha}_i(0)+1/2)}Q_i^*E_i^{(-)}(z)^{-1}E_i^{(+)}(z)^{-1}, \end{aligned} \tag{7.2.11}$$

and from these relations it is clear that the operators Q_i and Q_i^*, which were defined in the lemma as "integration constants," change the total charge of the fermions of type i by $+1$ respectively -1. In the case of the homogeneous HSA the operators $T_i := Q_iQ_{i+1}^*$ $1 \leqq i \leqq n-1$ are called translation operators (see [3]), because they are closely related to the translation subgroup of the affine Weyl group. For these reasons we will call the Q_i's and Q_i^*'s fermionic translation operators.

7.3. The Operators Q_i and Q_i^*. In this subsection we will prove the following theorem, which determines completely the action of the fermionic translation operators on the semi-infinite wedge space.

Theorem 7.3.1. *Let $Q_i, Q_i^*: \wedge^{\infty}\mathbf{C}^{\infty} \to \wedge^{\infty}\mathbf{C}^{\infty}$ be the operators defined by (7.2.6), then the following relations hold:*

$$\begin{aligned} Q_i\psi_j(z) &= -\psi_j(z)Q_i \quad \text{if} \quad i \neq j, \\ Q_i\psi_j^*(z) &= -\psi_j^*(z)Q_i \quad \text{if} \quad i \neq j, \\ Q_i\psi_i(z) &= z^{-N/n_i}\psi_i(z)Q_i, \\ Q_i\psi_i^*(z) &= z^{N/n_i}\psi_i^*(z)Q_i, \end{aligned} \tag{7.3.1}$$

$$\begin{aligned} Q_i|0\rangle &= \psi_i(1)|0\rangle, \\ Q_i^*|0\rangle &= \psi_i^*(0)|0\rangle. \end{aligned} \tag{7.3.2}$$

Moreover, these operators are unitary –

$$Q_i^* = Q_i^{-1} \tag{7.3.3}$$

– and they satisfy the anticommutation relations:

$$\{Q_i, Q_j\} = 0 = \{Q_i^*, Q_j^*\} \quad \text{if} \quad i \neq j \quad \{Q_i, Q_j^*\} = 2\delta_{ij}. \tag{7.3.4}$$

Note that the relations (7.3.1) can also be written as:

$$\begin{aligned} Q_i\psi_j(k) &= -\psi_j(k)Q_i \quad \text{if} \quad i \neq j, \\ Q_i\psi_j^*(k) &= -\psi_j^*(k)Q_i \quad \text{if} \quad i \neq j, \\ Q_i\psi_i(k) &= \psi_i(k+1)Q_i, \\ Q_i\psi_i^*(k) &= \psi_i^*(k+1)Q_i, \end{aligned} \tag{7.3.5}$$

while the Hermitian conjugates of these relations read:

$$\begin{aligned} Q_i^*\psi_j(k) &= -\psi_j(k)Q_i^* \quad \text{if} \quad i \neq j, \\ Q_i^*\psi_j^*(k) &= -\psi_j^*(k)Q_i^* \quad \text{if} \quad i \neq j, \\ Q_i^*\psi_i(k) &= \psi_i(k-1)Q_i^*, \\ Q_i^*\psi_i^*(k) &= \psi_i^*(k-1)Q_i^*. \end{aligned} \tag{7.3.6}$$

Together with (7.3.2) this indeed determines the action of Q_i and Q_i^*.

In the sequel we will make an extensive use of the calculus of formal variables developed in [22]. In particular we will use the following formal power series:

$$\frac{1}{1-z} := \sum_{k \geqq 0} z^k, \tag{7.3.7}$$

$$\log(1-z) := -\sum_{k>0} \frac{1}{k} z^k, \tag{7.3.8}$$

$$\exp z := \sum_{k \geqq 0} \frac{1}{k!} z^k, \tag{7.3.9}$$

$$\delta(z) := \sum_{k \in \mathbf{Z}} z^k. \tag{7.3.10}$$

With these definitions one clearly has the following formal identities:

$$(1-z)\frac{1}{1-z} = 1, \tag{7.3.11}$$

$$\exp(\log(1-z)) = 1-z, \tag{7.3.12}$$

$$P(z)\delta(z) = P(1)\delta(z) \quad \forall P \in \mathbf{C}[z, z^{-1}]. \tag{7.3.13}$$

We start to prove a lemma.

Lemma 7.3.2. *We have the following formal identities*:

$$\begin{aligned} E_i^{(+)}(z)\psi_i(y)E_i^{(+)}(z)^{-1} &= (1-(y/z)^{N/n_i})\psi_i(y), \\ E_i^{(+)}(z)\psi_i^*(y)E_i^{(+)}(z)^{-1} &= \frac{1}{1-(y/z)^{N/n_i}}\psi_i^*(y), \\ E_i^{(-)}(z)\psi_i(y)E_i^{(-)}(z)^{-1} &= \frac{1}{1-(z/y)^{N/n_i}}\psi_i(y), \\ E_i^{(-)}(z)\psi_i^*(y)E_i^{(-)}(z)^{-1} &= (1-(z/y)^{N/n_i})\psi_i^*(y), \end{aligned} \tag{7.3.14}$$

$$E_i^{(+)}(z)E_i^{(-)}(y)^{-1} = \frac{1}{1-(y/z)^{N/n_i})} E_i^{(-)}(y)^{-1}E_i^{(+)}(z),$$

$$E_i^{(-)}(z)E_i^{(+)}(y)^{-1} = (1-(z/y)^{N/n_i})E_i^{(+)}(y)^{-1}E_i^{(-)}(z), \tag{7.3.15}$$

$$\{\psi_i(y), \psi_i^*(z)\} = \delta((y/z)^{N/n_i}). \tag{7.3.16}$$

Proof. We have

$$\begin{aligned} E_i^{(+)}(z)\psi_i(y)E_i^{(+)}(z)^{-1} &= \exp\left(-\mathrm{ad}\sum_{k>0}\frac{1}{k}z^{-(N/n_i)k}\hat{\alpha}_i\left(\frac{N}{n_i}k\right)\right)\psi_i(y) \\ &= \exp\left(-\sum_{k>0}\frac{1}{k}(y/z)^{(N/n_i)k}\right)\psi_i(y) \\ &= \exp(\log(1-(y/z)^{N/n_i}))\psi_i(y) \\ &= (1-(y/z)^{N/n_i})\psi_i(y). \end{aligned} \tag{7.3.17}$$

The second relation of (7.3.14) can be proved analogously. The third and fourth ones can be derived from the first two by Hermitian conjugacy.

To prove (7.3.15), we remark that the products of exponentials in the oscillators are all of the form $e^A e^B$ while the commutator $[A, B]$ is a multiple of the identity. In that case one has:

$$e^A e^B = e^{[A,B]} e^B e^A. \tag{7.3.18}$$

So for the first relation of (7.3.15) we only have to make the following computation:

$$\begin{aligned} \left[-\sum_{k>0}\frac{1}{k}z^{-(N/n_i)k}\hat{\alpha}_i\left(\frac{N}{n_i}k\right), \sum_{l<0}\frac{1}{l}y^{-(N/n_i)l}\hat{\alpha}_i\left(\frac{N}{n_i}l\right)\right] &= \sum_{k>0}\frac{1}{k}(y/z)^{(N/n_i)k} \\ &= -\log(1-(y/z)^{N/n_i}), \end{aligned}$$

and the result follows. The second relation of (7.3.15) is proved by inversion of the first.

Finally, (7.3.16) is immediate from the anticommutation relations (5.4.9) and the definition of the δ-function. □

Let us solve Q_i and Q_i^* from (7.2.3) and (7.2.6);

$$\begin{aligned} Q_i &= E_i^{(-)}(z)^{-1}\psi_i(z)E_i^{(+)}(z)^{-1}z^{-(N/n_i)(\hat{\alpha}_i(0)+1/2)}, \\ Q_i^* &= E_i^{(-)}(z)\psi_i^*(z)E_i^{(+)}(z)z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}. \end{aligned} \tag{7.3.19}$$

With these relations and the lemma we can prove the following lemma.

Lemma 7.3.3.

$$Q_i^* Q_i = Q_i Q_i^* \tag{7.3.20}$$

Proof. With (7.3.19) we can write:

$$\begin{aligned} Q_i^* Q_i &= E_i^{(-)}(z)\psi_i^*(z)E_i^{(+)}(z)z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}E_i^{(-)}(y)^{-1}\psi_i(y)E_i^{(+)}(y)^{-1}y^{-(N/n_i)(\hat{\alpha}_i(0)+1/2)} \\ &= E_i^{(-)}(z)\psi_i^*(z)E_i^{(+)}(z)E_i^{(-)}(y)^{-1}\psi_i(y)E_i^{(+)}(y)^{-1}(z/y)^{(N/n_i)(\hat{\alpha}_i(0)+1/2)}. \end{aligned}$$

Using the relations (7.3.14) and (7.3.15), this can be rewritten as:

$$Q_i^* Q_i = (1-(y/z)^{N/n_i})E_i^{(-)}(y)^{-1}E_i^{(-)}(z)\psi_i^*(z)\psi_i(y)E_i^{(+)}(y)^{-1}E_i^{(+)}(z)(z/y)^{(N/n_i)(\hat{\alpha}_i(0)+1/2)}.$$

Because of the factor $(1-(y/z)^{N/n_i})$ in this expression and the relation

$$(1-(y/z)^{N/n_i})\delta((y/z)^{N/n_i})=0$$

we can anticommute the fields $\psi_i(y)$ and $\psi_i^*(z)$. We now use (7.3.14) and (7.3.15) again and obtain:

$$\begin{aligned}
Q_i^*Q_i &= -(1-(y/z)^{N/n_i})E_i^{(-)}(y)^{-1}E_i^{(-)}(z)\psi_i(y)\psi_i^*(z)E_i^{(+)}(y)^{-1}E_i^{(+)}(z)(z/y)^{(N/n_i)(\hat{\alpha}_i(0)+1/2)}\\
&= -\frac{1-(y/z)^{N/n_i}}{1-(z/y)^{N/n_i}}E_i^{(-)}(y)^{-1}\psi_i(y)E_i^{(+)}(y)^{-1}E_i^{(-)}(z)\psi_i^*(z)E_i^{(+)}(z)(z/y)^{(N/n_i)(\hat{\alpha}_i(0)+1/2)}\\
&= E_i^{(-)}(y)^{-1}\psi_i(y)E_i^{(+)}(y)^{-1}E_i^{(-)}(z)\psi_i^*(z)E_i^{(+)}(z)(z/y)^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}\\
&= E_i^{(-)}(y)^{-1}\psi_i(y)E_i^{(+)}(y)^{-1}(y)^{-(N/n_i)(\hat{\alpha}_i(0)+1/2)}E_i^{(-)}(z)\psi_i^*(z)E_i^{(+)}(z)(z)^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}\\
&= Q_iQ_i^*. \quad \square
\end{aligned}$$

We are now able to prove the relations (7.3.1) of the theorem. The first two are easy; substitute the expressions (7.3.19) for Q_i and Q_i^* and the result follows immediately. The third one is proved by:

$$\begin{aligned}
Q_i\psi_i(z) &= Q_i z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_iE_i(z)\\
&= z^{-(N/n_i)}z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_iE_i(z)Q_i\\
&= z^{-(N/n_i)}\psi_i(z)Q_i.
\end{aligned} \tag{7.3.21}$$

Finally, for the fourth relation of (7.3.1) we need the lemma above;

$$\begin{aligned}
Q_i^*\psi_i(z) &= Q_i^* z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_iE_i(z)\\
&= z^{(N/n_i)}z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_i^*Q_iE_i(z)\\
&= z^{(N/n_i)}z^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_iQ_i^*E_i(z)\\
&= z^{(N/n_i)}\psi_i(z)Q_i^*.
\end{aligned} \tag{7.3.22}$$

Note that these relations imply that the operator $Q_i^*Q_i$ commutes with the fermionic fields $\psi_i(z)$ and $\psi_i^*(z)$. Therefore, if we want to prove that the Q_i's are unitary operators, we only have to show that $Q_i^*Q_i$ stabilizes the vacuum $|0\rangle$.

We first prove the relations (7.3.2);

$$\begin{aligned}
Q_i|0\rangle &= E_i^-(z)^{-1}\psi_i(z)E_i^+(z)^{-1}z^{-(N/n_i)(\hat{\alpha}_i(0)+1/2)}|0\rangle\\
&= z^{-(N/2n_i)}E_i^-(z)^{-1}\psi_i(z)|0\rangle.
\end{aligned} \tag{7.3.23}$$

The right-hand side of this relation is a power series in z^{N/n_i}. Extracting the coefficient of the constant term, we obtain:

$$Q_i|0\rangle = \psi_i(1)|0\rangle. \tag{7.3.24}$$

The second relation of (7.3.2) is proved analogously.

Now that we have proved (7.3.2), we calculate:

$$Q_i^*Q_i|0\rangle = Q_i^*\psi_i(1)|0\rangle = \psi_i(0)Q_i^*|0\rangle = \psi_i(0)\psi_i^*(0)|0\rangle = |0\rangle. \tag{7.3.25}$$

This completes the proof of the unitarity of the fermionic translation operators. The anticommutation relations (7.3.4) will be left to the reader.

7.4. Vertex Operators for A_∞ and $\hat{gl}_n(\mathbf{C})$

Here we will study the normal ordered operator product $:\psi_i(u)\psi_j^*(v):$ in two formal variables u and v. Note that the coefficients of $u^k v^{-l}$, $k, l \in \mathbf{Z}$ and the identity operator I span an algebra of operators isomorphic to A_∞. Therefore, this operator product can be called a vertex operator for A_∞. Of course our aim is to replace the formal variables u and v by $\omega^p z$ and $\omega^q z$ respectively, to obtain the vertex operators for $\hat{gl}_n(\mathbf{C})$. The justification for this kind of manipulation will be given below.

With the formal variables u and v we can write

$$:\psi_i(u)\psi_j^*(v):= \psi_i(u)\psi_j^*(v) - \langle 0|\psi_i(u)\psi_j^*(v)|0\rangle, \tag{7.4.1}$$

where the "contraction" $\langle 0|\psi_i(u)\psi_j^*(v)|0\rangle$ is formally well defined;

$$\langle 0|\psi_i(u)\psi_j^*(v)|0\rangle = \delta_{ij}\sum_{k\geq 0}(v/u)^{(N/n_i)(k+1/2)} = \delta_{ij}\frac{(v/u)^{N/2n_i}}{1-(v/u)^{N/n_i}}. \tag{7.4.2}$$

For the ordinary product $\psi_i(u)\psi_j^*(v)$ we write, using (7.3.15):

$$\begin{aligned}
&\psi_i(u)\psi_j^*(v)\\
&= u^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_i E_i^{(-)}(u)E_i^{(+)}(u)E_j^{(-)}(v)^{-1}E_j^{(+)}(v)^{-1}Q_j^{-1}v^{-(N/n_j)(\hat{\alpha}_j(0)-1/2)}\\
&= \frac{1}{1-\delta_{ij}(v/u)^{N/n_i}}u^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}Q_iQ_j^{-1}E_i^{(-)}(u)E_j^{(-)}(v)^{-1}E_i^{(+)}(u)E_j^{(+)}(v)^{-1}v^{-(N/n_j)(\hat{\alpha}_j(0)-1/2)}\\
&= \frac{u^{(N/n_i)(1-\delta_{ij})}}{1-\delta_{ij}(v/u)^{N/n_i}}Q_iQ_j^{-1}u^{(N/n_i)(\hat{\alpha}_i(0)-1/2)}v^{-(N/n_j)(\hat{\alpha}_j(0)-1/2)}E_i^{(-)}(u)E_j^{(-)}(v)^{-1}E_i^{(+)}(u)E_j^{(+)}(v)^{-1}.
\end{aligned} \tag{7.4.3}$$

With this calculation we have proved the following lemma:

Lemma 7.4.1.

$$\begin{aligned}
:\psi_i(u)\psi_j^*(v): &= \frac{v^{N/2n_j}u^{-(N/2n_i)}u^{(N/n_i)(1-\delta_{ij})}}{1-\delta_{ij}(v/u)^{N/n_i}}\\
&\quad\cdot\{Q_iQ_j^{-1}u^{(N/n_i)(\hat{\alpha}_i(0))}v^{-(N/n_j)(\hat{\alpha}_j(0))}E_i^{(-)}(u)E_j^{(-)}(v)^{-1}E_i^{(+)}(u)E_j^{(+)}(v)^{-1} - \delta_{ij}I\}.
\end{aligned} \tag{7.4.4}$$

In order to obtain an expression for the vertex operators $\hat{A}_{pq}^{ij}(z)$ for $\hat{gl}_n(\mathbf{C})$ we would like to replace the formal variables u and v in this formula by $\omega^p z$ and $\omega^q z$ respectively. Of course this must be done with some care, because of the formal power series $(1-\delta_{ij}(v/u)^{N/n_i})^{-1} = 1 + \delta_{ij}\sum_{k>0}(v/u)^{(N/n_i)k}$ in the right-hand side. Note however, that it is allowed to multiply both sides of (7.4.4) with the polynomial $(1-\delta_{ij}(v/u)^{N/n_i})$. This justifies the substitution $u \mapsto \omega^p z$, $v \mapsto \omega^q z$ for the case $i \neq j \vee (i=j \wedge p \neq q)$. Using (5.5.4), we find after a short calculation:

Theorem 7.4.2. *For $i \neq j \vee (i=j \wedge p \neq q)$ we have:*

$$\begin{aligned}
\hat{A}_{pq}^{ij}(z) &= \frac{1}{\sqrt{n_in_j}}\frac{\omega^{p(N/n_i)(1-\delta_{ij})}}{1-\delta_{ij}(\omega^{q-p})^{N/n_i}}z^{N((1/2n_j)+(1/2n_i)-\delta_{ij}(1/n_i))}Q_iQ_j^{-1}\\
&\quad\cdot\omega^{p(N/n_i)\hat{\alpha}_i(0)-q(N/n_j)\hat{\alpha}(0)}z^{(N/n_i)\hat{\alpha}_i(0)-(N/n_j(0)}\\
&\quad\cdot E_i^{(-)}(\omega^p z)E_j^{(-)}(\omega^q z)^{-1}E_i^{(+)}(\omega^p z)E_j^{(+)}(\omega^q z)^{-1}.
\end{aligned} \tag{7.4.5}$$

7.5. Bosonic Realization of $\wedge^\infty \mathbf{C}^\infty$ *and* $\wedge_k^\infty \mathbf{C}^\infty$. Recall from Sect. 4 that the HSA $\hat{s}_{\underline{n}}$ has a basis $\{p_i, q_i\}_{i\in\mathbf{N}} \cup \{c\}$ whose elements satisfy the commutation relations $[p_i, q_j] = \delta_{ij} c$. The p_j's correspond to oscillators $\hat{\alpha}_i\left(\frac{N}{n_i} k\right)$ with positive k. Hence they annihilate the vacuum vector v_0. With these remarks it is easy to see that the set

$$\{q_1^{k_1} \cdots q_m^{k_m} \cdot v_0 \,|\, k_i \in \mathbf{Z}_{\geqq 0},\ m \in \mathbf{N}\} \tag{7.5.1}$$

is independent. In other words: the linear mapping $f: \mathscr{U}(\hat{s}_{\underline{n}}) \cdot v_0 \to \mathbf{C}[x_1, x_2, \ldots]$ defined by

$$f(q_1^{k_1} \cdots q_m^{k_m} \cdot v_0) := x_1^{k_1} \cdots x_m^{k_m} \tag{7.5.2}$$

is an isomorphism between vector spaces. It is clear that on the polynomial ring $\mathbf{C}[x_i]$ the p_i's and q_i's are represented by $\frac{\partial}{\partial x_i}$ and x_i respectively.

Next we consider the vectors $v_{k_1, \ldots k_s} := Q_1^{k_1} \cdots Q_s^{k_s} \cdot v_0$, $k_i \in \mathbf{Z}$. This collection contains all vacua $v_k, k \in \mathbf{Z}$; one easily derives that for $k = n_1 + \cdots n_{i-1} + j + nl$, $1 \leqq i \leqq s$, $1 \leqq j \leqq n_i$, $l \in \mathbf{Z}$:

$$v_k = \pm Q_1^{n_1(1+l)} \cdots Q_{i-1}^{n_{i-1}(1+l)} Q_i^{j+n_i l} Q_{i+1}^{n_{i+1} l} \cdots Q_s^{n_s l} v_0. \tag{7.5.3}$$

Notice that these vectors have different eigenvalues with respect to $\bigoplus_{i=1}^{s} \mathbf{C}\hat{\alpha}_i(0)$:

$$\hat{\alpha}_i(0) \cdot v_{k_1, \ldots k_s} = k_i v_{k_1, \ldots k_s}. \tag{7.5.4}$$

For this reason the collection $\{v_{k_1, \ldots k_s} | k_i \in \mathbf{Z}\}$ is independent. Let Ω be the linear span of all $v_{k_1, \ldots k_s}$. It has the following charge decomposition:

$$\begin{aligned} \Omega &= \bigoplus_{k\in\mathbf{Z}} \Omega_k, \\ \Omega_k &:= \bigoplus_{k_1 + \cdots k_s = k} \mathbf{C}(Q_1^{k_1} \cdots Q_s^{k_s} \cdot v_0) \\ &= \bigoplus_{k_1 + \cdots k_s = 0} \mathbf{C}(Q_1^{k_1} \cdots Q_s^{k_s} \cdot v_k). \end{aligned} \tag{7.5.5}$$

Now consider the space $\mathscr{U}(\hat{s}_{\underline{n}})\Omega_k$, which can be identified with the tensor product $\Omega_k \otimes \mathbf{C}[x_i]$. From the expressions (7.4.4) and (7.4.5) for the vertex operators we see that this space is invariant under the actions of A_∞ and $\hat{gl}_n(\mathbf{C})$. It is also clear that this space cannot contain any invariant subspaces. Therefore it must coincide with the k^{th} wedge space;

$$\wedge_k^\infty \mathbf{C}^\infty \cong \Omega_k \otimes \mathbf{C}[x_i]. \tag{7.5.6}$$

Similarly:

$$\wedge^\infty \mathbf{C}^\infty \cong \Omega \otimes \mathbf{C}[x_i]. \tag{7.5.7}$$

The space Ω_k can also be described in a somewhat different manner. Define for $1 \leqq i \leqq s-1$ the operators $T_i := Q_i Q_{i+1}^{-1}$ and let $\hat{T}_{\underline{n}}$ be the group generated by these operators. A short calculation yields

$$T_i T_j T_i^{-1} T_j^{-1} = (-)^{a_{ij}}, \tag{7.5.8}$$

where $a_{ij} := 2\delta_{ij} - \delta_{i+1,j} - \delta_{j+1,i}$ is the Cartan matrix for the Lie algebra $sl_s(\mathbf{C})$.

This relation expresses that $\hat{T}_{\underline{n}}$ is a central extension of the (co)root lattice of $sl_s(\mathbf{C})$ by the two element group $\mathbf{Z}_2 \cong \{\pm 1\}$ (cf. [3] for the homogeneous case). The space Ω_k can be identified with (a copy of) the group algebra $\mathbf{C}[\hat{T}_{\underline{n}}]$.

8. Remarks

8.1. Restriction to $\hat{sl}_n(\mathbf{C})$. In Sect. 7.5 we have seen that the $\hat{gl}_n(\mathbf{C})$-module $\wedge_k^\infty \mathbf{C}^\infty$ is generated by the action of the creation operators, i.e., the elements $\hat{\alpha}_i\left(\frac{N}{n_i}k\right)$ for $k<0$ and of the operators $T_i = Q_i Q_{i+1}^{-1}$ on the k^{th} vacuum v_k. Restricted to $\hat{sl}_n(\mathbf{C})$ this module does not remain irreducible. This is caused by the fact that not all elements of the HSA $\hat{s}_{\underline{n}}$ belong to $\hat{sl}_n(\mathbf{C})$. If we want to describe the $\hat{sl}_n(\mathbf{C})$-module $L(\Lambda_{k \bmod n})$, we have to throw out certain variables from the polynomial ring $\mathbf{C}[x_i]$.

Define $\beta_i(l)$, $l \in \mathbf{Z} - \{0\}$ as follows:

$$\beta_i\left(\frac{N}{n_i}k\right) := \hat{\alpha}_i\left(\frac{N}{n_i}k\right) \quad \text{for} \quad k \notin n_i \mathbf{Z},$$

$$\beta_i(Nk) := \frac{1}{n_i}\hat{\alpha}_i(Nk) - \frac{1}{n_{i+1}}\hat{\alpha}_{i+1}(Nk) \quad \text{for} \quad 1 \leqq i \leqq s-1,$$

$$\beta_s(Nk) := \hat{\alpha}_1(Nk) + \hat{\alpha}_2(Nk) + \cdots \hat{\alpha}_s(Nk). \tag{8.1.1}$$

Let $\hat{t}_{\underline{n}}$ be the intersection $\hat{s}_{\underline{n}} \cap \hat{sl}_n(\mathbf{C})$, then:

$$\hat{t}_{\underline{n}} = \left\{ \bigoplus_{k \in \mathbf{Z}-\{0\}} \bigoplus_{i=1}^{s-1} \mathbf{C}\beta_i\left(\frac{N}{n_i}k\right) \right\} \oplus \left\{ \bigoplus_{k \in \mathbf{Z} - n_s \mathbf{Z}} \mathbf{C}\beta_s\left(\frac{N}{n_s}k\right) \right\}. \tag{8.1.2}$$

In terms of the variables x_i (see Lemma 4.3.2) this means that we define new variables y_i by:

$$y_{n_1+n_2+\cdots n_{i-1}+j+nl} := x_{n_1+n_2+\cdots n_{i-1}+j+nl} \quad \text{for} \quad j \neq n_i$$

$$y_{n_1+n_2+\cdots n_{i-1}+nl} := \frac{1}{n_i} x_{n_1+n_2+\cdots n_{i-1}+nl} - \frac{1}{n_{i+1}} x_{n_1+n_2+\cdots n_i+nl} \quad \text{for} \quad 1 \leqq i \leqq s-1$$

$$y_{nl} := x_{n_1+nl} + x_{n_1+n_2+nl} + \cdots x_{n+nl}. \tag{8.1.3}$$

The $\hat{sl}_n(\mathbf{C})$-module $L(\Lambda_{k \bmod n})$ is generated by the action of $\hat{t}_{\underline{n}}$ and the group $\hat{T}_{\underline{n}}$ on the vacuum vector v_k; we find:

$$L(\Lambda_{k \bmod n}) \cong \mathbf{C}[\hat{T}_{\underline{n}}] \otimes \mathbf{C}[y_i;\, i \neq nl]. \tag{8.1.4}$$

8.2. q-Dimension Formulas. Recall the operator D_0 (see (6.3.8)). Using (7.5.4), one finds:

$$D_0(Q_1^{k_1} \cdots Q_s^{k_s} \cdot v_0) = \left\{ \frac{1}{2}|H_{\underline{n}}|^2 + \frac{1}{2}\left(\frac{k_1^2}{n_1} + \cdots \frac{k_s^2}{n_s} \right) \right\} Q_1^{k_1} \cdots Q_s^{k_s} \cdot v_0. \tag{8.2.1}$$

Combining this with (6.3.9) and (7.5.6), we find the following explicit form of the

"q-dimension" formulas from [5]:

$$\operatorname{trace}_{\Lambda_k^\infty \mathbf{C}^\infty} q^{D_0} = \frac{q^{1/2|H_n|^2} \sum_{k_1+k_2+\cdots k_s=k} q^{1/2((k_1^2/n_1)+\cdots(k_s^2/n_s))}}{\prod_{i=1}^{s} \prod_{j \geq 1} (1-q^{j/n_i})} \tag{8.2.2}$$

and

$$\operatorname{trace}_{L(\Lambda_k)} q^{D_0} = \prod_{l \geq 1} (1-q^l) \operatorname{trace}_{\Lambda_k^\infty \mathbf{C}^\infty} q^{D_0}. \tag{8.2.3}$$

8.3. Hierarchies of Soliton Equations. Kac and Peterson [15] have shown that the KP-hierarchy of soliton equations can be defined in terms of the 1-component Clifford algebra:

$$\sum_{k \in \mathbf{Z}} \psi(k)\tau \otimes \psi^*(k)\tau = 0 \tag{8.3.1}$$

or, equivalently:

$$\oint \frac{dz}{z} \psi(z)\tau \otimes \psi^*(z)\tau = 0. \tag{8.3.2}$$

Here τ is an element of $\Lambda_0^\infty \mathbf{C}^\infty$, which, in the 1-component case, is realized as a polynomial ring. Equations (8.3.1) and (8.3.2) describe the orbit of the group $\widehat{GL}_\infty$ through the vacuum vector v_0.

It is clear that (8.3.1) can be rewritten in terms of s-component fermions;

$$\sum_{i=1}^{s} \sum_{k \in \mathbf{Z}} \psi_i(k)\tau \otimes \psi_i^*(k)\tau = 0, \tag{8.3.3}$$

where we now think of τ as a multicomponent polynomial, its components being labeled by the (co)root lattice of $sl_s(\mathbf{C})$. This equation is called the s-component KP-hierarchy in [23]. It is interesting to consider reductions of this hierarchy to the $\widehat{SL}_n$-orbit. The case $n=2$, $s=1$ is well known; one finds the KdV-family of p.d.e.'s. In the case $n=2$, $s=2$ the 2-component KP-hierarchy reduces to the so-called Toda-AKNS hierarchy (see [10, 13]).

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Note. After completion of the manuscript we learned of a paper by Dodd [24], in which the realizations of level one highest weight representations of $\widehat{sl}_n(\mathbf{C})$ associated to different HSA's are studied from a somewhat different point of view.

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