Commun. Math. Phys. 134, 347–370 (1990)

Communications in
Mathematical
Physics
© Springer-Verlag 1990

Quantum and Classical Pseudogroups. Part I. Union Pseudogroups and Their Quantization

S. Zakrzewski

Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Hoza 74. PL-00-682 Warszawa, Poland

Received December 14, 1989

Abstract. Union pseudogroups (structures analogical to pseudogroups in the sense of [1]) are defined using the category dual to the category of groupoids instead of the category of pseudospaces in the sense of [2]. It is shown that these structures are equivalent to double groups (in the sense of [3]). Moreover, it is shown that a quantization procedure associates with each finite union pseudogroup a (quantum) pseudogroup. Therefore for each finite double group there is a finite pseudogroup.

Introduction

Many attempts have been made in order to create a theory of objects more general than groups, essentially by considering not necessarily commutative spaces. Those efforts are valuable because of the following two features of such generalizations:
1) a principle of duality, which, in particular, can be applied to noncommutative groups,
2) a description of new kind of symmetries.

In spite of the same fundamental idea, there exist many different approaches using even completely different words denoting the objects generalizing groups, such as pseudogroups [1], quantum groups [4], Kac algebras, Hopf algebras, etc. The fundamental idea of all these approaches consists in using the category of linear maps as a basic language. One formulates the notion of a group in this language and then one observes that there is room for "groups with noncommutative space." In the sequel we propose to call such objects "quantum pseudogroups" in order to emphasize the existence of approaches based on categories other than linear (= quantum).

In this paper we study one such approach based on binary relations. Since binary relations are union maps (Sect. 1), the category they form is said to be the union category and we obtain a notion of a union pseudogroup (Sect. 6). A quantization procedure (Sect. 4) applied to each finite union pseudogroup produces a finite quantum pseudogroup (Sect. 8). In Sect. 9 and Sect. 10 we show that union pseudogroups coincide with double groups. Therefore double groups provide union pseudogroups with a "second face." It is interesting that the "union pseudogroup picture" is appropriate for quantization, while the "double group picture" has a very simple structure and allows to find examples.

In our next paper we study differential pseudogroups and symplectic pseudogroups (using differentiable relations and symplectic relations, respectively). Their quantization will be studied in subsequent publications.

1. Binary Relations, Union Spaces and Algebras

A *binary relation* is a triple $r=(R;\, Y, X)$, where X and Y are sets (X is the *domain* of r, Y is the *codomain* of r) and R is a subset of $Y \times X$ (R is the *graph* of r and is denoted by $\mathscr{G}(r)$). We say that $r=(R;\, Y, X)$ is a relation *from* X *to* Y and we denote it by $r: X \to Y$.

Two relations, $r: Y \to Z$ and $s: X \to Y$ can be composed in a standard way, the composed relation rs being a relation from X to Z. Binary relations (with this composition) form a category. The *transpose relation* of a relation $r: X \to Y$ is a relation $r^T: Y \to X$ such that $\mathscr{G}(r^T)=\{p \in X \times Y : p=tq,\ q \in \mathscr{G}(r)\}$, where $t: Y \times X \to X \times Y$ is the flip map defined by $t(y,x)=(x,y)$. The *transposition* of relations, i.e. the assignment $r \mapsto r^T$ is an involutive contravariant functor.

A useful fact about the category of binary relations is that it is isomorphic to a concrete category (i.e. such category whose morphisms are mappings and the composition is the composition of mappings). Let $\mathscr{R}(X, Y)$ denote the set of all relations from X to Y. Let $\{1\}$ denote a distinguished set consisting of one point, 1. We shall identify the set $\mathbb{B}X=\mathscr{R}(\{1\}, X)$ with the set of all subsets in X. To each relation $r: X \to Y$ there corresponds a map $\mathbb{B}r: \mathbb{B}X \to \mathbb{B}Y$, defined by

$$(\mathbb{B}r)(A)=rA \quad \text{for} \quad A \subset X .$$

A map $F: \mathbb{B}X \to \mathbb{B}Y$ is said to be a *union map* if it commutes with taking unions, i.e.

$$F\Big(\bigcup_{A \in \mathscr{A}} A\Big) = \bigcup_{A \in \mathscr{A}} F(A) \quad \text{for any } \mathscr{A} \subset \mathbb{B}X ,$$

and preserves the empty set: $F(\emptyset)=\emptyset$. Union maps form a (concrete) category. The assignment $r \mapsto \mathbb{B}r$ is a functor from the category of binary relations to the category of union maps. This functor is actually an isomorphism.

A *union space* is an object of the category of union maps, i.e. the set $\mathbb{B}X$ of all subsets in a set X, equipped with its obvious "union structure" (see appendix on *abstract* union spaces).

Union spaces with union maps resemble to some extent vector spaces with linear maps. This allows to introduce several structures based on union spaces, by imitating some familiar structures based on vector spaces.

A map $F: \mathbb{B}X \times \mathbb{B}Y \to \mathbb{B}Z$ is said to be a *bi-union map*, if $F(A \times I): \mathbb{B}Y \to \mathbb{B}Z$ and $F(I \times B): \mathbb{B}X \to \mathbb{B}Z$ are union maps for all $A \subset X$, $B \subset Y$ (we shall always denote the identity map by I). A *tensor product* $\mathbb{B}X \otimes \mathbb{B}Y$ of two union spaces is defined as in the case of vector spaces (by a universal bi-union map $\mathbb{B}X \times \mathbb{B}Y \to \mathbb{B}X \otimes \mathbb{B}Y$) and can be identified with $\mathbb{B}Z$, where Z is the set-theoretic direct product of X and Y (note also that a *direct product* exists in the category of union spaces and corresponds to the set-theoretic disjoint union of the underlying sets).

Now let us compare the following two definitions.
Definition. A *vector algebra* is a pair (V, μ), where V is a vector space and $\mu: V \otimes V \to V$ is a linear map, which is associative, i.e. $\mu(\mu \otimes I) = \mu(I \otimes \mu)$.

Definition. A *union algebra* is a pair (U, M), where U is a union space and $M: U \otimes U \to U$ is a union map which is associative, i.e. $M(M \otimes I) = M(I \otimes M)$.

Example 1.1. The intersection $A \cap B$ of two subsets of X defines an associative bi-union map $\cap: \mathbb{B}X \times \mathbb{B}X \to \mathbb{B}X$, hence a structure of a union algebra on $\mathbb{B}X$. This is the usual *algebra of subsets* of X.

Example 1.2. The product $AB = \{xy : x \in A,\ y \in B\}$ of two subsets A, B in a group $X = G$ defines an associative bi-union map from $\mathbb{B}X \times \mathbb{B}X$ to $\mathbb{B}X$. The corresponding union algebra is said to be the *union group algebra* of G.

Example 1.3. The composition of relations in a set Z defines a union map $M: \mathbb{B}X \otimes \mathbb{B}X \to \mathbb{B}X$, where $X = Z \times Z$, such that $M(\mathscr{G}(r) \otimes \mathscr{G}(s)) = \mathscr{G}(rs)$ for any relations r, $s: Z \to Z$. (X, M) is said to be the *algebra of all relations in* Z.

For simplicity, we shall work with binary relations rather than union maps. Therefore we rewrite the above definition of a union algebra in terms of relations. Since our basic category is the category of relations, we use the tensor notation $X \otimes X$ for the set-theoretic product $X \times X$. A *tensor product of relations* is naturally defined.

Definition. A *union algebra* is a pair (X, m), where X is a set and $m: X \otimes X \to X$ is a relation which is associative, i.e.

$$m(m \otimes I) = m(I \otimes m). \tag{1}$$

In the above examples, the multiplication map M corresponds to a *multiplication relation* $m: X \otimes X \to X$, given (respectively) by
1. $m = d^T$, where $d: X \to X \otimes X$ is the *diagonal map*
2. m is the group multiplication
3. $$m((x, y), (z, t)) = \begin{cases} \emptyset, & \text{if} \quad y \neq z \\ (x, t) & \text{if} \quad y = z. \end{cases}$$

In the last point we have used the following simplified notation for the *image* $r\{x\}$ *of a point* $x \in X$ *under a relation* $r: X \to Y$. We shall write $r(x)$ instead of $r\{x\}$, if there is no danger of a confusion.

2. Union Algebras with Unit

A union algebra (X, m) *has a unit* if there exists a relation $e: \{1\} \to X$ (the *unit*) such that

$$m(e \otimes I) = I = m(I \otimes e). \tag{2}$$

In this case e is unique and we have the following three lemmas.
We set $E = e(1)$.

Lemma 2.1. *If* $a, b \in E$ *then* $m(a, b) \neq \emptyset$ *if and only if* $a = b$ *and in this case* $m(a, b) = a$.

Proof. If $m(a, b) \neq \emptyset$ then $a = m(a, b) = b$. □

Lemma 2.2. *There exist unique two mappings* $e_L, e_R: X \to E$ *such that* $m(e_L(x), x) \neq \emptyset$ $\neq m(x, e_R(x))$ *for* $x \in X$. *These mappings have the following properties:*
(i) $m(e_L(x), x) = x = m(x, e_R(x))$ *for* $x \in X$,
(ii) $e_L(a) = a = e_R(a)$ *for* $a \in E$.

Proof. For each $x \in X$ there exists $a \in E$ such that $m(a, x) \neq \emptyset$ and then $m(a, x) = x$. If $b \in E$ is such that $m(b, x) \neq \emptyset$ then $\emptyset \neq m(b, m(a, x)) = m(m(b, a), x)$, hence $m(b, a) \neq \emptyset$ and, by Lemma 2.1, $b = a$. This shows the existence and uniqueness of e_L. A similar argument works for e_R. Property (ii) follows from Lemma 2.1.

Definition. The mapping e_L (respectively e_R) is said to be *left* (respectively *right*) *projection*. For any $a \in E$, the set ${}_aX = e_L^{-1}(a)$ (respectively $X_a = e_R^{-1}(a)$) is said to be the *left* (respectively *right*) *fiber* over a.

Lemma 2.3. *For any* $x, y \in X$, $m(x, y) \neq \emptyset$ *implies* $e_R(x) = e_L(y)$, $e_L(m(x, y)) = e_L(x)$ *and* $e_R(m(x, y)) = e_R(y)$.

Proof. If $m(x, y) \neq \emptyset$ then $\emptyset \neq m(m(x, e_R(x)), y) = m(x, m(e_R(x), y))$, hence $m(e_R(x), y) \neq \emptyset$ and $e_R(x) = e_L(y)$. We have

$$m(e_L(x), m(x, y)) = m(m(e_L(x), x), y) = m(x, y) \neq \emptyset$$

and, similarly, $m(m(x, y), e_R(y)) \neq \emptyset$. □

A *union algebra with unit* is a triple (X, m, e), where (X, m) is a union algebra having a unit denoted by e. Let (X, m, e) and (X', m', e') be two union algebras with unit. A *morphism* from (X, m, e) to (X', m', e') is a relation $h: X \to X'$ such that

$$hm = m'(h \otimes h) \tag{3}$$

and

$$he = e'. \tag{4}$$

Let h be as above. We set $E = e(1)$, $E' = e'(1)$. Let $h_0: E \to E'$ denote the *base relation* of h, defined by $\mathcal{G}(h_0) = \mathcal{G}(h) \cap (E' \times E)$. Then we have two following lemmas.

Lemma 2.4. $(e'_L \times e_L)\mathcal{G}(h) = \mathcal{G}(h_0) = (e'_R \times e_R)\mathcal{G}(h)$.

Proof. We prove the first equality. If $(x', x) \in \mathcal{G}(h)$ then

$$x' \in h(x) = hm(e_L(x), x) = m'(h_0(e_L(x)), h(x)),$$

hence $e'_L(x') \in e'_L(h_0(e_L(x))) = h_0(e_L(x))$ and $(e'_L(x'), e_L(x)) \in \mathcal{G}(h_0)$. The converse inclusion follows from the definition of h_0. □

Lemma 2.5. $h_0^T: E' \to E$ *is a map.*

Proof. From (4) we have $h_0(E) = E'$. If a, $b \in h_0^T(a')$ then $hm(a, b) = m'(h(a), h(b))$ $\supset m'(a', a') \neq \emptyset$, hence $a = b$. □

3. Union Star Algebras

A bijection $s: X \to X$ is said to be a *star operation* in a union algebra (X, m) if $s^2 = I$ and $sm = m(s \otimes s)t$.

It is easy to see that if (X, m) has, additionally, a unit e, then $se = e$ and $e_L(sx)$ $= e_R(x)$ for $x \in X$.

A *union star algebra with unit* is a quadruple (X, m, e, s), where (X, m, e) is a union algebra with unit and s is a star operation in (X, m).

An important example of a union star algebra with unit is the algebra of all relations in a set Z, $\operatorname{End} Z = (Z \times Z, m, \mathscr{G}(I), t)$ with m defined in Sect. 1.

The inverse $x \mapsto x^{-1}$ in a group is a star operation in the union group algebra (Example 1.2).

Let (X, m, e, s) and (X', m', e', s') be two union star algebras with unit. A *morphism* from (X, m, e, s) to (X', m', e', s') is a relation $h: X \to X'$ satisfying (3), (4) and

$$hs = s'h. \tag{5}$$

A *representation* of (X, m, e, s) *in a set* Z is a morphism from (X, m, e, s) to $\operatorname{End} Z$. Such a representation is said to be *faithful* if the underlying relation h is a union monomorphism (i.e. $\mathbb{B}h$ is injective).

A very important class of star algebras in the vector case is the class of C^*-algebras. A finite-dimensional star algebra $\mathscr{A}$ is a C^*-algebra if and only if the following positivity condition is satisfied:

$$\textit{if} \quad A \in \mathscr{A}, A \neq 0 \quad \text{then} \quad A^*A \neq 0$$

(cf. [5]). In the union case this corresponds to the following condition

$$\textit{if} \quad A \subset X, A \neq \emptyset \quad \text{then} \quad m(sA \otimes A) \neq \emptyset,$$

which is equivalent to the following statement

$$m(sx, x) \neq \emptyset \quad \text{for each } x \in X. \tag{6}$$

This property holds in particular in union star algebras with unit having a faithful representation.

Lemma 3.1. *If condition* (6) *is satisfied, then* $sa = a$ *for each* $a \in E$.

Proof. For $a \in E$ we have $sa \in E$ and $m(sa, a) \neq \emptyset$, hence $sa = a$. $\square$

Note that in Example 1.3 and Example 1.1 there exists only one star-operation satisfying (6) and then

$$m(sx, x) \subset E \quad \text{for} \quad x \in X. \tag{7}$$

Also the inverse in the group case (Example 1.2) satisfies (7). In the differentiable case (Part II of this work) we shall be interested in smooth deformations of our fundamental examples, and it is reasonable to expect that the star invariant set $m(sx, x)$ will remain in E (in our examples there exists a neighborhood of E in which E is the set of fixed points of s).

Definition. A *U^*-algebra* is a union star algebra with unit (X, m, e, s) such that

$$\emptyset \neq m(sx, x) \subset E \quad \text{for} \quad x \in X. \tag{8}$$

We refer to condition (8) as to *strong positivity condition.* We consider U^*-algebras as union counterparts of C^*-algebras. Morphisms of U^*-algebras are defined as before as relations satisfying (3), (4), and (5).

Lemma 3.2. *If* (X, m, e, s) *is a* U^**-algebra, then for* $x, y \in X$ *we have*

(i) $m(sx, x) = e_R(x)$, $m(x, sx) = e_L(x)$,

(ii) $e_R(x) = e_L(y)$ *implies* $m(x, y) \neq \emptyset$,

(iii) $m(x, y) \cap E \neq \emptyset$ *implies* $y = sx$,

(iv) $m(x, y)$ *is either empty, or consists of one point.*

Proof. (i) $m(sx, x) = e_R(m(sx, x)) = e_R(x)$.
(ii) $\emptyset \neq m(x, e_R(x)) = m(x, e_L(y)) = m(x, m(y, sy)) = m(m(x, y), sy)$.
(iii) If $a \in m(x, y) \cap E$ then $e_R(sx) = e_L(x) = e_L(a)$ and we have

$$sx = m(sx, a) \subset m(sx, m(x, y)) = m(m(sx, x), y) = m(e_R(x), y) = y.$$

(iv) If z, $t \in m(x, y)$ then $e_R(sz) = e_L(z) = e_L(t)$, and we have

$$\emptyset \neq m(sz, t) \subset m(sm(x, y), m(x, y)) = m(m(sy, sx), m(x, y))$$
$$= m(sy, m(m(sx, x), y)) = m(sy, m(e_R(x), y)) = m(sy, y) = e_R(y),$$

and from (iii) it follows that $z = t$. □

Point (iii) of the above lemma shows that the star operation in a U^*-algebra is determined by m:

$$m^T E = \mathscr{G}(s) = (I \otimes s)\mathscr{G}(I). \tag{9}$$

Example 3.1. A U^*-algebra such that m is a mapping is necessarily a group. It is easy to see that morphisms of such U^*-algebras are group homomorphism (cf. Lemma 3.3 below).

Example 3.2. Let (X, m, e, s) be a U^*-algebra and let $a \in E$. Then $\emptyset \neq m(x, y) \in {}_aX \cap X_a$ for all x, $y \in {}_aX \cap X_a$. The relation m defines on ${}_aX \cap X_a$ a structure of a group.

Lemma 3.2 shows that U^*-algebras coincide with groupoids [6, 7]. We prefer to use here "nonstandard" terminology for two reasons:
1° we choose a terminology consistent with general scheme of "algebras,"
2° our approach to morphisms of these objects is completely different from the standard one (we are talking about different categories!).

Lemma 3.3. *Let $h: X \to X'$ be a morphism from a U^*-algebra (X, m, e, s) to a U^*-algebra (X', m', e', s'). For $b \in E'$, let ${}_bh: {}_aX \to {}_bX'$ (respectively $h_b: X_a \to X'_b$) be a relation defined by*

$$\mathscr{G}({}_bh) = \mathscr{G}(h) \cap ({}_bX' \times {}_aX) \quad (\textit{respectively } \mathscr{G}(h_b) = \mathscr{G}(h) \cap (X'_b \times X_a)),$$

where $a = h_0^T(b)$. Then ${}_bh$ and h_b are mappings.

Proof. If $x \in {}_aX$, then

$$b \in ha = he_L(x) = hm(x, sx) = m'(hx, s'hx).$$

It follows that there exists $y \in hx$ such that $b \in e'_L(y)$. Such y is unique: if $z \in hx$, $b \in e'_L(z)$, then $\emptyset \neq m'(s'y, z) \in m'(s'hx, hx) = hm(sx, x) = he_R(x) \subset E'$, hence $y = z$. □

Definition. A *U^*-coalgebra* is a quadruple (X, d, c, r), where X is a set, d, c, r are relations and (X, d^T, c^T, r^T) is a U^*-algebra. *Morphisms* of U^*-coalgebras are defined in a way dual to morphisms of U^*-algebras, by "reversing the arrows in suitable diagrams" (cf. [5]). U^*-coalgebras are also said to be *union pseudospaces* or *U^*-spaces.*

A union pseudospace (X, d, c, r) is said to be finite, if X is a finite set.

There is a complete symmetry between the theory of U^*-algebras and the theory of union pseudospaces, passage from one point of view to another being given by the transposition functor. If $M = (X, m, e, s)$ is a U^*-algebra ($D = (X, d, c, r)$ is a union pseudospace) then $M^T = (X, m^T, e^T, s^T)$ is a union pseudospace ($D^T = (X, d^T, c^T, r^T)$ is a U^*-algebra). The choice of a particular point of view depends what aspects we are interested in, algebraic or geometric. It is often easier to work with U^*-algebras, because such structures as multiplication, unit, and the star operation (the inverse of a groupoid) are familiar from the theory of algebras (groups).

The correspondence $h \mapsto h^T$ defines a contravariant functor from the category of U^*-algebras to the category of union pseudospaces and establishes a duality between these categories. The following example shows that the category of sets is a (full) subcategory of the category of union pseudospaces.

Example 3.3. Let (X, d, c, r) be a union pseudospace such that d is a mapping. Then d has to be the usual diagonal map, c has to be the unique (constant) map from X to $\{1\}$ and $r = I$. Such a union pseudospace will be denoted by D_X. A relation $f: X \to Y$ is a morphism from D_X to D_Y if and only if f is a map (f coincides with its base map, f_0).

If $D = (X, d, c, r)$ is a union pseudospace, then the left (respectively right) projection in D^T will be denoted by c_L (respectively c_R). We set also $C = c^T(1)$.

A *point* of a union pseudospace D is a morphism from $D_{\{1\}}$ to D. Let $p: \{1\} \to X$ be a point of $D = (X, d, c, r)$. Then the base map $p_0: \{1\} \to C$ may be identified with the point $p_0 = p_0(1) \in C \cap P$, where $P = p(1)$.

Lemma 3.4. $P = {}_{p_0}X = X_{p_0}$. *In particular, P equipped with $d^T|_{P \times P}$ is a group.*

Proof. If $c_L(x) = p_0$ then $\emptyset \neq p^T c_L(x) = p^T d^T(x, rx) = p^T(x) \otimes p^T(rx)$, hence $x \in P$. If $x \in P$, then $c_L(x) = c_L(p(1)) = p_0(1)$.

Corollary. *A point $p: \{1\} \to X$ of a union pseudospace (X, d, c, r) is uniquely determined by its base point p_0.*

Lemma 3.5. $(p^T \otimes I)d = pp^T = (I \otimes d^T)d$.

Proof. If $x \in P$, then $d^T(p \otimes I)(x) = P = pp^T(x)$. If $x \notin P$, then $d^T(p \otimes I)(x) = \emptyset = pp^T(x)$. $\square$

A *character* of a U^*-algebra $M = (X, m, e, s)$ is a morphism from M to $D^T_{\{1\}}$. If $q: X \to \{1\}$ is a character of M then Lemma 3.5 says that

$$m(q^T \otimes I) = q^T q = m(I \otimes q^T)$$

(cf. [1], Proposition A.2.1, point 1).

4. Quantization of Binary Relations

A binary relation is said to be *finite* if it has a finite domain and a finite codomain.

For any finite set X we denote by $\mathbb{C}X$ the vector space of formal linear combinations of elements of X with coefficients in $\mathbb{C}$ (the field of complex numbers). For any finite relation $r: X \to Y$ we denote by $\mathbb{C}r$ a linear map from $\mathbb{C}X$ to $\mathbb{C}Y$ defined as follows

$$(\mathbb{C}r)x = \sum_{y \in r(x)} y \quad \text{for} \quad x \in X$$

(X is a basis in $\mathbb{C}X$).

The correspondence $r \mapsto \mathbb{C}r$ (*quantization*) is not a functor. For two finite relations, $r: Y \to Z$, $s: X \to Y$ we have $\mathbb{C}(rs) = \mathbb{C}(r)\mathbb{C}(s)$ if and only if

$$\text{for each } (z, x) \in \mathscr{G}(rs) \text{ there exists only one } y \in Y \text{ such that } y \in s(x) \text{ and } z \in r(y). \tag{10}$$

Definition. Two binary relations r and s are said to *have a simple composition* if condition (10) is satisfied. In this case we shall write $r \rfloor s$.

For general relations $r: Y \to Z$, $s: X \to Y$ the composed relation rs is a disjoint sum

$$rs = (r,s)_1 \cup (r,s)_2 \cup \ldots,$$

where for any cardinal number n, $(r,s)_n$ is a relation from X to Z such that

$$\mathscr{G}((r,s)_n) = \{(z,x) \in Z \times X : r^T(z) \cap s(x) \text{ has } n \text{ elements}\}.$$

Then we have (if the relations are finite)

$$\mathbb{C}(r)\mathbb{C}(s) = \mathbb{C}((r,s)_1) + 2\mathbb{C}((r,s)_2) + \ldots = \mathbb{C}(rs) + \mathbb{C}(rs \backslash (r,s)_1) + \ldots.$$

Condition (10) means that $rs = (r,s)_1$.

Definition. A relation $s: X \to Y$ is said to be *simple* if for each relation r with domain Y, r and s have a simple composition. A relation r is said to be *co-simple*, if r^T is simple.

It is easy to see that simple relations from X to Y are exactly relations of the form $r = fi^T$, where $i: X_0 \to X$ is the inclusion of a subset X_0 in X and $f: X_0 \to Y$ is a map.

There is a *canonical scalar product* on $\mathbb{C}X$ with respect to which the set $X \subset \mathbb{C}X$ is an orthonormal basis. For each finite relation r we have $\mathbb{C}(r^T) = \mathbb{C}(r)^\dagger$ where $\dagger$ denotes the hermitian conjugation of (continuous) linear maps of Hilbert spaces.

There is a canonical antilinear map $J: \mathbb{C}X \to \mathbb{C}X$ called the *complex conjugation*, such that $Jx = x$ for $x \in X$. The complex conjugation J is antiunitary and commutes with linear maps of the form $\mathbb{C}(r)$.

Linear maps $\varrho: V_1 \to V_2$ of finite-dimensional vector spaces are in one-to-one correspondence with their *kernels* $\mathscr{K}(\varrho) \in V_2 \otimes V_1^*$. The correspondence is given by

$$\langle \alpha \otimes v, \mathscr{K}(\varrho) \rangle = \langle \alpha, \varrho v \rangle \quad \text{for} \quad \alpha \in V_2^*,\ v \in V_1.$$

If $r: X \to Y$ is a finite relation, then $\mathscr{K}(\mathbb{C}(r))$ is related to $\mathbb{C}(\mathscr{G}(r))$ by the following formula

$$\mathscr{K}(\mathbb{C}(r)) = (1 \otimes FJ)\mathbb{C}(\mathscr{G}(r)), \tag{11}$$

where $F: \mathbb{C}X \to (\mathbb{C}X)^*$ is the Frechet-Riesz anti-isomorphism (defined for each Hilbert space).

Remark. Binary relations with simple composition form a WP-category in the sense of [8, 9]. The quantization $r \mapsto \mathbb{C}r$ is a functor from the WP-category of finite binary relations with simple composition to the category of linear maps of finite-dimensional Hilbert spaces. This functor commutes with canonical involutions. It is possible to consider a quantization of infinite relations. To each relation $r: X \to Y$ there corresponds a linear continuous map $\mathbb{C}r: C_0(X) \to C(Y)$, where $C_0(X)$ is the space of formal linear combinations of elements of X (the inductive limit of $\mathbb{C}X_0$ over finite parts X_0 of X) and $C(X)$ denotes the space of all complex functions on X (the projective limit of $\mathbb{C}X_0$ over finite parts X_0 of X). The space of linear continuous maps from $C_0(X)$ to $C(Y)$ is isomorphic to $C(Y \times X)$. Elements of this space are said to be *infinite matrices*. Infinite matrices $a \in C(Z \times Y)$, $b \in C(Y \times X)$ are said to be *composable* if for each $(z,x) \in Z \times X$, the sum $\sum_{y \in Y} a(z,y)b(y,x)$ is absolutely convergent. By Fubini theorem, infinite matrices form a WP-category and the correspondence $r \mapsto \mathbb{C}r$ is a functor from the WP-category of binary relations with simple composition to the above WP-category.

A quantization of differentiable relations and symplectic relations requires in general some additional structure, typically a measure (see [10, 11] etc.).

5. Quantization of U^*-Algebras and Union Pseudospaces

Definition. A finite-dimensional *Hilbert algebra* is a quadruple $\mathscr{A}=(H,\mu,\varepsilon,\sigma)$, where H is a finite-dimensional Hilbert space, $\mu: H\otimes H\to H$, $\varepsilon:\mathbb{C}\to H$ are linear maps and $\sigma: H\to H$ is an antiunitary map such that

$$\mu(\mu\otimes I)=\mu(I\otimes\mu), \tag{12}$$

$$\mu(\varepsilon\otimes I)=I=\mu(I\otimes\varepsilon), \tag{13}$$

$$\sigma^2=I, \tag{14}$$

$$\sigma\mu=\mu(\sigma\otimes\sigma)\tau, \tag{15}$$

$$\mu^\dagger\varepsilon=(I\otimes\sigma F^{-1})\mathscr{K}(I), \tag{16}$$

where $\tau: H\otimes H\to H\otimes H$ is a linear map defined by $\tau(a\otimes b)=b\otimes a$, for a, $b\in H$. Conditions (12)–(15) mean that $\mathscr{A}$ is an involutive algebra with unit. Condition (16) is equivalent to the following property:

$$(\mathbb{1}\,|\,ab)=(b^\sigma\,|\,a) \quad \text{for} \quad a,b\in H,$$

where $\mathbb{1}=\varepsilon(1)$, $ab=\mu(a\otimes b)$ and $b^\sigma=\sigma(b)$. Using this notation we have

$$(ab\,|\,c)=(\mathbb{1}\,|\,(ab)^\sigma c)=(\mathbb{1}\,|\,b^\sigma a^\sigma c)=(b\,|\,a^\sigma c),$$

hence

$$(ab\,|\,c)=(b\,|\,a^\sigma c). \tag{16'}$$

The last condition is used in a standard definition of a Hilbert algebra instead of condition (16) (see [5]). Condition (16′) says that the left regular representation $\lambda: H\to \operatorname{End} H$, defined by $\lambda(a)b=\mu(a\otimes b)$ for a, $b\in H$, is a $*$-representation of $\mathscr{A}$ in H:

$$\lambda(a^\sigma)=\lambda(a)^\dagger \quad \text{for} \quad a\in H.$$

Since λ is faithful, it follows that $\mathscr{A}$ is a C^*-algebra.

Any quadruple $(H,\mu^\dagger,\varepsilon^\dagger,\sigma^\dagger)$, where $(H,\mu,\varepsilon,\sigma)$ is a finite dimensional Hilbert algebra is said to be a finite-dimensional *Hilbert coalgebra* (in fact $\sigma^\dagger=\sigma$, since σ is an unitary involution). Any finite-dimensional Hilbert coalgebra is (in particular) a C^*-coalgebra (see [5] for a definition).

Proposition 5.1. *Let $M=(X,m,e,s)$ be a U^*-algebra such that X is a finite set (*i.e. *M^T is a finite union pseudospace). Then $C^*(M)=(\mathbb{C}X, \mathbb{C}m, \mathbb{C}e, J\mathbb{C}s)$ is a finite-dimensional Hilbert algebra.*

Proof. The projections e_L and e_R are unique and m is simple (by Lemma 3.2(iv)), hence we have only simple compositions in (1) and (2). Applying quantization, we obtain (12) and (13) with $\mu=\mathbb{C}m$, $\varepsilon=\mathbb{C}e$. It is also clear that $\sigma=J\mathbb{C}s$ is antiunitary and satisfies (14) and (15). From (9) we have

$$\mu^\dagger\varepsilon=(I\otimes\mathbb{C}s)\cdot\mathbb{C}\mathscr{G}(I).$$

Using (11) we have

$$\mu^\dagger\varepsilon=(I\otimes\mathbb{C}sJF^{-1})(I\otimes FJ)\cdot\mathbb{C}\mathscr{G}(I)=(I\otimes J\mathbb{C}s\cdot F^{-1})\mathscr{K}(I). \quad \square$$

$C^*(M)$ is said to be the Hilbert algebra of M. By the dual construction, to each union pseudospace $D=(X,d,c,r)$ there corresponds a Hilbert coalgebra $C^*(D)=(\mathbb{C}X, \mathbb{C}d, \mathbb{C}c, J\mathbb{C}r)$.

Lemma 5.2. *If $h: X \to X'$ is a morphism from a U^*-algebra (X, m, e, s) to a U^*-algebra (X', m', e', s'), then all compositions in* (3), (4), *and* (5) *are simple.*

Proof. Let y, $z \in X$ and $x' \in X'$ be such that $x' \in hm(y, z)$. Let y', $z' \in X'$ be some intermediate points for the right-hand side of (3), i.e. $x' \in m'(y', z')$, $y' \in h(y)$ and $z' \in h(z)$. It follows that $e'_L(y') = e'_L(x')$ and, by Lemma 3.3, y' is uniquely determined by x' and y. Similarly we prove that z' is unique. □

Lemma 5.3. *Composition of morphisms of U^*-algebras is always simple.*

Proof. It follows from the structure of morphisms, as described by Lemma 3.3. If $h: X \to Y (k: Y \to Z)$ is a morphism from M to M' (M' to M''), $z \in k(y)$ and $y \in h(x)$, then $e'_L(y) = k_0^T(e''_L(z))$ and y is uniquely determined by x and z. □

By Lemma 5.2 we can quantize not only finite U^*-algebras and union pseudospaces, but also their morphisms. By a morphism of finite-dimensional C^*-algebras we mean a $*$-homomorphism preserving the unit. A dual definition is assumed for morphisms of finite-dimensional C^*-coalgebras (these morphisms form precisely what is meant by the category of finite quantum pseudospaces). By Lemma 5.2, to each morphism h of U^*-algebras (f of union pseudospaces) there corresponds a morphism $\mathbb{C}h$ of C^*-algebras ($\mathbb{C}f$ of pseudospaces). By Lemma 5.3, the assignments $h \mapsto \mathbb{C}h$, $f \mapsto \mathbb{C}f$ are functors.

6. Union Pseudogroups

A *product* of two union pseudospaces $D_1 = (X_1, d_1, c_1, r_1)$ and $D_2 = (X_2, d_2, c_2, r_2)$ is a union pseudospace defined as follows:

$$D_1 \otimes D_2 = (X_1 \otimes X_2, (I \otimes t \otimes I)(d_1 \otimes d_2), c_1 \otimes c_2, r_1 \otimes r_2).$$

Definition. A *union pseudogroup* or *U^*-group* is a pair $P = (D, m)$, such that
(i) $D = (X, d, c, r)$ is a union pseudospace,
(ii) $m: X \otimes X \to X$ is a morphism from $D \otimes D$ to D which is associative:

$$m(m \otimes I) = m(I \otimes m),$$

(iii) there exist relations e, $e': \{1\} \to X$ such that

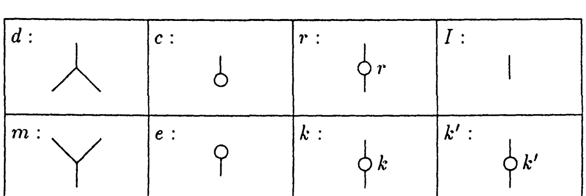
$$m(e \otimes I) = I = m(I \otimes e')$$

(in this case $e = e'$ is unique),
(iv) there exist relations k, $k': X \to X$ such that

$$m(k \otimes I)d = ec = m(I \otimes k')d. \tag{17}$$

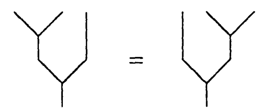
P is said to be *finite* if X is a finite set.

In order to simplify some proofs in the sequel, we introduce graphic symbols denoting the elementary relations d, c, r, I, m, e, k, k', as given by the following table:

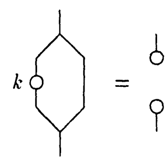


We believe that the graphic representation of a complicated formula is more clear than the standard linear one. Here are examples of some formulas written graphically:

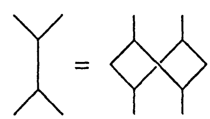
$m(m \otimes I) = m(I \otimes m)$:



$m(k \otimes I)d = ec$:



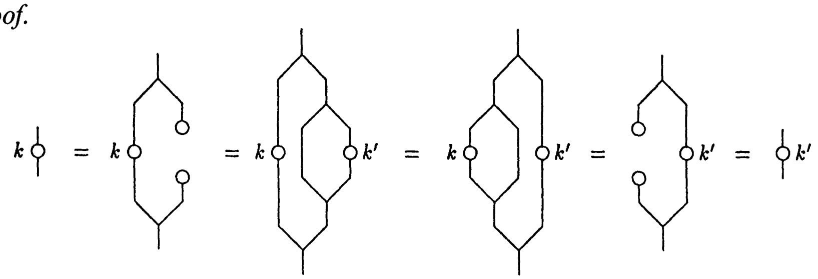
$dm = (m \otimes m)(I \otimes t \otimes I)(d \otimes d)$:



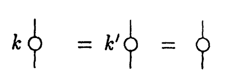
All diagrams represent relations acting **downward**.

Lemma 6.1. *$k = k'$ and k is unique.*

Proof.



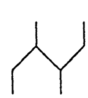
We introduce the following notation:



Lemma 6.2. *Let $s = kr$. Then $s^2 = I$ and, in particular, k is bijection.*

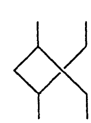
Proof. We introduce two relations

U :

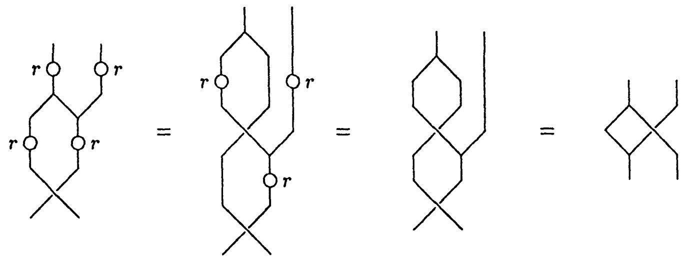


and

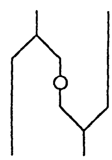
V :



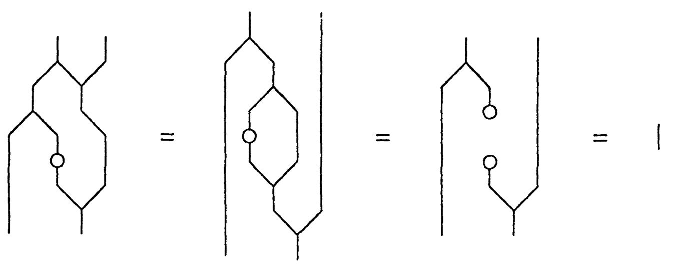
We have $V=t(r\otimes r)U(r\otimes r)$, because



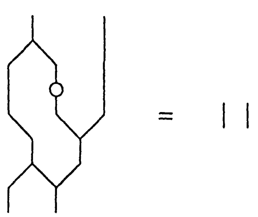
We shall show that U is invertible, and U^{-1} is given by the following diagram:



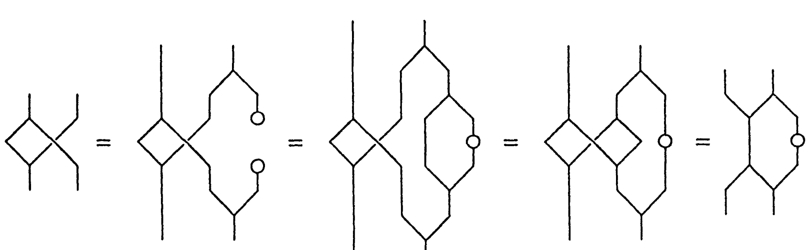
In fact,



and, similarly,

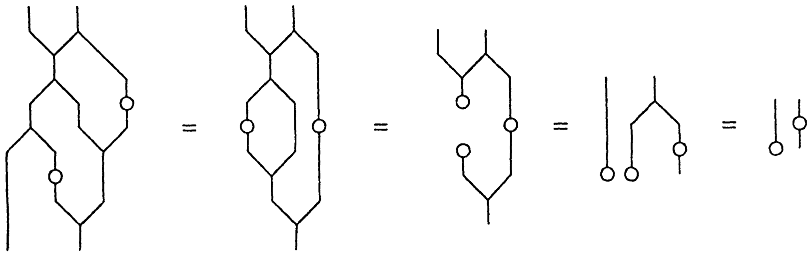


We shall use another form of V:



(18)

We have $(c\otimes I)U^{-1}V=c\otimes k$, because



If we set $W=U^{-1}V(r\otimes r)$, then

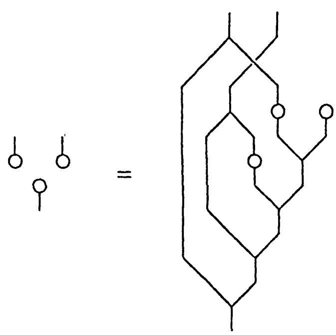
$$\begin{aligned}W^2&=U^{-1}V(r\otimes r)U^{-1}V(r\otimes r)=U^{-1}t(r\otimes r)UU^{-1}V(r\otimes r)\\&=U^{-1}t(r\otimes r)V(r\otimes r)=U^{-1}U=I,\end{aligned}$$

and $(c\otimes I)W=cr\otimes kr=c\otimes s$, hence

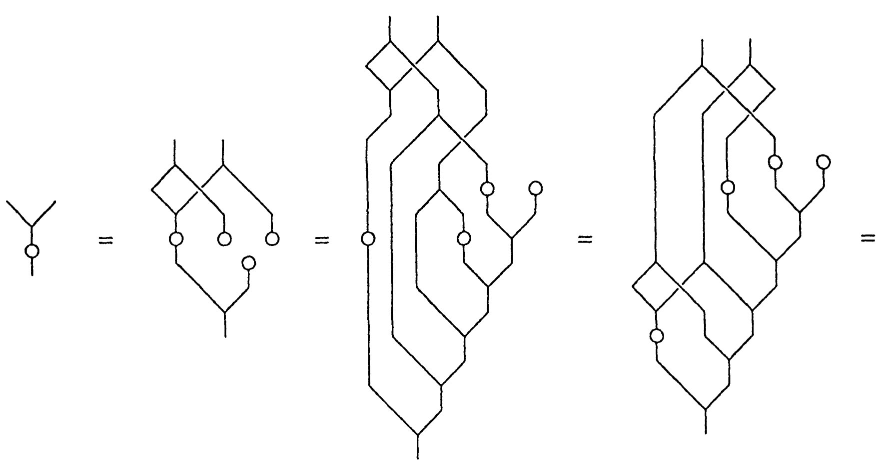
$$c\otimes I=(c\otimes I)W^2=(c\otimes s)W=s(c\otimes I)W=s(c\otimes s)=c\otimes s^2. \quad \square$$

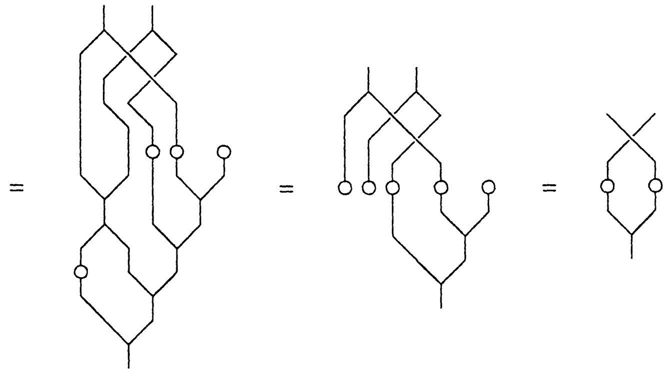
Lemma 6.3. *k is antimultiplicative*, i.e. $km=m(k\otimes k)t$.

Proof. Using the following formula



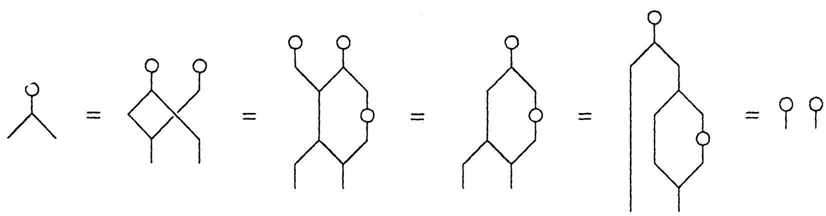
we obtain





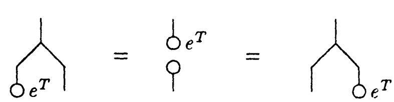
Lemma 6.4. *e is a point of D.*

Proof.
(i) We have



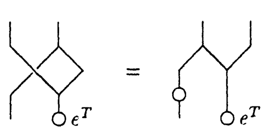
(see (18)), hence $de = e \otimes e$.
(ii) We have $c = cI = cm(e \otimes I) = (c \otimes c)(e \otimes I) = ce \otimes c$, hence $ce = I$.
(iii) $m(re \otimes I) = m(re \otimes rr) = m(r \otimes r)\ (e \otimes r) = rm(e \otimes I)r = rr = I$, hence $re = e$ by the uniqueness of e. □

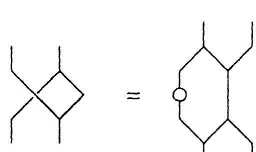
Corollary. *If we denote e^T graphically by*  *e^T, then we have*

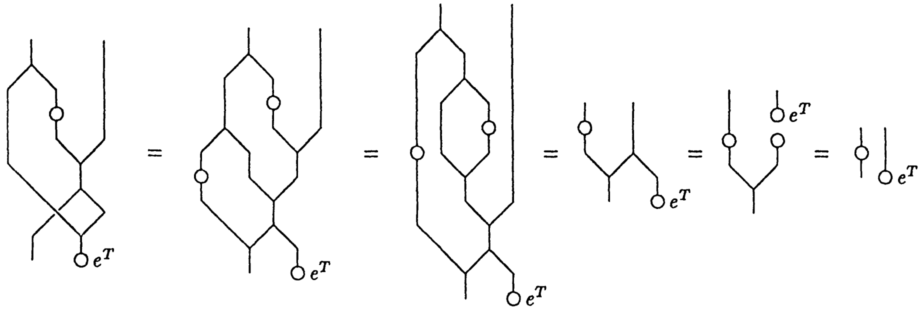


(see Lemma 3.5).

Lemma 6.5.



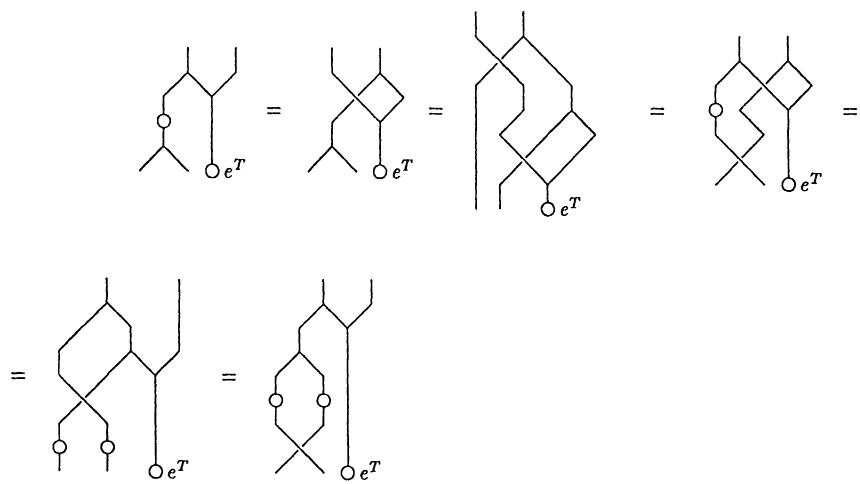
Proof. Since  like in (18), we have



and the statement follows from the form of U and U^{-1} in Lemma 6.2. □

Lemma 6.6. *k is anti-comultiplicative*, i.e. $dk = t(k \otimes k)d$.

Proof. Applying three times Lemma 6.5, we have



Proposition 6.7. *$U^*(P) = (X, m, e, s)$ is a U^*-algebra.*

Proof. From Lemma 6.3 and 6.6, s is antimultiplicative. We shall show that condition (6) is satisfied. In fact, if x_0 is such that $m(sx_0, x_0) = \emptyset$ then $s' : X \to X$, defined by

$$s'(x) = \begin{cases} s(x) & \text{for} \quad x \neq x_0 \\ \emptyset & \text{for} \quad x = x_0 , \end{cases}$$

satisfies $m(s'r \otimes I)d = ec$, but this contradicts the uniqueness of k. From (17) it follows that condition (7) is satisfied. □

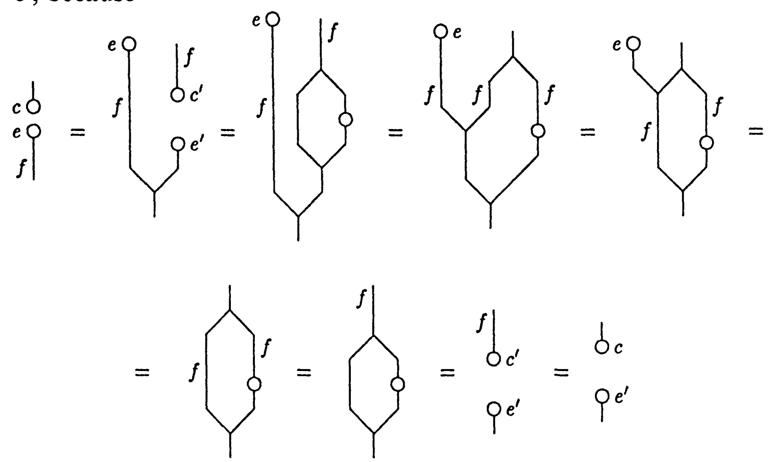
$U^*(P)$ is said to be the *union group algebra* of P.

Definition. If $P = (D, m)$, $P' = (D', m')$ are two union pseudogroups, $D = (X, d, c, r)$, $D' = (X', d', c', r')$, then a relation $f : X \to X'$ is a *morphism* from P to P' if f is a morphism from D to D' and

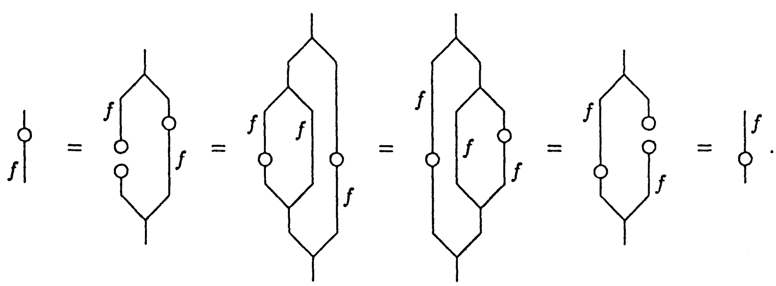
$$fm = m'(f \otimes f).$$

Lemma 6.8. *A morphism from P to P' is also a morphism from $U^*(P)$ to $U^*(P')$.*

Proof.
(i) $fe = e'$, because



(we have introduced a new line $f|$, which denotes f and we have labelled some other relations explicitly for the reader's convenience).
(ii) $fk = k'f$:



7. Union Kac Algebras and Duality

A *union Kac algebra* is a pair (D, M), where $D = (X, d, c, r)$ is a union pseudospace and $M = (X, m, e, s)$ is a U^*-algebra (with the same set X) such that

$$dm = (m\otimes m)(I\otimes t\otimes I)(d\otimes d), \tag{19}$$

$$cm = c\otimes c, \tag{20}$$

$$rm = m(r\otimes r), \tag{21}$$

$$de = e\otimes e, \tag{22}$$

$$ce = I, \tag{23}$$

$$re = e, \tag{24}$$

$$ds = (s\otimes s)d, \tag{25}$$

$$cs = c, \tag{26}$$

and

$$m(sr\otimes I)d = ec = m(I\otimes sr)d. \tag{27}$$

Above conditions are not independent.

Lemma 7.1. *If $D=(X,d,c,r)$ is a union pseudospace. $M=(X,m,e,s)$ is a U^*-algebra and condition* (27) *is satisfied, then three following conditions are equivalent:*
(i) *(D,M) is a union Kac algebra,*
(ii) *m is a morphism from $D\otimes D$ to D,*
(iii) *d is a morphism from M to $M\otimes M$.*

Proof. We shall prove that (ii) implies (i). Since (D,m) is a union pseudogroup, it follows from Lemma 6.4 that (22), (23), and (24) are fulfilled. From Lemma 6.6 we have (25). Formula (26) follows from (25) exactly in the same way as (24) follows from (22) in the proof of Lemma 6.4. □

Proposition 7.2. *There is a bijective correspondence between union pseudogroups and union Kac algebras, given by*

$$P=(D,m)\mapsto(D,U^*(P)).$$

Proof. It follows from Proposition 6.7 and Lemma 7.1. □

Morphisms of union Kac algebras are relations (between the underlying sets) which are both morphisms of the underlying pseudospaces and morphisms of the underlying U^*-algebras. By Lemma 6.8, the category of union pseudogroups is isomorphic to the category of union Kac algebras.

Applying the transposition functor to all ingredients of a union Kac algebra (D,M), we obtain a union Kac algebra (M^T,D^T), which is said to be *dual* to (D,M). This allows (by Proposition 7.2) to associate with each union pseudogroup a *dual union pseudogroup.* By Lemma 6.8, to each morphism of union pseudogroups corresponds (via $f\mapsto f^T$) a morphism of dual union pseudogroups. We have therefore an involutive contravariant functor expressing a *principle of duality* for union pseudogroups.

8. Quantization of Union Pseudogroups

A *finite quantum pseudogroup* is a pair $\mathscr{P}=(\mathscr{D},\mu)$, such that
(i) $\mathscr{D}=(V,\delta,\gamma,\varrho)$ is a finite quantum pseudospace (cf. Sect. 5),
(ii) $\mu: V\otimes V\to V$ is a morphism from $\mathscr{D}\otimes\mathscr{D}$ to $\mathscr{D}$, which is associative (Eq. (12)),
(iii) there exist linear maps ε, $\varepsilon': \mathbb{C}\to V$ such that

$$\mu(\varepsilon\otimes I)=I=\mu(I\otimes\varepsilon').$$

In this case $\varepsilon=\varepsilon'$ is unique.
(iv) there exist linear maps κ, $\kappa': V\to V$ such that

$$\mu(\kappa\otimes I)\delta=\varepsilon\gamma=\mu(I\otimes\kappa')\delta.$$

All lemmas of Sect. 6 are valid in the case of finite quantum pseudogroups, because of the universal character of the proofs given there (the validity of Corollary after Lemma 6.4 follows from [1], Proposition A.2.1, point 1). In particular, $\kappa'=\kappa$, $\kappa\varrho\kappa\varrho=I$ and κ is anti-comultiplicative. Passing to the dual space, $A=V^*$ and dual maps, $\Phi=\mu^*$ etc., and choosing an arbitrary vector basis $(a_k)_{k=1,\dots,N}$ in A, we can introduce an $N\times N$ matrix of elements of A, $u=(u_{kl})_{k,l=1,\dots,N}$ as follows:

$$\Phi(a_k)=\sum_{i=1}^{N} a_i\otimes u_{ik}.$$

Then the elements of u generate A:

$$a_k=(\varepsilon^*\otimes I)\Phi(a_k)=\sum_i \varepsilon^*(a_i)u_{ik}$$

and

$$a_k = (I \otimes \varepsilon^*)\Phi(a_k) = \sum_i a_i \varepsilon^*(u_{ik}),$$

hence $\varepsilon^*(u_{ik}) = \delta_{ik}$, where δ_{ik} is the Kronecker symbol. From the co-associativity of Φ one can easily deduce that

$$\sum_i a_i \otimes \Phi(u_{ik}) = \sum_{ij} a_i \otimes u_{ij} \otimes u_{jk},$$

hence $\Phi(u_{ik}) = u_{ij} \otimes u_{jk}$. We have also

$$\sum_r \delta^*(\kappa^*(u_{kr}) \otimes u_{rl}) = \delta^*(\kappa^* \otimes I)\Phi(u_{kl}) = \gamma^* \varepsilon^*(u_{kl}) = \delta_{kl}\gamma^*,$$

and, similarly,

$$\sum_r \delta^*(u_{kr} \otimes \kappa^*(u_{rl})) = \delta_{kl}\gamma^*.$$

It follows that (A, u) is a finite matrix pseudogroup in the sense of [1].

Proposition 8.1. *Let $P = (D, m)$ be a finite union pseudogroup. Then $(C^*(D), \mathbb{C}m)$ is a finite quantum pseudogroup.*

Proof. It suffices to prove that compositions in (17) (or (27)) are simple. We shall show that

$$m(s \otimes I) 1 (r \otimes I) d.$$

Since $m(I \otimes s)(I \otimes r)d = ec$, we have for $x, y \in X$,

$$m(x, sy) \neq \emptyset \neq d^T(x, ry)$$

if and only if $E \ni m(x, sy) \neq \emptyset \neq d^T(x, ry) \in C$, i.e. $y = x$. It follows that

$$e_R(x) = e_L(sy) \quad \text{and} \quad c_R(x) = c_L(ry) \quad \text{implies} \quad x = y. \tag{28}$$

Let $a \in E$, $b \in C$ and let x, y be such that

$$n(sx, x) = m(sy, y) \quad \text{and} \quad d^T(rx, x) = d^T(ry, y)$$

(i.e. (x, x), (y, y) are two intermediary points when composing $m(s \otimes I)$ and $(r \otimes I)d$), hence $e_R(x) = e_R(y)$ and $c_R(x) = c_R(y)$. It follows from (28) that $x = y$. □

9. Double Groups

Let G be a group and let B be a subgroup of G. Let A be a subset of G which has at most one common point with each left and each right coset of B in G. In other words, each element x of $X = AB \cap BA$ (the product AB is defined in Example 1.2) has a unique decomposition

$$x = a_L(x) b_R(x) = b_L(x) a_R(x),$$

where $a_L(x)$, $a_R(x) \in A$ and $b_L(x)$, $b_R(x) \in B$. Maps a_L, $a_R : X \to A (b_L, b_R : X \to B)$ are said to be the *left* and the *right projection* on A (on B).

We define a relation $\alpha : X \otimes X \to X$ as follows:

(i) if $a_R(x) \neq a_L(y)$ then $\alpha(x, y) = \emptyset$,

(ii) if $a_R(x) = a_L(y)$ then $\alpha(x, y) = b_L(x) y = x b_R(y)$

(the last equality follows from $b_L(x) y = b_L(x) a_L(y) b_R(y) = b_L(x) a_R(x) b_R(y) = x b_R(y)$).

We have

$$\alpha(A \otimes I) = I = \alpha(I \otimes A).$$

(A is identified with a relation from $\{1\}$ to X with the image equal to A.) In fact, for each $x\in X$ there exists exactly one element $a\in A$ such that $\alpha(a,x)\neq\emptyset$, namely $a=a_L(x)$, and $\alpha(a_L(x),x)=b_L(a_L(x))x=x$.

Let $s_A: X\to X$ be a map defined as follows:

$$s_A(x)=b_L(x)^{-1}a_L(x)=a_R(x)b_R(x)^{-1}.$$

We have $s_A^2=I$.

Proposition 9.1. *(X,α,A,s_A) is a U^*-algebra.*

Proof. We have $a_L(\alpha(x,y))=a_L(x)$, $a_R(\alpha(x,y))=a_R(y)$, $b_L(\alpha(x,y))=b_L(x)b_L(y)$ and $b_R(\alpha(x,y))=b_R(x)b_R(y)$ if $\alpha(x,y)\neq\emptyset$. Nowe we have
(i) the associativity of α: it is easy to see that

$$\alpha(\alpha(x,y),z)\neq\emptyset\Leftrightarrow a_R(x)=a_L(y),\, a_R(y)=a_L(z)\Leftrightarrow\alpha(x,\alpha(y,z))\neq\emptyset$$

and in this case

$$\alpha(\alpha(x,y),z)=b_L(x)yb_R(z)=\alpha(x,\alpha(y,z)).$$

(ii) antimultiplicativity of s_A:

$$\alpha(s_Ay,s_Ax)\neq\emptyset\Leftrightarrow a_R(s_Ay)=a_L(s_Ax)\Leftrightarrow a_L(y)=a_R(x)\Leftrightarrow\alpha(x,y)\neq\emptyset,$$

and in this case

$$\alpha(s_Ay,s_Ax)=b_L(s_Ay)b_L(x)^{-1}a_L(x)=b_L(\alpha(x,y))^{-1}a_L(\alpha(x,y))=s_A\alpha(x,y).$$

(iii) $\alpha(s_Ax,x)=b_L(x)^{-1}x=a_R(x)$. □

Now let A and B be two subgroups in G such that $A\cap B=\{o\}$, where o is the neutral element of G. In this case there is a relation $\beta: X\otimes X\to X$ and an involution s_B, defined analogously as α and s_A, by interchanging the role of A and B.

Proposition 9.2. *With the above assumptions we have*

$$\beta^T\alpha\supset(\alpha\otimes\alpha)(I\otimes t\otimes I)(\beta^T\otimes\beta^T), \tag{29}$$

$$\alpha(s_As_B\otimes I)\beta^T\subset AB^T, \tag{30}$$

$$\alpha(I\otimes s_As_B)\beta^T\subset AB^T. \tag{31}$$

Moreover, the following conditions are equivalent:
(i) $AB=BA$,
(ii) *equality in* (30),
(iii) *equality in* (31),
(iv) *equality in* (29).

Proof. If x, y, z, t are such that $\alpha(x,y)\neq\emptyset\neq\alpha(z,t)$ and $\beta(x,z)\neq\emptyset\neq\beta(y,t)$, then

$$b_R(\alpha(x,y))=b_R(x)b_R(y)=b_L(z)b_L(t)=b_L(\alpha(z,t)),$$
$$a_R(\beta(x,z))=a_R(x)a_R(z)=a_L(y)a_L(t)=a_L(\beta(y,t)),$$

and

$$\beta(\alpha(x,y),\alpha(z,t))=\alpha(x,y)a_R(t)=b_L(x)ya_R(t)=b_L(x)\beta(y,t)=\alpha(\beta(x,z),\beta(y,t)).$$

This implies (29). The graph of the left-hand side of (30) consists of pairs $(\alpha(s_Ax,y)$, $\beta(s_Bx,y))$, where $x,y\in X$ are such that $a_L(x)=a_L(y)$ and $b_L(x)=b_L(y)$. The last two equalities imply $x=y$, since $x=s_A(b_L(x)^{-1}a_L(x))$. It follows that the graph consists of pairs $(a_R(x),\, b_R(x))$, $x\in X$, hence (30). The proof of (31) is similar.

Now we prove the equivalences.
(i) ⇒ (ii).
If $AB=BA$, then for each $a\in A$, $b\in B$ there exists $x\in X$ such that $a_R(x)=a$ and $b_R(x)=b$, namely $x=s_A(ab^{-1})$.
(ii) ⇒ (i).
If $a\in A$, $b\in B$, then there exists $x\in X$ such that $a_R(x)=a$, $b_R(x)=b^{-1}$. Then $a_L(s_A x)=a$ and $b_R(s_A x)=b$, hence $s_A x=ab$. It follows that $ab\in X$.
(i) ⇔ (iii) is proved similarly.
(iv) ⇒ (i).
For each $a\in A$, $b\in B$ we have $(b^{-1},b)\in\beta^T\alpha(a,a^{-1})$, hence there exists $z\in X$ such that $a_L(z)=a$, $b_R(z)=b$, hence $ab\in X$.
(i) ⇒ (iv).
Let p, q, u, v be such that $\alpha(p,q)=\beta(u,v)\neq\emptyset$. We have to show that there exist x, y, z, t such that

$$u=\alpha(x,y), \quad v=\alpha(z,t), \quad p=\beta(x,z), \quad q=\beta(y,t). \tag{32}$$

Equations $a_L(x)=a_L(u)$, $b_L(x)=b_L(p)$ have a solution $x=s_A(b_L(p)^{-1}a_L(u))$. Equations $a_R(y)=a_R(u)$, $b_L(y)=b_L(q)$ imply that $y=b_L(q)a_R(u)$, similarly $z=a_L(v)b_R(p)$. Equations $a_R(t)=a_R(v)$, $b_R(t)=b_R(q)$ imply that $t=s_A(a_R(v)b_R(q)^{-1})$. We have

$$y=b_L(p)^{-1}b_L(u)a_R(u)=b_L(p)^{-1}u,$$
$$z=a_L(v)b_R(p)=a_L(u)^{-1}a_L(p)b_R(p)=a_L(u)^{-1}p,$$

and

$$a_R(v)b_R(q)^{-1}=u^{-1}ua_R(v)b_R(q)^{-1}=u^{-1}\beta(u,v)b_R(q)^{-1}=u^{-1}p.$$

The right-hand sides in (32) are non-empty, because

$$a_R(x)=a_L(b_L(p)^{-1}a_L(u))=a_L(b_L(p)^{-1}u)=a_L(y),$$
$$a_R(z)=a_R(a_L(u)^{-1}p)=a_R(u^{-1}p)=a_L(s_A(u^{-1}p))=a_L(t),$$
$$b_R(x)=[b_R(b_L(p)^{-1}a_L(u))]^{-1}=b_L(a_L(u)^{-1}b_L(p))=b_L(a_L(u)^{-1}p)=b_L(z),$$
$$b_R(y)=b_R(b_L(p)^{-1}u)=[b_L(u^{-1}b_L(p))]^{-1}=[b_L(u^{-1}p)]^{-1}=b_L(s_A(u^{-1}p))=b_L(t),$$

and

$$\alpha(x,y)=b_L(x)y=b_L(p)b_L(p)^{-1}u=v,$$
$$\alpha(z,t)=a_L(u)^{-1}pb_R(t)=a_L(u)^{-1}pb_R(q)=a_L(u)^{-1}\alpha(p,q)=a_L(u)^{-1}\beta(u,v)=v,$$
$$\beta(x,z)=a_L(x)z=a_L(u)a_L(u)^{-1}p=p,$$
$$\beta(y,t)=ya_R(t)=b_L(p)^{-1}ua_R(v)=b_L(p)^{-1}\alpha(p,q)=q. \qquad \square$$

Definition (cf. [3]). A *double group* is a triple $(G;A,B)$, where G is a group and A, B are subgroups of G such that $A\cap B=\{o\}$, $AB=G$.

Proposition 9.3. *Let $(G;A,B)$ be a double group. If $\mathscr{A}=(G,\alpha,A,s_A)$ and $\mathscr{B}=(G,\beta,B,s_B)$, then $(\mathscr{A}^T,\mathscr{B})$ is a union Kac algebra.*

Proof. It follows from Proposition 9.1 and Proposition 9.2. Conditions (20)–(26) are easy to prove. □

$(\mathscr{A}^T,\mathscr{B})$ is said to be the union Kac algebra *associated* with $(G;A,B)$.

If $(G;A,B)$ is a double group, then each element $g\in G$ has a unique decomposition of the form $g=ab$, where $a\in A$, $b\in B$. We denote by ${}^b a\in A$ and $b^a\in B$ factors obtained from the decomposition of ba:

$$ba={}^b a b^a.$$

With this notation we have the following lemma.

Lemma 9.4. *Mappings*

$$B \times A \ni (b, a) \mapsto {}^b a \in A,$$
$$B \times A \ni (b, a) \mapsto b^a \in B,$$

define a left action of B on A and a right action of A on B, respectively. A and B act by "twisted automorphisms":

$${}^b(a_1 a_2) = {}^b a_1 \cdot {}^{b^{a_1}} a_2,$$
$$(b_1 b_2)^a = b_1{}^{b_2 a} \cdot b_2{}^a.$$

Proof. It follows directly from the definition of the mappings. □

Using the map $(a, b) \mapsto ab$, we can identify G and $A \times B$ equipped with the following multiplication:

$$(a_1, b_1)(a_2, b_2) = (a_1{}^{b_1} a_2, b_1{}^{a_2} b_2). \tag{33}$$

Conversely, given two groups A, B and two actions satisfying conditions listed in Lemma 9.4, formula (33) defines on $A \times B$ the structure of a group. This group is usually denoted by $A \bowtie B$ (cf. [3]).

In the next section we show that each union Kac algebra is a double group. In terms of double groups, the duality principle is simple: it consists in interchanging the role of subgroups in a double group.

Example 9.1. $(G; G, \{o\})$ and $(G; \{o\}, G)$ are mutually dual double groups. The first is identified with the group G and the second is the object dual to G (in the category of union pseudogroups). After quantization we obtain the group G and the object dual (in the usual sense) to G.

Example 9.2. If $G = A \times B$ is a direct product of its subgroups A and B then the quantization of $(G; A, B)$ leads to the direct product of the group A and the object dual to B.

Example 9.3. Let $(G; A, B)$ be a double group. Then α is commutative (i.e. $\alpha t = \alpha$) if and only if B is an abelian normal subgroup in G. The quantization of $(G; A, B)$ is a semidirect product of A and the group dual to B.

10. Union Kac Algebras as Double Groups

Let (D, M) be a union Kac algebra, $D = (X, d, c, r)$, $M = (X, m, e, s)$. Then e is a point of D and c is a character of M, hence $E = e(1)$ and $C = c^T(1)$ are groups with the group structure defined by d^T and m, respectively (cf. Lemma 3.4). From the proof of Proposition 8.1 it follows that

$$X \ni x \mapsto (e_R(x), c_R(x)) \in E \times C,$$
$$X \ni x \mapsto (e_L(x), c_L(x)) \in E \times C$$

are bijections. Moreover, since

$$X \ni x \mapsto (e_R(sx), c_R(sx)) = (e_L(x), sc_R(x)) = (e_L(x), c_R(x)^{-1}) \in E \times C$$

is a bijection,

$$X \ni x \mapsto (e_L(x), c_R(x)) \in E \times C$$

and (similarly)

$$X \ni x \mapsto (e_R(x), c_L(x)) \in E \times C$$

are bijections.

This allows to define a *product* $x \cdot y$ of two elements of X as follows:

$$e_L(x \cdot y) = e_L(x) e_L(x \natural y), \quad c_R(x \cdot y) = c_R(x \natural y) c_R(y),$$

where $x \natural y$ is such an element of X that

$$e_R(x \natural y) = e_L(y), \quad c_L(x \natural y) = c_R(x).$$

Proposition 10.1. *With the product introduced above. X is a group, $(X; C, E)$ is a double group and (D, M) is its associated union Kac algebra.*

Proof. For any $a \in E$, $b \in C$ we have $e_L(a \cdot b) = a = e_R(b \cdot a)$ and $c_R(a \cdot b) = b = c_L(b \cdot a)$. It follows that for each $x \in X$,

$$x = e_L(x) \cdot c_R(x) = c_L(x) \cdot e_R(x). \tag{34}$$

We set ${}^b a = e_L(b \cdot a)$, $b^a = c_R(b \cdot a)$ for $a \in E$, $b \in C$. We have $b \cdot a = {}^b a \cdot b^a$. Now, since $(a_1 \cdot b_1) \natural (a_2 \cdot b_2) = b_1 \cdot a_2 = {}^{b_1} a_2 \cdot b_1^{a_2}$, we have

$$(a_1 \cdot b_1) \cdot (a_2 \cdot b_2) = a_1 {}^{b_1} a_2 \cdot b_1^{a_2} b_2.$$

Comparing this with formula (33) we see that the product is a group multiplication if and only if all conditions listed in Lemma 9.4 are satisfied. We shall prove one half of these conditions (the second half is proved in the same way). Since

$$e_R(b_1 \cdot {}^{b_2} a) = {}^{b_2} a = e_L(b_2 \cdot a),$$

there exists $x \in X$ such that $x = m(b_1 \cdot {}^{b_2} a,\ b_2 \cdot a)$. We have

$$\begin{aligned} {}^{(b_1 b_2)} a \cdot (b_1 b_2)^a &= (b_1 b_2) a = c_L(b_1 \cdot {}^{b_2} a) c_L(b_2 \cdot a) \cdot e_R(b_2 \cdot a) = c_L(x) \cdot e_R(x) = x \\ &= e_L(x) c_R(x) = e_L(b_1 \cdot {}^{b_2} a) \cdot c_R(b_1 \cdot {}^{b_2} a) c_R(b_2 \cdot a) = {}^{b_1}({}^{b_2} a) \cdot b_1{}^{{}^{b_2} a} b_2^a. \end{aligned}$$

hence ${}^{b_1 b_2} a = {}^{b_1}({}^{b_2} a)$ and $(b_1 b_2)^a = b_1{}^{{}^{b_2} a} \cdot b_2^a$. By (34), projections in the double group coincide with projections in (D, M). If $e_R(x) = e_L(y)$, then

$$e_L(m(x, y)) = e_L(x) = e_L(x \cdot c_R(y)),$$

$$c_R(m(x, y)) = c_R(x) c_R(y) = c_R(x \cdot c_R(y)),$$

hence $m(x, y) = x \cdot c_R(y)$. Similarly, we have $d^T(x, y) = x \cdot e_R(y)$ if $c_R(x) = c_L(y)$. $\square$

Corollary. *The inverse in a union pseudogroup is an involution (because $k = sr$ is the inverse in the double group).*

We conclude that union pseudogroups, union Kac algebras and double groups are in fact the same objects. Now we characterize morphisms of union pseudogroups in terms of double groups.

Let $(G; A, B)$, $(G'; A', B')$ be two double groups and let $f: G \to G'$ be a relation. We introduce *base relations* $f_0: A \to A'$, $f^0: B' \to B$ as follows

$$\mathscr{G}(f_0) = \mathscr{G}(f) \cap (A' \times A), \quad \mathscr{G}(f^0) = \mathscr{G}(f^T) \cap (B \times B').$$

Proposition 10.2. *The following conditions are equivalent:*

(i) *f is a morphism of union Kac algebras (from $(\mathscr{A}^T, \mathscr{B})$ to $(\mathscr{A}'^T, \mathscr{B}')$),*

(ii) *f_0 and f^0 are group homomorphisms such that*

$$a' \cdot b' \in f(a \cdot b) \Leftrightarrow a' = f_0(a), b = f^0(b'), \tag{35}$$

and

$$b' \cdot a' \in f(b \cdot a) \Leftrightarrow a' = f_0(a), b = f^0(b'), \tag{36}$$

for $a \in A$, $a' \in A'$, $b \in B$, $b' \in B'$.
(iii) f_0 *and* f^0 *are group homomorphisms such that* (35) *holds and*

$$f_0({}^{f^0(b')}a) = {}^{b'}f_0(a) \quad \textit{and} \quad f^0(b'^{f_0(a)}) = f^0(b')^a \tag{37}$$

for $a \in A$, $b' \in B'$.
(iv) f_0 *and* f^0 *are group homomorphisms such that* (36) *holds and*

$$f_0(a^{f^0(b')}) = f_0(a)^{b'} \quad \textit{and} \quad f^0({}^{f_0(a)}b') = {}^{a}f^0(b') \tag{38}$$

for $a \in A$, $b' \in B'$.
(v) $\mathscr{G}(f)$ *is a subgroup of* $G' \times G$, f_0 *and* f^0 *are maps, and* $f(B) = B'$, $f(A') = A$.

Proof. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). It follows from equalities such as $ba = {}^b a \cdot b^a$ and $ab = {}^a b \cdot a^b$.
(i) $\Rightarrow$ (ii). It follows from Lemma 3.3 and the fact that f_0 and f^0 are base relations of morphisms of U^*-algebras.
(ii) $\Rightarrow$ (i). We shall prove for instance that $f\beta = \beta'(f \otimes f)$. It follows from the following equivalences:

$$a'' \cdot b'' \in f\beta(ab, a'b') \Leftrightarrow b = {}^{a'}b', a'' = f_0(aa'), b' = f^0(b'')$$

and

$$a'' \cdot b'' \in \beta'(f \otimes f)(ab, a'b') \Leftrightarrow$$

there exist $c, c' \in B'$ such that $a'' \cdot b'' \in \beta'(f_0(a)c,\ f_0(a')c')$, $b = f^0(c)$, $b' = f^0(c') \Leftrightarrow$

$$a'' = f_0(a) f_0(a'), b' = f^0(a'') \quad \text{and} \quad b = f^0({}^{f_0(a')}b''),$$

The equivalence of (v) with remaining conditions follows easily from formulas (35)–(38).

Appendix: Abstract Union Spaces

Let $i_X : X \to \mathbb{B}X$ denote the natural inclusion $i_X(x) = \{x\}$. Then $U_{\mathbb{B}X} = \mathbb{B}i_X^T : \mathbb{B}\mathbb{B}X \to \mathbb{B}X$ is the operation of taking union of a family of subsets in X. A map $F : \mathbb{B}X \to \mathbb{B}Y$ is a union map if and only if $F(\emptyset) = \emptyset$ and $FU_{\mathbb{B}X} = U_{\mathbb{B}Y}\mathbb{B}F$.

An *upper bound space* (u.b.s., in short) is a triple (S, v, n), *where* n is a distinguished element of S, $v : \mathbb{B}S \to S$ is a surjective map such that $v(\emptyset) = n$ and $vU_{\mathbb{B}S} = v(\mathbb{B}v)$.

A *morphism* from one u.b.s. (S, v, n) to another u.b.s. (S', v', n') is a map $F : S \to S'$ such that $F(n) = n'$ and $Fv = v'\mathbb{B}F$.

A union space $(\mathbb{B}X, U_{\mathbb{B}X}, \emptyset)$ is an example of a u.b. space.

Lemma A.1. $vi_S = I$.

Proof. $vi_S = vU_{\mathbb{B}S}i_{\mathbb{B}S}i_S = v(\mathbb{B}v)i_{\mathbb{B}S}i_S = v(i_S v i_S)$. For each $s \in S$ there exists $A \subset S$ such that $s = v(A)$, hence $s = v(A) = vU_{\mathbb{B}S}(\{A\}) = v(\{v(A)\}) = v(\{s\})$. □

Lemma A.2. *If a bijection* $F : S \to S'$ *is a morphism from* (S, v, n) *to* (S', v', n'), *then* F^{-1} *is also a morphism.*

Proof. $Fv = v'\mathbb{B}F$ implies $F^{-1}v' = v\mathbb{B}F^{-1}$. □

Let (S, v, n) be an u.b.s. Let $i: X \to S$ be the inclusion map of a subset X in S. Then we say that an element $s \in S$ *has a decomposition on elements of* X if there exists $A \subset X$ such that $s = v\mathbb{B}i(A)$. X is said to be *union independent* if all such possible decompositions are unique, i.e. the map $v\mathbb{B}i$ is injective. X is said to *generate* S if each $s \in S$ has a decomposition on elements of X, i.e. the map $v\mathbb{B}i$ is surjective. X is said to be a *union basis* in (S, v, n) if X is union independent and generates S, i.e. the map $v\mathbb{B}i$ is bijective.

Definition. An *abstract union space* is an u.b.s. (S, v, n) having at least one union basis.

Lemma A.3. *In any abstract union space* (S, v, n) *there exists only one union basis* $X \subset S$ *and* $S \cong \mathbb{B}X$ *as* u.b. *spaces.*

Proof. Let $i: X \to S$ be the inclusion of a union basis. Then $v\mathbb{B}i: \mathbb{B}X \to S$ is an isomorphism of u.b. spaces. If $j: Y \to S$ is the inclusion of another union basis, then $(v\mathbb{B}j)^{-1}(v\mathbb{B}i)$ is an isomorphism, hence there exists a bijection $f: X \to Y$ such that $v\mathbb{B}i = (v\mathbb{B}j)\mathbb{B}f$. It follows that $i = vi_S i = v\mathbb{B}i i_X = (v\mathbb{B}j)(\mathbb{B}f)i_X = (v\mathbb{B}j)i_Y j = vi_S j f = jf$, hence $Y = X$. $\square$

In view of Lemma A.3, we do not have to distinguish between abstract union spaces and union spaces.

Example. Let (S, v, n) be a union space. Let $S' = \mathrm{Map}(X, S)$ be the set of all maps from a set X to S. Then (S', v', n') is a union space, if we set $n'(x) = n$, $v'(\mathscr{A})(x) = v\{A(x), A \in \mathscr{A}\}$ for $x \in X$, $\mathscr{A} \subset S'$.

Acknowledgements. The author is very much indebted to Prof. S.L. Woronowicz for his clarifying remarks concerning quantum groups.

References

1. Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**, 613–665 (1987)
2. Woronowicz, S.L.: Pseudospaces, pseudogroups and Pontryagin duality. Proceedings of the International Conference on Mathematics and Physics, Lausanne 1979. Lecture Notes in Physics, vol. 116 Berlin, Heidelberg, New York: Springer
3. Lu, Jiang-Hua, Weinstein, A.: Poisson Lie groups, dressing transformations and Bruhat decompositions. J. Diff. Geom. **31**, 501–526 (1990)
4. Drinfeld, V.G.: Quantum groups. Proc. ICM, Berkeley, 1986, vol. 1, 789–820
5. Kac, G.I., Paljutkin, V.G.: Finite ring groups (Russian). Trudy Moskov. Mat. Obsc. **15**, 224–261 (1986)
6. Coste, A., Dazord, P., Weinstein, A.: Groupoïdes symplectiques. Publ. du Dep. de Math. de l'Universite de Lyon 1, 2/A-1987
7. Mackenzie, K.: Le groupoids and Lie algebroids in Differential Geometry. Lond. Math. Soc. Lect. Note series 124
8. Tulczyjew, W.M., Zakrzewski, S.: Partial categories. Atti Accad. Sci. Torino, **121**, 46–52 (1987)
9. Tulczyjew, W.M., Zakrzewski, S.: Composition classes and partial categories. Att Accad. Sci. Torino, **122**, 148–154 (1988)
10. Weinstein, A.: The symplectic "category". In: Lecture Notes in Mathematics, vol. 905, pp. 45–51. Berlin, Heidelberg, New York: Springer
11. Tulczyjew, W.M., Zakrzewski, S.: The category of Fresnel kernels. J. Geom. and Phys. **1**, 79–120 (1984)

Communicated by H. Araki