Periodic and Flat Irreducible Representations of $SU(3)_q$

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Abstract. We construct all the periodic irreducible representations of $\mathscr{U}(SU(3))_q$ for q a m-root of unity. Their dimensions are $k(2m)^2$ for $k = 1, \ldots, m$ $\left(\text{only } k = 1, \ldots, \dfrac{m}{2}\right.$ for even $m\left.\right)$. Their interest is that they could be a tool to generalize the chiral Potts model. By truncation of these representations, we construct "flat representations" of $\mathscr{U}(SU(3))_q$, in which all the multiplicities of the weights are set to 1.

I. Introduction

In [1], M. Rosso classified the finite dimensional irreducible representations of the quantum analogue $\mathscr{U}(\mathscr{G})_q$ of the enveloping algebra of a complex simple Lie algebra when the parameter of deformation q is not a root of unity. He proved that they were deformations of the finite dimensional irreducible representations of the classical $\mathscr{U}(\mathscr{G})$. They are in particular characterized by a highest weight λ corresponding to a classical representation of $\mathscr{U}(\mathscr{G})$ and by $\omega \in \{1, -1, i, -i\}$ characterizing the average (the center value) of the eigenvalues of the generators h_i of the Cartan torus.

In [2], the finite dimensional irreducible representations of $\mathscr{U}(SU(2))_q$ for q a root of unity are classified. The new fact is that the dimensions of these representations is bounded by m, if $q^m = 1$. The $d < m$ representations are called regular and correspond to unitary representations of the WZW theory based on affine $SU(2)$ level $m - 2$. Furthermore, the m-dimensional irreducible representations can be periodic, in the sense that the generators J^+ and J^- are not nilpotent and act as $\mathbb{Z}_m$. Continuous parameters also enter in their definition. In [3], the composition of regular representations is studied. It leads to a sum of irreducible and indecomposable representations, an explicit truncation being possible to recover the sum over regular representations provided by the WZW theory. This result is generalized in [4] to all the quantum analogues of simple Lie algebra.

The periodic representations of $\mathcal{U}(SU(2))_q$ are used in [5] and [6] for a connection to the chiral Potts model.

In this paper, we classify the periodic irreducible representations of $\mathcal{U}(SU(3))_q$ for q a mth root of unity and prove that their dimensions are $k(2m)^2$ with $k = 1, \ldots, m^*$, $\left(\text{where } m^* = m \text{ if } m \text{ is odd and } m^* = \dfrac{m}{2} \text{ if } m \text{ is even}\right)$. These representations may play a role in a generalization of the chiral Potts model, with a method inspired by that of [5,6].

In Sect. II, we derive an auxiliary algebra $\mathcal{A}$ whose finite dimensional irreducible representations are the fundamental tool to construct the periodic irreducible representations of $\mathcal{U}(SU(3))_q$. In Sect. III, we classify the irreducible representations of $\mathcal{A}$. In Sect. IV, we perform a truncation of the $(2m)^2$-dimensional periodic representation of $\mathcal{U}(SU(3))_q$ and obtain a new type of highest weight representations, which we call "flat" since all their weights have multiplicity 1. In Sect. V, we study a subtlety that appeared in II when m is a multiple of 3, and prove that this is indeed not a particular case.

We conclude the introduction with the following remark: each simple link of the Dynkin diagram of a simply laced algebra provides a constraint, via the Serre relations, corresponding to the constraint of a single $SU(3)$. So it seems that the knowledge of periodic representations of $\mathcal{U}(SU(3))_q$ will be the basic tool for the construction of the periodic representations of the quantum analogues of a simply laced algebra. The generalization of this work to the quantum analogue of simply laced algebras will be the subject of a further publication. Note however that the results of Sect. IV on flat representations are immediately generalizable to the $\mathcal{U}(SU(N))_q$ case.

II. Derivation of the Auxiliary Algebra $\mathcal{A}$

The quantum group $\mathcal{U}(SU(3))_q$ is defined by the generators $q^{\pm h_i/2}, e_i, f_i$ $(i = 1, 2)$ and the following relations:

$$\begin{cases} q^{h_i/2} \cdot q^{-h_i/2} = q^{-h_i/2} \cdot q^{h_i/2} = 1, \\ q^{h_i/2} \cdot q^{h_j/2} = q^{h_j/2} \cdot q^{h_i/2}, \\ q^{h_i/2} \cdot e_j \cdot q^{-h_i/2} = q^{a_{ij}/2} e_j, \\ q^{h_i/2} \cdot f_j \cdot q^{-h_i/2} = q^{-a_{ij}/2} f_j, \\ [e_i, f_j] = \delta_{ij} \dfrac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \equiv \delta_{ij}(h_i)_q, \end{cases}$$

$$(S) \quad \begin{cases} e_i^2 e_{i\pm 1} - (q + q^{-1}) e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0 \\ f_i^2 f_{i\pm 1} - (q + q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0 \end{cases}$$

where $(a_{ij})_{i,j=1,2}$ is the Cartan matrix of $SU(3)$, i.e.

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We shall not use the coalgebra structure in the following. The representations we

will construct will be representations of the algebra structure only. The coalgebra structure is then the tool to construct the analogue of the tensor products of representations corresponding to the composition of cinetic momenta.

We shall suppose in the following that m is the smallest integer such that $q^m = 1$.

Lemma. *As a consequence of the commutation relations, $(e_i)^{2m}$, $(f_i)^{2m}$ and $(q^{h_i/2})^{2m}$ are in the center of the algebra.*

Note that this is true for $\mathcal{U}(SU(N))_q$ for general N. But for $N = 2$, the mth power is enough [2] since in that case the Cartan matrix contains no odd integers.

Lemma. *Let M be a finite dimensional simple module over $\mathbb{C}$. Then $q^{h_1/2}$ and $q^{h_2/2}$ are simultaneously diagonalizable and*

$$M = \bigoplus_{\substack{p_1 \in \mathbb{Z}_{2m} \\ p_2 \in \mathbb{Z}_{2m}}} M_{\mu_1/2 - p_1 + p_2/2, \mu_2/2 - p_2 + p_1/2}, \tag{1}$$

where M_{λ_1, λ_2} is the common eigenspace of $q^{h_1/2}$ and $q^{h_2/2}$ associated to the eigenvalues q^{λ_1} and q^{λ_2}.

There is a subtlety here if m is a multiple of 3, since in this case the sum is not a direct sum. For reasons which will be explained in the last section, it is nevertheless possible not to distinguish this case in the following.

Proof of the Lemma. Since $q^{h_1/2}$ and $q^{h_2/2}$ commute, let v be an eigenvector of both of them, associated to $q^{\mu_1/2}$ and $q^{\mu_2/2}$. Then $M = \mathcal{U}(SU(3))_q \cdot v$ since M is simple. Because of the commutation relations, every $A \cdot v$, where A is a word made of e_i's, f_i's and $q^{h_i/2}$'s, is an eigenvector of $q^{h_1/2}$ and $q^{h_2/2}$ associated to the eigenvalues $q^{\mu_1/2 - p_1 + p_2/2}$ and $q^{\mu_2/2 - p_2 + p_1/2}$, where p_i is the number (modulo $2m$) of f_i's minus the number of e_i's in A.

We now make the assumption that f_1 and e_2 act injectively on M, i.e., since f_1^{2m} and e_2^{2m} are in the center and since M is a simple module

$$f_1^{2m} \cdot v = \alpha_1^{2m} \cdot v \qquad \forall v \in M$$
$$e_2^{2m} \cdot v = \alpha_2^{2m} \cdot v$$

with $\alpha_1 \in \mathbb{C}^*$ and $\alpha_2 \in \mathbb{C}^*$.

Let $M_{\mu_1/2, \mu_2/2}$ be a common eigenspace of $q^{h_1/2}$ and $q^{h_2/2}$ associated to $q^{\mu_1/2}$ and $q^{\mu_2/2}$. Then f_1 (respectively e_2) defines an isomorphism (of vector spaces) from $M_{\mu_1/2, \mu_2/2}$ to $M_{\mu_1/2 - 1, \mu_2/2 + 1/2}$ (respectively $M_{\mu_1/2 - 1/2, \mu_2/2 + 1}$) which consequently has the same dimension. Since f_1 and e_2 commute, any basis of $M_{\mu_1/2, \mu_2/2}$ can be carried to every $M_{\mu_1'/2, \mu_2'/2}$. With such correlated bases, f_1 and e_2 can be defined to act as a multiple of identity, i.e.

$$f_1 \cdot V_{p_1, p_2} = \alpha_1 \, \text{Id} \cdot V_{p_1 + 1, p_2},$$
$$e_2 \cdot V_{p_1, p_2} = \alpha_2 \, \text{Id} \cdot V_{p_1, p_2 - 1},$$

where V_{p_1, p_2} denotes a vector in

$$M_{\mu_1/2 - p_1 + p_2/2, \mu_2/2 - p_2 + p_1/2}$$

and $V_{p_1 + 1, p_2}$ a vector in

$$M_{\mu_1/2 - (p_1 + 1) + p_2/2, \mu_2/2 - p_2 + (p_1 + 1)/2}$$

with the same coordinates.

Action of e_1 and f_2. The commutation relations $[e_i, f_i] = (h_i)_q$ imply

$$e_1 \cdot V_{p_1, p_2} = \left[\frac{1}{\alpha_1} (p_1)_q (\mu_1 - p_1 + p_2 + 1)_q \operatorname{Id} + \beta_{p_2} \right] \cdot V_{p_1 - 1, p_2},$$

$$f_2 \cdot V_{p_1, p_2} = \left[\frac{1}{\alpha_2} (p_2 + 1)_q (\mu_2 - p_2 + p_1)_q \operatorname{Id} + \gamma_{p_1} \right] \cdot V_{p_1, p_2 + 1}, \tag{2}$$

where β_{p_2} (respectively γ_{p_1}) is an operator that does not depend on p_1 (respectively p_2). This solves the constraints given by the two $\mathcal{U}(SU(2))_q$ subalgebras generated by $(q^{\pm h_1/2}, e_1, f_1)$ and $(q^{\pm h_2/2}, e_2, f_2)$.

Lemma. *The dependence of β_{p_2} and γ_{p_1} on p_2 and p_1 respectively is given by*

$$\begin{cases} \beta_{p_2} = q^{p_2} \beta + q^{-p_2} \beta' \\ \gamma_{p_1} = q^{p_1} \gamma + q^{-p_1} \gamma' \end{cases}.$$

Proof. This is a direct consequence of two of the Serre relations (S) applied to V_{p_1, p_2}:

$$\begin{cases} (e_2^2 e_1 - (q + q^{-1}) e_2 e_1 e_2 + e_1 e_2^2) \cdot V_{p_1, p_2} = 0 \Rightarrow \beta_{p_2} - (q + q^{-1}) \beta_{p_2 - 1} + \beta_{p_2 - 2} = 0 \\ (f_1^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 f_1^2) \cdot V_{p_1, p_2} = 0 \Rightarrow \gamma_{p_1} - (q + q^{-1}) \gamma_{p_1 + 1} + \gamma_{p_1 + 2} = 0 \end{cases}.$$

(Note that $(a + 1)_q - (q + q^{-1})(a)_q + (a - 1)_q = 0 \ \forall a$.)

The two other Serre relations now provide the following constraints on the operators β, β', γ and γ':

$$\text{(E)} \quad -\frac{1}{\alpha_1} q^{-\mu_1 - 1} \beta - \frac{1}{\alpha_1} q^{\mu_1 + 1} \beta' + (1 - q^{-2}) \beta \beta' + (1 - q^2) \beta' \beta = 0$$

$$\text{(F)} \quad -\frac{1}{\alpha_2} q^{-\mu_2 - 1} \gamma - \frac{1}{\alpha_2} q^{\mu_2 + 1} \gamma' + (1 - q^2) \gamma \gamma' + (1 - q^{-2}) \gamma' \gamma = 0$$

All the relations of definition of $\mathcal{U}(SU(3))_q$ are now satisfied on the module but $[e_1, f_2] = 0$. This leads to the following four relations

$$\begin{cases}
(++) \quad \frac{1}{\alpha_1 \alpha_2 (q - q^{-1})^2} (\mu_1 + \mu_2 + 1)_q + \frac{1}{\alpha_1 (q - q^{-1})} q^{\mu_1} \gamma \\
\qquad + \frac{1}{\alpha_2 (q - q^{-1})} q^{\mu_2} \beta + (\beta \gamma - q^{-2} \gamma \beta) = 0, \\[2mm]
(+-) \quad -\frac{1}{\alpha_1 \alpha_2 (q - q^{-1})^2} (\mu_1 + \mu_2 + 1)_q + \frac{1}{\alpha_1 (q - q^{-1})} q^{-\mu_1} \gamma' \\
\qquad - \frac{1}{\alpha_2 (q - q^{-1})} q^{\mu_2} \beta + [\beta', \gamma] = 0, \\[2mm]
(-+) \quad -\frac{1}{\alpha_1 \alpha_2 (q - q^{-1})^2} (\mu_1 + \mu_2 + 1)_q - \frac{1}{\alpha_1 (q - q^{-1})} q^{\mu_1} \gamma \\
\qquad + \frac{1}{\alpha_2 (q - q^{-1})} q^{-\mu_2} \beta' + [\beta, \gamma'] = 0,
\end{cases}$$

$$\left|\quad (--)\quad \frac{1}{\alpha_1\alpha_2(q-q^{-1})^2}(\mu_1+\mu_2+1)_q - \frac{1}{\alpha_1(q-q^{-1})}q^{-\mu_1}\gamma'\right.$$

$$-\frac{1}{\alpha_2(q-q^{-1})}q^{-\mu_2}\beta' + (\beta'\gamma' - q^2\gamma'\beta') = 0,$$

where the equation $(\varepsilon_1\varepsilon_2)_{\varepsilon_i = \pm 1}$ is the coefficient of $q^{\varepsilon_1 p_1 + \varepsilon_2(p_2+1)}$ in

$$[e_1, f_2]\cdot V_{p_1,p_2} = 0.$$

Let us now define the algebra $\mathscr{A}$ by the generators

$$\begin{cases} u = q^{1/3(\mu_1-\mu_2)}[\alpha_1 q^{\mu_2}(q-q^{-1})^2\beta + q^{\mu_1+\mu_2+1}] \\ u' = q^{-1/3(\mu_1-\mu_2)}[\alpha_1 q^{-\mu_2}(q-q^{-1})^2\beta' + q^{-\mu_1-\mu_2-1}] \\ v = q^{-1/3(\mu_1-\mu_2)}[\alpha_2 q^{\mu_1}(q-q^{-1})^2\gamma + q^{\mu_1+\mu_2+1}] \\ v' = q^{1/3(\mu_1-\mu_2)}[\alpha_2 q^{-\mu_1}(q-q^{-1})^2\gamma' + q^{-\mu_1-\mu_2-1}] \end{cases}$$

and the relations provided by (E), (F), $(\pm, \pm)$

$$\text{(E)}\quad \frac{qu'u - q^{-1}uu'}{q-q^{-1}} = 1 \qquad \text{(F)}\quad \frac{qvv' - q^{-1}v'v}{q-q^{-1}} = 1$$

$$(++)\quad \frac{quv - q^{-1}vu}{q-q^{-1}} = 1 \qquad (--)\quad \frac{qv'u' - q^{-1}u'v'}{q-q^{-1}} = 1$$

$$(+-)\quad v' - u + \frac{1}{q-q^{-1}}[u',v] = 0,$$

$$(-+)\quad v - u' - \frac{1}{q-q^{-1}}[u,v'] = 0.$$

Note the similar form of (E), (F), consequence of the Serre relations (S) and $(++), (--)$, due to $[e_1, f_2] = 0$.

$(-+)$ is not independent and can be derived for example from (E), $(+-)$ and $(++)$.

Since f_1 and e_2 provide an identification of all the common eigenspaces of $q^{h_1/2}$ and $q^{h_2/2}$, the classification of all the irreducible representations of $\mathscr{U}(SU(3))_q$ with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ reduces to the classification of the irreducible representations of the algebra $\mathscr{A}$ generated by u, u', v and v'. The first ones will then have a dimension $(2m)^2$ times bigger than the second.

The expressions of the quadratic and cubic Casimirs of $\mathscr{U}(SU(3))_q$ are given by [3] $C_{2,3} = (C_+ \pm C_-)/(q \pm q^{-1})$, where

$$C_+ = \frac{1}{(q-q^{-1})^2}(q^{2+(4h_1+2h_2)/3} + q^{-2-(2h_1+4h_2)/3} + q^{-2(h_1-h_2)/3} - 3)$$

$$+ q^{1+(h_1+2h_2)/3}f_1e_1 + q^{-1-(2h_1+h_2)/3}f_2e_2$$

$$- q^{(h_1-h_2)/3}(f_1f_2 - qf_2f_1)(e_1e_2 - q^{-1}e_2e_1),$$

$$C_- = \text{idem with } q \leftrightarrow q^{-1}. \tag{Cas.}$$

In terms of elements of $\mathcal{A}$, they write

$$(q - q^{-1})^2 C_+ = qu + q^{-1}v' + vu' - 3,$$
$$(q - q^{-1})^2 C_- = qv + q^{-1}u' + v'u - 3.$$

III. Classification of the Irreducible Representations of the Algebra $\mathcal{A}$

Lemma. u^m, v^m, v'^m and u'^m are in the center of $\mathcal{A}$. If m is even, $u^{m/2}, v^{m/2}, v'^{(m/2)}$ and $u'^{(m/2)}$ are also in the center of $\mathcal{A}$.

This is a consequence of the relations of definition of $\mathcal{A}$.

Since $\dfrac{m}{2}$ will often occur instead of m when m is even, let us define

$$m^* = \begin{cases} m & \text{if } m \text{ is odd} \\[2mm] \dfrac{m}{2} & \text{if } m \text{ is even} \end{cases}.$$

Let $\mathcal{M}$ be a simple module on $\mathcal{A}$. Let us first suppose that one of these operators, say u, is such that $u^{m^*} = \lambda^{m^*} \neq 0$ on $\mathcal{M}$. The operator u is then invertible and diagonalizable on $\mathcal{M}$ (since its minimal polynomial has only simple roots) and its eigenvalues can be $\lambda, \lambda q^2, \lambda q^4 \cdots \lambda q^{2(m^*-1)}$.

Theorem 1. *If* $u^{m^*} \neq 0$, *then* $\dim \mathcal{M} \leq m^*$.

Proof. Let us decompose u, u', v and v' in blocks:

$$\mathcal{M} = \bigoplus_i \mathcal{M}_i,$$

$$u_{ij} \colon \mathcal{M}_j \to \mathcal{M}_i \quad v_{ij} \colon \mathcal{M}_j \to \mathcal{M}_i \cdots.$$

(Assume that $x_{ij} = 0$ if $\mathcal{M}_i$ or $\mathcal{M}_j$ is empty.)

Then let

$$u_{ij} = \delta_{ij} \lambda q^{2i} \, \text{Id}.$$

From (E) and $(++)$:

$$u'_{ii} = u_{ii}^{-1} = \lambda^{-1} q^{-2i} \, \text{Id},$$
$$v_{ii} = u_{ii}^{-1} = \lambda^{-1} q^{-2i} \, \text{Id},$$

and

$$(u' - u^{-1})_{ij} = 0 \quad \text{unless} \quad j = i - 1 \quad (\text{modulo } m^*),$$
$$(v - u^{-1})_{ij} = 0 \quad \text{unless} \quad j = i + 1 \quad (\text{modulo } m^*).$$

Let us define

$$u'_- = u' - u^{-1},$$
$$v_+ = v - u^{-1}$$

in order to write the algebra $\mathcal{A}$ in terms of u, u'_-, v_+ and relations.

Obviously, $\quad \begin{cases} [u, u'_- v_+] = 0 \\ [u, v_+ u'_-] = 0 \end{cases}.$

Inserting the expression of v' given by $(+-)$ into (F) and $(--)$ leads to the following relations:

$$qv_+^2 u'_- - (q+q^{-1})v_+ u'_- v_+ + q^{-1}u'_- v_+^2 = -(q-q^{-1})^2 (2)_q v_+ [q^{-1}u - q^2 u^{-2}] \quad \text{(F')}$$

and

$$qv_+ u'^2_- - (q+q^{-1})u'_- v_+ u'_- + q^{-1}u'^2_- v_+ = -(q-q^{-1})^2 (2)_q u'_- [qu - q^{-2}u^{-2}] \quad (--').$$

Combining these two relations leads to

$$[u'_- v_+, v_+ u'_-] = 0.$$

They also allow to check that

$$(u'_-)^{m^*} \in \text{center of } \mathscr{A},$$
$$(v_+)^{m^*} \in \text{center of } \mathscr{A}.$$

Let now x be a common eigenvector of $u, u'_- v_+$ and $v_+ u'_-$, associated to the eigenvalues $\lambda q^{2i}, a_+$ and a_-.
   Then

$$u \cdot (v_+ x) = q^{-2}\lambda q^{2i}(v_+ x),$$
$$v_+ u'_- \cdot (v_+ x) = a_+ (v_+ x),$$
$$u'_- v_+ \cdot (v_+ x) = [-q^2 a_- + q(q+q^{-1})a_+ - q(q-q^{-1})^2 (2)_q$$
$$\cdot (q^{-1}\lambda q^{2i} - q^2 \lambda^{-2}q^{-4i})](v_+ x).$$

$(v_+ x)$ is then a common eigenvector of $u, u'_- v_+$ and $v_+ u'_-$. Similarly,

$$u \cdot (u'_- x) = q^2 \lambda q^{2i}(u'_- x),$$
$$v_+ u'_- \cdot (u'_- x) = [-q^{-2}a_+ + q^{-1}(q+q^{-1})a_- - q^{-1}(q-q^{-1})^2 (2)_q$$
$$\cdot (q\lambda q^{2i} - q^{-2}\lambda^{-2}q^{-4i})](u'_- x),$$
$$u'_- v_+ \cdot (u'_- x) = a_- (u'_- x).$$

$(u'_- x)$ is also a common eigenvector of $u, u'_- v_+$ and $v_+ u'_-$.
   We are now ready to prove our Theorem 1: let us consider two cases:

– If u'_- is not injective, let x be a common eigenvector of $u, u'_- v_+$ and $v_+ u'_-$ with $a_- = 0$. Then $\text{Vect}\{v_+^k x\}_{k=0,...,m^*-1}$ is stable since $u'_- v_+^k x$ is proportional to $v_+^{k-1}x$ and since $v_+^{m^*}x$ is proportional to x. So $\text{Vect}\{v_+^k x\}_{k=0,...,m^*-1}$ is a submudule of $\mathscr{M}$, so is equal to $\mathscr{M}$ since $\mathscr{M}$ is simple. Hence $\dim \mathscr{M} \leqq m^*$.

– If u'_- is injective, let x be a common eigenvector of $u, u'_- v_+$ and $v_+ u'_-$. Then $\text{Vect}\{u'^k_- x\}_{k=0,...,m^*-1}$ is stable and hence equal to $\mathscr{M}$. So $\dim \mathscr{M} = m^*$.

We check then that the relations (F') and $(--')$ are compatible.

   Theorem 1 is then proved. A representation $\mathscr{M}$ of dimension $k \leqq m^*$ of $\mathscr{A}$ is then characterized by

● its dimension k
● the eigenvalues $\lambda \neq 0, a_+$ and a_- of $u, u'_- v_+$ and $v_+ u'_-$ on one of the commn eigenvectors. These values are related to the quadratic and cubic Casimirs (Cas) by

$$(q - q^{-1})^2 C_+ = (q + q^{-1})\lambda + \lambda^{-2} + \frac{qa_- - q^{-1}a_+}{q - q^{-1}}$$

$$(q - q^{-1})^2 C_+ = (q + q^{-1})\lambda^{-1} + \lambda^2 - \lambda\frac{a_+ - a_-}{q - q^{-1}}$$

- the value of $(u'_-)^{m^*}$ and $(v_+)^{m^*}$ on $\mathcal{M}$.

Let $\mathcal{M}$ be a simple module on $\mathcal{A}$, and let us now suppose that $u^{m^*} = 0$, $v^{m^*} = 0$, $u'^{m^*} = 0$ and $v'^{m^*} = 0$ on $\mathcal{M}$. We shall now prove the

Theorem 2. *There is no finite dimensional representation of $\mathcal{A}$ on which all the generators u, v, u' and v' are nilpotent, but for $m = 4$ $(m^* = 2)$, in which case there are four two-dimensional irreducible representations of $\mathcal{A}$.*

Proof. u is not yet diagonalizable. (Unless if $u = 0$ which contradicts (E) and $(+ +)$.) We shall first prove the

Lemma. *There is a basis of $\mathcal{M}$ on which u is written as a $m^* \times m^*$ matrix of blocks of size $N \times N$.*

$$u = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \text{Id} & 0 & \cdots & 0 & 0 \\ 0 & \text{Id} & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \text{Id} & 0 \end{pmatrix}$$

(Id *and* 0 *are* $N \times N$ *matrices.*)

Proof of the Lemma. We first choose a basis of $\mathcal{M}$ such that u takes a Jordan form, i.e. a matrix with zeroes everywhere but just under the diagonal where there can be either 0 or 1. Denote this basis

$$(x_k^{(i)})_{\substack{k = 0, \ldots, m_i - 1 \\ i = 1, \ldots, N}}.$$

It satisfies

$$ux_k^{(i)} = x_{k+1}^{(i)}$$

and

$$ux_{m_i - 1}^{(i)} = 0$$

such that $\mathcal{M} = \bigoplus_{i=1}^{N} \mathcal{M}_i$, where $\mathcal{M}_i = \text{Vect}\{x_k^{(i)}\}_{k = 0, \ldots, m_i - 1}$ is stable under the action of u.

Then decompose v, u' and v' in blocks, i.e.

$$u'_{ij}: \mathcal{M}_j \to \mathcal{M}_i \quad v_{ij}: \mathcal{M}_j \to \mathcal{M}_i \cdots.$$

Now $(+ +)$ applied on $\mathcal{M}_j$ and projected on $\mathcal{M}_i$ implies

- $m_i \neq 1$
-

$$v_{ii} = \begin{pmatrix} * & 1-q^2 & 0 & \cdots & 0 & 0 \\ * & * & 1-q^4 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ * & * & * & \cdots & * & 1-q^{2(m_i-1)} \\ * & * & * & \cdots & * & * \end{pmatrix} \quad \text{and} \quad (m_i) = 0$$

and hence $m_i = m^*$.

•

$$v_{ij} = \begin{pmatrix} * & 0 & 0 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ * & * & * & \cdots & * & 0 \\ * & * & * & \cdots & * & * \end{pmatrix} \quad \text{for} \quad i \neq j.$$

So $m_i = m^* \;\forall i \in \{1, \ldots, N\}$, and the lemma is proved after an inversion of the indexes k and i of our original basis

$$\mathcal{M} = \text{Vect}\{x_k^{(i)}\}_{\substack{i=1,\ldots,N \\ k=0,\ldots,m^*-1}} = \bigoplus_{k=0}^{m^*-1} \mathcal{N}_k.$$

Now uv is a triangular matrix with $(1 - q^{2(i-1)})\text{Id}$ in the diagonal, and is hence diagonalizable. Furthermore, all the eigenspaces have the same dimension N. We eventually choose the basis where

$$uv = \text{Diag}[(1 - q^{2(i-1)})\,\text{Id}],$$

where Id is the $N \times N$ identity matrix.

$$\text{Since} \quad \begin{cases} (1 - uv)v = q^{-2}v(1 - uv) \\ (1 - uv)u = q^2 u(1 - uv) \end{cases},$$

u and v in this basis are such that

$$u_{ij} = 0 \quad \text{unless} \quad j = i - 1 \quad (\text{modulo } m^*),$$
$$v_{ij} = 0 \quad \text{unless} \quad j = i + 1 \quad (\text{modulo } m^*),$$

and $u_{i,i-1}v_{i-1,i} = (1 - q^{2(i-1)})\text{Id}$, so that we can write

$$u = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \text{Id} & 0 & \cdots & 0 & 0 \\ 0 & \text{Id} & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \text{Id} & 0 \end{pmatrix},$$

$$v = \begin{pmatrix} 0 & (1-q^2)\text{Id} & 0 & \cdots & 0 & 0 \\ 0 & 0 & (1-q^4)\text{Id} & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (1-q^{2(m_i-1)})\text{Id} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

$$\text{(E)} \quad \text{implies} \quad \begin{cases} u'_{i,i+1} = 1 - q^{-2i}\,\text{Id} \\ u'_{ij} = 0 & \text{for} \quad j > i+1. \\ u'_{i+1,j+1} = q^{-2} u'_{ij} & \text{for} \quad j \leqq i \end{cases}$$

$$\text{(--)} \quad \text{implies} \quad \begin{cases} v'_{i+1,i} = -q^{-2i}\,\text{Id} \\ v'_{ij} = 0 & \text{for} \quad j < i-1, \\ v'_{i+1,j+1} = q^{-2}\dfrac{1-q^{2j}}{1-q^{2i}} v'_{ij} & \text{for} \quad j \geqq i \end{cases}$$

but $(+\,-)$ also allows to compute v', and in particular $v'_{i+1,i}$. If $m \neq 4$ $(m^* \neq 2)$, the two expressions of $v'_{i+1,i}$ are incompatible. In this case, there is no finite dimensional representation of $\mathscr{A}$ with u, v, v' and u' nilpotent. Theorem 2 is then almost proved.

Finally, if $m = 4$ $(q = i)$

$$u = \begin{pmatrix} 0 & 0 \\ \text{Id} & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 2\,\text{Id} \\ 0 & 0 \end{pmatrix}, \quad u' = \begin{pmatrix} u'_{11} & 2\,\text{Id} \\ u'_{21} & -u'_{11} \end{pmatrix}, \quad v' = \begin{pmatrix} v'_{11} & v'_{12} \\ \text{Id} & -v'_{22} \end{pmatrix},$$

$$\text{with} \quad \begin{cases} u'_{21} = -u'^2_{11}/2 & \text{since} \quad u'^2 = 0 \\ v'_{11} = i u'^2_{11}/2 \\ v'_{12} = 2i u'_{11} \end{cases}$$

u'_{11} is then the only non-trivial operator. If x is an eigenvector of u'_{11} with eigenvalue λ, then $\text{Vect}\{x\} \oplus \text{Vect}\{x\} \subset \mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ is stable. We hence obtain a two dimensional representation of $\mathscr{A}$. Note that λ satisfies $\lambda^4 = 8i\lambda$ since $v'^2 = 0$, which allows four distinct values for λ.

So our Theorem 2 is proved and we summarize this section with

Theorem. *The finite dimensional irreducible representations of $\mathscr{A}$ have their dimension between 1 and m $\left(\text{respectively } \dfrac{m}{2} \text{ for } m \text{ even}\right)$.*

Hence, back to $\mathscr{U}(SU(3))_q$:

Theorem. *The finite dimensional irreducible representations of $\mathscr{U}(SU(3))_q$, where all the generators are injective are of dimension $k(2m)^2$ with $k = 1, \ldots, m$ $\left(\text{respectively } \dfrac{m}{2} \text{ for even } m\right)$.*

Note that f_1 and e_2 injective is actually the only hypothesis we have made.

Note also that the non-trivial generators e_1 and f_2 (2) are $2m$-idempotent on irreducible representations, which is equivalent to the fact that u, v, v' and u' are m^*-idempotent on irreducible representations of $\mathscr{A}$.

IV. Truncation and Flat Representations of $\mathscr{U}(SU(3))_q$

In this section, we perform a truncation of the representation defined above, in which the operators β, β', γ and γ' (or u, v, u' and v') are chosen to be scalars (i.e.

in their one dimensional representation). This $(2m)^2$ dimensional representation M is described by $(v_{p_1,p_2})_{p_1,p_2=0,\ldots,2m-1}$,

$$f_1 \cdot v_{p_1,p_2} = \alpha_1 \cdot v_{p_1+1,p_2},$$

$$e_2 \cdot v_{p_1,p_2} = \alpha_2 \cdot v_{p_1,p_2-1},$$

$$e_1 \cdot v_{p_1,p_2} = \left[\frac{1}{\alpha_1}(p_1-x)_q(\mu_1-x-p_1+p_2+1)_q\right] \cdot v_{p_1-1,p_2},$$

$$f_2 \cdot v_{p_1,p_2} = \left[\frac{1}{\alpha_2}(p_2+1-y)_q(\mu_2-y-p_2+p_1)_q\right] \cdot v_{p_1,p_2+1},$$

where

$$u = q^{+1/3(\mu_1-\mu_2)+(\mu_1+\mu_2+1)-2x} = u'^{-1},$$

$$v = q^{-1/3(\mu_1-\mu_2)+(\mu_1+\mu_2+1)-2y} = v'^{-1},$$

and

$$x + y = \mu_1 + \mu_2 + 1 \quad \text{since} \quad uv = 1.$$

The truncation is obtained by choosing particular values of x, y, μ_1 and μ_2, and forgetting the constraint that f_1 and e_2 should be idempotent.

Let us first fix x and y to be integers, say 0. (Hence $\mu_1 + \mu_2 + 1 = 0$.) Then

$$e_1 v_{0,p_2} = 0 \quad \text{and} \quad e_1 v_{m,p_2} = 0$$

$$e_1 v_{m/2,p_2} = 0 \quad \text{and} \quad e_1 v_{3m/2,p_2} = 0 \quad \text{for even } m$$

$$f_2 v_{p_1,m-1} = 0 \quad \text{and} \quad f_2 v_{p_1,2m-1} = 0$$

$$f_2 v_{p_1,(m/2)-1} = 0 \quad \text{and} \quad f_2 v_{p_1,(3m/2)-1} = 0 \quad \text{for even } m.$$

M still remains a module when we perform the following change on f_1 and e_2:

$$f_1 v_{m^*-1,p_2} = 0,$$

$$e_2 v_{p_1,0} = 0.$$

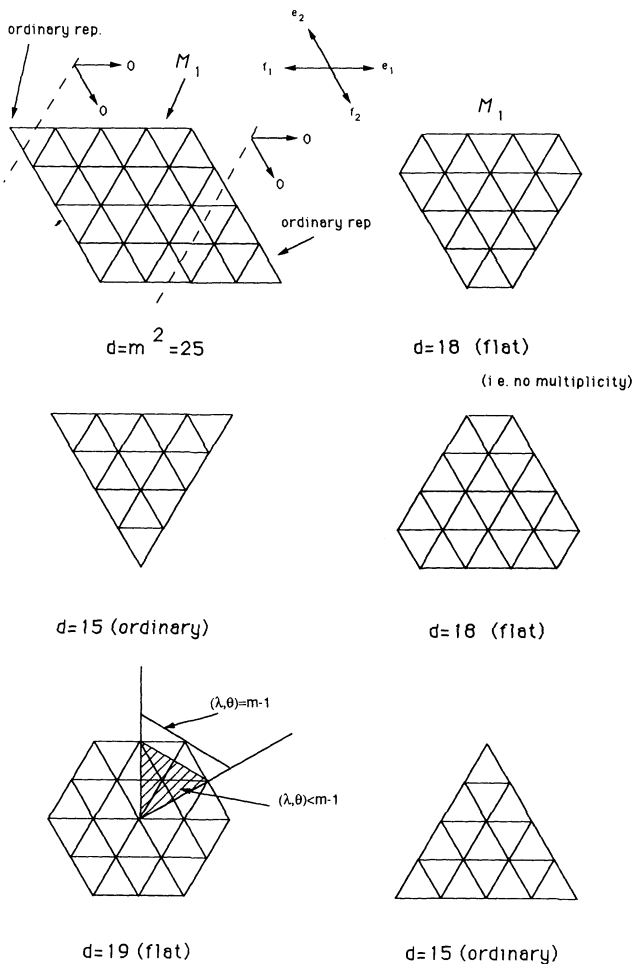α_1 and α_2 can then be set equal to 1 by a change of basis since there is no periodicity left.

But now M is no longer irreducible and $M_0 = \text{Vect}\{v_{p_1,p_2}\}_{p_1,p_2=0,\ldots,m^*-1}$ is a submodule of M. The periodicity of the set of weights is lost, so there is no invariance under the action of the Weyl group. But the sets of weights provided by the action of the Weyl group correspond to another truncation, obtained by setting for example $\mu_1 - x$ and $\mu_2 - y$ to be integers instead of x_1 and x_2.

We finally choose integer values for μ_1 (and hence μ_2) so that

$$(\mu_1 - x - p_1 + p_2 + 1)_q = -(\mu_2 - y - p_2 + p_1)_q$$

appearing in the expressions of e_1 and f_2 can also vanish. M_0 is no longer irreducible (but indecomposable). The upper-left part of M_0 (see fig.) is a submodule of M_0.

This submodule enters in the category of ordinary irreducible representations of $\mathcal{U}(SU(3))_q$. Its highest weight λ is in the authorized sector (i.e. the part of the first Weyl chamber in which $(\lambda, \theta) < m^* - 1$, where θ is the highest root of $SU(3)$. This sector is painted in the case $m = 5$ on the figure representing the $d = 19$ representation.) Note nevertheless that the values of $q^{h_1/2}$ and $q^{h_2/2}$ on this submodule are not necessarily centered at 1, but at (ω_1, ω_2), where $\omega_i^4 = 1$ [1, 2].

d=m^2=25   d=18 (flat)

(i e. no multiplicity)

d=15 (ordinary)   d=18 (flat)

d=19 (flat)   d=15 (ordinary)

Figures for m=5

All the multiplicities of its weights are 1 (its Young tableau is ▭▭▭···▭ with strictly less than $m^* - 1$ ▢'s and corresponds to the completely symmetric case).

Quotienting M_0 by this upper-left submodule leads to another indecomposable module, whose upper-left part is again a submodule we call M_1. Quotienting by M_1 leads to an irreducible representation of $\mathcal{U}(SU(3))_q$, this time corresponding to the Young Tableau ⊞⊞···⊟ with stricly less than $m^* - 1$ columns. All its weights have multiplicity 1 and it also corresponds to a pair (ω_1, ω_2) not necessarily equal to $(1,1)$.

Back to M_1, the most interesting part of M_0. This module corresponds to a highest weight λ on the line just after the edge of the authorized zone of the first Weyl chamber, i.e. $(\lambda, \theta) = m^* - 1$. It does not correspond for example to an integrable representation in a WZW theory. Its particular feature is that it has

exactly the shape of an ordinary representation of $SU(3)$ or $\mathcal{U}(SU(3))_q$ with generic q, but all the multiplicities of its weights are 1 by construction unlike the ordinary case, so that we call it a "flat representation." This means that the representations of $\mathcal{U}(SU(3))_q$ with highest weight λ such that $(\lambda, \theta) = m^* - 1$ do not remain irreducible when $q^m = 1$ (unless they are triangular ones). If $m = 3$ for example, there exists a $d = 7$ irreducible representation of $\mathcal{U}(SU(3))_q$, arising from the decomposition of the ordinary $d = 8$ representation. This is another new fact when comparing to the $\mathcal{U}(SU(2))_q$ case [2].

The sets of weights of the representations satisfying $(\lambda, \theta) = m^* - 1$ are drawn on the figure in the case $m = 5$.

The whole content of this section is immediately generalizable to $\mathcal{U}(SU(N))_q$: there exist $(2m)^{N-1}$-dimensional irreducible periodic representations which can be truncated (or partially truncated) to flat representations. This is nevertheless not true for the quantum analogue of the other simply laced algebras D or E. This remark and further investigations on $\mathcal{U}(SU(N))_q$ will be the subject of a different publication.

V. Subtlety for m Multiple of 3

When m is a multiple of 3, we have

$$[q^{h_1/2}, f_1^{2m/3} e_2^{2m/3}] = [q^{h_2/2}, f_1^{2m/3} e_2^{2m/3}] = 0.$$

The sum (1) cannot consequently be a direct sum, since (p_1, p_2), $\left(p_1 + \dfrac{4m}{3}, p_2 + \dfrac{2m}{3}\right)$, and $\left(p_1 + \dfrac{2m}{3}, p_2 + \dfrac{4m}{3}\right)$ correspond to the same common eigenspace of $q^{h_1/2}$ and $q^{h_2/2}$. We then have to restrict the sum (1) to

$$M = \bigoplus_{\substack{p_1 = 0,\ldots,2m-1 \\ p_2 = 0,\ldots,2m/3-1}} M_{\mu_1/2 - p_1 + p_2/2, \mu_2/2 - p_2 + p_1/2}, \tag{3}$$

and e_2 has to be redefined on the boundary $p_2 = 0$ of this domain by

$$e_2 \colon M_{\mu_1/2 - p_1, \mu_2/2 + p_1/2} \to M_{\mu_1/2 - (p_1 + 4m/3) + (m/3 - 1/2), \mu_2/2 + (p_1 + 4m/3)/2 - (2m/3 - 1)},$$
$$V_{p_1, 0} \mapsto \alpha_2 A_{p_1} V_{p_1 + 4m/3, 2m/3 - 1}.$$

The operator A_{p_1} cannot be chosen to be proportional to the identity (i.e. preserving the choice of basis) since $f_1^{2m/3} e_2^{2m/3}$ is not in the center of the algebra.

Since $[f_1, e_2^{2m/3}] = 0$, we have $A_{p_1} = A_{p_1 + 1}$ so $A_{p_1} = A$ does not depend on p_1.

Since $e_2^{2m} = \alpha_2^{2m} \mathrm{Id}$, we have $A^3 = \mathrm{Id}$.

The relations $[e_i, f_i] = (h_i)_q$ and two of the Serre relations still allow the derivation of the expressions of e_1 and f_2 and the definition of β, β', γ and γ' (or u, u', v and v'). But another consequence of $[e_2, f_2] = (h_2)_q$ on the boundary of the domain $\left(p_2 = 0 \text{ or } \dfrac{2m}{3} - 1\right)$ is

$$\begin{aligned} Av &= q^{4m/3} vA, \\ Av' &= q^{2m/3} v'A, \end{aligned} \tag{Av}$$

whereas the Serre relation $(e_2^2 e_1 - \cdots = 0)$ implies

$$Au = q^{2m/3}uA,$$
$$Au' = q^{4m/3}u'A. \tag{Au}$$

u, v, u' and v' still have to satisfy the relations of definition of $\mathscr{A}$, which are compatible with (Au) and (Av).

Since at least one of the operators u, v, u' and v' is invertible, (Au) and (Av) associated with $A^3 = \mathrm{Id}$ imply that the three eigenspaces of A have the same dimension N. So A can be written as a permutation of three N-dimensional vector spaces

$$A = \begin{pmatrix} 0 & 0 & \mathrm{Id} \\ \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \end{pmatrix}.$$

Each common eigenspace of $q^{h_1/2}$ and $q^{h_2/2}$ splits then in three parts of dimension N and we write

$$M_{\mu_1/2 - p_1 + p_2/2, \mu_2/2 - p_2 + p_1/2} = \mathscr{N}_{p_1, p_2} \oplus \mathscr{N}_{p_1 + 4m/3, p_2 + 2m/3} \oplus \mathscr{N}_{p_1 + 2m/3, p_2 + 4m/3}$$

so that

$$M = \bigoplus_{\substack{p_1 \in \mathbb{Z}_{2m} \\ p_2 \in \mathbb{Z}_{2m}}} \mathscr{N}_{p_1, p_2}.$$

The general strategy can then be followed, and the periodic irreducible representations of $\mathscr{U}(SU(3))_q$ with m multiple of 3 will also have the dimensions $k(2m)^2$ with $k = 1, \ldots, m^*$. In particular, no further reducibility dividing the dimension by 3 will occur as suggested by (3).

VI. Conclusion

The eventually known irreducible representations of $\mathscr{U}(SU(3))_q$ for q a root of unity are the following:

– the highest weight representations characterized by a highest weight λ such that $(\lambda, \theta) < m^* - 1$

– the periodic representations, of dimensions $k(2m)^2$ with $k = 1, \ldots, m^*$, also characterized by continuous parameters. This paper exhausts this type of representatations.

– the flat representations, characterized by a highest weight λ such that $(\lambda, \theta) = m^* - 1$. All the weights of these representations have multiplicity one.

We might think that this gives a complete classification of the irreducible finite dimensional representations of $\mathscr{U}(SU(3))_q$.

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