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# An Application of Aomoto–Gelfand Hypergeometric Functions to the $SU(n)$ Knizhnik–Zamolodchikov Equation

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**Abstract.** Solutions to the Knizhnik–Zamolodchikov equation for Verma modules of the Lie algebra $\mathfrak{sl}(n+1, \mathbb{C})$ are explicitly given by certain integrals called Aomoto–Gelfand hypergeometric functions.

## 1. Introduction

The starting point of our study was the result of Christe and Flume [4], which gave explicit integral representations of the 4-point functions of the $SU(2)$ Wess–Zumino–Witten model as solutions to the Knizhnik–Zamolodchikov equation. Similar results were previously obtained by Zamolodchikov and Fateev [13]. On the other hand, Aomoto [1], [2] studied the integrals of the following kind and derived a system of differential equations for them with respect to variables $z_1, \ldots, z_N$:

$$\int \Phi \varphi \, dt_1 \cdots dt_m,$$
$$\Phi = \prod_{i,a} (t_i - z_a)^{\lambda_{ai}} \prod_{i,j} (t_i - t_j)^{\nu_{ij}} \prod_{a,b} (z_a - z_b)^{\mu_{ab}}. \tag{1.1}$$

Here $\varphi$ are rational functions whose poles are contained in the diagonal set $\bigcup_{i,a} \{t_i = z_a\} \cup \bigcup_{i,j} \{t_i = t_j\} \cup \bigcup_{a,b} \{z_a = z_b\}$, and $\lambda_{ai}, \nu_{ij}, \mu_{ab}$ are complex parameters. This kind of integrals are generalizations of hypergeometric function, and Gelfand and others studied a class of generalized hypergeometric functions including (1.1) ([12]). We call them 'Aomoto–Gelfand hypergeometric functions'.

If the parameters $\lambda_{ai}, \nu_{ij}, \mu_{ab}$ take certain values, then the integral (1.1) reduces to the one of Christe and Flume. In this case, Aomoto's differential equation is nothing but the Knizhnik–Zamolodchikov equation. A similar result on the $n$-point functions was obtained by Date et al. [6].

In this paper, we shall generalize the last result to the $SU(n)$ Knizhnik–Zamolodchikov equation. We briefly sketch our construction.

Let $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{C})$ and let $(,)$ be the Cartan–Killing form normalized by $(\alpha,\alpha)=2$ for any root $\alpha$. Let $V_{\lambda_1}^*,\dots,V_{\lambda_{N-1}}^*,V_{\lambda_N}^*$ be irreducible Verma modules of the lowest weight $-\lambda_1,\dots,-\lambda_N$. The Knizhnik–Zamolodchikov equation is the following system of differential equations for function $\Psi(z)$ with values in $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_N}^*,V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*)$:

$$d\Psi(z)=\nu\sum_{1\leqq a<b\leqq N-1}\Psi(z)\cdot\Omega_{ab}d\log(z_a-z_b). \tag{1.2}$$

The operator $\Omega_{ab}$ will be defined by (2.1). Note that we have set $z_N=\infty$ for simplicity.

We have the following isomorphism.

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_N}^*,V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*)\\&\quad\cong\{v^*\in V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*\,|\,v^*\text{ is a lowest weight vector of weight }-\lambda_N\}.\end{aligned} \tag{1.3}$$

We fix a basis $\{u^*(\vec{p})\}$ of weight space of weight $-\lambda_N$ of representation $V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*$. For each index $\vec{p}$, we assign an Aomoto–Gelfand hypergeometric function

$$I(\vec{p})=\int\Phi\varphi(\vec{p})dt$$

with the $\Phi$ determined by the data of the representations. We define

$$w^*(z)=\sum_{\vec{p}}I(\vec{p})\cdot u^*(\vec{p}).$$

For certain choice of $u^*(\vec{p})$ and $\varphi(\vec{p})$, $w^*(z)$ becomes a lowest weight vector in $V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*$ of weight $-\lambda_N$, and therefore determines $\Psi(z)$ in $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_N}^*,V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*)$ by (1.3). This $\Psi(z)$ satisfies the Knizhnik–Zamolodchikov equation.

In Sect. 2, we will give the detailed description of the result. Its proof will be given in Sect. 5, for which we will prepare Sects. 3 and 4.

After completing this work, the author received the announcement of Schechtman and Varchenko [9]. It covers any symmetrizable Kac–Moody Lie algebra using different expressions of the integrands from the present ones.

## 2. Statement of the Theorem

Let $\mathfrak{g}$ be a simple Lie algebra and $(,)$ a fixed invariant bilinear form of $\mathfrak{g}$. Let $X_i$ be a basis of $\mathfrak{g}$ and $X^i$ the dual basis with respect to $(,)$. Let $V_\lambda^*$ denote a lowest weight $\mathfrak{g}$-module with the lowest weight $-\lambda$. For a given sequence of weights $\lambda_1,\dots,\lambda_{N-1},\lambda_N$, we consider the following operator acting on $V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*$

$$\Omega_{ab}=\sum_i\rho_a(X_i)\otimes\rho_b(X^i),$$

where $\rho_a(x)$ signifies the action of $x\in\mathfrak{g}$ on the $a$-th component of the tensor product $V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*$. $\Omega_{ab}$ does not depend on the choice of $\{X_i\}$. The Knizhnik–Zamolodchikov equation is the following system of differential equations for a function $\Psi(z)=\Psi(z_1,\dots,z_{N-1})$ with values in $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_N}^*,V_{\lambda_1}^*\otimes\cdots\otimes V_{\lambda_{N-1}}^*)$:

$$d\Psi(z)=\nu\sum_{1\leqq a<b\leqq N-1}\Psi(z)\Omega_{ab}d\log(z_a-z_b). \tag{2.1}$$

In this paper, it is assumed that $V^*_{\lambda_i}$ is the Verma module with the lowest weight $-\lambda_i$ and $v$ is a generic complex parameter.

We have the following isomorphism of vector spaces which commutes with the action of $\Omega_{ab}$.

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g}}(V^*_{\lambda_N}, V^*_{\lambda_1}\otimes\cdots\otimes V^*_{\lambda_{N-1}})\\ &\quad\cong\{v^*\in V^*_{\lambda_1}\otimes\cdots\otimes V^*_{\lambda_{N-1}}\,|\,v^* \text{ is a lowest weight vector of weight } -\lambda_N\}.\end{aligned} \tag{2.2}$$

Note that this depends on the choice of the lowest weight vector in $V^*_{\lambda_N}$.

From now on, we assume that $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbf{C})$, the simple Lie algebra of type $A_n$, and $(,)$ is the Cartan–Killing form normalized by $(\alpha,\alpha)=2$ for any root $\alpha$.

Let $E_{ij}$ denote the elementary matrix:

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & \vdots & \\ \cdots\cdots & 1 & \cdots\cdots \\ & \vdots & \end{pmatrix}. \end{array}$$

We set

$$H_{ij}=E_{ii}-E_{jj},\quad F_{ij}=E_{ji}$$

for any $i, j$ such that $1\leqq i<j\leqq n+1$, and

$$H_i=H_{ii+1},\quad F_i=F_{ii+1},\quad E_i=E_{ii+1}$$

for any $i$ such that $1\leqq i\leqq n$. Let $\alpha_i$ be the simple root corresponding to $H_i$ and let $\alpha_{ij}=\alpha_i+\cdots+\alpha_{j-1}$, $(i<j)$ be the positive roots.

We set

$$P(n,N)=\{\vec{p}=\{p^a_{hj}\},\ 1\leqq a\leqq N-1,\ 1\leqq h<j\leqq n+1,\ p^a_{hj}\in\mathbf{Z}_{\geqq 0}\}.$$

Vector of the representation space shall be parametrized by elements of $P(n,N)$.

Fix a lowest weight vector $v^*_{\lambda_a}\in V^*_{\lambda_a}$. We define the ordering on $\{E_{ij}\}$ by

$$E_{ij}\succ E_{kl}\Leftrightarrow\begin{cases} j>l & \text{or}\\ j=l & \text{and}\quad i>k.\end{cases} \tag{2.4}$$

The following vectors form a basis of $V^*_{\lambda_a}$ by the Poincaré–Birkhoff–Witt theorem.

$$u^*_{\lambda_a}(\vec{p}^a)=\frac{E_{n\,n+1}^{p^a_{n\,n+1}}}{p^a_{n\,n+1}!}\cdots\frac{E_{ij}^{p^a_{ij}}}{p^a_{ij}!}\cdots\frac{E_{12}^{p^a_{12}}}{p^a_{12}!}v^*_{\lambda_a}.$$

Here $E_{ij}$ are arranged according to the ordering (2.4), and $\vec{p}^a=\{p^a_{ij}\}$ runs over all sequences of non-negative integers.

Suppose that there exist non-negative integers $m_i$ such that

$$\lambda_1+\cdots+\lambda_{N-1}-\lambda_N=\sum_{i=1}^{n}m_i\alpha_i. \tag{2.5}$$

We consider the following vector

$$u^*(\vec{p})=\bigotimes_{a=1}^{N-1}u^*_{\lambda_a}(\vec{p}^a)\in V^*_{\lambda_1}\otimes\cdots\otimes V^*_{\lambda_{N-1}}, \tag{2.6}$$

where $\vec{p} = \{p^a_{hj}\} \in P(n, N)$ and they satisfy

$$\sum_{1 \leqq a \leqq N-1} \sum_{h=1}^{i} \sum_{j=i+1}^{n+1} p^a_{hj} = m_i \tag{2.7}$$

for each $i$ $(1 \leqq i \leqq n)$. This condition means that the weight of $u^*(\vec{p})$ equals to $-\lambda_N$.

We shall describe certain integrals also parametrized by $\vec{p} \in P(n, N)$.

Let $S_i$ denote an index set with $m_i$ elements for each $i$ $(1 \leqq i \leqq n)$. We prepare the following integration variables:

$$\{t_i^{(s)}; 1 \leqq i \leqq n, s \in S_i\}. \tag{2.8}$$

Introduce a total ordering $\ll$ on the set $\{(i, s);\ 1 \leqq i \leqq n,\ s \in S_i\}$ and define

$$\Phi = \left\{ \prod_{(i,s) \ll (i',s')} (t_i^{(s)} - t_{i'}^{(s')})^{(\alpha_i, \alpha_{i'})} \prod_{(i,s)} \prod_{b=1}^{N-1} (t_i^{(s)} - z_b)^{-(\alpha_i, \lambda_b)} \right\}^{\nu}. \tag{2.9}$$

For each $\vec{p} \in P(n, N)$, consider the following sets.

$$\begin{aligned} S(\vec{p}) &= \{(a, h, j, q); 1 \leqq a \leqq N-1, 1 \leqq h < j \leqq n+1, 1 \leqq q \leqq p^a_{hj}\}, \\ S_i(\vec{p}) &= \{s = (a, h, j, q) \in S(\vec{p}); h \leqq i < j\}. \end{aligned} \tag{2.10}$$

If $\vec{p}$ satisfies (2.7), then $S_i(\vec{p})$ has $m_i$ elements. We therefore take a bijection

$$\beta_i : S_i(\vec{p}) \tilde{\to} S_i. \tag{2.11}$$

For each $s = (a, h, j, q) \in S(\vec{p})$, we set

$$\varphi^{(s)} = \left( \prod_{i=h}^{j-2} \frac{1}{t_i^{(s_i)} - t_{i+1}^{(s_{i+1})}} \right) \frac{1}{t_{j-1}^{(s_{j-1})} - z_a}, \quad s_i = \beta_i(s),$$

and define

$$\varphi(\vec{p}) = \prod_{s \in S(\vec{p})} \varphi^{(s)}. \tag{2.12}$$

The integral is defined by

$$I(\vec{p})(z) = \int_{\Gamma} \Phi \varphi(\vec{p}) dt, \quad dt = \prod_{(i,s)} dt_i^{(s)}.$$

Here the $\Gamma$, a closed contour as a cycle of the twisted homology defined by the $\Phi$, is assumed to satisfy the following condition. Let $\mathfrak{S}_{m_i}$ be the group of all permutations on the variables $\{t_i^{(s)}; s \in S_i\}$ for each $i$. It is the symmetric group of $m_i$-th order. We define $\mathfrak{S}(m_1, \ldots, m_n) = \prod_{i=1}^{n} \mathfrak{S}_{m_i}$, then it acts on the set of variables $\{t_i^{(s)}; 1 \leqq i \leqq n, s \in S_i\}$. Let $D$ denote the diagonal set:

$$D = \bigcup_{i,a} \{t_i = z_a\} \cup \bigcup_{i,j} \{t_i = t_j\}. \tag{2.13}$$

Then the condition for the contour $\Gamma$ is given by

**Assumption 2.1.** *For any rational function $\varphi$ with poles in $D$, the integral $\int_{\Gamma} \Phi \varphi dt$ is invariant with respect to any permutation in $\mathfrak{S}(m_1, \ldots, m_n)$.*

*Remark 2.2.* The first product in the right hand side of (2.9) is invariant with respect to $\mathfrak{S}(m_1, \ldots, m_n)$, because if $i = i'$ then $(\alpha_i, \alpha_{i'}) = 2$. The second one is also invariant obviously. Hence we have no ambiguity of choice of branch of $\Phi$ under the permutation. The integral $I(\vec{p})$ is uniquely determined by $\vec{p}$ and $\Gamma$; it does not depend on the choice of the bijection (2.11).

We finally define

$$w^*(z) = \prod_{a<b} (z_a - z_b)^{(\lambda_a, \lambda_b)\nu} \sum_{\vec{p}} I(\vec{p}) u^*(\vec{p}), \tag{2.14}$$

where the summation is taken over all sequences $\vec{p}$ satisfying (2.7).

**Proposition 2.3.** *$w^*(z)$ is a lowest weight vector, that means that the equality*

$$F_i \cdot w^*(z) = 0$$

*holds for any $i$ $(1 \leqq i \leqq n)$.*

*It determines an element $\Psi(z) \in \mathrm{Hom}_{\mathfrak{g}}(V^*_{\lambda_N}, V^*_{\lambda_1} \otimes \cdots \otimes V^*_{\lambda_{N-1}})$, because the weight of $w^*(z)$ equals to $-\lambda_N$. Then we obtain the theorem.*

**Theorem 2.4.** *$\Psi(z)$ satisfies the Knizhnik–Zamolodchikov Eq. (2.1).*

## 3. Lemmas on Partial Fraction Expansion

The results in this section is independent of the $\vec{p} \in P(n, N)$. We will define two equivalence relations in the set of rational functions whose poles are contained in the diagonal set $D$ defined by (2.13). Let $\varphi, \varphi'$ be two such rational functions. We write $\varphi \sim \varphi'$ if $\varphi - \varphi'$ is anti-symmetric under a permutation of $\mathfrak{S}(m_1, \ldots, m_n)$, and write $\varphi \approx \varphi'$ if there exists a rational form $\eta$ with poles in $D$ such that $\Phi(\varphi - \varphi')dt = d(\Phi\eta)$. Here $d$ denotes the exterior differentiation with respect to the variables $\{t_i^{(s)}\}$. We consider the equivalence relation generated by $\sim$ and write it by the same symbol. The equality

$$\int_\Gamma \Phi\varphi\, dt = \int_\Gamma \Phi\varphi'\, dt$$

is then implied by these equivalence relations. Note that for $\sim$ this follows from Assumption 2.1, and for $\approx$ from the assumption that the contour $\Gamma$ is closed. For example, let

$$\varphi = \frac{1}{t_i^{(s)} - t_{i'}^{(s')}},$$

then $\varphi \sim 0$ and we have $\int_\Gamma \Phi\varphi\, dt = 0$.

For any $a, h, j$ such that $1 \leqq a \leqq N-1$, $1 \leqq h < j \leqq n+1$ and for any set of variables $\{t_i^{(s_i)}\}_{h \leqq i < j}$, we set

$$\varphi_{hj}^a(t_h^{(s_h)}, \ldots, t_{j-1}^{(s_{j-1})}) = \left( \prod_{i=h}^{j-2} \frac{1}{t_i^{(s_i)} - t_{i+1}^{(s_{i+1})}} \right) \frac{1}{t_{j-1}^{(s_{j-1})} - z_a}.$$

*Remark 3.1.* Consider the equivalence class of the product $\prod_{k=1}^{m} \varphi_{h_k j_k}^{a_k}(t_{h_k}^{(s_{k,h_k})}, \ldots, t_{j_k-1}^{(s_{k,j_k-1})})$ with respect to $\sim$ where the $s_{k,i} \in S_i (k = 1, \ldots, m)$ are distinct for each $i$.

Since it does not depend on the choice of $s_{k,i}$, it will be written by $\prod_{k=1}^{m} \varphi^{a_k}_{h_k j_k}$ in the sequel. If $h=j$ then we understand $\varphi^a_{hj}=1$.

We set

$$\theta(C):=\begin{cases}1 & \text{if } C \text{ is true,}\\ 0 & \text{otherwise,}\end{cases}$$

for any statement $C$.

**Lemma 3.2.** *For any indices $a, b$, and $h_a, j_a, h_b, j_b$ such that $1 \leqq a \neq b \leqq N-1$, $1 \leqq h_a < j_a \leqq n+1$, $1 \leqq h_b < j_b \leqq n+1$, we take $s_i \in S_i$* for $h_a \leqq i < j_a$ and $s'_i \in S_i$ for $h_b \leqq i < j_b$ *such that* $\{s_i\}$ and $\{s'_i\}$ are disjoint. Then we have

$$\left(\sum_{i=h_a}^{j_a-1}\sum_{i'=h_b}^{j_b-1}\frac{(\alpha_i,\alpha_{i'})}{t_i^{(s_i)}-t_{i'}^{(s'_{i'})}}\right)\varphi^a_{h_a j_a}(t_{h_a}^{(s_{h_a})},\dots,t_{j_a-1}^{(s_{j_a-1})})\varphi^b_{h_b j_b}(t_{h_b}^{(s'_{h_b})},\dots,t_{j_b-1}^{(s'_{j_b-1})})$$

$$\sim \{\theta(j_a<j_b)\theta(h_b\leqq j_a)K_{ab}+\theta(j_b<j_a)\theta(h_a\leqq j_b)K_{ba}+\theta(j_a=j_b)L_{ab}\}\frac{1}{z_a-z_b},$$

*where*

$$K_{ab}=-\sum_{r\geqq 0}(-1)^r\sum_{\substack{i_0=h_a,\, i_0<i_1\\ j_a\leqq i_1<\cdots<i_r\leqq j_b}}\varphi^a_{h_b j_a}\varphi^a_{i_0 i_1}\cdots\varphi^a_{i_{r-1}i_r}\varphi^b_{i_r j_b},$$

$$L_{ab}=-(\varphi^a_{h_a j_a}-\varphi^b_{h_a j_a})(\varphi^b_{h_b j_a}-\varphi^a_{h_b j_a}).$$

*Here* (,) *denotes the Cartan–Killing form and $\alpha_i$ are simple roots.*

*Remark 3.3.* $K_{ab}$ and $L_{ab}$ are explicitly written by

$$K_{ab}=\left(\prod_{i=h_b}^{j_a-2}\frac{1}{t_i^{(s'_i)}-t_{i+1}^{(s'_{i+1})}}\right)\frac{1}{t_{j_a-1}^{(s'_{j_a-1})}-z_a}$$

$$\cdot\left(\prod_{i=h_a}^{j_a-2}\frac{1}{t_i^{(s_i)}-t_{i+1}^{(s_{i+1})}}\right)\left(\frac{1}{t_{j_a-1}^{(s_{j_a-1})}-t_{j_a}^{(s'_{j_a})}}\frac{1}{t_{j_a-1}^{(s_{j_a-1})}-z_a}\right)$$

$$\cdot\left\{\prod_{i=j_a}^{j_b-2}\left(\frac{1}{t_i^{(s'_i)}-t_{i+1}^{(s'_{i+1})}}-\frac{1}{t_i^{(s'_i)}-z_a}\right)\right\}\left(\frac{1}{t_{j_b-1}^{(s'_{j_b-1})}-z_a}-\frac{1}{t_{j_b-1}^{(s'_{j_b-1})}-z_b}\right),$$

$$L_{ab}=\left(\prod_{i=h_b}^{j_a-2}\frac{1}{t_i^{(s'_i)}-t_{i+1}^{(s'_{i+1})}}\right)\left(\frac{1}{t_{j_a}^{(s'_{j_a})}-z_a}-\frac{1}{t_{j_a}^{(s'_{j_a})}-z_b}\right)$$

$$\cdot\left(\prod_{i=h_a}^{j_a-2}\frac{1}{t_i^{(s_i)}-t_{i+1}^{(s_{i+1})}}\right)\left(\frac{1}{t_{j_a}^{(s_{j_a})}-z_a}-\frac{1}{t_{j_a}^{(s_{j_a})}-z_b}\right).$$

Lemma 3.2 can be proved by easy calculation using partial fraction expansion.

Following two lemmas are corollaries to the Lemma 3.2.

**Lemma 3.4.** *For any $h,j$ and $i$ such that $1\leqq h<j\leqq n+1$, $1\leqq i\leqq n$, we have*

$$\left(\sum_{k=h}^{j-1}\frac{(\alpha_i,\alpha_k)}{t_i^{(s'_i)}-t_k^{(s_k)}}\right)\varphi^a_{hj}(t_h^{(s_h)},\dots,t_{j-1}^{(s_{j-1})})\sim\begin{cases}-\varphi^a_{ij} & \text{if } i=h-1,\\ \varphi^a_{hi+1}\varphi^a_{ii+1} & \text{if } i=j-1,\\ \varphi^a_{hi+1}-\varphi^a_{hi}\varphi^a_{ii+1} & \text{if } i=j,\\ 0 & \text{otherwise,}\end{cases}$$

*provided $s'_i\neq s_i$.*

*Proof*. We apply Lemma 3.2 to the following:

$$\left(\sum_{k=h}^{j-1} \frac{(\alpha_i, \alpha_k)}{t_i^{(s_i')} - t_k^{(s_k)}}\right) \varphi_{hj}^a(t_h^{(s_h)}, \ldots, t_{j-1}^{(s_{j-1})}) \varphi_{ii+1}^b(t_i^{(s_i')}).$$

We multiply the both side by $z_a - z_b$ and let $z_b$ tend to infinity, then we obtain the result.

**Lemma 3.5.** *For any $h, i, j, a, b$ such that $1 \leqq h \leqq i < j \leqq n+1$ and $1 \leqq a \neq b \leqq N-1$, we have*

$$\frac{1}{t_i^{(s_i)} - z_a} \varphi_{hj}^b(t_h^{(s_h)}, \ldots, t_{j-1}^{(s_{j-1})}) \sim \frac{1}{z_a - z_b} \sum_{r \geqq 0} (-1)^r \sum_{\substack{i_0 = h \\ i < i_1 < \cdots < i_r \leqq j}} \varphi_{i_0 i_1}^a \cdots \varphi_{i_{r-1} i_r}^a \varphi_{i_r j}^b.$$

*Proof*. The left hand side is written as:

$$\frac{1}{t_i^{(s_i)} - t_{i+1}^{(s_{i+1})}} \varphi_{hi+1}^a(t_h^{(s_h)}, \ldots, t_i^{(s_i)}) \varphi_{i+1j}^b(t_{i+1}^{(s_{i+1})}, \ldots, t_{j-1}^{(s_{j-1})}).$$

Applying Lemma 3.2 to this, we obtain Lemma 3.5.

## 4. Lemmas on Action of $\mathfrak{g} = \mathfrak{sl}(n+1, \mathscr{C})$

In this section, we shall describe the action of elements of the Lie algebra $\mathfrak{g}$ on the vector $u^*(\vec{p})$ defined by (2.6). For any $a, h, j$ such that $1 \leqq a \leqq N-1$, $1 \leqq h < j \leqq n+1$ let $\vec{\varepsilon}_{hj}^{\,a}$ be the element of $P(n. N)$ such that its ${}_{hj}^{a}$-th component is 1 and the others are 0. If $h \not< j$, we understand that all components of $\vec{\varepsilon}_{hj}^{\,a}$ are 0. We can prove the following lemma by straightforward calculation using the following relations.

$$\begin{aligned}
[E_{ij}, E_{kl}] &= \theta(j=k)E_{il} - \theta(i=l)E_{jk}, \\
[F_{ij}, E_{kl}] &= \theta(i=k)\{\theta(j<l)E_{jl} + \theta(j>l)F_{lj}\} \\
&\quad - \theta(j=l)\{\theta(i<k)F_{ik} + \theta(i>k)E_{ki}\} \\
&\quad + \theta(i=k)\theta(j=l)H_{ij}, \\
[H_{ij}, E_{kl}] &= \{\theta(i=k) - \theta(i=l) - \theta(j=k) + \theta(j=l)\}E_{kl}.
\end{aligned}$$

**Lemma 4.1.** *For any $a, h, i$ and $\vec{p} \in P(n, N)$ such that $1 \leqq a \leqq N-1$, $1 \leqq h < i \leqq n+1$, we have*

$$\begin{aligned}
&\rho_a(E_{hi}) \cdot u^*(\vec{p}) \\
&\quad = \sum_{j=1}^{n} (p_{hj}^a + 1) \cdot u^*(\vec{p} + \vec{\varepsilon}_{hj}^{\,a} - \vec{\varepsilon}_{ij}^{\,a}), \\
&\rho_a(F_{hi}) \cdot u^*(\vec{p}) \\
&\quad = \sum_{j=i+1}^{n} (p_{ij}^a + 1) \cdot u^*(\vec{p} - \vec{\varepsilon}_{hj}^{\,a} + \vec{\varepsilon}_{ij}^{\,a}) \\
&\qquad + \sum_{r>0} (-1)^r \sum_{\substack{i_1 < \cdots < i_r = i \\ h' < h \leqq j < i_1}} (p_{h'j}^a + 1) \cdot u^*(\vec{p} + \vec{\varepsilon}_{h'j}^{\,a} - \vec{\varepsilon}_{hj}^{\,a} - \vec{\varepsilon}_{h'i_1}^{\,a} - \vec{\varepsilon}_{i_1 i_2}^{\,a} - \cdots - \vec{\varepsilon}_{i_{r-1} i_r}^{\,a})
\end{aligned}$$

$$+\sum_{r>0}(-1)^r \sum_{\substack{i_1<\cdots<i_r=i\\ h<h'<j<i_1}} (p^a_{h'j}+1)\cdot u^*(\vec{p}+\vec{\varepsilon}^a_{h'j}-\vec{\varepsilon}^a_{hj}-\vec{\varepsilon}^a_{h'i_1}-\vec{\varepsilon}^a_{i_1i_2}-\cdots-\vec{\varepsilon}^a_{i_{r-1}i_r})$$

$$+\sum_{r>0}(-1)^r \sum_{\substack{i_1<\cdots<i_r=i\\ h'<h\leqq i_1}} p^a_{h'i_1}\cdot u^*(\vec{p}-\vec{\varepsilon}^a_{hi_1}-\vec{\varepsilon}^a_{i_1i_2}-\cdots-\vec{\varepsilon}^a_{i_{r-1}i_r})$$

$$+\sum_{r>0}(-1)^r \sum_{\substack{i_1<\cdots<i_r=i\\ h'<h\leqq i_1}} p^a_{h'i_1}\cdot u^*(\vec{p}-\vec{\varepsilon}^a_{hi_1}-\vec{\varepsilon}^a_{i_1i_2}-\cdots-\vec{\varepsilon}^a_{i_{r-1}i_r})$$

$$+\sum_{r>0}(-1)^r \sum_{\substack{i_1<\cdots<i_r=i\\ h<i_1}} (p^a_{hi_1}-1)\cdot u^*(\vec{p}-\vec{\varepsilon}^a_{hi_1}-\vec{\varepsilon}^a_{i_1i_2}-\cdots-\vec{\varepsilon}^a_{i_{r-1}i_r})$$

$$-\sum_{r>0}(-1)^r \sum_{h<i_1\cdots<i_r=i} (\alpha_{hi_1},\lambda_a)\cdot u^*(\vec{p}-\vec{\varepsilon}^a_{hi_1}-\vec{\varepsilon}^a_{i_1i_2}-\cdots-\vec{\varepsilon}^a_{i_{r-1}i_r}).$$

*Here if $\vec{p}'\notin P(n,N)$, then we understand $u^*(\vec{p}')=0$.*

*As a corollary of this lemma, we obtain Lemma 4.2.*

**Lemma 4.2.** *For any $i$ and $\vec{p}=\{p^a_{hj}\}$* such that $1\leqq i\leqq n$, *we have*

$$\rho_a(F_i)\cdot u^*(\vec{p})=\sum_{j=i+2}^{n}(p^a_{i+1j}+1)\,u^*(\vec{p}+\vec{\varepsilon}^a_{i+1j}-\vec{\varepsilon}^a_{ij})$$
$$-\sum_{j=1}^{i-1}(p^a_{ji}+1)\,u^*(\vec{p}+\vec{\varepsilon}^a_{ji}-\vec{\varepsilon}^a_{ji+1})$$
$$+\left\{(\alpha_i,\lambda_a)+\sum_{h=1}^{i-1}p^a_{hi}-\sum_{h=1}^{i}p^a_{hi+1}+1\right\}u^*(\vec{p}-\vec{\varepsilon}^a_{ii+1}).$$

## 5. Proof of the Theorem

*Proof of Proposition 2.3.* For each $i(1\leqq i\leqq n)$, let $\vec{p}'=\{p'^a_{hj}\}\in P(n,N)$ satisfy

$$\sum_a\sum_{h=1}^{k}\sum_{j=k+1}^{n+1}p'^a_{hj}=m_k \quad\text{if}\quad k\neq i,$$
$$\sum_a\sum_{h=1}^{i}\sum_{j=i+1}^{n+1}p'^a_{hj}=m_i-1. \tag{5.1}$$

We set $S_k(\vec{p}')$ as (2.10). Take bijections $\beta_k:S_k(\vec{p})\tilde{\to}S_k$ for $k\neq i$, and an injection $\beta_i:S_i(\vec{p}')\hookrightarrow S_i$. Let $s'_i\in S_i$ be the unique element such that $s'_i\notin\beta_i(S_i(\vec{p}))$. We define

$$\varphi(\vec{p}')=\sum_{s\in S(\vec{p}')}\varphi^{(s)}$$

similarly to (2.12). Then the variable $t_i^{(s'_i)}$ is not contained in $\varphi(\vec{p}')$. We have

$$\int_\Gamma\frac{\partial}{\partial t_i^{(s'_i)}}(\Phi\varphi(\vec{p}'))dt=\int_\Gamma\Phi\nabla_{t_i^{(s'_i)}}\varphi(\vec{p}')dt,$$

where

$$\nabla_{t_i^{(s'_i)}}\varphi(\vec{p}')=\left(\frac{\partial}{\partial t_i^{(s'_i)}}+\frac{\partial}{\partial t_i^{(s'_i)}}\log\Phi\right)\varphi(\vec{p}')$$
$$=\left(-\frac{(\alpha_i,\lambda_a)\nu}{t_i^{(s'_i)}-z_a}+\sum_{k=1}^{n}\sum_{s_k\in S_k}\frac{(\alpha_i,\alpha_k)\nu}{t_i^{(s'_i)}-t_k^{(s_k)}}\right)\varphi(\vec{p}').$$

To the first term of the right hand side, we apply the following obvious identity:

$$\frac{1}{t_i^{(s_i')} - z_a}\varphi(\vec{p}') = \varphi(\vec{p}' + \vec{\varepsilon}^a_{ii+1}).$$

To the second term, we apply Lemma 3.4. Then we have

$$\nabla_{t_i^{(s_i')}}\varphi(\vec{p}') \sim -\nu\sum_a\Bigg\{\Bigg((\alpha_i,\lambda_a) - \sum_{h=1}^{i} p'^a_{hi+1} + \sum_{h=1}^{i-1} p'^a_{hi}\Bigg)\varphi(\vec{p}' + \vec{\varepsilon}^a_{ii+1})$$
$$- \sum_{h=1}^{i-1} p'^a_{hi}\varphi(\vec{p}' - \vec{\varepsilon}^a_{hi} + \vec{\varepsilon}^a_{hi+1}) + \sum_{j=i+1}^{n+1} p'^a_{ij}\varphi(\vec{p}' - \vec{\varepsilon}^a_{i+1j} + \vec{\varepsilon}^a_{ij})\Bigg\}.$$

We therefore obtain

$$\sum_{\vec{p}'}\int_\Gamma \frac{\partial}{\partial t_i^{(s_i')}}(\Phi\varphi(\vec{p}'))dt\cdot u^*(\vec{p}')$$
$$= -\nu\sum_{\vec{p}'}\sum_a\Bigg[\Bigg\{(\alpha_i,\lambda_a) - \sum_{h=1}^{i} p'^a_{hi+1} + \sum_{h=1}^{i-1} p'^a_{hi}\Bigg\}I(\vec{p}' + \vec{\varepsilon}^a_{ii+1})$$
$$- \sum_{h=1}^{i-1} p'^a_{hi}I(\vec{p}' - \vec{\varepsilon}^a_{hi} + \vec{\varepsilon}^a_{hi+1}) + \sum_{j=i+1}^{n+1} p'^a_{ij}I(\vec{p}' - \vec{\varepsilon}^a_{i+1j} + \vec{\varepsilon}^a_{ij})\Bigg]\cdot u^*(\vec{p}').$$

Here the first summation is over $\vec{p}' = \{p'^a_{hj}\}$ satisfying (5.1). We put $\vec{p} = \vec{p}' + \vec{\varepsilon}^a_{ii+1}$ in the first term of the right hand side, $\vec{p} = \vec{p}' - \vec{\varepsilon}^a_{hi} + \vec{\varepsilon}^a_{hi+1}$ in the second, and $\vec{p} = \vec{p}' - \vec{\varepsilon}^a_{i+1j} + \vec{\varepsilon}^a_{ij}$ in the last. Altogether, the right hand side reads as

$$-\nu\sum_{\vec{p}}\sum_a I(\vec{p})\Bigg[\Bigg\{(\alpha_i,\lambda_a) - \sum_{h=1}^{i} p^a_{hi+1} + \sum_{h=1}^{i-1} p^a_{hi} + 1\Bigg\}\cdot u^*(\vec{p} - \vec{\varepsilon}^a_{ik+1})$$
$$- \sum_{h=1}^{i-1}(p^a_{hi}+1)\cdot u^*(\vec{p} + \vec{\varepsilon}^a_{hi} - \vec{\varepsilon}^a_{hi+1}) + \sum_{j=i+1}^{n+1}(p^a_{ij}+1)\cdot u^*(\vec{p} + \vec{\varepsilon}^a_{i+1j} - \vec{\varepsilon}^a_{ij})\Bigg]. \tag{5.2}$$

The summation is over $\vec{p} = \{p^a_{hj}\}$ satisfying (2.7). Comparing (5.2) with the action of $F_i$ on $u^*(\vec{p})$ described in Lemma 4.2, we conclude that

$$0 = \prod(z_a - z_b)^{(\lambda_a,\lambda_b)}\sum_{\vec{p}'}\int_\Gamma \frac{\partial}{\partial t_i^{(s_i')}}(\Phi\varphi(\vec{p}'))dt\cdot u^*(\vec{p}')$$
$$= -\nu\prod(z_a - z_b)^{(\lambda_a,\lambda_b)}\sum_{\vec{p}} I(\vec{p})F_i\cdot u^*(\vec{p})$$
$$= -\nu\cdot F_i\cdot w^*(z).$$

*Proof of Theorem 2.4*. For simplicity, we write $t_i^{(s)}$ instead of $t_i^{(\beta_i(s))}$ for any $s = (a,h,j,q)\in S_i(\vec{p}) \subset S(\vec{p})$. We begin with

$$\nabla_{z_a}\varphi(\vec{p}) = \Bigg\{\frac{\partial}{\partial z_a}\log\Phi + \frac{\partial}{\partial z_a}\log\varphi(\vec{p})\Bigg\}\varphi(\vec{p})$$
$$= \Bigg\{\sum_{(i_b,s_b)}\frac{(\alpha_{i_b},\lambda_a)\cdot\nu}{t_{i_b}^{(s_b)} - z_a} + \sum_{\substack{s_a = (a,h_a,j_a,q_a)\\ s_a\in S(\vec{p})}}\frac{1}{t_{j_a-1}^{(s_a)} - z^a}\Bigg\}\varphi(\vec{p}). \tag{5.3}$$

On the other hand, we have the following relation for any $s_a=(a,h_a,j_a,q_a)\in S(\vec{p})$ and $i_a(h_a\leqq i_a<j_a)$.

$$\begin{aligned}0&\approx\nabla_{t^{(s_a)}_{i_a}}\varphi(\vec{p})\\&=\left\{\frac{\partial}{\partial t^{(s_a)}_{i_a}}\log\Phi+\frac{\partial}{\partial t^{(s_a)}_{i_a}}\log\varphi(\vec{p})\right\}\varphi(\vec{p})\\&=\left\{-\sum_b\frac{(\alpha_i,\lambda_b)\cdot\nu}{t^{(s_a)}_{i_a}-z_b}+\sum_{(i_b,s_b)(\neq(i_a,s_a))}\frac{(\alpha_{i_a},\alpha_{i_b})\cdot\nu}{t^{(s_a)}_{i_a}-t^{(s_b)}_{i_b}}-\theta(i_a=h_a)\frac{1}{t^{(s_a)}_{h_a}-t^{(s_a)}_{h_a+1}}\right.\\&\quad+\theta(h_a<i_a<j_a-1)\left(\frac{1}{t^{(s_a)}_{i_a-1}-t^{(s_a)}_{i_a}}-\frac{1}{t^{(s_a)}_{i_a}-t^{(s_a)}_{i_a+1}}\right)\\&\quad\left.+\theta(i_a=j_a-1)\left(\frac{1}{t^{(s_a)}_{j_a-2}-t^{(s_a)}_{j_a-1}}-\frac{1}{t^{(s_a)}_{j_a-1}-z_a}\right)\right\}\varphi(\vec{p}).\end{aligned}$$

Summing up the right hand side of this relation with respect to $i_a$, we obtain

$$\frac{1}{t^{(s_a)}_{j_a-1}-z_a}\varphi(\vec{p})\approx-\sum_{i_a=h_a}^{j_a-1}\sum_{b=1}^{N-1}\frac{(\alpha_{i_a},\lambda_b)\cdot\nu}{t^{(s_a)}_{i_a}-z_b}\varphi(\vec{p})+\sum_{i_a=h_a}^{j_a-1}\sum_{b(\neq a)}\sum_{(i_b,s_b)}\frac{(\alpha_{i_a},\alpha_{i_b})\cdot\nu}{t^{(s_a)}_{i_a}-t^{(s_b)}_{i_b}}\varphi(\vec{p}).\tag{5.4}$$

From (5.3) and (5.4), we obtain

$$\nabla_{z_a}\varphi(\vec{p})\approx\sum_{b(\neq a)}\left\{-\sum_{(i_b,s_b)}\frac{(\alpha_{i_b},\lambda_a)\cdot\nu}{t^{(s_b)}_{i_b}-z_a}+\sum_{(i_a,s_b)}\frac{(\alpha_{i_a},\lambda_b)\cdot\nu}{t^{(s_a)}_{i_a}-z_b}-\sum_{(i_a,s_a)}\sum_{(i_b,s_b)}\frac{(\alpha_{i_a},\alpha_{i_b})\cdot\nu}{t^{(s_a)}_{i_a}-t^{(s_b)}_{h_b}}\right\}\varphi(\vec{p}).\tag{5.5}$$

The following identity follows from Lemma 3.5.

$$\begin{aligned}\sum_{b(\neq a)}\sum_{(i_b,s_b)}\frac{1}{t^{(s_b)}_{i_b}-z_a}\varphi(\vec{p})&\sim\frac{1}{z_a-z_b}\sum_{b(\neq a)}\sum_{\substack{s_b=(b,h_b,j_b,q_b)\\h_b\leqq i_b<j_b}}\sum_{r\geqq 0}(-1)^r\sum_{\substack{i_0=h_b\\i_b<i_1<\cdots<i_r\leqq j_b}}\\&\quad\cdot\varphi(\vec{p}-\vec{\varepsilon}^{\,b}_{h_bj_b}+\vec{\varepsilon}^{\,a}_{i_0i_1}+\cdots+\vec{\varepsilon}^{\,a}_{i_{r-1}i_r}+\vec{\varepsilon}^{\,b}_{i_rj_b}).\end{aligned}$$

Applying this identity to the first and the second term of (5.5), and applying Lemma 3.2 to the third term by the same way, we see that the right hand side of (5.5) is equivalent to

$$\nu\sum_{b(\neq a)}\frac{1}{z_a-z_b}(A_{ab}-A_{ba}),$$

where

$$\begin{aligned}A_{ab}&=\sum_{h_a,h_b,j_a,j_b}p^a_{h_aj_a}p^b_{h_bj_b}\{\theta(h_b\leqq j_a<j_b)B_1+\theta(j_a=j_b)\theta(h_a>h_b)B_2\\&\quad+\theta(j_a=j_b)\theta(h_a=h_b)B_3\}+\sum_{h_b<j_b}\sum_{i_b=h_b}^{j_b}p^b_{h_bj_b}(\alpha_{i_b},\lambda_a)B_4,\end{aligned}$$

$$\begin{aligned}B_1&=\sum_{r\geqq 0}(-1)^r\sum_{\substack{i_0=h_a,i_0<i_1\\j_a\leqq i_1<\cdots<i_r\leqq j_b}}\varphi(\vec{p}-\vec{\varepsilon}^{\,b}_{h_bj_b}+\vec{\varepsilon}^{\,a}_{h_aj_a}+\vec{\varepsilon}^{\,a}_{h_bj_a}+\vec{\varepsilon}^{\,a}_{i_0i_1}+\cdots\\&\quad+\vec{\varepsilon}^{\,a}_{i_{r-1}i_r}+\vec{\varepsilon}^{\,b}_{i_rj_b}),\end{aligned}$$

$$B_2 = -\varphi(\vec{p} - \vec{\varepsilon}^{\,b}_{h_b j_b} + \vec{\varepsilon}^{\,a}_{h_b j_a}) + \varphi(\vec{p} - \vec{\varepsilon}^{\,b}_{h_b j_b} - \vec{\varepsilon}^{\,a}_{h_a j_a} + \vec{\varepsilon}^{\,b}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{h_b j_b})$$
$$+ \varphi(\vec{p}) - \varphi(\vec{p} - \vec{\varepsilon}^{\,a}_{h_a j_a} + \vec{\varepsilon}^{\,b}_{h_a j_a}),$$
$$B_3 = -\varphi(\vec{p} + \vec{\varepsilon}^{\,b}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{h_a j_a}) + \varphi(\vec{p}),$$
$$B_4 = -\sum_{r \geqq 0} (-1)^r \sum_{\substack{i_0 = h_b \\ i_b < i_1 < \cdots < i_r \leqq j_b}} \varphi(\vec{p} - \vec{\varepsilon}^{\,b}_{h_b j_b} + \vec{\varepsilon}^{\,a}_{i_0 i_1} + \cdots + \vec{\varepsilon}^{\,a}_{i_{r-1} i_r} + \vec{\varepsilon}^{\,b}_{i_r j_b}).$$

Hence we have

$$d \sum_{\vec{p}} I(\vec{p}) \cdot u^*(\vec{p}) = \nu \sum_{\vec{p}} \sum_{a \neq b} d \log(z_a - z_b) \cdot u^*(\vec{p}) \int_{\Gamma} \Phi A_{ab} dt. \tag{5.6}$$

The coefficient of $\nu\, d\log(z_a - z_b) I(\vec{p})$ is given by

$$\sum_{h_a, j_a, h_b, j_b} \{(p^a_{h_a j_a} + 1)(p^b_{h_b j_b} + 1)(C_1 + C_2) + p^a_{h_a j_a}(p^b_{h_b j_b} + 1)(C_3 + C_4)$$
$$+ (p^a_{h_a j_a} + 1) p^b_{h_b j_b} C_5 + (p^a_{h_a j_a} - 1)(p^b_{h_b j_b} + 1) C_6 + p^a_{h_a j_a} p^b_{h_b j_b} C_7\}$$
$$+ \sum_{h_b, j_b} \{(p^b_{h_b j_b} + 1) D_1 + p^b_{h_b j_b} D_2\},$$

where

$$C_1 = \theta(h_b \leqq j_a < j_b)\theta(h_a \neq h_b) \sum_{r > 0} (-1)^r \sum_{j_a < i_1 < \cdots < i_r \leqq j_b}$$
$$\cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_b} + \vec{\varepsilon}^{\,a}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{h_b j_a} - \vec{\varepsilon}^{\,a}_{h_a i_1} - \cdots - \vec{\varepsilon}^{\,a}_{i_{r-1} i_r} - \vec{\varepsilon}^{\,b}_{i_r j_b}) \qquad \text{(from } B_1)$$
$$C_2 = \theta(h_b \leqq j_a < j_b)\theta(h_a \neq h_b) \cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_a} + \vec{\varepsilon}^{\,a}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{h_b j_a} - \vec{\varepsilon}^{\,b}_{h_a j_b}) \qquad \text{(from } B_1)$$
$$+ \theta(j_a = j_b)\theta(h_a > h_b) \cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_a} + \vec{\varepsilon}^{\,a}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{h_b j_a} - \vec{\varepsilon}^{\,b}_{h_a j_b}) \qquad \text{(from } B_2),$$
$$C_3 = \theta(j_a < j_b)\theta(h_a = h_b) \sum_{r > 0} (-1)^r \sum_{j_a < i_1 < \cdots < i_r \leqq j_b}$$
$$\cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_b} - \vec{\varepsilon}^{\,a}_{h_a i_1} - \cdots - \vec{\varepsilon}^{\,a}_{i_{r-1} i_r} - \vec{\varepsilon}^{\,b}_{i_r j_b}) \qquad \text{(from } B_1)$$
$$C_4 = \theta(h_b \leqq j_a < j_b)\theta(h_a \neq h_b) \sum_{r > 0} (-1)^r \sum_{j_a = i_1 < \cdots < i_r \leqq j_b}$$
$$\cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_b} - \vec{\varepsilon}^{\,a}_{h_b j_a} - \vec{\varepsilon}^{\,a}_{i_1 i_2} - \cdots - \vec{\varepsilon}^{\,a}_{i_{r-1} i_r} - \vec{\varepsilon}^{\,b}_{i_r j_b}) \qquad \text{(from } B_1)$$
$$- \theta(j_a = j_b)\theta(h_a < h_b) \cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_b} - \vec{\varepsilon}^{\,a}_{h_b j_a}) \qquad \text{(from } B_2),$$
$$C_5 = -\theta(j_a = j_b)\theta(h_a < h_b) \cdot u^*(\vec{p} + \vec{\varepsilon}^{\,a}_{h_a j_a} - \vec{\varepsilon}^{\,b}_{h_a j_a}) \qquad \text{(from } B_2),$$
$$C_6 = \theta(j_a < j_b)\theta(h_a = h_b) \sum_{r > 0} (-1)^r \sum_{j_a = i_1 < \cdots < i_r \leqq j_b}$$
$$\cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_b} - \vec{\varepsilon}^{\,a}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{i_1 i_2} - \cdots - \vec{\varepsilon}^{\,a}_{i_{r-1} i_r} - \vec{\varepsilon}^{\,b}_{i_r j_b}) \qquad \text{(from } B_1)$$
$$- \theta(j_a = j_b)\theta(h_a = h_b) \cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_a j_a} - \vec{\varepsilon}^{\,a}_{h_a j_a}) \qquad \text{(from } B_3),$$
$$C_7 = \{\theta(j_a < j_b)\theta(h_a = h_b) + \theta(j_a = j_b)\theta(h_a > h_b) + \theta(j_a = j_b)\theta(h_a = h_b)\} \cdot u^*(\vec{p})$$
$$\text{(from } B_1, B_2, B_3),$$
$$D_1 = -\sum_{r > 0} (-1)^r \sum_{h_b < i_1 < \cdots < i_r \leqq j_b} \sum_{i_b = h_b}^{i_1 - 1} (\alpha_{i_b}, \lambda_a)$$
$$\cdot u^*(\vec{p} + \vec{\varepsilon}^{\,b}_{h_b j_b} - \vec{\varepsilon}^{\,a}_{h_b i_1} - \vec{\varepsilon}^{\,a}_{i_1 i_2} - \cdots - \vec{\varepsilon}^{\,a}_{i_{r-1} i_r} - \vec{\varepsilon}^{\,b}_{i_r j_b}) \qquad \text{(from } B_4),$$
$$D_2 = -(\alpha_{h_b j_b}, \lambda_a) \cdot u^*(\vec{p}) \qquad \text{(from } B_4).$$

Note that the following identities can be proved by using the symmetry with respect to $a$ and $b$:

$$\sum_{a\neq b}\sum_{h_a,j_a,h_b,j_b}(p^a_{h_aj_a}+1)(p^b_{h_bj_b}+1)C_2=\sum_{a\neq b}\sum_{h_a,j_a,h_b,j_b}(p^a_{h_aj_a}+1)(p^b_{h_bj_b}+1)C'_2,$$

$$\begin{aligned}&\sum_{a\neq b}\sum_{h_a,j_a,h_b,j_b}\{p^a_{h_aj_a}(p^b_{h_bj_b}+1)C_4+(p^a_{h_aj_a}+1)p^b_{h_bj_b}C_5\}\\&\quad=\sum_{a\neq b}\sum_{h_a,j_a,h_b,j_b}p^a_{h_aj_a}(p^b_{h_bj_b}+1)C'_{45},\end{aligned}$$

where

$$\begin{aligned}C'_2&=\theta(h_a>h_b)\cdot u^*(\vec{p}+\vec{\varepsilon}^{\,b}_{h_bj_a}+\vec{\varepsilon}^{\,a}_{h_aj_a}-\vec{\varepsilon}^{\,a}_{h_bj_a}-\vec{\varepsilon}^{\,b}_{h_aj_b})\\C'_{45}&=\theta(h_a\neq h_b)\sum_{r>0}(-1)^r\sum_{j_a=i_1<\cdots<i_r\leqq j_b}\\&\qquad\cdot u^*(\vec{p}+\vec{\varepsilon}^{\,b}_{h_bj_b}-\vec{\varepsilon}^{\,a}_{h_bj_a}-\vec{\varepsilon}^{\,a}_{i_1i_2}-\cdots-\vec{\varepsilon}^{\,a}_{i_{r-1}i_r}-\vec{\varepsilon}^{\,b}_{i_rj_b}).\end{aligned}$$

We rewrite (5.6) by these identities. Comparing the terms, except $C_7$ and $D_2$, with the terms of $\rho_a(F_{h_bi})\otimes\rho_b(E_{h_bi})\cdot u^*(\vec{p})$ described in Lemma 4.1, we obtain

$$\begin{aligned}d\sum_{\vec{p}}I(\vec{p})\cdot u^*(\vec{p})&=\nu\sum_{\vec{p}}\sum_{a>b}d\log(z_a-z_b)I(\vec{p})\Bigg\{\sum_{h_b<i_r}\rho_a(F_{h_bi_r})\otimes\rho_b(E_{h_bi_r})\\&\quad+\sum_{h_b<i_r}\rho_a(E_{h_bi_r})\otimes\rho_b(F_{h_bi_r})-\sum_{h_b<j_b}p^b_{h_bj_b}(\alpha_{h_bj_b},\lambda_a)\\&\quad-\sum_{h_a<j_a}p^a_{h_aj_a}(\alpha_{h_aj_a},\lambda_b)+\Bigg(\sum_{h_a=h_b}+\sum_{j_a=j_b}\Bigg)p^a_{h_aj_a}p^b_{h_bj_b}\Bigg\}\cdot u^*(\vec{p}).\qquad(5.7)\end{aligned}$$

Note that we have changed the range of the summation from $a\neq b$ to $a<b$. We finally consider the action of Cartan subalgebra. Let $\{H^i\}$ denote the dual basis to $\{H_i\}$ with respect to (,). We have

$$\begin{aligned}\sum_i\rho_a(H_i)\otimes\rho_b(H^i)\cdot u^*(\vec{p})&=\Bigg(-\lambda_a+\sum_{h_a\leqq i_a<j_a}p^a_{h_aj_a}\alpha_{i_a},-\lambda_b+\sum_{h_b\leqq i_b<j_b}p^b_{h_bj_b}\alpha_{i_b}\Bigg)\cdot u^*(\vec{p})\\&=\Bigg\{(\lambda_a,\lambda_b)-\sum_{h_a\leqq i_a<j_a}p^b_{h_aj_a}(\alpha_{i_a},\lambda_a)-\sum_{h_b\leqq i_b<j_b}p^a_{h_bj_b}(\alpha_{i_b},\lambda_b)\\&\quad+\Bigg(\sum_{h_a\leqq i_a<j_a}p^a_{h_aj_a}\Bigg)\Bigg(\sum_{h_b\leqq i_b<j_b}p^b_{h_bj_b}\Bigg)(\alpha_{i_a},\alpha_{i_b})\Bigg\}\cdot u^*(\vec{p}).\end{aligned}\qquad(5.8)$$

The following relation is easily obtained by simple calculation using the explicit values of $(\alpha_{i_a},\alpha_{i_b})$.

$$\begin{aligned}&\Bigg(\sum_{h_a=h_b}+\sum_{j_a=j_b}\Bigg)p^a_{h_aj_a}p^b_{h_bj_b}\cdot u^*(\vec{p})\\&\quad=\Bigg(\sum_{h_a\leqq i_a<j_a}p^a_{h_aj_a}\Bigg)\Bigg(\sum_{h_b\leqq i_b<j_b}p^b_{h_bj_b}\Bigg)(\alpha_{i_a},\alpha_{i_b})\cdot u^*(\vec{p}).\end{aligned}\qquad(5.9)$$

Comparing (5.7), (5.8), (5.9), we obtain

$$d\sum_{\vec{p}} I(\vec{p})\cdot\mu^*(\vec{p}) = v\sum_{\vec{p}}\sum_{a<b} d\log(z_a - z_b) I(\vec{p})\left\{\sum_{h<i}\rho_a(F_{hi})\otimes\rho_b(E_{hi})\right.$$
$$\left. + \sum_{h<i}\rho_a(E_{hi})\otimes\rho_b(F_{hi}) + \sum_{i=1}^{n}\rho_a(H_i)\otimes\rho_b(H^i) - (\lambda_a, \lambda_b)\right\}\cdot u^*(\vec{p}).$$

This is nothing but the Knizhnik–Zamolodchikov equation (2.1).

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