Commun. Math. Phys. 134, 555–585 (1990)

Communications in
Mathematical
Physics

Isospectral Hamiltonian Flows in Finite and Infinite Dimensions

II. Integration of Flows*

M. R. Adams[1], J. Harnad[2] and J. Hurtubise[3]

[1] Department of Mathematics, University of Georgia, Athens, GA 30602, USA
[2] Centre de Recherches Mathématiques, Université de Montréal, C. P. 6128-A, Montréal, Qué, H3C 3J7, Canada and Department of Mathematics, Concordia University, Montréal, Qué, Canada
[3] Department of Mathematics, McGill University, Montréal, Qué, H3A 2K6, Canada

Received May 30, 1989; in revised form March 12, 1990

Abstract. The approach to isospectral Hamiltonian flow introduced in part I is further developed to include integration of flows with singular spectral curves. The flow on finite dimensional Ad*-invariant Poisson submanifolds of the dual $(\widetilde{gl}(r)^+)^*$ of the positive part of the loop algebra $\widetilde{gl}(r)$ is obtained through a generalization of the standard method of linearization on the Jacobi variety of the invariant spectral curve S. These curves are embedded in the total space of a line bundle $T \to \mathbb{P}_1(\mathbb{C})$, allowing an explicit analysis of singularities arising from the structure of the image of a moment map $\tilde{J}_r: M_{N,r} \times M_{N,r} \to (\widetilde{gl}(r)^+)^*$ from the space of rank-r deformations of a fixed $N \times N$ matrix A. It is shown how the linear flow of line bundles $E_t \to \tilde{S}$ over a suitably desingularized curve $\tilde{S}$ may be used to determine both the flow of matricial polynomials $L(\lambda)$ and the Hamiltonian flow in the space $M_{N,r} \times M_{N,r}$ in terms of θ-functions. The resulting flows are proved to be completely integrable. The reductions to subalgebras developed in part I are shown to correspond to invariance of the spectral curves and line bundles $E_t \to \tilde{S}$ under certain linear or anti-linear involutions. The integration of two examples from part I is given to illustrate the method: the Rosochatius system, and the CNLS (coupled non-linear Schrödinger) equation.

Introduction

In [1] it was shown how isospectral Hamiltonian flows in the space of rank r perturbations, $\mathcal{M}_A$, of an $N \times N$ matrix A can be derived from the Adler–Kostant–

* Research supported in part by the Natural Sciences and Engineering Research Council of Canada and by U.S. Army grant DAA L03-87-K-0110

Symes (AKS) theorem through the use of a moment map from $\mathcal{M}_A$ into the dual, $(\widetilde{gl}(r)^+)^*$, of the positive part of the loop algebra $\widetilde{gl}(r)$ (cf. [13] for the rank 2 case). These systems were shown to be completely integrable under special assumptions on the spectrum of A and the resulting matricial polynomial $L(\lambda)\in(\widetilde{gl}(r)^+)^*$. For these cases, standard linearization methods ([3, 4, 5, 11, 12, 15, 17]) yield solutions in terms of θ-functions for the associated spectral curves S. To treat the general case these methods must be extended to allow for singularities occurring in S when the assumptions on A and $L(\lambda)$ are removed. The purpose of this paper is to provide a more unified, streamlined formulation allowing A and $L(\lambda)$ to have more general spectra. Such a generalization is necessary in order to cover important examples of integrable systems, e.g. the coupled nonlinear Schrödinger equation, discussed in [1]. The construction we consider expands on work of Hitchin [9].

Since this work is a sequel to [1], we shall adopt the same notational conventions, which are briefly summarized, together with some of the main results, in Sect. 1. In Sect. 2 we describe an embedding of the spectral curve S into the line bundle $O(n)\to\mathbb{P}_1(\mathbb{C})$ which is used to integrate the flows on the Jacobian of an appropriate desingularization of S. The geometry of Sect. 2, along with a simple dimension count, is used in Sect. 3 to prove complete integrability in the generic case. In Sect. 4 we obtain the explicit realization of the geometric solutions from Sect. 2 in terms of θ-functions. In Sect. 5 we derive the properties of S, and the restriction to special classes of line bundles in the Jacobian of S, following from the reductions of the generic system to flows in $(\widetilde{gl}(r)^+)^*$ which are invariant under finite automorphism groups as introduced in [1]. Finally, in Sect. 6 we illustrate the general constructions in this paper by explicitly solving two examples introduced in [1]; the Rosochatius system, and the coupled nonlinear Schrödinger equation (CNLS).

1. Notation and Summary

Following [1], we let $\mathcal{M}_0\subset M_{N,r}\times M_{N,r}$ denote the open, dense submanifold of pairs (F,G) of $N\times r$ complex matrices, $F,G\in M_{N,r}$, defined by

$$\mathcal{M}_0=\{(F,G)\in M_{N,r}\times M_{N,r}\,|\,F \text{ and } G \text{ have rank } r\}. \tag{1.1}$$

This space inherits the natural symplectic structure on $M_{N,r}\times M_{N,r}$:

$$\omega=\operatorname{tr}(dF\wedge dG^T). \tag{1.2}$$

For a fixed $N\times N$ matrix A, let

$$\mathcal{M}_A=\{A+FG^T\,|\,(F,G)\in\mathcal{M}_0\} \tag{1.3}$$

denote the space of rank r perturbations of A. The group $GL(r,\mathbb{C})$ acts freely and properly on $\mathcal{M}_0$ by

$$g\cdot(F,G)=(Fg^{-1},Gg^T), \tag{1.4}$$

which preserves the symplectic structure (1.2). Therefore one has a natural Poisson manifold structure on the quotient space $\mathcal{M}_0/GL(r,\mathbb{C})$, with Poisson bracket given by

$$\{f,g\}=\{\pi^*f,\pi^*g\}, \tag{1.5}$$

where $\pi: \mathcal{M}_0 \to \mathcal{M}_0/GL(r, \mathbb{C})$ is the natural projection. Since the projection from $\mathcal{M}_0$ to $\mathcal{M}_A$ has as its fibers the $GL(r, \mathbb{C})$ orbits, we may identify $\mathcal{M}_A$ with $\mathcal{M}_0/GL(r, \mathbb{C})$.

Let $\widetilde{gl}(r) = gl(r, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ be the loop algebra of semi-infinite formal Laurent series in λ with coefficients in $gl(r, \mathbb{C})$; i.e. with elements of the form

$$X(\lambda) = \sum_{i=-\infty}^{m} a_i \lambda^i, \quad a_i \in gl(r, \mathbb{C}).$$

Let $\widetilde{gl}(r)^+$ denote the subalgebra of $\widetilde{gl}(r)$ given by the matricial polynomials in λ and $\widetilde{gl}(r)^-$ the subalgebra whose elements are sums of terms involving only strictly negative powers of λ. Then $\widetilde{gl}(r)$ is the vector space direct sum

$$\widetilde{gl}(r) = \widetilde{gl}(r)^+ \oplus \widetilde{gl}(r)^-. \tag{1.6}$$

The algebra $\widetilde{gl}(r)$ has a nondegenerate, ad-invariant, inner product given by

$$\langle X(\lambda), Y(\lambda) \rangle = \operatorname{tr}((X(\lambda) Y(\lambda))_0), \tag{1.7}$$

where $(X(\lambda) Y(\lambda))_0$ denotes the constant term in the formal series $X(\lambda) Y(\lambda)$. This pairing gives the identification

$$(\widetilde{gl}(r)^+)^* \sim (\widetilde{gl}(r)^-)^\perp = \widetilde{gl}(r)_0^-, \tag{1.8}$$

where $\widetilde{gl}(r)_0^- = \lambda \widetilde{gl}(r)^-$.

Assuming A to be diagonal, with eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ of multiplicity $\{k_i\}_{i=1,\dots,n}$, write $F, G \in M_{N,r}$ in block form

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_i \\ \vdots \\ F_n \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ \vdots \\ G_i \\ \vdots \\ G_n \end{bmatrix},$$

where F_i, G_i are $k_i \times r$ matrices. Define the map

$$\tilde{J}_r : M_{N,r} \times M_{N,r} \to (\widetilde{gl}(r)^+)^* \sim \widetilde{gl}(r)_0^-$$

by

$$\tilde{J}_r(F, G) = \sum_{i=1}^{n} \frac{\lambda G_i^T F_i}{\alpha_i - \lambda}. \tag{1.9}$$

Theorem 1.1 ([1], Ch. 2). *$\tilde{J}_r$ defines an equivariant moment map for an infinitesimal $\widetilde{gl}(r)^+$ action on $M_{N,r} \times M_{N,r}$.*

Let $I(\widetilde{gl}(r)^*)$ denote the ring of coadjoint-invariant functions on $\widetilde{gl}(r)^*$ and let $\mathscr{F}_+$ be the ring of functions on $(\widetilde{gl}(r)^+)^*$ given by restricting $I(\widetilde{gl}(r)^*)$. More generally, if $a \in GL(r, \mathbb{C})$, $Y = a^{-1} - I$, and $\hat{\phi} \in C^\infty(\widetilde{gl}(r)^*)$, we define $\hat{\phi}_Y \in C^\infty(\widetilde{gl}(r)^*)$ by

$$\hat{\phi}_Y(X(\lambda)) = \hat{\phi}(X(\lambda) + \lambda Y) \tag{1.10}$$

and set

$$\mathscr{F}_+^Y = \{\phi_Y \in C^\infty((\widetilde{gl}(r)^+)^*) \mid \phi_Y = \hat{\phi}_Y|_{(\widetilde{gl}(r)^+)^*}, \hat{\phi} \in I(\widetilde{gl}(r)^*)\}. \tag{1.11}$$

The generalization of the AKS theorem given in [18] implies that the functions in $\mathscr{F}^Y_+$ Poisson commute in the Lie-Poisson structure of $(\widetilde{gl}(r)^+)^*$. Let $\mathscr{F}^Y$ denote the pullback of $\mathscr{F}^Y_+$ to $M_{N,r} \times M_{N,r}$ by $\tilde{J}_r$. The functions in $\mathscr{F}^Y$ are invariant under the stabilizer subgroup $GL(r, \mathbb{C})_a$ of a in $GL(r, \mathbb{C})$.

Theorem 1.2 ([1], Thm 3.6). *The functions in $\mathscr{F}^Y$ Poisson commute on $M_{N,r} \times M_{N,r}$. Moreover, the Hamiltonian flows are isospectral for $A + FaG^T$, i.e., if $(F(t), G(t))$ describes the Hamiltonian flow of $h \in \mathscr{F}^Y$, then $\mathrm{spec}(A + F(t)aG(t)^T)$ is independent of t.*

Remark. In the case that $Y = 0$, $a = I$, $GL(r, \mathbb{C})_a = GL(r, \mathbb{C})$ so the functions in $\mathscr{F} = \mathscr{F}^0$ reduce to functions on $\mathscr{M}_A$, i.e. we can consider the flows on $\mathscr{M}_A$ as Hamiltonian with respect to the natural Poisson structure there.

In [1] it was shown that when $k_i = r - 1$ for all i then $\mathscr{F} = \mathscr{F}^0$ is a completely integrable ring of functions. One of the purposes of the present paper is to generalize this result to $Y \neq 0$ and more general $\{k_i\}$. In [1] the problem was reduced to one on $(\widetilde{gl}(r)^+)^*$ symplectic leaves by quotienting $M_{N,r} \times M_{N,r}$ by the natural symplectic action of the subgroup H of $GL(N, \mathbb{C})$ stabilizing A under conjugation.

Let $\mathscr{M}^{\mathbf{k}}$ denote the open dense submanifold of $M_{N,r} \times M_{N,r}$ on which F_i, G_i have rank k_i. The group H acts freely and properly on $\mathscr{M}^{\mathbf{k}}$, again preserving the symplectic structure, so $\mathscr{M}^{\mathbf{k}}/H$ is a Poisson manifold.

Theorem 1.3 ([1], Corollary 2.5). *$\tilde{J}_r$ is invariant under the H action on $\mathscr{M}^{\mathbf{k}}$, hence it reduces to an equivariant moment map*

$$\tilde{J}_{r,0} \colon \mathscr{M}^{\mathbf{k}}/H \to (\widetilde{gl}(r)^+)^*$$

for an infinitesimal $\widetilde{gl}(r)^+$ action on $\mathscr{M}^{\mathbf{k}}/H$. The map $\tilde{J}_{r,0}$ is one-to-one and maps symplectic leaves of $\mathscr{M}^{\mathbf{k}}/H$ onto symplectic leaves in $(\widetilde{gl}(r)^+)^*$.

Now let $\mathscr{M} = \mathscr{M}^{\mathbf{k}} \cap \mathscr{M}_0$, i.e. $(F, G) \in \mathscr{M}$ if F and G have rank r and F_i and G_i have rank k_i, all i. $\mathscr{M}$ is invariant under the actions of H and $GL(r, \mathbb{C})_a$, and the functions in $\mathscr{F}^Y$ are invariant under both of these actions. Hence we may consider $\mathscr{F}^Y$ as a ring of functions on the Poisson manifold $\mathscr{M}/(H \times GL(r, \mathbb{C})_a) = \mathscr{M}/(H \times SL(r, \mathbb{C})_a)$ and it is this ring of functions on $\mathscr{M}/(H \times GL(r, \mathbb{C})_a)$ which will be proved to be completely integrable. It then follows that the ring of functions $\mathscr{F}^Y$ extends to a completely integrable ring on an open dense subset of $\mathscr{M}$ by the arguments of Theorem 4.2 in [1], (cf. [8]).

Because of Theorem 1.3, in order to study the ring of functions $\mathscr{F}^Y$ reduced to $\mathscr{M}/H$ it suffices to consider the functions $\mathscr{F}^Y_+$ on finite dimensional symplectic leaves of $(\widetilde{gl}(r)^+)^*$. There is a natural $GL(r, \mathbb{C})$ action on $(\widetilde{gl}(r)^+)^*$, given by conjugation, which corresponds to the $GL(r, \mathbb{C})$ action on $\mathscr{M}/H$, i.e. the actions are intertwined by $\tilde{J}_{r,0}$. To study the ring of functions $\mathscr{F}^Y$ reduced to $\mathscr{M}/(H \times GL(r, \mathbb{C})_a)$ it suffices to consider the ring of functions $\mathscr{F}^Y_+$ on certain finite dimensional symplectic leaves of $(\widetilde{gl}(r)^+)^*$ reduced by the $GL(r, \mathbb{C})_a$ action.

With the identification $(\widetilde{gl}(r)^+)^* \sim \widetilde{gl}(r)_0^-$, if $\hat{\phi} \in I(\widetilde{gl}(r)^*)$, the Hamiltonian flow for $\phi_Y \in \mathscr{F}^Y_+$ on $\widetilde{gl}(r)_0^-$ is described in the AKS theorem as given by

$$\frac{d}{dt} X(\lambda) = [d\hat{\phi}((X(\lambda) + \lambda Y)^\flat)_+, X(\lambda) + \lambda Y], \tag{1.12}$$

where $X(\lambda) \in \widetilde{gl}(r)_0^-$, $(X(\lambda) + \lambda Y)^\flat$ is the element of $\widetilde{gl}(r)^*$ corresponding to

$X(\lambda) + \lambda Y \in \widetilde{gl}(r)$, $d\hat{\phi}((X(\lambda) + \lambda Y)^\flat)$ is the differential of $\hat{\phi}$ at $(X(\lambda) + \lambda Y)^\flat$ considered as an element of $\widetilde{gl}(r) \sim \widetilde{gl}(r)^{**}$, and the subscript $+$ denotes the projection to $\widetilde{gl}(r)^+$ along $\widetilde{gl}(r)^-$.

Since we are interested in these flows only on the finite dimensional orbits in the image of $\tilde{J}_r$ we can manipulate Eq. (1.12) to a more convenient form. First recall the expression for $\tilde{J}_r$ given in (1.9). We multiply away the poles at the α_i's to convert this to a matricial polynomial, i.e. let

$$L(\lambda) = \lambda^{-1} a(\lambda)(\tilde{J}_r(F, G) + \lambda Y), \tag{1.13}$$

where $a(\lambda) = \sum_{i=1}^{n} (\lambda - \alpha_i)$. We can thus rewrite (1.12) as a flow of matricial polynomials by

$$\frac{d}{dt} L(\lambda) = [d\hat{\phi}((X(\lambda) + \lambda Y)^\flat)_+, L(\lambda)]. \tag{1.14}$$

Finally, since we are working on finite dimensional orbits in $\widetilde{gl}(r)_0^-$, it is enough to consider the case when (see Proposition 3.1) $d\hat{\phi}(X(\lambda) + \lambda Y)$ can be written in the form $P(L(\lambda), \lambda^{-1})$ where $P(z, \lambda^{-1})$ is a complex polynomial in λ^{-1} and z. Thus we have reduced our study of the flows on orbits in $\mathscr{M}/H$ to the study of the flows

$$\frac{d}{dt} L(\lambda) = [P(L(\lambda), \lambda^{-1})_+, L(\lambda)], \tag{1.15}$$

where $P(z, \lambda^{-1})$ is an arbitrary complex polynomial in λ^{-1} and z. In the next section we study these flows.

2. Geometric Solutions of Matricial Polynomial Lax Equations

In this section we derive geometric solutions to Lax equations of the form

$$\frac{d}{dt} L(\lambda; t) = [P(L(\lambda; t), \lambda^{-1})_+, L(\lambda; t)], \tag{2.1}$$

where

$$L(\lambda; t) = L(\lambda) = L_0 \lambda^m + L_1 \lambda^{m-1} + \cdots + L_m \tag{2.2}$$

is a matricial polynomial in λ, i.e. $L_i \in gl(r, \mathbb{C})$, $P(z, \lambda^{-1})$ is an arbitrary complex polynomial in λ^{-1} and z, and the subscript $+$ denotes taking the positive part with respect to the splitting $\widetilde{gl}(r) = \widetilde{gl}(r)^+ \oplus \widetilde{gl}(r)^-$.

Define the spectral curve $S_0 \subset \mathbb{C}^2$ *by*

$$S_0 = \{(\lambda, z) | \det(zId - L(\lambda; t)) = 0\}, \tag{2.3}$$

It follows from (2.1) that S_0 is independent of t. On S_0 the value $z = z(\lambda)$ is an eigenvalue of the matrix $L(\lambda; t)$.

According to the techniques of [5, 12, 11, 15, 4, 3, 17], one defines, for generic $L(\lambda; t)$, *a line bundle (or its associated divisor class) over a compactification* S *of* S_0, with fibre at (λ, z) the z-eigenspace of $L(\lambda; t)$, and one obtains from the flow $L(\lambda; t)$ a linear flow in the Jacobian of S.

The approach in terms of line bundles must be generalized if the eigenspaces over the points $(\lambda, z) \in S_0 \subset \mathbb{C}^2$ are not everywhere one-dimensional. This is the case at the points $(\lambda, z) = (\alpha_i, 0)$ for $L(\lambda; t)$ of the form (1.13) arising from rank-r perturbations of A if the multiplicity k_i of its eigenvalues α_i is less than $r-1$. To handle such cases, we adopt a construction of Hitchin [9] and consider the sheaf $\bar{E}_t^0$ over $\mathbb{C}^2$, with support $S_0 \subset \mathbb{C}^2$ defined by the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{C}^2}^{\oplus r} \xrightarrow{zId - L(\lambda;t)} \mathcal{O}_{\mathbb{C}^2}^{\oplus r} \to \bar{E}_t^0 \to 0,$$

which is well defined for any $L(\lambda; t)$. In the case when the eigenspaces of $L^T(\lambda; t)$ remain of dimension one, $\bar{E}_t^0$ is just the sheaf of sections of the dual of the eigenvector bundle of $L^T(\lambda; t)$.

In subsections (a)–(d), the corresponding sheaves $\bar{E}_t$ and E_t will be constructed on a compactified curve S and its desingularization $\pi: \tilde{S} \to S$. A key feature of the construction (cf. [10, 9]) is an embedding of S into a surface T which is a partial compactification of $\mathbb{C}^2$. (T is just the total space of the m^{th} power of the hyperplane section bundle of $\mathbb{P}_1(\mathbb{C})$.) The flow of line bundles over $\tilde{S}$ is then seen to be induced by a flow of line bundles over T which, in turn, is linked explicitly to the Lax pair flow. If the Lax equation is of the form (2.1) the flow of line bundles over T corresponds to a flow of transition functions $e^{tP(z,\lambda^{-1})}$ over T.

a) *The Embedding.* To describe the embedding, consider $\mathbb{P}_1(\mathbb{C})$ with the standard coordinate charts (V_0, λ) and $(V_1, \lambda' = (1/\lambda))$. We consider the line bundle $\pi: T \to \mathbb{P}_1(\mathbb{C})$ which is the m^{th} power of the hyperplane line bundle over $\mathbb{P}_1(\mathbb{C})$. The transition function from V_0 to V_1 for T is $1/\lambda^m$, thus T can be covered by two coordinate patches $U_i = \pi^{-1}(V_i)$ with coordinates (λ, z) on U_0 and $(\lambda', z') = (1/\lambda, z/\lambda^m)$ on U_1.

The equation

$$0 = \det(L(\lambda) - zId) = (-1)^r z^r + a_1(\lambda) z^{r-1} + \cdots + a_r(\lambda) \tag{2.4}$$

embeds S_0 into U_0. From (2.2) we see that $a_i(\lambda)$ is a polynomial of degree im. Therefore, by switching to (λ', z') coordinates, Eq. (2.4) becomes (away from $\lambda = 0$)

$$0 = (-1)^r (z')^r + \tilde{a}_1(\lambda')(z')^{r-1} + \cdots + \tilde{a}_r(\lambda'), \tag{2.5}$$

where $\tilde{a}_i(\lambda')$ is a polynomial of degree im. From this it follows that S_0 extends to a (possibly singular) compact curve S in T. Via the map $\pi: T \to \mathbb{P}_1(\mathbb{C})$, S is an r-fold branched cover of $\mathbb{P}_1(\mathbb{C})$. The adjunction formula ([6], p. 146) gives the virtual genus of S to be

$$g = \tfrac{1}{2}(r-1)(rm-2). \tag{2.6}$$

When $L(\lambda)$ arises from a rank r perturbation, and hence has the form given by Eq. (1.13), the curve S has specific geometric features. In this case the degree m is either n, if $Y \neq 0$, or $n-1$, if $Y = 0$. Evaluating (1.13) at $\lambda = \alpha_i$ yields

$$L(\alpha_i) = -\left(\prod_{j \neq i} (\alpha_i - \alpha_j)\right) G_i^T F_i. \tag{2.7}$$

Hence, generically (i.e. on $\mathcal{M}^{\mathbf{k}} \subset M_{N,r} \times M_{N,r}$) $L(\alpha_i)$ has rank k_i. Thus the characteristic polynomial of $L(\alpha_i)$ vanishes *at least* to order $r - k_i$ at $z = 0$. (It may vanish to higher order if the Jordan form of $L(\alpha_i)$ is not diagonal.) It follows that the curve S has at least an $(r-k_i)$-fold intersection at the point $(\lambda, z)=(\alpha_i, 0)$. In the generic situation $L(\alpha_i)$ is diagonalizable and S has *exactly* an *ordinary* $(r-k_i)$-fold intersection at $(\alpha_i, 0)$.

One can now see geometrically that the flow (2.1) preserves the spectrum of an element $A+FG^T$ of $\mathscr{M}_A$. The spectrum $\{q_i\}$ of $A+FG^T$ is given by the λ-coordinates of the intersection of S with the rational curve $z=a(\lambda)$, away from $\lambda=\alpha_i$. This follows from the general formula

$$\det(A+FG^T-\lambda Id_N)=\det(A-\lambda Id_N)\det(Id_r-G^T(\lambda Id_N-A)^{-1}F). \tag{2.8}$$

Finally, note that the nonzero z-coordinates p_{ij}, $j=1,\dots,k_i$, of the points in S over $\lambda=\alpha_i$ are determined by

$$0=\det(zId_r-L(\alpha_i))=\left(\prod_{j=1}^{k_i}(z-p_{ij})\right)z^{r-k_i}. \tag{2.9}$$

Hence, by (2.7) the p_{ij}'s are given $\left(\text{up to constant factor } -\prod_{j\neq i}(\alpha_i-\alpha_j)\right)$ by the eigenvalues of the $k_i\times k_i$ diagonal block $F_iG_i^T$ of FG^T. These eigenvalues are therefore also constants of the motion.

b) *Line Bundles.* One can describe the line bundles on S using the embedding of S into T.

Proposition 2.1. *The line bundles of degree zero on S are all given by restrictions of line bundles on T with first Chern class $c_1=0$.*

Proof. Since T is simply connected it follows from the exponential exact sequence that the line bundles with $c_1=0$ on T are given by the cohomology group $H^1(T,\mathcal{O}_T)$, where $\mathcal{O}_T$ denotes the sheaf of holomorphic functions on T. Likewise the degree zero line bundles on S are given by $H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z})$. Thus it suffices to prove that $H^1(T,\mathcal{O}_T)$ surjects onto $H^1(S,\mathcal{O}_S)$.

Let $\mathcal{O}(i)$ denote the i^{th} power of the hyperplane bundle on $\mathbb{P}_1(\mathbb{C})$ and $\mathcal{O}_T(i)$ the pull back of $\mathcal{O}(i)$ to T via $\pi\colon T\to\mathbb{P}_1(\mathbb{C})$. By considering transition functions, it is easy to see that, for $i>0$, in the U_0 trivialization $H^0(T,\mathcal{O}_T(i))$ is generated by monomials $z^k\lambda^l$, $k\geqq 0, l\geqq 0$ and $km+l\leqq i$. From this and Eq. (2.4) it follows that S is the divisor of a section of $\mathcal{O}_T(rm)$, thus we have the exact sequence

$$0\to\mathcal{O}_T(-rm)\to\mathcal{O}_T\to\mathcal{O}_S\to 0, \tag{2.10}$$

where, by the standard abuse of notation, we have let $\mathcal{O}_T(i)$ denote both the line bundle and its sheaf of local sections.

From the long exact sequence for (2.10) it follows that $H^1(T,\mathcal{O}_T)$ surjects onto $H^1(S,\mathcal{O}_S)$ as long as $H^2(T,\mathcal{O}_T(-rm))=0$. But this is the case since T has a Leray open cover by the two open sets U_0, U_1. $\square$

To make use of Proposition 2.1 we need to study the line bundles on T. As mentioned in the above proof, the line bundles on T with $c_1=0$ are given by $H^1(T,\mathcal{O}_T)$. Since U_0, U_1 is a Leray cover of T we can compute $H^1(T,\mathcal{O}_T)$ by

$$H^1(T,\mathcal{O}_T)=H^0(U_0\cap U_1,\mathcal{O}_T)/(r_0(H^0(U_0,\mathcal{O}_T)\oplus r_1(H^0(U_1,\mathcal{O}_T)),$$

where r_i is the restriction map from U_i to $U_0\cap U_1$.

Proposition 2.2. *$H^1(T, \mathcal{O}_T)$ is generated by monomials of the form $z^i\lambda^j$, $i>0$, $-\mathrm{im}<j<0$. The corresponding line bundles have transition functions* $\exp(z^i\lambda^j)$ *from U_0 to U_1.*

From Propositions 2.1 and 2.2, along with the equation for S, we conclude

Corollary 2.3. *$H^1(S, \mathcal{O}_S)$ is generated by the monomials $z^i\lambda^j$, $0<i\leqq r-1$, $-\mathrm{im}<j<0$.*

c) *The Desingularization.* In subsection e) we describe how to pass from a linear flow of line bundles to a matricial polynomial satisfying a Lax pair equation. In order to arrive at a matricial polynomial with the required behavior at the α_i's we will have to consider line bundles on a partial desingularization of S.

Recall that we have assumed that S has an ordinary $(r-k_i)$-fold intersection at $(\alpha_i, 0)$, $i=1,\dots,n$. Desingularize S at these points to get a curve $\tilde{S}$ with a mapping $\psi: \tilde{S}\to S$ which is an isomorphism away from $\psi^{-1}((\alpha_i, 0))$ and has the property that $\psi^{-1}((\alpha_i,0))$ consists of $r-k_i$ distinct points at which $\tilde{S}$ is smooth. (Note: we allow that $\tilde{S}$ may have singularities elsewhere.) In part e) we shall see that line bundles on $\tilde{S}$ correspond to matricial polynomials with the desired behavior at $\lambda=\alpha_i$.

The (virtual) genus $\tilde{g}$ of $\tilde{S}$ is given by the formula (see [6] p. 505, or [19])

$$\begin{aligned}\tilde{g} &= g-\sum_{i=1}^{n}\frac{(r-k_i)(r-k_i-1)}{2}\\ &= \tfrac{1}{2}\left[N(2r-1)-\sum k_i^2+r(r-1)(m-n)-2r+2\right],\end{aligned} \tag{2.11}$$

where $m=n$ if $Y\neq 0$ and $m=n-1$ if $Y=0$.

The meromorphic functions λ and z on S lift via ψ to $\tilde{S}$. Likewise the line bundles on S lift to $\tilde{S}$. By standard results ([19]) $H^1(S, \mathcal{O}_S)$ surjects onto $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$, hence, using Proposition 2.1, we conclude that all line bundles of degree zero on $\tilde{S}$ are lifts of line bundles on T with $c_1=0$.

Recall that $\mathcal{O}_T(i)$ denotes the pullback to T of the i^{th} power of the hyperplane bundle on $\mathbb{P}_1(\mathbb{C})$. Pulling back by the maps $\tilde{S}\xrightarrow{\psi} S\xrightarrow{i} T$, we may consider these to be line bundles $\mathcal{O}_{\tilde{S}}(i)$ over $\tilde{S}$. Since $\tilde{S}$ is an r-fold branched cover of $\mathbb{P}_1(\mathbb{C})$, it follows that $\mathcal{O}_{\tilde{S}}(i)$ has degree ri. If E is a line bundle on $\tilde{S}$, we denote $E\otimes\mathcal{O}_{\tilde{S}}(i)$ by $E(i)$.

Remark. In the case that $Y\neq 0$, note that $\lambda^{-m}L(\lambda)$ equals Y at $\lambda=\infty$. We will assume that Y has r distinct eigenvalues, so the curve is nonsingular at $\lambda=\infty$. More general cases can be dealt with by appropriately desingularizing the curve at $\lambda=\infty$. (See Sect. 6.)

d) *From Matricial Polynomials to Line Bundles.* The first step in the integration of Lax pair equations of the form Eq. (2.1) consists of solving the "direct problem"; namely, given a matricial polynomial $L(\lambda)$, as defined e.g. by Eq. (1.13), construct a corresponding line bundle over S. The idea of this construction is by now quite standard ([3, 4, 5, 11, 12, 15, 17]). The modification that we make is in principle quite simple, and consists essentially in taking the dual approach. However, in the cases when one has degenerate spectrum (e.g. when one has singular curves) it provides a more transparent treatment of the different Jordan canonical forms which can occur.

We define a sheaf $\bar{E}$ over T by the exact sequence:

$$0 \to \mathcal{O}_T(-m)^{\oplus r} \xrightarrow{z - L(\lambda)} \mathcal{O}_T^{\oplus r} \to \bar{E} \to 0. \tag{2.12}$$

This sheaf is supported over S. When S is reduced and when the kernel of L is everywhere one dimensional over S, $\bar{E}$ is a line bundle of degree $g-1+r$ (see e.g. [17]). This is the case when S is smooth. At points where the kernel of $(z-L(\lambda))$ is of dimension greater than one (which are of necessity singular points of S), $\bar{E}$ may not be a line bundle. In fact, in the generic case for our problem, in which there is an $(r-k_i)$ – fold ordinary point at $\lambda = \alpha_i$ with $\dim(\ker(L(\alpha_i)) = r - k_i$, $\bar{E}$ is just the pushdown $\psi_*(E)$ of a line bundle of degree $\tilde{g}-1+r$ on $\tilde{S}$. Note that by (2.12),

$$\mathbb{C}^r \approx H^0(S, \mathcal{O}^{\oplus r}) = H^0(S, \bar{E}) \cong H^0(\tilde{S}, E).$$

e) *From Line Bundles to Matricial Polynomials.* We now turn to the "inverse" problem; namely given a line bundle E over $\tilde{S}$ with degree $\tilde{g}-1+r$, to construct the associated matricial polynomial $L(\lambda)$. The line bundle $E(-1)$ has degree $\tilde{g}-1$ and hence, generically, has no sections. Throughout this paper we assume the line bundles are generic in this sense, i.e.

$$H^0(\tilde{S}, (E(-1)) = 0. \tag{2.13}$$

In fact our solutions will have poles where this assumption fails. Assuming (2.13) it follows (by Riemann–Roch) that $H^0(\tilde{S}, E)$ has dimension r. We will make use of the following specific isomorphism of $H^0(\tilde{S}, E)$ with $\mathbb{C}^r$.

For $\lambda_0 \in \mathbb{C}$ let D_{λ_0} denote the zero divisor of the meromorphic function $(\lambda - \lambda_0)$ on $\tilde{S}$, and let $D_{\lambda_0}^{(j)}$ be the j^{th} formal neighborhood of D_{λ_0} (cut out by $(\lambda-\lambda_0)^{j+1}$). Since $\tilde{S}$ is an r-fold cover D_{λ_0} is a set of r points in $\tilde{S}$, counting multiplicity.

Proposition 2.4. *Via the restriction map one has the isomorphism*

$$H^0(\tilde{S}, E) \cong H^0(D_{\lambda_0}, E) \cong \mathbb{C}^r.$$

More generally

$$H^0(\tilde{S}, E(j)) \cong H^0(D_{\lambda_0}^{(j)}, E(j)) \cong \mathbb{C}^{r(j+1)}.$$

Proof. One has over $\tilde{S}$ the exact sequence of sheaves

$$0 \to \mathcal{O}_{\tilde{S}}(-1) \to \mathcal{O}_{\tilde{S}}(j) \to \mathcal{O}_{D_{\lambda_0}^{(j)}}(j) \to 0.$$

Tensoring with E and taking the long exact sequence yields the result as long as $H^1(\tilde{S}, E(-1)) = 0$. But this follows from Riemann–Roch and the assumption $H^0(\tilde{S}, E(-1)) = 0$. The dimensions of the spaces follows from the fact that the $D_{\lambda_0}^{(j)}$'s are sets of points with multiplicity. $\square$

Now recalling that the z coordinate corresponds to the eigenvalue of the matricial polynomial we define a linear map

$$\mathbf{L}(\lambda): H^0(\tilde{S}, E) \to H^0(\tilde{S}, E)$$

by the commuting diagram

$$\begin{array}{ccc} H^0(\tilde{S}, E) & \longrightarrow & H^0(D_\lambda, E) \\ {\scriptstyle \mathbf{L}(\lambda)} \downarrow & & \downarrow {\scriptstyle Z} \\ H^0(\tilde{S}, E) & \longrightarrow & H^0(D_\lambda, E) \end{array} \tag{2.14}$$

where Z denotes the map $H^0(D_\lambda, E) \to H^0(D_\lambda, E)$ given by multiplication by z. If $s \in H^0(D_\lambda, E)$ and p is a point of D_λ with multiplicity one then $(Zs)(p) = z(p)s(p)$. At points p of higher multiplicity the section s is given by a truncated power series in $(z - z(p))$, and Z gives the truncated series of zs.

To get the matrix $L(\lambda)$ corresponding to the linear transformation $\mathbf{L}(\lambda)$ we choose a basis of $H^0(\tilde{S}, E)$. Fixing a basis for all λ determines $L(\lambda)$ and a change of basis changes $L(\lambda)$ by conjugation with an element of $GL(r, \mathbb{C})$. If D_λ consists of r distinct points $D_\lambda = \{p_i; i = 1, \ldots, r\}$, then $\mathbf{L}(\lambda)$ has the eigenvalues $z(p_i)$ with a corresponding eigenvector given by the section of E which vanishes at $p_j, j \neq i$. This section naturally corresponds to the fiber of the dual bundle E^* at p_i. Thus $L(\lambda)$ is diagonalizable with eigenvalues $z(p_i)$. In particular, since we have desingularized at α_i, $L(\alpha_i)$ is diagonalizable, with rank k_i. If D_λ does not consist of distinct points then $\mathbf{L}(\lambda)$ is determined by multiplication by z on truncated power series in z. This gives a nilpotent part so that the Jordan form of $L(\lambda)$ will not be diagonal. *This fact explains why we have desingularized S at the α_i's.* If we want to study more general cases in which $L(\alpha_i)$ has nondiagonal Jordan form we should desingularize S only the appropriate amount.

We still need to check that $L(\lambda)$ is a matricial polynomial.

Proposition 2.5. $\mathbf{L}(\lambda) = \sum_{i=0}^{m} \mathbf{L}_i \lambda^i$, *where* $\mathbf{L}_i : H^0(\tilde{S}, E) \to H^0(\tilde{S}, E)$ *is linear.*

Proof. On U_0 the space $H^0(\tilde{S}, \mathcal{O}(m))$ is generated by the $m + 2$ sections $1, \lambda, \ldots, \lambda^m, z$. Hence, if we consider the Taylor expansion $\mathbf{L}(\lambda) = \sum_{i=0}^{m} \mathbf{L}_i \lambda^i + R(\lambda)$, where $R(\lambda)$ has order λ^{m+1}, we see that $\sum_{i=0}^{m} \mathbf{L}_i \lambda^i$ defines a map $H^0(\tilde{S}, E) \to H^0(\tilde{S}, E(m))$. Likewise, multiplication by z defines a map $H^0(\tilde{S}, E) \to H^0(\tilde{S}, E(m))$. Composing these maps with the restriction to $D_0^{(m)}$ gives two maps $H^0(\tilde{S}, E) \to H^0(D_0^{(m)}, E(m))$. By definition, these two maps coincide. Applying Proposition 2.4, multiplication by z and $\sum_{i=0}^{m} \mathbf{L}_i \lambda^i$ coincide over all of $\tilde{S}$. $\square$

Thus from a line bundle E of degree $\tilde{g} - 1 + r$ we have constructed a matricial polynomial $L(\lambda)$ with the desired Jordan form at $\lambda = \alpha_i$. Comparing (2.14) and (2.12) one sees that (2.12) with this definition gives back $\psi_*(E)$. Because the construction depends on a choice of basis, $L(\lambda)$ is determined up to conjugation by an element of $GL(r, \mathbb{C})$. Recall however, that if we desire $L(\lambda)$ to come from a shifted isospectral flow, then the value at $\lambda = \infty$ of $L(\lambda)$ is fixed (i.e. $L(\infty) = Y$). In this case, the only allowed basis changes are those which leave Y fixed, i.e. $L(\lambda)$ is determined up to conjugation by $GL(r, \mathbb{C})_a$.

f) *From Linear Flows of Line Bundles to Lax Pairs.* Finally, we need to see how a linear flow of line bundles E_t of degree $\tilde{g} - 1 + r$ gives rise to a Lax equation of matricial polynomials. The bundle E_0 can be trivialized with respect to the covering U_0, U_1 with a transition function $f(z,\lambda)$ from U_0 to U_1. Also, $E_0^* \otimes E_t$ has degree zero so, by Corollary 2.3, its transition function from U_0 to U_1 can be given by the exponential of a polynomial in z and λ^{-1}. Since the flow is linear the transition function must be of the form $\exp(t\mu(z,\lambda))$, $\mu(z,\lambda)$ a polynomial in z and λ^{-1}. Putting this together, E_t has transition functions $f(z,\lambda)\exp(t\mu(z,\lambda))$ which we henceforth use to fix trivializations of E_t.

From E_t, we obtain a family $V_t = H^0(\tilde{S}, E_t)$ of vector spaces, which we will think of as a vector bundle V over $\mathbb{C}$, and a family of maps $\mathbf{L}(\lambda; t)$. For fixed t we can get a matrix $L(\lambda; t)$ for the map $\mathbf{L}(\lambda; t)$ by choosing a basis for V_t. However, without some canonical choice of basis for V_t, there is no hope that $L(\lambda; t)$ so chosen will satisfy a Lax equation. We therefore define a connection $\nabla_t: H^0(\mathbb{C}, V) \to H^0(\mathbb{C}, V)$ which, by parallel translation, smoothly extends a choice of basis at V_0 to all of V.

Let $ev_i: H^0(U_i \cap \tilde{S}, E_t) \to \mathcal{O}_{\tilde{S} \cap U_i}$ be maps evaluating sections with respect to the trivializations of E_t over U_i. Set

$$P(\mathbf{L})(\lambda) = \mu(\mathbf{L}(\lambda), \lambda), \tag{2.15}$$

then, if $\psi \in H^0(\tilde{S}, E_t)$ one has

$$ev_0(\mathbf{L}\psi) = z\, ev_0(\psi), \tag{2.16a}$$

$$ev_0(P(\mathbf{L})\psi) = \mu(z,\lambda)\, ev_0(\psi), \tag{2.16b}$$

and also the relation linking the two trivializations:

$$ev_0(\psi) = f^{-1}(z,\lambda) e^{-t\mu(z,\lambda)} ev_1(\psi). \tag{2.17}$$

Differentiating (2.17), then using (2.16b), we obtain

$$\partial_t(ev_0\psi) + ev_0(P(\mathbf{L})_+\psi) = f^{-1}e^{-t\mu}(\partial_t(ev_1\psi) - ev_1(P(\mathbf{L})_-\psi)), \tag{2.18}$$

where $P(\mathbf{L}) = P(\mathbf{L})_+ + P(\mathbf{L})_-$ and $P(\mathbf{L})_+$ is the polynomial part of $P(\mathbf{L})$. Therefore $(\partial_t ev_0 + ev_0 P(\mathbf{L})_+)\psi$, $(\partial_t ev_1 - ev_1 P(\mathbf{L})_-)\psi$ are the evaluation of a section of $H^0(\tilde{S}, E_t)$ over U_0, U_1 respectively; we use this to define a connection ∇_t acting on sections of V; it is given over U_0 by $ev_0^{-1}(\partial_t ev_0 + ev_0 P(\mathbf{L})_+)$.

Remark. As $P(\mathbf{L})_+$ is of the form $P(\mathbf{L})_0 + \lambda P(\mathbf{L})_1 + \lambda^2 P(\mathbf{L})_2 + \cdots$, Eq. (2.21) implies that ∇_t just corresponds to taking derivatives over $\lambda = \infty$. Thus we have

Proposition 2.6. *A basis e_i of $H^0(\tilde{S}, E_t)$ satisfying $\nabla_t e_i = 0$ is given by choosing a basis with constant values at the r points over $\lambda = \infty$.*

Now assume that one has a basis of sections (trivialization) $e_i = e_i(t)$ of V such that $\nabla_t e_i = 0$. Let $\psi = \psi^i e_i$ be any section of V over $\mathbb{C}$; let $\partial_t \psi$ denote "naive" differentiation: $\partial_t \psi = (\partial_t \psi^i) e_i$. Since $\nabla_t e_i = 0$, $\nabla_t \psi = \partial_t \psi$. Let $L = L(\lambda, t)$ be the matrix of $\mathbf{L}$ with respect to this basis. Since $L\psi$ gives the components of a vector in the e_i basis we again have $\nabla_t(L\psi) = \partial_t(L\psi)$.

Theorem 2.7. *$L(\lambda, t)$ satisfies the Lax equation $\partial_t L = [P(L)_+, L]$.*

Proof. Let ψ be a section of V; then

$$ev_0(\nabla_t \psi) = \partial_t ev_0(\psi) + ev_0(P(\mathbf{L})_+\psi) = ev_0(\partial_t \psi). \tag{2.19}$$

Applying (2.19), (2.16a) to ψ and $L\psi$ and combining, we get

$$-ev_0(P(\mathbf{L})_+L\psi)+ev_0((\partial_t L)\psi+L(\partial_t\psi))=-zev_0(P(L)_+\psi)+zev_0(\partial_t\psi); \tag{2.20}$$

applying (2.16a) to $\partial_t\psi$ and $P(L)_+\psi$ instead of ψ, (2.20) becomes

$$ev_0((\partial_t L-[P(L)_+,L])\psi)=0. \tag{2.21}$$

Since ev_0 is injective and the above is true for all ψ, the theorem is proved. □

g) *From Line Bundles to Rank r Perturbations.* We can recover the $N\times r$ matrices F and G (modulo the reduction by H) directly from the line bundle E on $\tilde{S}$.

Since the curve $\tilde{S}$ is the desingularization of S over the points $\lambda=\alpha_i$, we can write

$$\tilde{D}_{\alpha_i}=x_{i,1}+\cdots+x_{i,r}, \tag{2.22}$$

where for $j>k_i$, $z(x_{i,j})=0$ and for $j=1,\dots,k_i$, $z(x_{i,j})=p_{ij}$, the eigenvalues of the $k_i\times k_i$ block $F_iG_i^T$.

Choose sections $s_{ij}\in H^0(\tilde{S},E)$ such that, in the U_0 trivialization,

$$\begin{aligned} s_{ij}(x_{i,k})&=0 \quad\text{for}\quad k\neq j,\\ s_{ij}(x_{i,j})&=-p_{ij}\Big/\Big(\prod_{l\neq i}(\alpha_i-\alpha_l)\Big), \end{aligned} \tag{2.23}$$

and define maps

$$\mathbf{F}_i: H^0(\tilde{S},E)\to\mathbb{C}^{k_i} \tag{2.24}$$

by

$$s\to(s(x_{i,1}),\dots,s(x_{i,k_i}))$$

and

$$\mathbf{G}_i^T:\mathbb{C}^{k_i}\to H^0(\tilde{S},E) \tag{2.25}$$

by

$$(r_1,\dots,r_{k_i})\to\sum_{j=1}^{k_i} r_j s_{i,j}.$$

Choosing a basis of $H^0(\tilde{S},E)$ to determine the matricial polynomial $L(\lambda)$ from the map $\mathbf{L}(\lambda): H^0(\tilde{S},E)\to H^0(\tilde{S},E)$ also yields matrices F_i and G_i^T for the maps $\mathbf{F}_i$ and $\mathbf{G}_i^T$. Using F_i and G_i as the $k_i\times r$ blocks of $N\times r$ matrices F and G it is then straightforward to check that

$$\frac{a(\lambda)}{\lambda}(G^T(\lambda I-A)^{-1}F+\lambda Y)$$

coincides with $L(\lambda)$ at $\lambda=\alpha_i$, and hence at all λ.

Remark. By fixing the s_{ij}'s we have fixed the value of the H moment map, namely $F_iG_i^T$. The reduction by H still leaves an ambiguity in F_i and G_i up to an action of the stabilizer group of $F_iG_i^T$ in $GL(k_i,\mathbb{C})$. We have, in effect divided out this ambiguity by our choice of definitions (2.24) and (2.25).

3. Integrability

In Sect. 1 we constructed a ring of functions $\mathscr{F}^Y$ on $\mathscr{M}$ which Poisson commute, produce isospectral Hamiltonian flows through $A+FaG^T$, and are invariant under the action of $H \times SL(r, \mathbb{C})_a$. We now show that these functions are completely integrable on generic leaves of the Poisson manifold $\mathcal{M}/(H \times SL(r, \mathbb{C})_a)$.

Since the symplectic leaf Γ through $N(\lambda) = \tilde{J}_r(F, G)$ in $\widetilde{gl}(r)_0^-$ is identified with the symplectic leaf in $\mathcal{M}/H$ through the H orbit $[(F, G)]$ of (F, G), it follows that in order to prove complete integrability of the ring $\mathcal{F}^Y$ on symplectic leaves in $\mathcal{M}/(H \times SL(r, \mathbb{C})_a)$ it suffices to prove complete integrability of the ring of functions $\mathcal{F}_+^Y$ on the symplectic leaves of $\Gamma/SL(r, \mathbb{C})_a$.

In Sect. 2 we have seen that there is a one-to-one correspondence between flows of matricial polynomials (modulo $SL(r, \mathbb{C})_a$) of the form (2.1) and linear flows in the Jacobian of the desingularized spectral curve $\tilde{S}$. To relate the flows of matricial polynomials to rank r isospectral perturbations we use the map given in Eq. (1.13). A slight variation of the argument in [1], Proposition 4.9, shows that the map

$$\widetilde{gl}(r)_0^- \to \widetilde{gl}(r)^+$$

given by

$$N(\lambda) \to \lambda^{-1} a(\lambda)(N(\lambda) + \lambda Y) \tag{3.1}$$

gives a correspondence between the flows of type (2.1) in $\widetilde{gl}(r)^+$ and Hamiltonian flows of shifted AKS type (1.12) through $N(\lambda) = \tilde{J}_r(F, G)$ in $\widetilde{gl}(r)_0^-$. Indeed consider $N(\lambda) = \tilde{J}_r(F, G) \in \widetilde{gl}(r)_0^-$ and let

$$L(\lambda) = \lambda^{-1} a(\lambda)(N(\lambda) + \lambda Y) \in \widetilde{gl}(r)^+ . \tag{3.2}$$

Proposition 3.1. *Let $L(\lambda; t)$ denote the flow through $L(\lambda)$ given by a Lax equation of the form* (2.1), i.e.

$$\frac{d}{dt} L(\lambda; t) = [(P(L(\lambda; t), \lambda^{-1}))_+, L(\lambda; t)] \tag{3.3}$$

and let $N(\lambda; t) = \dfrac{\lambda}{a(\lambda)} L(\lambda; t) - \lambda Y$. Then there is a $\phi \in \mathcal{F}_+^Y$ such that $N(\lambda; t)$ describes the hamiltonian flow for ϕ on $\widetilde{gl}(r)_0^- \sim (\widetilde{gl}(r)^+)^*$ through the point $N(\lambda)$.

Proof. It is enough to prove the proposition in the case that $P(L(\lambda; t), \lambda^{-1}) = \lambda^{-j}(L(\lambda; t))^k$. Multiplying (3.3) by $\lambda/a(\lambda)$ we get

$$\frac{d}{dt}(N(\lambda; t) + \lambda Y) = [(\lambda^{-j}(L(\lambda; t))^k)_+, N(\lambda; t) + \lambda Y]$$

or

$$\frac{d}{dt} N(\lambda; t) = \left[\left(\lambda^{-j} \left(\frac{1}{\lambda} a(\lambda)(N(\lambda; t) + \lambda Y) \right)^k \right)_+ , N(\lambda; t) + \lambda Y \right].$$

Thus by (1.12), the proposition is true if we can find $\phi \in I(\widetilde{gl}(r)^*)$ such that

$$d\phi(X(\lambda)) = \lambda^{-j-k}(a(\lambda))^k X(\lambda)^k$$

for $X(\lambda) \in \widetilde{gl}(r) \sim \widetilde{gl}(r)^*$. Since $\phi^{m,n} \in I(\widetilde{gl}(r)^*)$ given by

$$\phi^{m,n}(X(\lambda)) = \operatorname{tr}(\lambda^m X(\lambda)^n)_0$$

has derivative $\dfrac{1}{n} \lambda^m X(\lambda)^{n-1}$, it follows that ϕ is given by a finite linear combination of the $\phi^{m,n}$'s. $\square$

From Proposition 3.1 it follows that the flows of type 2.1 from Chap. 2 give a family of $\tilde{g} = \frac{1}{2}[N(2r-1) - \sum k_i^2 + r(r-1)(m-n) - 2r + 2]$ independent hamiltonian flows on $\Gamma/SL(r, \mathbb{C})_a$, ($N = \sum k_i$, and $m = n$ if $Y \neq 0$, $m = n-1$ if $Y = 0$). To prove complete integrability we need only show that the generic leaf of $\Gamma/SL(r, \mathbb{C})_a$ has twice this dimension, i.e.

$$N(2r-1) - \sum k_i^2 + r(r-1)(m-n) - 2r + 2. \tag{3.4}$$

Proposition 4.3 of [1] yields the fact that Γ is given by

$$\Gamma = \left\{ \sum_{i=1}^{n} \lambda \frac{g_i \mu_i g_i^{-1}}{\alpha_i - \lambda}, g_i \in GL(r, \mathbb{C}) \right\}, \tag{3.5}$$

where $\mu_i = F_i G_i^T$ is a fixed matrix of rank k_i. Let us assume that μ_i is diagonalizable with k_i *distinct* nonzero eigenvalues. From this it follows that the $GL(r, \mathbb{C})$ orbit through μ_i has dimensions

$$(r^2 - r) - [(r - k_i)^2 - (r - k_i)] = 2k_i r - k_i - k_i^2.$$

Summing over i, we get

$$\dim \Gamma = 2Nr - N - \sum k_i^2. \tag{3.6}$$

Finally, for the reduction by the $SL(r, \mathbb{C})_a$ action we discuss two important cases:

i) $Y = 0$, $a = I$.
ii) a is diagonalizable with r distinct eigenvalues.

For case i) we compute the dimension of the generic symplectic leaf in $\Gamma/SL(r, \mathbb{C})$ by $\dim \Gamma - \dim(SL(r, \mathbb{C})) - \operatorname{rank}(SL(r, \mathbb{C}))$, i.e. $N(2r-1) - \sum k_i^2 - (r^2 - 1) - (r-1)$. This agrees with 3.4 when $m = n - 1$.

For case ii) $SL(r, \mathbb{C})_a$ is simply an $(r-1)$ dimensional abelian subgroup of $SL(r, \mathbb{C})$. Thus the dimension of the generic symplectic leaf in $\Gamma/SL(r, \mathbb{C})_a$ in this case is $\dim \Gamma - 2(r-1) = N(2r-1) - \sum k_i^2 - 2r + 2$ which agrees with (3.4) with $m = n$.

Remarks. 1) More general μ_i's can be considered. For instance we may allow repetition of nonzero eigenvalues and also nontrivial Jordan block structure. In this case the curve must be desingularized the appropriate amount so that $L(\alpha_i)$ has the correct Jordan structure. The genus of the desingularized curve can then be compared to the dimensions of the symplectic leaves of $\Gamma/SL(r, \mathbb{C})$ to study integrability.

2) Similarly, more general a's may be considered. Here also, one must desingularize the curve, at $\lambda = \infty$, so that $L(\infty) = Y$ has the correct Jordan form. (It is easy to check that when a is diagonalizable the dimension count still yields complete integrability.)

4. Theta Functions

In Sect. 2 we discussed the relation between linear flows of line bundles on $\tilde{S}$ and Lax pair flows of matricial polynomials given by Eq. (2.1). In this section we use this relation to construct explicitly the flow of matricial polynomials satisfying

(2.1) in terms of theta functions on the Jacobi variety of $\tilde{S}$. As in Sect. 2 we do this in two steps; first for a fixed line bundle on $\tilde{S}$ we describe the corresponding matricial polynomial in terms of theta functions, then, using the connection of Sect. 2f, we describe the dynamics in terms of generalized Baker–Akhiezer functions. At the end of the chapter we also write the coefficients of the matrix FG^T in terms of theta functions. Throughout this section we assume that $\tilde{S}$ is smooth so that we may make use of the standard theta function theory.

a) We start by making more explicit the link between line bundles E on $\tilde{S}$, positive divisors, and matricial polynomials $L(\lambda)$. In Sect. 2d we saw that E was defined by the exact sequence (2.12). A positive divisor Δ representing E (i.e. the zeroes of a section of E) could be obtained by fixing $v \in \mathbb{C}^r \approx H^0(S, \mathcal{O}^{\oplus r})$ and setting Δ to be the sum of points p in $\tilde{S}$ such that $v \in \operatorname{Im}(z - L(\lambda))$ at p. From the relation $(z - L(\lambda))_{\mathrm{adj}}(z - L(\lambda)) = \det(z - L(\lambda)) \cdot \mathrm{Id}$ (where $(z - L(\lambda))_{\mathrm{adj}}$ represents the classical adjoint matrix, i.e. the transposed matrix of cofactors) one sees that such points are given over S by the condition $(z - L(\lambda))_{\mathrm{adj}} v = 0$ away from the points in S where $\operatorname{corank}(z - L(\lambda)) \geqq 2$, i.e. the points where $(z - L(\lambda))_{\mathrm{adj}} = 0$.

Conversely, given a line bundle E, the discussion in Sect. 2 says that the matrix $L(\lambda)$ can be found as follows: Choose a basis $\psi^1, \ldots, \psi^r$ of $H^0(\tilde{S}, E)$. Choose $\lambda_0 \in \mathbb{C}$ over which lies exactly r distinct points in $\tilde{S}$, and let $p_1, \ldots, p_r$ be an ordering of these points $(\lambda(p_j) = \lambda_0)$. Let $\psi(\lambda_0)$ be the $r \times r$ matrix given by

$$(\psi(\lambda_0))_{ij} = \psi^j(p_i), \tag{4.1}$$

then

$$L(\lambda_0) = \psi(\lambda_0)^{-1} \begin{bmatrix} z(p_1) & & \\ & \ddots & \\ & & z(p_r) \end{bmatrix} \psi(\lambda_0). \tag{4.2}$$

For λ in a neighborhood of λ_0 there are r distinct points in $\tilde{S}$ mapping to λ. These points may be ordered continuously in that neighborhood, so a choice of ordering at λ_0 gives $L(\lambda)$ on a neighborhood of λ_0. But, since $L(\lambda)$ is a polynomial, its description on a neighborhood defines it globally.

To compute $L(\lambda)$ we need an explicit representation of sections of E, and an explicit choice of the basis ψ^i, $i = 1, \ldots, r$ of $H^0(\tilde{S}, E)$. We choose the following representation of sections of E. Let E_0 be an "initial value" line bundle of the appropriate degree $(g + r - 1)$. Let Δ be a positive divisor representing E_0. (For instance, Δ may be determined by the above prescription.) Sections of E_0 can be represented as meromorphic functions ψ on $\tilde{S}$ with poles allowed at Δ, i.e.

$$(\psi) \geqq -\Delta. \tag{4.3}$$

Other line bundles E_t of the same degree can be written as $E_0 \otimes F_t$, where $F_t = E_0^* \otimes E_t$ is of degree zero. F_t will be given the "exponential" transition function considered in Sect. 2, and sections of E_t will be meromorphic sections of F_t whose divisor D is greater than or equal to $-\Delta$.

We next consider a basis of $H^0(\tilde{S}, E_0)$. By Proposition 2.6, the dynamical problem will require that the ψ^i's be constant at $\lambda = \infty$, and so we choose the ψ^i's by fixing their values at ∞. Let $D_\infty = P_\infty^1 + \cdots + P_\infty^r$ be an ordering of the r points in D_∞. It follows from the genericity assumption (2.13) that

$$\dim(H^0(\tilde{S}, [\Delta - D_\infty + P_\infty^i])) = 1 \quad \forall i. \tag{4.4}$$

Here $[\Delta - D_\infty + P^i_\infty]$ denotes the line bundle represented by the divisor $\Delta - D_\infty + P^i_\infty$. Global sections of this bundle may be represented as meromorphic functions on $\tilde{S}$ with zeros at $D_\infty - P^i_\infty$ and poles allowed at Δ. By (4.4) such functions also represent sections of E_0. Equation (4.4) says that for each i, there is a meromorphic function ψ^i on $\tilde{S}$ such that

$$(\psi^i) \geqq -\Delta + D_\infty - P^i_\infty. \tag{4.5}$$

Since $\Delta - D_\infty$ is a divisor representing $E_0(-1)$ it follows from (2.13) that $\psi^i(P^i_\infty) \neq 0$, hence we may normalize by

$$\psi^i(P^i_\infty) = 1. \tag{4.6}$$

Equations (4.5) and (4.6) determine ψ^i uniquely, $i = 1, \ldots, r$, and these functions clearly give a basis for $H^0(\tilde{S}, E_0)$.

The final step to writing $L(\lambda)$ in terms of θ-functions is to write the ψ^i's in terms of θ-functions (see [14]).

Fix a basis of a-cycles and b-cycles and let $w_1, \ldots, w_{\tilde{g}}$ be a basis of holomorphic 1-forms dual to the a-cycles of that homology basis. Define $A: \tilde{S} \times \tilde{S} \to \mathrm{Jac}(\tilde{S})$ by

$$A(x, y) = \left(\int_x^y w_1, \ldots, \int_x^y w_{\tilde{g}} \right) \tag{4.7}$$

and let θ be the theta-function defined by the b-period matrix of $w_1, \ldots, w_{\tilde{g}}$.

Now $(\psi^i) \geqq -\Delta + D_\infty - P^i_\infty$, thus $(\psi^i) + \Delta - D_\infty + P^i_\infty$ is effective (and has degree $\tilde{g}$). Thus

$$(\psi^i) = Q_i^1 + \cdots + Q_i^{\tilde{g}} + D_\infty - P^i_\infty - \Delta, \tag{4.8}$$

i.e. (ψ^i) has polar divisor Δ and zero divisor $Q_i^1 + \cdots + Q_i^{\tilde{g}} + D_\infty - P^i_\infty$.

Let Δ be a sum of points $\Delta^1 + \cdots + \Delta^{\tilde{g}+r-1}$. One can find $e \in \mathbb{C}^{\tilde{g}}$ such that

$$\begin{gathered} \theta(e) = 0, \quad \theta(e + A(Q_i^j, y)) \not\equiv 0 \quad \forall j, \\ \theta(e + A(P^j_\infty, y)) \not\equiv 0 \quad \forall j, \quad \theta(e + A(\Delta^j, y)) \not\equiv 0 \quad \forall j. \end{gathered} \tag{4.9}$$

Fix $x \in \tilde{S}$ and set

$$\alpha' = \sum_{j=1}^{\tilde{g}} A(x, \Delta^j), \quad \alpha'' = \sum_{j=1}^{r-1} A(x, \Delta^{\tilde{g}+j}), \tag{4.10}$$

$$\beta = \sum_{j=1}^{r} A(x, P^j_\infty), \quad \beta_i = A(x, P^i_\infty), \tag{4.11}$$

$$\gamma_i = \sum_{j=1}^{\tilde{g}} A(x, Q_i^j). \tag{4.12}$$

Abel's theorem says that modulo periods

$$\gamma_i \equiv \alpha' + \alpha'' - \beta + \beta_i, \tag{4.13}$$

since $Q_i^1 + \cdots + Q_i^{\tilde{g}} + D_\infty - P^i_\infty - \Delta$ is the divisor of a meromorphic function. Now, if $x_1, \ldots, x_{\tilde{g}}$ are points on $\tilde{S}$ and δ denotes the Riemann constant, the function

$$y \to \theta\left(A(x, y) + \delta - \sum_{j=1}^{\tilde{g}} A(x, x_j) \right) \tag{4.14}$$

has zeros exactly at $y = x_j$, $j = 1, \ldots, \tilde{g}$, (see e.g. [6]). In particular the functions

$$y \to \theta(A(x, y) + \delta - \gamma_i), \tag{4.15}$$

$$y \to \theta(A(x, y) + \delta - \alpha') \tag{4.15b}$$

have zeros at the Q_i^j's and the Δ^j's, $j = 1, \ldots, \tilde{g}$, respectively. Using (4.14) it follows that the function

$$y \to \theta(A(x, y) + \delta - \alpha' - \alpha'' + \beta - \beta_i) \tag{4.16}$$

also vanishes exactly at the Q_i^j's, $j = 1, \ldots, \tilde{g}$. Hence, a single valued meromorphic function with the correct poles and zeros is given by

$$\tilde{F}^i(y) = \frac{\theta(A(x, y) + \delta - \alpha' - \alpha'' + \beta - \beta_i) \prod\limits_{j \neq i} \theta(A(P_\infty^j, y) + e)}{\theta(A(x, y) + \delta - \alpha') \prod\limits_{j=1}^{r-1} \theta(A(\Delta^{\tilde{g}+j}, y) + e)}, \tag{4.17}$$

and so

$$\psi^i(y) = \frac{\tilde{F}^i(y)}{\tilde{F}^i(P_\infty^i)}. \tag{4.18}$$

b) *The Dynamics.* We now consider a linear flow of line bundles E_t on $\tilde{S}$ with initial value E_0 as above. To produce the flow of matricial polynomials $L(\lambda; t)$ we must choose r independent sections

$$\psi_t^i \in H^0(\tilde{S}, E_t) \tag{4.19}$$

for each t. By Proposition 2.6 and Theorem 2.7, if $L(\lambda; t)$ is to satisfy a Lax pair equation, the ψ_t^i's must be chosen to be constant on D_∞. If we simply use Eq. (4.17) and (4.18) to describe ψ_t^i, the time dependence appears extremely complicated since it enters through the time dependence of Δ. However the time dependence of the ψ_t^i's can be made more transparent by considering the time dependence of the transition functions for the line bundles. (See e.g. [14, 16]).

Let F_t denote the degree zero line bundle on $\tilde{S}$ given by $F_t = E_0^* \otimes E_t$. Thus ψ_t^i is a section of $E_0 \otimes F_t$. Now, as we saw above, the section ψ^i of E_0 can be represented by a meromorphic function with zeros at $D_\infty - P_\infty^i$ and poles permitted at Δ. Since F_t is defined by exponential transition functions between U_0 and U_1 (see Sect. 2b), its sections in the U_1 trivialization are functions with an exponential singularity at D_0. Thus, the section ψ_t^i of $E_0 \otimes F_t$ may be represented as a function with zeros at $D_\infty - P_\infty^i$, poles permitted at Δ, and an exponential singularity at D_0, i.e. ψ_t^i is represented by a *generalized Baker–Akhiezer function.*

To produce ψ_t^i explicitly, let the transition function for F_t from U_0 to U_1 be given by $\exp t\mu(z, \lambda)$, where $\mu(z, \lambda)$ is a polynomial in z and λ^{-1}. Let η be the unique differential of the second kind on $\tilde{S}$ which is holomorphic away from D_0, has the same polar part at D_0 as $d\mu$, and is normalized by having zero a-cycle integrals. Let $U \in \mathbb{C}^{\tilde{g}}$ be given by

$$2\pi\sqrt{-1}\,U = \left(\int_{b_1} \eta, \ldots, \int_{b_{\tilde{g}}} \eta \right). \tag{4.20}$$

Fix $x \in \tilde{S}$ and set

$$h_t^i(y) = \exp\left(t \int_x^y \eta\right) \frac{\theta(A(x,y) + tU + \delta - \gamma_i)}{\theta(A(x,y) + \delta - \gamma_i)}, \tag{4.21}$$

where γ_i is given by (4.12). From (4.14) and the transformation properties of θ functions, we have that $h_t^i(y)$ is a single valued function on $\tilde{S}$ with poles at the Q_i^j's, defined by (4.8), and with the appropriate exponential singularity at D_0. Thus

$$\psi_t^i(y) = \frac{h_t^i(y)\tilde{F}^i(y)}{h_t^i(P_\infty^i)\tilde{F}_i(P_\infty^i)} \tag{4.22}$$

has the desired properties.

c) *The Matrix FG^T*. We now turn to the problem of presenting the coefficients of FG^T in terms of theta functions. Recall from Sect. 2g that the ij coefficient $(F_m G_k^T)_{ij}$ of the mk block $F_m G_k^T$ of FG^T is given by:

$$(F_m G_k^T)_{ij} = s_{ki}(x_{mj}). \tag{4.23}$$

Thus, in order to present the coefficients $(F_m G_k^T)_{ij}$ in terms of theta functions, it suffices to describe the sections s_{ij} in terms of theta functions. As above, s_{ij} may be represented by a meromorphic function on $\tilde{S}$ which vanishes on $\tilde{D}_{\alpha_i} - x_{ij}$ and is allowed to have poles on Δ. Let α', α'' be as in (4.10) and define ζ_{ij} and ζ_i by

$$\zeta_{ij} = A(x, x_{ij}), \quad \zeta_i = \sum_{j=1}^{r} \zeta_{ij}. \tag{4.24}$$

Then proceeding as above, we define

$$\tilde{R}_{ij} = \exp\left(t \int_x^y \eta\right) \frac{\theta(A(x,y) + tU + \delta - \alpha' - \alpha'' + \zeta_i - \zeta_{ij}) \prod\limits_{n \neq 1} \theta(A(x_{in}, y) + e)}{\theta(A(x,y) + \delta - \alpha') \prod\limits_{n=1}^{r-1} \theta(A(\Delta^{\tilde{g}+n}, y) + e)} \tag{4.25}$$

and then,

$$s_{ij}(y,t) = \frac{-p_{ij}}{\prod\limits_{n \neq i} (\alpha_i - \alpha_n)} \frac{\tilde{R}_{ij}(y,t)}{\tilde{R}_{ij}(x_{ij}, t)} \tag{4.26}$$

5. Reduction to Subalgebras

In the previous sections we have dealt only with the case that the loop algebra is $\widetilde{gl}(r, \mathbb{C})$. This gives Hamiltonian isospectral flows in the space of complex rank r perturbations of the complex $N \times N$ matrix $A + FG^T$. For many applications it is necessary to repeat the above study for a subalgebra of $\widetilde{gl}(r, \mathbb{C})$. For instance, if we wish to study real rank-r perturbations a study of $\widetilde{gl}(r, \mathbb{R})$ is necessary.

In [1] a description of the appropriate finite rank perturbation spaces was given for various subalgebras of $\widetilde{gl}(r, \mathbb{C})$. Here we wish to discuss the effect on the linearization process (Sect. 2) when we reduce to a subalgebra of $\widetilde{gl}(r, \mathbb{C})$. In general, we will see that requiring the matricial polynomial $L(\lambda)$ to lie in a subalgebra of $\widetilde{gl}(r, \mathbb{C})$ is equivalent to assuming various extra structures on the spectral curve, $\det(L(\lambda) - z) = 0$, and on the eigenvector line bundle for $L(\lambda)$. The requirement that the Lax pair flow (2.1) leave the subalgebra invariant is then translated into the requirement that the flows in the Jacobi variety satisfy these extra conditions.

The simplest case is the reduction to $\widetilde{sl}(r, \mathbb{C}) \subset \widetilde{gl}(r, \mathbb{C})$. Requiring that $L(\lambda)$ is traceless gives a condition on the curve (and its embedding). Namely, that for fixed λ, the sum of the z values must be zero. This is reflected in the equation for the curve in that the coefficient $a_1(\lambda)$ of z^{r-1} must vanish, i.e.

$$\det(L(\lambda) - zI) = z^r + a_2(\lambda) z^{r-2} + \cdots + a_r(\lambda). \tag{5.1}$$

Another situation we consider is given when the subalgebra is of the form $\tilde{k} \subset \widetilde{gl}(r, \mathbb{C})$, where $k \subset gl(r, \mathbb{C})$ is given by the fixed points of an involutive automorphism σ on $gl(r, \mathbb{C})$. (Thus $\tilde{k}$ is the fixed point set of the automorphism $\tilde{\sigma}$ on $\widetilde{gl}(r, \mathbb{C})$ given by $\tilde{\sigma}(X(\lambda)) = \sum \sigma(X_j)\lambda^j$, where $X(\lambda) = \sum X_j \lambda^j$.) Since the details differ for each subalgebra, we shall demonstrate this situation only with the cases where k is either $gl(r, \mathbb{R})$, $so(r, \mathbb{C})$, $sp(r, \mathbb{C})$, $u(r)$, or $u(p, q)$.

Let $\tilde{k}^+ = \tilde{k} \cap \widetilde{gl}(r, \mathbb{C})^+$, $\tilde{k}^- = \tilde{k} \cap \widetilde{gl}(r, \mathbb{C})^-$, and $\tilde{k}_0^- = \tilde{k} \cap \widetilde{gl}(r, \mathbb{C})_0^-$. There is a natural identification of $(\tilde{k}^+)^*$ with $\tilde{k}_0^-$ (see [1], Chap. 5). Also, for the real subalgebras we assume that the polynomial $\acute{a}(\lambda)$ is real, thus, as in Sect. 2, we may consider the flows in $(\tilde{k}^+)^*$ as flows through matricial polynomials

$$L(\lambda) = L_0 \lambda^m + L_1 \lambda^{m-1} + \cdots + L_m \quad \text{with} \quad L_i \in k. \tag{5.2}$$

Case i). $k = gl(r, \mathbb{R})$. In this case, the spectral curve S, given by $\det(L(\lambda) - z) = 0$ is equipped with the involution $\imath: (\lambda, z) \to (\bar{\lambda}, \bar{z})$ since $\overline{L(\lambda)} = L(\bar{\lambda})$. This involution lifts to the desingularized curve $\tilde{S}$. Furthermore, the eigenvector line bundle for $L(\lambda)$ over $\tilde{S}$ is invariant under this involution, i.e. $\imath^* E = E$.

Conversely, we claim that this invariance of the line bundle implies that the induced matricial polynomial $L(\lambda)$ is in $\widetilde{gl}(r, \mathbb{R})$. Indeed, assume that the curve $\tilde{S}$ is invariant under the involution $\imath: (\lambda, z) \to (\bar{\lambda}, \bar{z})$ and that the line bundle E is also invariant under this involution. Then $H^0(\tilde{S}, E)$ is equipped with a natural real structure where $s \in H^0(\tilde{S}, E)$ is said to be real if $\imath^* s = s$. Recall that for $\lambda \in \mathbb{C}$, $\mathbf{L}(\lambda)$ is given by the commuting diagram

$$\begin{array}{ccc} H^0(\tilde{S}, E) & \xrightarrow{ev_\lambda} & \mathbb{C}^r \\ {\scriptstyle \mathbf{L}(\lambda)} \downarrow & & \downarrow {\scriptstyle \operatorname{diag}(z_i)} \\ H^0(\tilde{S}, E) & \xrightarrow{ev_\lambda} & \mathbb{C}^r \end{array} \tag{5.3}$$

where ev_λ evaluates a section of E at the points of D_λ and $\operatorname{diag}(z_i)$ denotes multiplication by z on D_λ (see Sect. 2e). For real λ, the involution on $H^0(\tilde{S}, E)$ induces an involution $\imath$ on $\mathbb{C}^r$ satisfying

$$\imath \circ \operatorname{diag}(z_i) = \operatorname{diag}(\bar{z}_i) \circ \imath. \tag{5.4}$$

Hence for real λ, it follows that

$$\imath \circ \mathbf{L}(\lambda) = \overline{\mathbf{L}(\lambda)} \circ \imath. \tag{5.5}$$

Therefore, if we use a real basis of $H^0(\tilde{S}, E)$ to construct a matrix $L(\lambda)$ for $\mathbf{L}(\lambda)$, we have that $\overline{L(\lambda)} = L(\bar{\lambda})$.

Thus, we see that in order to produce flows of type (2.1) with $L(\lambda) \in \widetilde{gl}(r, \mathbb{R})^+$ we must consider curves invariant under the involution ι and restrict to the flows of line bundles which are invariant under the involution. Concretely, this means that the divisor Δ of E should be chosen to be ι-invariant and the transition functions, $\exp \mu(z, \lambda)$, describing the flow must also be ι-invariant.

Case ii). $k = so(r, \mathbb{C})$ *or* $sp(p, \mathbb{C})$, $2p = r$. In this case, the matrix $L(\lambda)$ is required to be compatible with a nondegenerate bilinear complex form $\langle , \rangle$, i.e.

$$0 = \langle L(\lambda)v, w \rangle + \langle v, L(\lambda)w \rangle, \quad v, w \in \mathbb{C}^r. \tag{5.6}$$

To interpret the structure in terms of the data of Sect. 2 we require a form on $H^0(\tilde{S}, E)$ which is compatible with the map $\mathbf{L}(\lambda): H^0(\tilde{S}, E) \to H^0(\tilde{S}, E)$ in the sense of (5.6).

Assume $\lambda \in \mathbb{C}$ is such that $\mathbf{L}(\lambda)$ has distinct eigenvalues $z_1, \ldots, z_r$. Let $e_i \in H^0(\tilde{S}, E)$ be the eigenvector of $\mathbf{L}(\lambda)$ corresponding to z_i. Then (5.6) implies

$$(z_i + z_j)\langle e_i, e_j \rangle = 0 \text{ for all } i, j. \tag{5.7}$$

Since the bilinear form $\langle , \rangle$ is nondegenerate, the z_i's must either come in pairs, or vanish. Thus we can order the e_i's so that $z_{2j} = -z_{2j-1}$ and $z_r = 0$ if r is odd. From this we conclude that the curve is fixed under the involution $\iota: z \to -z$. (This fact is also easily seen directly from the equation for the curve and the involution which fixes $L(\lambda)$.) Furthermore, in the basis $e_1, \ldots, e_r$, the matrix for $\langle , \rangle$ is given by

$$\left[\begin{array}{c|c|c} \begin{matrix} 0 & \pm p_1 \\ p_1 & 0 \end{matrix} & 0 & 0 \\ \hdashline 0 & \begin{matrix} 0 & \pm p_2 \\ p_2 & 0 \end{matrix} & 0 \\ \hdashline 0 & 0 & \ddots \end{array}\right]. \tag{5.8}$$

Using the isomorphism $ev_\lambda: H^0(\tilde{S}, E) \to \mathbb{C}^r$, the bilinear form $\langle \quad \rangle$ on $H^0(\tilde{S}, E)$ induces a bilinear form $\langle \quad \rangle_\lambda$ on $\mathbb{C}^r$ which is given by (5.8) in the basis $ev_\lambda(e_i)$, $i = 1, \ldots, r$, of $\mathbb{C}^r$. Defining an involution $\iota: \mathbb{C}^r \to \mathbb{C}^r$ by

$$\iota(a_1, \ldots, a_r) = (a_2, a_1, a_4, a_3, \ldots) \tag{5.9}$$

(leaving a_r fixed if r is odd) we see that

$$\langle a, b \rangle_\lambda = F_\lambda(a \circ \iota(b)) \quad \text{for} \quad a, b \in \mathbb{C}^r, \tag{5.10}$$

where $F_\lambda: \mathbb{C}^r \to \mathbb{C}^r$ is a linear map and $o: \mathbb{C}^r \otimes \mathbb{C}^r \to \mathbb{C}^r$ is the map given by componentwise multiplication. Since the diagram

$$\begin{array}{ccc} H^0(\tilde{S}, E) \otimes H^0(\tilde{S}, E) & \xrightarrow{ev_\lambda} & \mathbb{C}^r \otimes \mathbb{C}^r \\ \downarrow & & \downarrow \\ H^0(\tilde{S}, E \otimes \iota^* E) & \xrightarrow{ev_\lambda} & \mathbb{C}^r \end{array} \tag{5.11}$$

commutes when the map $\mathbb{C}^r \otimes \mathbb{C}^r \to \mathbb{C}^r$ is given by $(a, b) \to a \circ \imath(b)$, we may define a linear map $F: H^0(\tilde{S}, E \otimes \imath^* E) \to \mathbb{C}$ by

$$F(s) = F_\lambda(ev_\lambda(s)), \quad s \in H^0(\tilde{S}, E \otimes \imath^* E) \tag{5.12}$$

so that

$$\langle \alpha, \beta \rangle = F(\alpha \otimes \imath^* \beta) \quad \text{for} \quad \alpha, \beta \in H^0(\tilde{S}, E). \tag{5.13}$$

Since $\langle \alpha, \beta \rangle$ is independent of λ, it follows that F must also be. Notice that if $s \in H^0(\tilde{S}, E \otimes \imath^* E)$ satisfies $ev_\lambda(s) = 0$ for any λ, then $F(s) = 0$.

Now, as in Sect. 4, let D_0 denote the zero divisor of the meromorphic function λ on $\tilde{S}$ and let D_∞ denote its polar divisor. By (5.12),

$$F(H^0(\tilde{S}, (E \otimes \imath^* E)(-D_0)) + H^0(\tilde{S}, (E \otimes \imath^* E)(-D_\infty))) = 0. \tag{5.14}$$

Using the Riemann–Roch theorem, and Serre duality, one can compute the following dimensions:

$$\begin{aligned} h^0(\tilde{S}, E \otimes \imath^* E) &= \tilde{g} - 1 + 2r, \\ h^0(\tilde{S}, (E \otimes \imath^* E)(-D_0)) &= \tilde{g} - 1 + r, \\ h^0(\tilde{S}, (E \otimes \imath^* E)(-D_\infty)) &= \tilde{g} - 1 + r \end{aligned} \tag{5.15}$$

and

$$h^0(\tilde{S}, (E \otimes \imath^* E)(-D_0 - D_\infty)) = \begin{cases} \tilde{g} \text{ if } (E \otimes \imath^* E)(-D_0 - D_\infty) \approx K_{\tilde{S}} \\ \tilde{g} - 1 \text{ otherwise} \end{cases}, \tag{5.16}$$

where $K_{\tilde{S}}$ denotes the canonical line bundle on $\tilde{S}$ and $\approx$ denotes linear equivalence. Now F vanishes on $H^0(\tilde{S}, (E \otimes \imath^* E)(-D_0)) + H^0(\tilde{S}, (E \otimes \imath^* E)(-D_\infty))$ but must be nonzero on $H^0(\tilde{S}, E \otimes \imath^* E)$. Counting dimensions, it follows that the intersection, $H^0(\tilde{S}, (E \otimes \imath^* E)(-D_0 - D_\infty))$ must have dimension at least $\tilde{g}$ for F to be nonzero. Since D_0 and D_∞ are both divisors representing the hyperplane bundle $\mathcal{O}(1)$, we conclude from (5.16) that

$$E \otimes \imath^* E \approx K_{\tilde{S}}(2) \tag{5.17}$$

if we are to have a nondegenerate bilinear form given by (5.13).

To complete the discussion we shall

i) Describe the line bundles E satisfying (5.17).
ii) Describe the admissible maps $F: H^0(\tilde{S}, K_{\tilde{S}}(2)) \to \mathbb{C}$.
iii) Determine in which cases the form given is symmetric or anti-symmetric

To describe the line bundles satisfying (5.17) it is enough to display the existence of one line bundle E_0 with $E_0 \otimes \imath^* E_0 \approx K_{\tilde{S}}(2)$ and then to describe the degree zero line bundles X with $X \otimes \imath^* X \approx \mathcal{O}$, the trivial bundle. To find E_0, we first need to describe $K_{\tilde{S}}(2)$ more explicitly.

First, note that if the curve S in T were smooth, one could obtain the canonical sheaf over S by Poincaré residues, i.e.

$$\mathrm{PR}: K_T[S]|_S \xrightarrow{\approx} K_S, \tag{5.18}$$

which is given in local coordinates over S by

$$g(z, \lambda) \frac{dz \wedge d\lambda}{p(z, \lambda)} \to g(z, \lambda) \frac{dz}{\partial p / \partial \lambda} = -g(z, \lambda) \frac{d\lambda}{\partial p / \partial z}, \tag{5.19}$$

where $p(z,\lambda)=0$ is a local equation for S. When S is singular K_S can be defined as the sheaf of Poincaré residues. This gives rise to meromorphic 1-forms with poles at the singularities of S. In our case, since $K_T \approx \mathcal{O}_T(-m-2)$ and S is the divisor of a section of $\mathcal{O}_T(rm)$ we conclude

Lemma 5.1. $K_S \approx \mathcal{O}_S((r-1)m-2)$.

To describe $K_{\tilde{S}}$, let D_i denote the $r-k_i$ points of $\tilde{S}$ which project to the ordinary $(r-k_i)$-fold intersection of S at $(\alpha_i, 0)$. The forms on S pull back to forms on $\tilde{S}$ with poles at D where

$$D=\sum_i (r-k_i-1)D_i. \tag{5.20}$$

Thus we have an injective sheaf homomorphism

$$0 \to \psi^* K_S \to K_{\tilde{S}}(D). \tag{5.21}$$

Combining a degree computation with Lemma 5.1 we conclude

Proposition 5.2. $K_{\tilde{S}}(2) \approx \mathcal{O}_{\tilde{S}}((r-1)m)(D)$.

We are now ready to prove

Theorem 5.3. *There exists a line bundle E_0, over $\tilde{S}$, with $E_0 \otimes \iota^* E_0 \approx K_{\tilde{S}}(2)$.*

Proof. To construct E_0, we first find $\tilde{E}_0$ with $\tilde{E}_0 \otimes \iota^* \tilde{E}_0 \approx \mathcal{O}_{\tilde{S}}((r-1)m)$. This construction breaks into two cases:

i) If $(r-1)m$ is even, choose

$$\tilde{E}_0 \approx \mathcal{O}_{\tilde{S}}\left(\frac{(r-1)m}{2}\right). \tag{5.22}$$

Since the bundles $\mathcal{O}_{\tilde{S}}(k)$ are invariant under the involution ι, it follows that $\tilde{E}_0 \otimes \iota^* \tilde{E}_0 \approx \mathcal{O}_{\tilde{S}}((r-1)m)$.

ii) If $(r-1)m$ is odd let $\lambda=\infty$ intersect S at the points $\pm z_i$, $i=1,\ldots,r/2$ and let Q be the divisor $z_1+z_2+\cdots+z_{r/2}$. Set

$$\tilde{E}_0 = \mathcal{O}_{\tilde{S}}\left(\frac{(r-1)(m+1)}{2}\right)(-Q). \tag{5.23}$$

Then since $Q \otimes \iota^* Q \approx \mathcal{O}(-1)$ we have $\tilde{E}_0 \otimes \iota^* \tilde{E}_0 \approx \mathcal{O}((r-1)m)$.

To construct E_0 we will find a divisor C with $C \otimes \iota^* C \approx D$ and then

$$E_0 \approx \tilde{E}_0(C). \tag{5.24}$$

To find C note that if $r-k_i$ is even, the involution must pair up the points in D_i, thus we can find C_i with $C_i \otimes \iota^* C_i \approx D_i$. In this case the contribution to C is $(r-k_i-1)C_i$. If $r-k_i$ is odd, the contribution to C is $\frac{1}{2}(r-k_i-1)D_i$. □

The next step is to study the line bundles X, of degree zero, satisfying

$$X \otimes \iota^* X = \mathcal{O}. \tag{5.25}$$

Recall that the degree zero line bundles on $\tilde{S}$ are isomorphic to $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})/H^1(\tilde{S}, \mathbb{Z})$ and that there is a surjection $H^1(S, \mathcal{O}_S) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$. Hence from Corollary 2.3 we conclude that $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$ is generated by the monomials $z^i \lambda^j$, $0<i<r$, $-\mathrm{im}<j<0$

(modulo relations describing the kernel of the above surjection). The action of $\imath^*$ splits $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$ into ± 1 eigenspaces

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = H^1_+ \otimes H^1_-, \tag{5.26}$$

where H^1_+ is generated by the monomials $z^i\lambda^j$ with i even, and H^1_- is generated by those with i odd.

Let $\Gamma \approx H^1(\tilde{S}, \mathbb{Z})$ denote the lattice in $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$. For $Z \in \Gamma$ write

$$Z = Z_+ + Z_-, \tag{5.27}$$

where $Z_\pm \in H^1_\pm$. Then $\imath^* Z = Z_+ - Z_-$, and since $\imath^*\Gamma \subset \Gamma$, letting $\Gamma_\pm = \Gamma \cap H^1_\pm$, we conclude

$$\Gamma_+ + \Gamma_- \subset \Gamma \subset \tfrac{1}{2}\Gamma_+ + \tfrac{1}{2}\Gamma_-, \tag{5.28}$$

where addition of lattices denotes the lattice generated by the elements of the two addends. Let $\pi_+ : \Gamma \to H^1_+$ be the projection given by (5.26). From (5.28) we have

$$\pi_+(\Gamma) \subset \tfrac{1}{2}\Gamma_+. \tag{5.29}$$

Let **A** denote the set of degree zero line bundles X satisfying (5.25). Representing X by an element $X \in H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$ we have

$$X + \imath^* X \in \Gamma. \tag{5.30}$$

With the splitting (5.26) we write $X = X_+ + X_-$ and conclude from (5.30) that

$$2X_+ \in \Gamma. \tag{5.31}$$

The lifting of X is defined only up to addition of an element of Γ. Thus the map

$$X \to X_+ \in \tfrac{1}{2}\Gamma_+ \tag{5.32}$$

induces a well defined map

$$\rho : A \to (\tfrac{1}{2}\Gamma_+)/\pi_+(\Gamma). \tag{5.33}$$

In conclusion, we have

Theorem 5.4. *The map ρ gives a bijection between the connected components of* **A** *and the elements of $(\frac{1}{2}\Gamma_+)/\pi_+(\Gamma)$. Each of these components lifts to an affine copy of H^1_- in $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$.*

To give an explicit description of the map $F : H^0(\tilde{S}, K_{\tilde{S}}(2)) \to \mathbb{C}$ we recall from (5.15) and (5.16) that F vanishes on a fixed hyperplane in $H^0(\tilde{S}, K_{\tilde{S}}(2))$ so it is unique up to a constant. Using Proposition 5.2 we see that a section of $K_{\tilde{S}}(2)$ is *uniquely expressible in the form*

$$a(z, \lambda) = a_0 z^{r-1} + a_1(\lambda) z^{r-2} + \cdots + a_{r-1}(\lambda), \tag{5.34}$$

where $\deg(a_i(\lambda)) = im$, and $a(z, \lambda)$ is required to vanish appropriately at the singular points. If at $\lambda = \lambda_0$, the section vanishes at the r points of D_{λ_0} (counted with multiplicity) then the polynomial $a(z, \lambda_0)$ must vanish identically. In particular $a_0 = 0$. Thus from (5.14) we conclude

Proposition 5.5. *Up to a constant multiple, the map*

$$F : H^0(\tilde{S}, \mathcal{O}((r-1)m)(D)) \to \mathbb{C}$$

is given by $F(s) = a_0$ *in the above trivialization.*

At this point we know that if $E = E_0 \otimes X$, $X \in \mathbf{A}$, then F induces a bilinear form on $H^0(\tilde{S}, E)$ of the form (5.13). We now wish to determine in which cases this form is symplectic, and in which cases it is orthogonal. To this end we define a signature which is constant on the components of $\mathbf{A}$.

For $X \in \mathbf{A}$ we have $\imath^* X \otimes X \approx \mathcal{O}$. The action of $\imath^*$ on the sections of X induces an involution $\imath^* \otimes \imath^*$ on the sections of $\mathcal{O}$. There are exactly 2 involutions on the sections of $\mathcal{O}$, namely $\pm I$, where I is the identity. We define

$$\mathrm{Sign}(X) = \begin{cases} +1 & \text{if } \quad \imath^* \otimes \imath^* = I \\ -1 & \text{if } \quad \imath^* \otimes \imath^* = -I \end{cases}. \tag{5.35}$$

Since this depends only on X_+, it follows that this induces a homomorphism

$$\sigma: (\tfrac{1}{2}\Gamma_+)/\pi_+(\Gamma) \to \mathbb{Z}/2. \tag{5.36}$$

A computation with respect to our local trivializations then yields:

Proposition 5.6. *Let* $E = E_0 \otimes X$, $X \in \mathbf{A}$. *For* $a, b \in H^0(\tilde{S}, E)$ *we have*

i) $\langle a, b \rangle = \langle b, a \rangle$ iff $\mathrm{sign}(X) = (-1)^{r-1}$.
ii) $\langle a, b \rangle = -\langle b, a \rangle$ iff $\mathrm{sign}(X) = (-1)^r$.

Proposition 5.7. *Assume* $\tilde{S}$ *is connected. If* r *is odd, the form constructed above is symmetric and hence* $L(\lambda)$ *belongs to* $\widetilde{so}(r, \mathbb{C})$. *If* r *is even and the involution has fixed points, then the form is antisymmetric and* $L(\lambda)$ *belongs to* $\widetilde{sp}\left(\frac{r}{2}, \mathbb{C}\right)$.

Proof. If the involution on $\tilde{S}$ has a fixed point then $\mathrm{sign}(X) = +1$ for all $X \in \mathbf{A}$. In this case the symmetry or antisymmetry of the form is determined by $(-1)^{r-1}$. When r is odd the involution always has fixed points. □

Case iii). $k = u(p, q)$. The arguments here are more or less the same as those for case ii) so we shall only state the results with a few brief remarks.

In this case, we require the coefficients of $L(\lambda)$ to be in $u(p, q)$ so from the equation for the spectral curve we find an antiholomorphic involution

$$\imath: (z, \lambda) \to (-\bar{z}, \bar{\lambda}). \tag{5.37}$$

This induces an involution on the line bundles, which at the level of transition functions is given by

$$T(z, \lambda) \to \overline{T(-\bar{z}, \bar{\lambda})}, \tag{5.38}$$

and at the level of sections in our choice of trivialization is given by

$$s(z, \lambda) \to \overline{s(-\bar{z}, \bar{\lambda})}. \tag{5.39}$$

Furthermore, we require an hermitian inner product on the space $H^0(\tilde{S}, E)$. As above, this inner product is given by

$$\langle a, b \rangle = F(a \otimes \imath^* b), \tag{5.40}$$

where, once more

$$E \otimes \iota^* E \approx K_{\tilde{S}}(2) \approx \mathcal{O}_{\tilde{S}}((r-1)m)(D). \tag{5.41}$$

In this case, however, one can normalize the choice of F so that $\langle a, b\rangle = \overline{\langle b, a\rangle}$. The signature of the form is determined partly by the geometry of the curve and partly by the line bundle.

For real λ, the involution fixes the fiber over λ, and hence we may label the eigenvalues in such a way that

$$z_{2i} = -\bar{z}_{2i-1}, \quad i = 1, \ldots, k \tag{5.42}$$

and

$$z_j = -\bar{z}_j, \quad j = 2k+1, \ldots, r. \tag{5.43}$$

Furthermore, we require that

$$\sqrt{-1}\, z_j \geqq \sqrt{-1}\, z_{j+1}, \quad j = 2k+1, \ldots, r-1. \tag{5.44}$$

For our "standard" bundle E_0, using the basis of sections determined by

$$s_i(\lambda, z_i) = \delta_{ij}, \tag{5.45}$$

one shows that the form is given (up to a constant multiple) by the matrix

$$\left[\begin{array}{c|c|c|c|c}
\begin{matrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{matrix} & & & & \\
\hdashline
 & \begin{matrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{matrix} & & & \\
\hdashline
 & & & & \\
\hdashline
 & & & \begin{matrix} 0 & \alpha_k \\ \beta_k & 0 \end{matrix} & \\
\hdashline
 & & & & \begin{matrix} \gamma_{2k+1} & & \\ & \ddots & \\ & & \gamma_r \end{matrix}
\end{array}\right] \tag{5.46}$$

where $\alpha_i \beta_i$ are real, positive numbers, and the γ_i's alternate in sign. Thus the signature of the form is $\left(\left|\frac{r}{2}\right|, r - \left|\frac{r}{2}\right|\right)$.

Now if L is a line bundle with $L \otimes \iota^* L \approx \mathcal{O}$, one can repeat the analysis for the form on $H^0(S, E_0 \otimes L)$, comparing with the results for E_0, we see that we get signatures within the range $(k, r-k)$ to $(r-k, k)$. Since these bounds hold for all real λ, the actual range possible is from $(k_m, r-k_m)$ to $(r-k_m, k_m)$, where k_m denotes the maximum k given as λ varies in $\mathbb{R}$.

6. Examples

a) *The Rosochatius System.* The Rosochatius system is a Hamiltonian dynamical system on T^*S^{n-1} which is most easily described by considering the Hamiltonian

$$H(x,y)=\frac{1}{2}\sum y_i^2+\sum\frac{\mu_i^2}{x_i^2}+\varepsilon\sum\alpha_i x_i^2 \tag{6.1}$$

on $T^*\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n$ and constraining the flow to the symplectic submanifold $T^*S^{n-1}\subset T^*\mathbb{R}^n$ which is given by

$$\sum x_i^2=1 \quad \text{and} \quad \sum x_i y_i=0. \tag{6.2}$$

The constrained equations of motion are given by

$$\begin{aligned}\dot{x}_i&=y_i,\\ \dot{y}_i&=-2\varepsilon\alpha_i x_i+\frac{2\mu_i^2}{x_i^3}+x_i\left(2\varepsilon\sum\alpha_j x_j^2-\sum y_j^2-2\sum\frac{\mu_j^2}{x_j^2}\right).\end{aligned} \tag{6.3}$$

In [1] it was shown how to realize this system as a flow of rank 2 isospectral perturbations generated by the Lax pair equation

$$\frac{d}{dt}N(\lambda)=\left[\left(\frac{-2a(\lambda)}{\lambda^n}\left(N(\lambda)+\varepsilon\lambda\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\right)_+, N(\lambda)+\varepsilon\lambda\begin{pmatrix}0&1\\0&0\end{pmatrix}\right], \tag{6.4}$$

where

$$N(\lambda)=\frac{1}{2}\sum_{i=1}^n\frac{\lambda}{\lambda-\alpha_i}\left\{\begin{pmatrix}x_iy_i&-y_i^2\\x_i^2&-x_iy_i\end{pmatrix}+\sqrt{-2}\mu_i\begin{pmatrix}1&0\\0&1\end{pmatrix}-2\left(\frac{\mu_i^2}{x_i^2}\right)\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}. \tag{6.5}$$

Setting

$$L(\lambda)=\frac{1}{\lambda}a(\lambda)\left(N(\lambda)+\varepsilon\lambda\begin{pmatrix}0&1\\0&0\end{pmatrix}\right) \tag{6.6}$$

we have that $L(\lambda)$ is a matricial polynomial satisfying an equation of the form (2.1) with

$$P(z,\lambda^{-1})=-2z\lambda^{1-n}. \tag{6.7}$$

We shall now apply the tools developed in Sects. 2, 4 and 5, to explicitly solve this system in terms of theta functions.

The spectral curve S for this system is hyperelliptic, has genus $g=n-1$, and is given by

$$z^2+z\left(\sqrt{-2}a(\lambda)\sum\frac{\mu_i}{\lambda-\alpha_i}\right)+\frac{1}{2}\left(\frac{1}{2}S^\varepsilon(\lambda)-a^2(\lambda)\left(\sum\frac{\mu_i}{\lambda-\alpha_i}\right)^2\right)=0, \tag{6.8}$$

where

$$\begin{aligned}S^\varepsilon(\lambda)=a^2(\lambda)\Bigg\{&\left(\sum\frac{x_i^2}{\lambda-\alpha_i}\right)\left(\sum\frac{y_i^2}{\lambda-\alpha_i}\right)-\left(\sum\frac{x_iy_i}{\lambda-\alpha_i}\right)^2\\&+2\left(\sum\frac{x_i^2}{\lambda-\alpha_i}\right)\left(\sum\frac{(\mu_i/x_i)^2}{\lambda-\alpha_i}\right)+2\varepsilon\sum\frac{x_i^2}{\lambda-\alpha_i}\Bigg\}.\end{aligned} \tag{6.9}$$

It is easily checked that for generic x, y the curve S is smooth, and so we need not perform any desingularization, i.e. $\tilde{S} = S$ and $\tilde{g} = g$.

The construction of Sect. 4 assumed that the points at $\lambda = \infty$ were distinct so that the sections ψ_i of the bundle E could be determined by their values at these points. That construction is easily generalized to the case that the divisor D_∞ has points with multiplicity by constraining not only the values of the ψ_i's at these multiple points, but also the values of an appropriate number of derivatives.

Let E be the line bundle on S obtained, as in Sect. 2, from $L(\lambda, 0)$. Assuming that E is generic in the sense of (2.13) it follows that one can choose a section ψ^1 of E which vanishes at the point ∞ to precisely first order. (There are no sections of E which vanish to second order at ∞.) We normalize ψ^1 by fixing the values of its derivatives in $\lambda^{-1/2}$ at ∞ in our chosen trivialization to be $\pm(2\varepsilon)^{-1/2}$. Similarly, we can choose ψ^2 which does not vanish at ∞. Normalizing we set $\psi^2(\infty) = 1$. There is an additional normalization of the derivative of ψ^2 at ∞, which is determined by the condition $x \cdot y = 0$ and is effected by the addition of a suitable multiple of ψ^1; we omit it as it is of no importance in the final description of the solution. Now ψ^1 and ψ^2 form a basis for the sections of E so we may proceed as in Sect. 4 with $L(\lambda)$ determined by (4.2).

The next step is to consider ψ^1 and ψ^2 as generalized Baker–Akhiezer functions, as in Sect. 4, and find expressions for them in terms of theta-functions. In particular, recall that a divisor corresponding to the line bundle given by the initial conditions is given by $(L(\lambda; 0) - z\,\mathrm{Id})_{\mathrm{adj}} v = 0$. Taking into account the normalizations at infinity which determine the choice of v, the zero divisor $\Gamma(0)$ of ψ^1 is then $q_1 + \dots + q_n$, where

$$L_{21}(\lambda(q_j)) = 0,$$
$$L_{22}(\lambda(q_j)) = z(q_j), \quad j = 1, \dots, n-1 \quad \text{and} \quad q_n = \infty. \tag{6.10}$$

Similarly, the polar divisor Δ is $p_1 + \dots + p_n$, with

$$L_{12}(\lambda(p_j)) = 0, \quad L_{11}(\lambda(p_j)) = z(p_j), \quad j = 1, \dots, n. \tag{6.11}$$

The results of Sect. 4 than give explicit formulae for the Baker–Akhiezer function $\psi^i(t) = \psi^i(\lambda, t)$ in terms of theta-functions and the "initial divisors" $\Gamma(0)$ and Δ, but the explicit determination of the x_i flows is still rather cumbersome since it requires the expansion of the $\psi^i(t)$'s in the local parameter λ' to high order. The x_i flows can, however, be derived implicitly. They are expressible in terms of hyperelliptic integrals as follows. Let $Q(t) = \sum q_i(t)$ be the zero divisor of $\psi^1(t)$ and let $a_i(t) = \lambda(q_i(t))$ be the corresponding set of λ-coordinates. From (6.10), we have

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{1}{a(\lambda)} \prod_{i=1}^{n-1} (\lambda - a_i), \tag{6.12}$$

i.e., the a_i's give hyperelliptic coordinates on the sphere. The form $\omega = d\ln(\psi^1)$ has simple poles at $Q(t)$, Δ and a pole of the form $d\left(\frac{2tz}{\lambda^{n-1}}\right)$ at ∞. Applying the reciprocity formula ([6], p. 240) for the periods of ω and those of the regular differentials $z^{-1}\lambda^{n-1-j} d\lambda$ we obtain

$$\sum_i \int_{x_0}^{a_i(t)} \frac{\lambda^{n-l-j}}{z(\lambda)} d\lambda = \begin{cases} 2t + c, & j = 1 \\ c_j, & j > 1 \end{cases}, \tag{6.13}$$

where c_j, $j=1,\ldots,n-1$ are constants. Up to normalization, this agrees with the integration of this system given in [7].

b) *The Coupled Nonlinear Schrödinger Equation* (CNLS). We next consider a real form of the coupled 2-mode nonlinear Schrödinger equation:

$$\begin{aligned}\sqrt{-1}u_t + u_{xx} &= 2u(|u|^2+|v|^2),\\ \sqrt{-1}v_t + v_{xx} &= 2v(|u|^2+|v|^2).\end{aligned} \tag{6.14}$$

As discussed in [1], Sect. 6F, this equation can be given as the commutation relation for t and x flows of matricial polynomials with coefficients in $su(1,2)$. Namely, let

$$L(\lambda)=\lambda^{n-1}L_0+\lambda^{n-2}L_1+\cdots+L_{n-1} \tag{6.15}$$

with $L_1 \in su(1,2)$ and

$$L_0=\frac{\sqrt{-1}}{3}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}, \tag{6.16}$$

$$L_1=\begin{bmatrix}0&\bar{u}&\bar{v}\\u&0&0\\v&0&0\end{bmatrix}. \tag{6.17}$$

Defining x and t flows by

$$\frac{d}{dx}L=[\lambda L_0+L_1,L], \tag{6.18}$$

$$\frac{d}{dt}L=[\lambda^2 L_0+\lambda L_1+L_2,L], \tag{6.19}$$

the commutativity of these flows requires u and v satisfy (6.14) and determines $L_2,\ldots,L_{n-1}$ in terms of u, v and their x derivatives, up to integration constants.

In [1] a space of rank 3 isospectral flows was related to CNLS as follows. Choosing distinct real numbers $\alpha_1,\ldots,\alpha_n$, a reduction of the moment map $\tilde{J}_3: M_{n,3}\times M_{n,3}\to \widetilde{gl}(r)^-$ gives a moment map $J:\mathbb{C}^{2n}\to\widetilde{su}(1,2)^-$ by

$$J(\eta,\zeta)=\sqrt{-1}\sum_{i=1}^{n}\frac{\lambda}{\lambda-\alpha_i}\begin{bmatrix}\rho_i^2&-\bar{\eta}_i\rho_i&-\bar{\zeta}_i\rho_i\\ \eta_i\rho_i&-|\eta_i|^2&-\eta_i\bar{\zeta}_i\\ \zeta_i\rho_i&-\bar{\eta}_i\zeta_i&-|\zeta_i|^2\end{bmatrix}\equiv N(\lambda), \tag{6.20}$$

where $\rho_i=\sqrt{|\eta_i|^2+|\zeta_i|^2}$.

With $L(\lambda)=\frac{1}{\lambda}a(\lambda)N(\lambda)$ the flows (6.18) and (6.19) are equivalent to the Hamiltonian flows on $\widetilde{su}(1,2)^-$ of the Poisson commuting functions,

$$H_x=\frac{1}{2}\left[\frac{a(\lambda)}{\lambda^{n-1}}\operatorname{tr}(N(\lambda)^2)\right]_0, \tag{6.21}$$

$$H_t=\frac{1}{2}\left[\frac{a(\lambda)}{\lambda^{n-2}}\operatorname{tr}(N(\lambda)^2\right]_0. \tag{6.22}$$

The value (6.16) of L_0 determines the value of the $su(1,2)$ moment map and is thus an invariant of the flows. It gives the constraints

$$\sum|\eta_i|^2 = \sum|\zeta_i|^2 = \tfrac{1}{3}, \tag{6.23}$$

$$\sum\eta_i\rho_i = \sum\zeta_i\rho_i = \sum\bar{\eta}_i\zeta_i = 0. \tag{6.24}$$

Under these constraints, the form of L_1 is also invariant, yielding

$$\sum\alpha_i|\eta_i|^2 = \sum\alpha_i|\zeta_i|^2 = \tfrac{1}{3}\sum\alpha_i, \quad \sum\alpha_i\bar{\eta}_i\zeta_i = 0 \tag{6.25}$$

and

$$u = -\sqrt{-1}\sum\alpha_i\bar{\eta}_i\rho_i, \quad v = -\sqrt{-1}\sum\alpha_i\bar{\zeta}_i\rho_i. \tag{6.26}$$

The spectral curve S for this problem is given by

$$\det(L(\lambda) - z) = -z^3 + za(\lambda)Q(\lambda) + a^2(\lambda)P(\lambda) = 0 \tag{6.27}$$

with

$$Q(\lambda) = Q_0 + Q_1\lambda + \cdots + Q_{n-2}\lambda^{n-2}, \tag{6.28}$$

$$P(\lambda) = P_0 + P_1\lambda + \cdots + P_{n-3}\lambda^{n-3}. \tag{6.29}$$

These $2n-3$ integrals Q_i and P_j give a completely integrable system on the $4n-6$ dimensional space given by $\mathbb{C}^{2n}$ with the 6 real constraints (6.24). Imposing all the constraints (6.23), (6.24) (i.e. assuming (6.16)) the curve has two points at $\lambda = \infty$, namely the regular point $\tilde{z} = \dfrac{2\sqrt{-1}}{3}$ and the double point $\tilde{z} = \dfrac{-\sqrt{-1}}{3}$. Assuming also (6.17) and changing to coordinates $\tilde{z} = z/\lambda^n$ and $\tilde{\lambda} = 1/\lambda$, an expansion near $\tilde{z} = \dfrac{-\sqrt{-1}}{3}$ yields that the curve (after an appropriate change of coordinates) is locally given at this double point by

$$u(u - v^3) = 0 \tag{6.30}$$

and thus separates into two sheets in the desingularization. Furthermore $L(\lambda)$ is generically of rank 1 at $\lambda = \alpha_i$ (the rank two case may also be considered but the explicit computations are more complicated) and so the curve also has double points when $\lambda = \alpha_i$. Resolving all of these singularities (including the one at ∞), the desingularized curve $\tilde{S}$ will have genus

$$\tilde{g} = 2n - 8. \tag{6.31}$$

The flows on the Jacobi variety of $\tilde{S}$ linearize the Hamiltonian flows on the $4n-16$ dimensional manifold given by $\mathbb{C}^{2n}$ with the 12 real constraints of (6.23), (6.24), and (6.25), divided by the 4 dimensional stabilizer group of L_0 in $su(1,2)$ (i.e. the flows generated by the first class constraints).

To solve for u and v in terms of θ-functions we choose sections ψ^1, ψ^2, ψ^3 of the line bundle E on $\tilde{S}$ such that

$$\psi_{ij} = \psi^i(\infty_j) = \delta_{ij}. \tag{6.32}$$

Inserting into (4.2), expanding near $\tilde{\lambda} = 0$, and discarding terms of order $\tilde{\lambda}^2$ yields

$$u = \sqrt{-1}\left.\frac{d}{d\tilde{\lambda}}\right|_{\tilde{\lambda}=0}\left(\frac{\psi_{21}}{\psi_{11}}\right), \tag{6.33a}$$

$$v = \sqrt{-1}\,\frac{d}{d\tilde{\lambda}}\bigg|_{\tilde{\lambda}=0}\left(\frac{\psi_{31}}{\psi_{11}}\right). \tag{6.33b}$$

Applying the results of Sect. 4, we have, with the notation given there,

$$\frac{\psi^i}{\psi^1}(y) = K_i \exp(t\mu_i + x\nu_i)\frac{\theta(A(p,y) + tU + xV + \delta - \gamma_i)\theta(A(\infty_1, y) + e)}{\theta(A(p,y) + tU + xV + \delta - \gamma_1)\theta(A(\infty_i, y) + e)}, \tag{6.34}$$

where $i = 2, 3$ and $\mu_i = \int_{\infty_i}^{\infty_1} \eta$, $\nu_i = \int_{\infty_i}^{\infty_1} \tilde{\eta}$, with $\eta, \tilde{\eta}$ being the forms corresponding as in (4.20) to the t, x flows respectively. The K_i are constants independent of x and t; p is a base point. Differentiation yields

$$u(x,t) = \tilde{K}_2 \exp(t\mu_2 + x\nu_2)\frac{\theta(A(p,\infty_1) + \delta + tU + xV - \gamma_2)}{\theta(A(p,\infty_1) + \delta + tU + xV - \gamma_1)}, \tag{6.35a}$$

$$v(x,t) = \tilde{K}_3 \exp(t\mu_3 + x\nu_3)\frac{\theta(A(p,\infty_1) + \delta + tU + xV - \gamma_3)}{\theta(A(p,\infty_1) + \delta + tU + xV - \gamma_1)}. \tag{6.35b}$$

Another method of solution, analogous to that given in [7] for the Rosochatius system may be used to obtain these solutions. This is based on finding a Liouville generating function for canonical transformations to an appropriate linearizing Darboux coordinate system. The general result integrating all isospectral flows corresponding to rank-r perturbations via such a Liouville generating function and details regarding these examples may be found in [2].

References

1. Adams, M. R., Harnad, J., Previato, E.: Isopectral Hamiltonian flows in finite and infinite dimensions, I. Generalized Moser systems and moment maps into loop algebras. Commun. Math. Phys. **117**, 451–500 (1988)
2. Adams, M. R., Harnad, J., Hurtubise, J.: Liouville generating functions for isospectral Hamiltonian flows in loop algebras. Integrable and Superintegrable Systems. Kupershmidt B. (ed.) Singapore: World Scientific (in press 1990); Darboux coordinates and Liouville–Arnold integration in loop algebras. (In preparation)
3. Adler, M., van Moerbeke, P.: Completely integrable systems, euclidean Lie algebras, and curves. Adv. Math. **38**, 267–317 (1980); Linearization of Hamiltonian systems, Jacobi varieties, and representation theory. Adv. Math. **38**, 318–379 (1980)
4. Dubrovin, B. A.: Theta Functions and non-linear equations. Russ. Math. Surv. **36**, 11–92 (1981)
5. Dubrovin, B. A., Matveev, V. B., Novikov, S. P.: Non-linear equations of Korteweg de Vries type, finite zone linear operators, and abelian varieties. Russ. Math. Surv. **31**, 59–146 (1976)
6. Griffiths, P., Harris, J.: Principles of algebraic geometry. New York: John Wiley 1978
7. Gagnon, L., Harnad, J., Hurtubise, J., Winternitz, P.: Abelian integrals and the reduction method for an integrable Hamiltonian system. J. Math. Phys. **26**, 1605–1612 (1985)
8. Guillemin, V., Sternberg, S.: The moment map and collective motion. Ann. Phys. **127**, 220–253 (1980)
9. Hitchin, N.: On the construction of monopoles. Commun. Math. Phys. **89**, 145–190 (1983)
10. Hurtubise, J.: Rank r perturbations, algebraic curves and ruled surfaces, preprint
11. Krichever, I. M., Novikov, S. P.: Holomorphic bundles over algebraic curves and non-linear equations. Russ. Math. Surv. **35**, 6, 53–79 (1980)

12. Krichever, I. M.: Algebraic curves and commuting matrix differential operators. Funct. Anal. Appl. **10**, 144–146 (1976); Methods of algebraic geometry in the theory of non-linear equations. Russ. Math. Surv. **32**, 6, 198–213 (1977)
13. Moser, J.: Geometry of quadrics and spectral theory. The Chern symposium, Berkeley, June 1979; pp. 147–188. Berlin, Heidelberg, New York: Springer 1980
14. Mumford, D.: Tata lectures on theta. II. Prog. Math. 43. Boston: Birkhäuser 1983
15. van Moerbeke, P., Mumford, D.: The spectrum of difference operators and algebraic curves. Acta Math. **143**, 93–154 (1979)
16. Previato, E.: Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation. Duke Math. J. **52**
17. Reyman, A. G., Semenov-Tian-Shansky, M. A.: Reduction of Hamiltonian systems, affine Lie algebras and Lax equations II. Invent. Math. **63**, 423–432 (1981)
18. Reyman, A. G., Semenov-Tian-Shansky, M. A., Frenkel, I. B.: Graded Lie algebras and completely integrable dynamical systems. Sov. Math. Doklady **20**, 811–814 (1979)
19. Serre, J. P.: Groupes algébriques et corps de classe. Paris: Hermann,

Communicated by A. Jaffe

Note added in proof. The following are errata for part I of this sequence [1].

On page 460, equation (2.40) the rank should be r.

In proposition 3.4, $d\hat{\phi}(\mathcal{N}) = (d\hat{\phi}(\mathcal{N}))_+ + (d\hat{\phi}(\mathcal{N}))_-$ and the N's in equation (3.11) should be script.

On page 468, the degree of the line bundle should be $-g-r+1$.

On the line above equation (6.4), $b-c=4\Delta$.

At the bottom of page 487 the subalgebras should be interchanged, so that (6.9a) corresponds to $su(1,1)$ and (6.9b) corresponds to $su(2)$.