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Supermoduli Spaces

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Abstract. The connection between different supermoduli spaces is studied. It is shown that the coincidence of the moduli space of $(1|1)$ dimensional complex manifolds and $N=2$ superconformal moduli space is connected with hidden $N=2$ superconformal symmetry in the superstring theory.

Let W denote the Lie superalgebra of vector fields on $\mathbb{C}^{1,1}$

$$\xi = P(z,\theta)\frac{\partial}{\partial z} + Q(z,\theta)\frac{\partial}{\partial \theta}$$

(here $P(z,\theta)$ and $Q(z,\theta)$ are finite linear combinations of $z^n, z^n\theta, n$ is an integer). It is proved that this Lie algebra is isomorphic to the Lie superalgebra $K(2)$ consisting of $N=2$ infinitesimal superconformal transformations [1,2]. One can show that this fact is closely related with hidden $N=2$ superconformal symmetry in the superstring theory [3]. For superghost system (and for a general B–C system) hidden $N=2$ supersymmetry was discovered in ref. 4. To understand the origin of the $N=2$ supersymmetry of the B–C system we recall that the fields B and C can be considered as sections of line bundles ω^k and ω^{1-k} correspondingly. However the line bundle ω and its powers can be determined not only for a superconformal manifold but also for arbitrary $(1|1)$ dimensional complex supermanifold M. (If $(\tilde{z},\tilde{\theta})$ and (z,θ) are co-ordinate systems in M, then the transition functions of the line bundle ω^k are equal to D^k, where $D=D(\tilde{z},\tilde{\theta}|z,\theta)$ denotes the superjacobian.)

Let us consider a $(1|1)$ dimensional compact complex supermanifold M, a point $m\in M$ and local complex co-ordinates (z,θ) in the neighbourhood of m. (Here z is even, $|z|\leqq 1$, and θ is odd.) The moduli space of such data will be denoted by $\mathscr{P}$.

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(In other words, a point of $\mathscr{P}$ is specified by a triple $(M, m, (z, \theta))$ and the triples connected with an analytic transformation are identified.) The Krichever construction permits us to build for every point $p \in \mathscr{P}$ an element $V_k(p) \in \mathrm{Gr}$. (Here Gr denotes the superGrassmannian manifold and $V_k(p)$ can be defined as space of functions on supercircle that can be extended to a holomorphic section of ω^k over the exterior of supercircle.) The element $V_k(p) \in \mathrm{Gr}$ specifies the state corresponding to the B–C system on the manifold M. Every vector field $\xi \in W = K(2)$ generates an infinitesimal transformation of $\mathscr{P}$; a corresponding change of $V_k(p)$ is given by the operator $\xi + k \operatorname{div} \xi$ acting in the space H of functions on the supercircle and therefore in Gr.

It is well known that the change of the state of B–C system by infinitesimal $N = 1$ superconformal transformations is governed by the energy tensor T (see [5]); in a similar way the variation of this state by other transformations for $W = K(2)$ is governed by the supercurrent J (see [3] for details).

The remarks above together with the results of [6] permit us to analyze the superbosonization problem [7]. The physical state can be represented as a section of the det* bundle over Gr. Let us fix an element $V \in \mathrm{Gr}$ and let us consider the restriction of the det* bundle over the set $\Gamma V \subset \mathrm{Gr}$, where Γ denotes the (super)group of invertible functions on the (super)circle. In the supercase this restriction is a trivial bundle [6] and we can assign to every state a section of this trivial bundle, i.e. a function on Γ. This realization of states by means of functions on Γ is closely related with superbosonization and with the results of ref. 8. In the usualcase the det* bundle is not trivial over ΓV, however it is trivial over $\Gamma_- V$ and the bosonization is connected with the representation of states by means of functions on Γ_-. $\left(\text{Here } \Gamma_- \text{ consists of functions on the circle } |z| = 1 \text{ having the form } \exp\left(\sum_{n \geq 1} x_n z^n\right).\right)$

It is shown in [6] that the super-Mumford form can be extended to the space $\mathscr{P}$; this follows also from the results of ref. 9. Moreover, it is proved in ref. 6 that the super-Mumford form can be extended to the "universal moduli space" $\mathrm{UMS} \subset \mathrm{Gr}$; the Krichever construction embeds $\mathscr{P}$ in UMS. (Note that B–C system can also be extended to UMS.) As in the case of B–C system the possibility to extend the super-Mumford form to the space $\mathscr{P}$ shows that this form has in some sense W-symmetry (or $N = 2$ superconformal symmetry).

It is well known that in the string theory $N = 1$ space-time supersymmetry is connected with $N = 2$ world-sheet superconformal symmetry [10]. In other words, we have to consider $N = 2$ superconformal theory living on the world-sheet and we obtain the string amplitudes integrating the vertex functions of the superconformal theory over the moduli space. The question arises what kind of moduli space we must use. The conjecture that one has to use $N = 2$ superconformal moduli space leads to anomalies and therefore it must be excluded. However, to integrate over $N = 1$ superconformal moduli space we have to explain how one can consider $N = 2$ superconformal theory living on the $N = 1$ superconformal world-sheet. To answer this question we use that the isomorphism $W = K(2)$ leads to the possibility of constructing for every $(1|1)$ dimensional complex supermanifold a $N = 2$ superconformal manifold. In particular, we can assign to every $N = 1$ superconformal manifold an $N = 2$ superconformal manifold; in such a way the

$N=1$ superconformal moduli space is embedded in $N=2$ superconformal moduli space.

We see that the symmetry with respect to the super Lie algebra $W=K(2)$ may play an important role in string theory. All questions discussed above will be analyzed in more detail in a forthcoming paper [3]. The present paper is devoted to the study of the connection between $(1|1)$ complex manifolds and $N=2$ superconformal manifolds and to related topics. In the main part of the paper we use a more elementary co-ordinate approach. In the appendix we show how one can simplify the proofs and go a little further by means of invariant considerations.

Let us consider a $(1|N)$ dimensional superdomain with co-ordinates $Z=(z,\theta_1,\dots,\theta_N)$. The covariant derivatives D_i,

$$D_i=\frac{\partial}{\partial\theta_i}+\theta_i\frac{\partial}{\partial z} \tag{1}$$

satisfy

$$\{D_i,D_j\}=2\delta_{ij}\frac{\partial}{\partial z}. \tag{2}$$

One says that the transformation

$$\tilde{Z}=F(Z) \tag{3}$$

is superconformal if

$$\tilde{D}_i=F_{ij}(Z)D_j \tag{4}$$

(i.e. $\tilde{D}_1,\dots,\tilde{D}_N$ can be considered as linear combinations of $D_1,\dots,D_N$). One can check [11] that the transformation (3) is superconformal if and only if

$$D_i\tilde{z}=\tilde{\theta}_jD_i\tilde{\theta}_j. \tag{5}$$

The infinitesimal transformation

$$\delta z=v-\tfrac{1}{2}\theta_jD_jv,\quad \delta\theta_i=\tfrac{1}{2}D_iv \tag{6}$$

is superconformal (here $v(Z)=v(z,\theta_1,\dots,\theta_N)$ is an arbitrary analytic field). Really, it is easy to check (4) with

$$F_{ij}(Z)=\delta_{ij}+\tfrac{1}{2}D_iD_jv. \tag{7}$$

Conversely, every infinitesimal superconformal transformation can be represented in the form (6). The matrices $F_{ij}(Z)$ in (7) satisfy

$$F_{ij}(Z)F_{kj}(Z)=\lambda(Z)\delta_{ik}, \tag{8}$$

where $\lambda(Z)=\dfrac{\partial\tilde{z}}{\partial z}+\tilde{\theta}_j\dfrac{\partial\tilde{\theta}_j}{\partial z}$ [11]. We see therefore that the matrix function $F(Z)=(F_{ij}(Z))$ takes on values in the group $O(N,\mathbb{C})\times\mathbb{C}^*$, where $O(N,\mathbb{C})$ denotes the complex orthogonal group and $\mathbb{C}^*$ denotes the group of non-zero complex numbers. (This fact follows also from (7).) We will consider here only the superconformal transformations satisfying $F(Z)\in SO(N,\mathbb{C})\times\mathbb{C}^*$. One can define the N-superconformal manifold as a $(1|N)$ dimensional complex supermanifold pasted from $(1|N)$ dimensional superdomains by means of superconformal transformations. Recall that we require $F \in SO(N,\mathbb{C}) \times \mathbb{C}^*$, and therefore our N-superconformal manifolds are untwisted super-Riemann surfaces in the sence of [11]. If T is a k-dimensional representation of the group $SO(N,\mathbb{C}) \times \mathbb{C}^*$ we define a field of type T on the superconformal manifold as a k-component field φ with the transformation law

$$\tilde{\varphi}(\tilde{Z}) = T(F)\varphi(Z) \tag{9}$$

(in other words, the value of φ in the patches connected with (3) satisfy (9).) In the case $N=2$ the group $SO(N,\mathbb{C}) \times \mathbb{C}^*$ is abelian; it has only one-dimensional irreducible representations. It is easy to check that $SO(2,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$ and therefore one-dimensional irreducible representations of $SO(2,\mathbb{C}) \times \mathbb{C}^* = \mathbb{C}^* \times \mathbb{C}^*$ are labelled by two integers.

It is convenient to consider instead of derivatives D_1, D_2 their linear combinations $D_+ = \frac{1}{\sqrt{2}}(D_1 + iD_2)$, satisfying

$$D_+^2 = D_-^2 = 0, \qquad [D_+, D_-] = \frac{\partial}{\partial z}. \tag{10}$$

In the co-ordinates $\theta_\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$ the derivatives D_+, D_- have the form $D_+ = \frac{\partial}{\partial \theta_-} + \frac{1}{2}\theta_+ \frac{\partial}{\partial z}$, $D_- = \frac{\partial}{\partial \theta_+} + \frac{1}{2}\theta_- \frac{\partial}{\partial z}$. The behaviour of D_+ by superconformal transformations is given by

$$\tilde{D}_+ = F_+ D_+, \qquad \tilde{D}_- = F_- D_-. \tag{11}$$

(Replacing the matrix $F \in SO(2,\mathbb{C}) \times \mathbb{C}^*$ by two numbers F_+, F_- we obtain the isomorphism $SO(2,\mathbb{C}) \times \mathbb{C}^* = \mathbb{C}^* \times \mathbb{C}^*$ mentioned above.)

The field of type (k,l) on $N=2$ superconformal manifold can be defined by means of the transformation law

$$\tilde{\varphi}^{k,l}(\tilde{Z}) = F_+^k(Z) F_-^l(Z) \varphi^{k,l}(Z), \tag{12}$$

such a field can be interpreted as a section of the line bundle $\omega_+^k \otimes \omega_-^l$, where ω_+ and ω_- are defined by means of transition functions F_+ and F_- correspondingly. It is easy to check that the transformation law for the field v entering (6) for $N=2$ is given by (12) with $k=l=-1$. Therefore infinitesimal superconformal transformations for $N=2$ are specified by analytic fields of the type $(-1,-1)$. In the general case v is

$$\tilde{v}(\tilde{Z}) = (\det F(Z))^{-2/N} v(Z). \tag{13}$$

One can also obtain [11] that the transformation law for the density f on the N-superconformal manifold is

$$\tilde{f}(\tilde{Z}) = (\det F(z))^{(N-2)/N} f(Z). \tag{14}$$

Let us prove that the moduli space $\mathcal{M}^2$ of $N=2$ superconformal manifolds can be identified with the moduli space $\mathcal{M}_{1,1}$ of all $(1|1)$ dimensional complex manifolds. The assertion was proved for the first time by Deligne in an unpublished letter to Yu. Manin [12]. Speaking about the moduli space one assumes usually that all manifolds under consideration are compact. However, this assumption will be irrelevant for our conclusions.

By definition the $N=2$ superconformal manifold X is patched together from $(1|2)$ dimensional superdomains by means of $N=2$ superconformal transformations. Let us replace the co-ordinates z, θ_1, θ_2 by

$$u = z - \frac{i}{2}\theta_1\theta_2 = z + \frac{1}{2}\theta_+\theta_-,$$

$$\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) = \theta_+,$$

$$\alpha = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) = \theta_-. \tag{15}$$

It is easy to check that the operators D_+, D_- in these co-ordinates have the form

$$D_+ = \frac{\partial}{\partial\alpha}, \quad D_- = \frac{\partial}{\partial\eta} + \alpha\frac{\partial}{\partial u}. \tag{16}$$

The general superconformal transformation in the co-ordinates z, θ_+, θ_- can be written as follows:

$$\begin{aligned}\tilde{z} &= q(z) + \tfrac{1}{2}\theta_-\varepsilon_+(z)q_-(z) + \tfrac{1}{2}\theta_+\varepsilon_-(z)q_+(z) + \tfrac{1}{4}\theta_+\theta_-(\varepsilon_+(z)\varepsilon_-(z))',\\ \tilde{\theta}_+ &= \varepsilon_+(z) + \theta_+q_+(z) + \tfrac{1}{2}\theta_+\theta_-\varepsilon'_+(z),\\ \tilde{\theta}_- &= \varepsilon_-(z) + \theta_-q_-(z) + \tfrac{1}{2}\theta_-\theta_+\varepsilon'_-(z)\\ &\cdot q_+(z)q_-(z) = q'(z) + \tfrac{1}{2}(\varepsilon_+(z)\varepsilon'_-(z) + \varepsilon_-(z)\varepsilon'_+(z)),\end{aligned} \tag{17}$$

where $q(z), q_+(z), q_-(z)$ are arbitrary even and $\varepsilon_+(z), \varepsilon_-(z)$ are arbitrary odd analytic functions. Corresponding formulas in the co-ordinates u, η, α are

$$\tilde{u} = (q + \tfrac{1}{2}\varepsilon_+\varepsilon_-)(u) + \eta q_+(u)\varepsilon_-(u), \tag{18}$$

$$\tilde{\eta} = \varepsilon_+(u) + \eta q_+(u), \tag{19}$$

$$\tilde{\alpha} = \varepsilon_-(u) + \alpha q_-(u) + \alpha\eta\varepsilon'_-(u). \tag{20}$$

It is important to stress that α does not enter the expression for u, η. This assertion can be proved without calculation. One has to use the fact that $d_+u = D_+\eta = 0$ and the superconformal conditions (5) which in the co-ordinates u, η, α take the form

$$\begin{aligned}D_+(\tilde{u} - \tfrac{1}{2}\tilde{\eta}\tilde{\alpha}) &= \tfrac{1}{2}\tilde{\eta}D_+\tilde{\alpha}, \quad D_+\tilde{\eta} = 0,\\ D_-(\tilde{u} - \tfrac{1}{2}\tilde{\eta}\tilde{\alpha}) &= \tfrac{1}{2}\tilde{\alpha}D_-\tilde{\eta}, \quad D_-\tilde{\alpha} = 0.\end{aligned}$$

For every $N=2$ superconformal manifold X we construct a $(1|1)$ dimensional complex manifold X' as the manifold obtained by patching together $(1|1)$ dimensional superdomains with co-ordinates (u, η) by means of (18) and (19). One can verify that this construction gives one-to-one correspondence between classes of $N=2$ superconformal manifolds and $(1|1)$ dimensional complex manifolds. This fact follows from the explicit formulas (18), (19) and (20). Really, let us take $(1|1)$ dimensional complex manifold obtained by patching together $(1|1)$ dimensional superdomains in an arbitrary way:

$$\tilde{u} = (S(u) + \eta V(u)\varphi(u),$$
$$\tilde{\eta} = \psi(u) + \eta V(u), \tag{21}$$

where $S(u)$, $V(u)(\varphi(u), \psi(u))$ are arbitrary even (odd) analytic functions. Then the corresponding $N=2$ superconformal manifold can be obtained in a similar way by means of (18), (19), (20), where

$$\varepsilon_+(u) = \psi(u), \quad q_+(u) = V(u)$$
$$\varepsilon_-(u) = \varphi(u), \quad q_-(u) = (S'(u) - \psi'(u)\varphi(u))V^{-1}(u),$$
$$q(u) = S(u) + \tfrac{1}{2}\varphi(u)\psi(u).$$

In other words, to obtain an $N=2$ superconformal manifold X from a $(1|1)$ dimensional complex manifold X' we add to u, η a new odd co-ordinate α and patch together $(1|2)$ dimensional superdomains with coordinates (u, η, α) by means of (21) and

$$\tilde{\alpha} = \varphi(u) + \alpha(S'(u) - \psi'(u)\varphi(u))V^{-1}(u) + \alpha\eta\varphi'(u).$$

In such a way we realize X as a $(0|1)$ dimensional fibering over X'.

The above construction can be generalized to the case $N>2$. Let us consider at first the subgroup $G, G \subset SO(N, \mathbb{C}) \times \mathbb{C}^*$ and define G-superconformal transformation as a transformation satisfying (3) with $F \in G$. We define a G-superconformal manifold as a manifold obtained by patching together $(1|N)$ dimensional superdomains by means of G-superconformal transformations. Let us denote by H the subgroup of $SO(2N, \mathbb{C}) \times \mathbb{C}^*$ consisting of transformations that leave invariant a maximal isotropic subspace in $\mathbb{C}^{2N}$. Introducing the operators $D_k^{\pm} = \frac{1}{\sqrt{2}}(D_k \pm D_{N+k})$ one can characterize the H-superconformal transformations as transformation satisfying

$$\tilde{D}_j^+ = F_{jk}^{++} D_k^+,$$
$$\tilde{D}_j^- = F_{jk}^{-+} D_k^+ + F_{jk}^{--} D_k^-. \tag{22}$$

In the co-ordinates u, η_j, α_j,

$$u = z - \frac{i}{2}\left(\sum_{k=1}^{N} \theta^k \theta_{N+k}\right) = z + \tfrac{1}{2}\theta_k^+ \theta_k^-,$$

$$\eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{N+j}) = \theta_j^+,$$

$$\alpha_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{N+j}) = \theta_j^-, \tag{23}$$

the operators D_j^+, D_j^- have the form

$$D_j^+ = \frac{\partial}{\partial \alpha^j}, \quad D_j^- = \frac{\partial}{\partial \eta^j} + \alpha_j \frac{\partial}{\partial u}. \tag{24}$$

It is easy to check that by the H-superconformal transformation

$$\tilde{u} = \psi(u, \eta_k), \tag{25}$$

$$\tilde{\eta}_j = \phi_j^+(u, \eta_k), \tag{26}$$

$$\tilde{\alpha}_j = \varphi_j^-(u, \eta_k, \alpha_k). \tag{27}$$

In other words only u, η_k enter the expressions for $\tilde{u}, \tilde{\eta}_j$. This remark permits us to construct for every $(1|2\mathrm{N})$ dimensional H-superconformal manifold a complex $(1|N)$ dimensional supermanifold (this manifold is pasted by means of (25), (26)). As in the case $N=2$ one can show that this construction gives an isomorphism between the moduli space $\mathscr{U}_{1,N}$ of complex $(1|N)$ dimensional supermanifolds and the moduli space $\mathscr{U}_H^{2N}$ of $(1|2N)$ dimensional H-superconformal manifolds. (One can see that in the case $N=2$ the notion of H-superconformal transformation coincides with the notion of $N=2$ superconformal transformation, i.e. automatically $F^{-+}=0$.)

It is evident that there exists one-to-one correspondence between the analytic vector fields on $(1|N)$ dimensional complex manifold X' and infinitesimal H-superconformal transformations of corresponding H-superconformal manifold X. Let f be a density on X'. Then the field $\hat{f}$ on X satisfying $\hat{f}(Z) = f(Z))$ is a chiral field on X (i.e. $D_k^+ \hat{f} = 0$) and the transformation law of $\hat{f}$ is

$$\tilde{\hat{f}}(\tilde{Z}) = (\det F_{--}(Z))^{(N-2)/N} \det F_{++}(Z)^{-2/N} \hat{f}(Z) \tag{28}$$

(the projection π transforms the point $x = (u, \eta_1, \ldots, \eta_N, \alpha_1, \ldots, \alpha_N) \in X$ into the point $x' = (u, \eta_1, \ldots, \eta_N) \in X'$). The transformation of X preserving the density f generates an H-superconformal transformation $\tilde{Z} = F(Z)$ on X satisfying

$$(\det F_{--}(Z))^{(N-2)/2} \cdot (\det F_{++}(Z))^{2/N} = f(\pi(\tilde{Z}(Z)))/f(\pi(Z)). \tag{29}$$

The infinitesimal H-superconformal transformation can be determined by means of the analytic field v satisfying

$$D_i^+ D_j v = 0.$$

It obeys (29) if

$$\left(\frac{\partial}{\partial z} - D_k^- D_k^+\right)(\hat{f} v) = 0. \tag{30}$$

Of course the constructions above give isomorphisms between different super-Lie algebras and super-Lie groups. For example, it is evident that the algebra $W = W(1)$ of vector fields in $\mathbb{C}^{1,1}$ is isomorphic to the superconformal algebra $K(2)$. (See [2] for rigorous definitions of algebras $W(N), S(N, f)$ and $K(N)$.)

In general our construction can be used to realize algebras $W(N), S(N, f)$ as subalgebras of $K(2N)$. Let us consider for instance the algebra $S(2, \alpha) = S(2, z^\alpha)$ consisting of $(1|2)$ dimensional vector fields preserving the volume element $z^\alpha dz\, d\theta_1\, d\theta_2$. The assertion [2] that the algebras from this family coincide with the so-called $SU(2)$ superconformal algebras follows from our arguments.

Appendix

In this appendix we give an invariant description of the above constructions. Let us consider $(1|N)$ dimensional complex supermanifold X and let us fix a $(0|N)$ distribution $\mathscr{V}$ on X. In other words, we suppose that for every point $x \in X$ a $(0|N)$ dimensional subspace $\mathscr{V}_x$ in tangent space $\mathscr{T}_x$ is specified. Let us consider a frame $v_1(x), \ldots, v_N(x)$ in $\mathscr{V}_x$. We suppose that the dependence of $\mathscr{V}_x$ on $x \in X$ is smooth and therefore at least locally we can assume that $v_1(x), \ldots, v_n(x)$ are smooth vector fields. (Anti commutator of these fields specifies a bilinear form v_x on $\mathscr{V}_x$ taking on values in $(1|0)$ dimensional space $\mathscr{T}_x/\mathscr{V}_x$. (It is easy to check that if $v_i(x_0) = v_i'(x_0)$, then $\{v_i', v_j'\}|_{x=x_0} - \{v_i, v_j\}|_{x=x_0} \in \mathscr{V}_{x_0}$ and therefore the image v_{x_0} depends only on the values of vector fields $v_k(x)$ at the point x_0.) If the form v_x is non-degenerate we will say that the distribution specifies an N-superconformal structure on x. One can prove that superconformal manifolds in this sense can be identified with the superconformal manifolds in the sense of [11]. In other words, in the neighbourhood of every point $x \in X$ we can introduce a co-ordinate system $(z, \theta_1, \ldots, \theta_N)$ in such a way that in these co-ordinates the subspace $\mathscr{V}_x$ is spanned by $D_i = \frac{\partial}{\partial \theta^i} + \theta_i \frac{\partial}{\partial z}, i = 1, \ldots, N$. The proof is based on the remark that the quadratic form v_x can be diagonalized. A superconformal transformation F can be defined as a transformation preserving the distribution $\mathscr{V}$ (i.e. F is superconformal if for $v \in \mathscr{V}_x$ we have $F_* v \in \mathscr{V}_{F(x)}$. The notion of an N-superconformal manifold is closely related with the notion of a contact manifold. One can define contact structure in $(M|N)$ dimensional manifold by means of $(M-1|N)$ dimensional distribution satisfying some non-degeneracy condition. Then for $N = 0$ we obtain the usual contact manifolds and for $M = 1$ the definition of contact structure coincides with the definition of N superconformal structure. (We consider always complex manifolds. Of course one can use similar definitions for real manifolds too). Some of the constructions below are similar to the constructions used in the theory of contact manifolds (see for example [13]).

Let us say that the superconformal manifold X is provided by an H-superconformal structure if a maximal isotropic subspace $\mathscr{W}_x$ is specified in every space $\mathscr{V}_x \subset \mathscr{T}_x, x \in X$. (Maximal isotropic subspace in $\mathscr{V}_x$ is a maximal subspace in which every two vectors are orthogonal with respect to the form v_x). Of course we assume that the subspace $\mathscr{W}_x$ depends continuously on $x \in X$. In the $N = 2$ superconformal manifold every tangent space contains exactly two maximal isotropic subspaces. To specify the H-superconformal structure on X we have to select continuously one of these subspaces. If this is possible then in terminology of [11] the $N = 2$ superconformal manifold is untwisted. Let us stress that in the main text we have considered only untwisted superconformal manifolds. It is evident that the untwisted $N = 2$ superconformal manifold has exactly two H-superconformal structures.

Let us consider the arbitrary $(1|N)$ dimensional manifold Y. We define $(1|2N)$ dimensional complex supermanifold $\hat{Y}$ as a manifold consisting of all $(0|N)$ dimensional subspaces in the tangent spaces $\mathscr{T}_y, y \in Y$. In such a way a point in Y is a pair (y, V), where V is a $(0|N)$ dimensional subspace in $\mathscr{T}_y$. The natural projection π of $\hat{Y}$ onto Y is a fibering with $(0|N)$ dimensional fibre. In local co-ordinates $(y, \eta_1, \ldots, \eta_N)$ on Y a $(0|N)$ dimensional subspace in tangent space can be written in the form

$$dy = \alpha^1 d\eta_1 + \cdots + \alpha^N d\eta_N. \tag{A.1}$$

The projection π transforms the point $(y, \eta_1, \ldots, \eta_N, \alpha^1, \ldots, \alpha^N) \in \hat{Y}$ into the point $(y, \eta_1, \ldots, \eta_N) \in Y$. The projection π generates a map $\hat{\pi}$ of the tangent space $\mathcal{T}_{y,v}$ to $\hat{Y}$ at the point $(y, V) \in \hat{Y}$ onto the tangent space $\mathcal{T}_y$ to Y at the point y.

Let us define $(0|N)$ dimensional subspace $\mathcal{V}_{y,V} \subset \mathcal{T}_{y,V}$ as the counterimage of $V \subset \mathcal{T}_y$ by the map $\hat{\pi}$. In the co-ordinates $(y, \eta_1, \ldots, \eta_N, \alpha^1, \ldots, \alpha^N)$ the map π transforms the vector $(dy, d\eta_1, \ldots, d\eta_N, d\alpha^1, \ldots, d\alpha^N)$ into the vector $(dy, d\eta_1, \ldots, d\eta_N)$ and therefore the space $\mathcal{V}_{y,V}$ consists of the vectors having the form

$$(\alpha^i d\eta_i, d\eta_1, \ldots, d\eta_N, d\alpha^1, \ldots, d\alpha^N).$$

In other words the space $\mathcal{V}_{y,V}$ is spanned by the vectors

$$e^{-1} = \alpha^1 \frac{\partial}{\partial y} + \frac{\partial}{\partial \eta_1}, \ldots e^{-N} = \alpha^N \frac{\partial}{\partial y} + \frac{\partial}{\partial \eta_N}, \quad e_1^+ = \frac{\partial}{\partial \alpha^1}, \ldots e_N^+ = \frac{\partial}{\partial \alpha^N}. \tag{A.2}$$

It is easy to check that the vector fields e^{-i}, e_i^+ satisfy

$$\{e^{-i}, e^{-j}\} = \{e_j^+, e_j^+\} = 0; \quad \{e^{-i}, e_j^+\} = \frac{\partial}{\partial y} \delta_j^i. \tag{A.3}$$

We see that these commutators determine a non-degenerate quadratic form on $\mathcal{V}_{y,V}$. In such a way the subspaces $\mathcal{V}_{y,V}$ specify a superconformal structure on Y. Moreover, every subspace $\mathcal{V}_{y,V}$ contains an isotropic subspace $\mathcal{W}_{y,V}$ spanned by the vectors $e_i^+, i = 1, \ldots, N$. In such a way for every $(1|N)$ dimensional complex manifold Y we have constructed an H-superconformal manifold $\hat{Y}$. Conversely, for every $(1|2N)$ dimensional H-superconformal manifold X we can construct a $(1|N)$ dimensional complex manifold X' in such a way that $(X') = X, (\hat{Y})' = Y$. (One has to apply Frobenius' theorem to the distribution specified by the isotropic subspaces.) An analytic transformation of the manifold Y generates an H-superconformal transformation of $\hat{Y}$ and vice versa. We see that the moduli space $\mathcal{U}_{1,N}$ of $(1|N)$ dimensional complex manifolds coincides with the moduli space $\mathcal{U}_H^{2N}$ of H-superconformal manifolds. As we have seen for every untwisted $N = 2$ superconformal manifold Y we can construct two different H-superconformal manifolds say Y_+ and Y_- and therefore two different $(1|1)$ complex manifolds $S_+ = (Y_+)', S = (Y_-)'$. In such a way we have two different isomorphisms P_+ and P_- between the spaces $\mathcal{U}^2$ and $\mathcal{U}_{1,1}$ defined by $P_+ Y = (Y_+)', P_- Y = (Y_-)'$. One can define an involution λ in $\mathcal{U}_{1,1}$ by the formula $\lambda(P_+ Y) = P_- Y$. There exists a simple geometric description of this involution: if S is an $(1|1)$ dimensional complex manifold then λS consists of all $(0|1)$ dimensional submanifolds in S. The $(1|1)$-dimensional manifolds S that can be provided with $N = 1$ superconformal structure satisfy $\lambda S = S$. In other words, $N = 1$ superconformal manifolds are fixed points of the involution λ.

For the proof let us recall that the $N = 2$ superconformal manifold $X = \hat{S}$ consists of $(0|1)$ dimensional subspaces of tangent spaces at all points of S. The natural projection of X onto S will be denoted by π_+; for the natural projection of X onto $\lambda(S)$ we use the notation π_-. For every $(0|1)$ dimensional submanifold

L of S we construct a point $\gamma(L)$ in $\lambda(S)$ by the formula

$$\gamma(L) = \pi_-(t_e),$$

where t_e denotes the tangent space to L at the point $l \in L$. One can verify that this definition is correct, i.e. the point $\gamma(L)$ does not depend on the choice of the point $l \in L$. To check it is convenient to use co-ordinates (v, η, d) on $X = S$ introduced above. In these co-ordinates $\pi_+(y, \eta, \alpha) = (y, \eta)$ and $\pi_-(y, \eta, \alpha) = (y - \alpha\eta, \alpha)$. If L is singled out by equation $y = y_0 + \sigma\eta$ then the co-ordinates of the point $t_l X$ are $(y + \sigma\eta, \eta, \sigma)$ and $\pi_-(t) = (y_0, \sigma)$ does not depend on the parameter η specifying the point on L. Conversely given a point $u \in \lambda(S)$ we can construct a $(0|1)$ dimensional submanifold $\rho(u) = \pi_+(\pi_-^{-1}u) \subset S$; if $u = \gamma(L)$ then $\rho(u) = L$. (Here $\pi_-^{-1}u$ denotes the fibre over u by the projection π_-.)

The equivalence between S and $\lambda(S)$ for the $N = 1$ superconformal manifold S is proved (in other terms) in ref. 14. Namely if a $(0|1)$ dimensional submanifold L of S satisfies the equation $f(y, \eta) = 0$ then one can single out a point of s by means of equations $f(y, \eta) = 0$, $Df(y, \eta) = 0$. We obtain in such a way one-to-one correspondence between $(0|1)$ dimensional submanifolds of S and points of S (i.e. equivalence between $\lambda(S)$ and S).

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