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Unitary Dressing Transformations and Exponential Decay Below Threshold for Quantum Spin Systems. Parts I and II

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Abstract. We consider a class of quantum spin systems defined on connected graphs of which the following Heisenberg XY-model with a variable magnetic field gives an example:

$$H_\lambda = \sum_{x \in \mathbb{Z}^d} h_x \sigma_x^{(3)} + \lambda \sum_{\langle x,y \rangle \subset \mathbb{Z}^d} (\sigma_x^{(1)}\sigma_y^{(1)} + \sigma_x^{(2)}\sigma_y^{(2)}).$$

We treat first the case in which $h_x = \pm 1$ for all sites x and we introduce a unitary dressing transformation to control the spectrum for λ small. Then, we consider a situation in which $|h_x|$ can be less than one for x in a finite set $\mathcal{S}$ and prove exponential decay away from $\mathcal{S}$ of dressed eigenfunctions with energy below the one-quasiparticle threshold. If the ground state is separated by a finite gap from the rest of the spectrum, this result can be strengthened and one can compute a second unitary transformation that makes the ground state of compact support. Finally, a case in which the singular set $\mathcal{S}$ is of finite density, is considered. The main technical tools we use are decay estimates on dressed Green's functions and variational inequalities.

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0. Introduction

In [1] and [2], a technique based on dressing transformations was introduced to study some problems of spectral perturbation theory for quantum many body systems. This enabled us to answer a few basic and quite elementary questions concerning the ground state properties, the existence of a gap, the definition and mutual interaction of quasiparticles and the decay of correlations. In the present article, such a technique is improved and more sophisticated questions are addressed. The improvement consists in the construction of a dressing transformation which is unitary. As remarked in [1] already, unitarity does not seem to be a property compatible with the commutativity of the algebra of the operators used to express the dressing transformation. Hence, the problem is how to allow a little noncommutativity to achieve unitarity without compromising the control of the cluster expansions defining the dressing transformation. The property of unitarity is essential in order to use variational methods to treat problems in which analyticity is missing and for which perturbative methods alone are not powerful enough. One such problem is considered in Part II, where we study the asymptotic behavior of eigenfunctions with energy below threshold, far away from the support of a local perturbation. "Below threshold" means, roughly speaking, that in such eigenstates there cannot be any quasiparticle in a scattering state, i.e. able to travel to infinity. In Part III, this is used to establish a result that represents one of the building blocks for the construction of the ground state of the random field quantum XY-model in dimension 2, [3]. Finally, in Part IV, we consider the ground state problem in a situation where the singular set $\mathscr{S}$ is the union of a finite density of small clusters separated by a large distance. Some of the physical literature on related problems is given in [5].

The present paper is subdivided into four parts, each depending on the preceding one. In each part, the first section is introductory and contains the description of the problem considered and the statement of the main results. The details of the proofs are deferred to the other sections. This paper has been split into two articles. The present article contains Parts I and II.

Part I. Unitary Dressing Transformations

1. The Model and the Results

The quantum spin systems considered in the present article, are defined on connected graphs Λ of finite but arbitrarily large size. The vertices of Λ will be called "sites" and denoted with letters like $x, y, \ldots$. If x is a site of Λ,we write $x \in \Lambda$. The model introduced in the abstract is defined on the cubic lattice $\mathbb{Z}^d$. In this particular case, $\mathbb{Z}^d$ has to be seen as an infinite graph in which each site is joined to its nearest neighbors and only to them by a line. In this case, Λ is an (arbitrarily large) connected subgraph of $\mathbb{Z}^d$.

The choice to work in the general setting of models defined on an arbitrary finite graph Λ, is motivated by our intention to apply the techniques developed here to the random field quantum XY-model in dimension 2; see [3]. This model is defined on $\mathbb{Z}^2$ but, since a multiscale analysis is required to construct the ground state, one needs to know how to deal with graphs obtained from $\mathbb{Z}^2$ by contracting to a point the singular sets. The quantum spin systems considered in [1] and [2] enjoy translation invariance and this property is used there to control the convergence of cluster expansions. One of the purposes of this paper is to show how to avoid such an assumption.

If $(N_x + 1) \geqq 2$ is the number of levels on the site x, the Hilbert space is

$$\mathscr{H}(\Lambda) = \bigotimes_{x \in \Lambda} \mathbb{C}^{N_x+1} \tag{1.1}$$

and the Hamiltonian operator has the form

$$H_\lambda = \sum_{x \in \Lambda} s_x + \sum_{\gamma_0 \subset \Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0}, \tag{1.2}$$

where λ is a small parameter. If $\gamma_0 \subset \Lambda$ is any subset, $|\gamma_0|_c$ is the volume of the smallest connected set containing γ_0. In order to avoid writing absolute values, we suppose that $\lambda \geqq 0$. s_x and t_{γ_0} are selfadjoint operators acting on the spins in x and γ_0, respectively. We suppose that $t_{\gamma_0} = 0$ if γ_0 consists of one single site. The basis of $\mathbb{C}^{N_x+1}$ in which s_x is diagonal, is denoted by

$$|0\rangle_x, |1\rangle_x, \ldots, |N_x\rangle. \tag{1.3}$$

In this first part, we assume that

$$s_x |0\rangle_x = 0 \tag{1.4}$$

and

$$\langle i | s_x | i \rangle \geqq 1 \quad \text{for all} \quad i = 1, \ldots, N_x. \tag{1.5}$$

In the second and third parts, this hypothesis is relaxed for x in a finite set. We also assume that

$$|s| = \sup_x \| s_x \| \tag{1.6}$$

is finite and fixed, so that any function depending only on $|s|$ can be called a constant. The condition (1.6) simplifies some of the arguments but is not essential and in the Appendix at the end of Sect. 2, we discuss how to remove it. Let

$$|0\rangle \equiv \bigotimes_{x \in \Lambda} |0\rangle_x \tag{1.7}$$

denote the ground state of H_λ for $\lambda = 0$.

Finally, we impose a normalization condition on the operators t_{γ_0} in (1.2). To state it, let us define an excitation to be a map γ defined on the sites of Λ and such that $\gamma(x)$ takes values in the set $\{0, 1, \ldots, N_x\}$. Let $|\gamma\rangle$ denote the state

$$|\gamma\rangle = \bigotimes_{x \in \Lambda} |\gamma(x)\rangle_x \tag{1.8}$$

and let the support $s(\gamma)$ of γ be defined as the following set:

$$s(\gamma) = \{x \in \Lambda \,|\, \gamma(x) \neq 0\}. \tag{1.9}$$

A state u of $\mathscr{H}$ can always be written in the form

$$u = \sum_{\gamma} u_\gamma |\gamma\rangle, \tag{1.10}$$

where the sum runs over all excitations. The L^1-norm of u is defined as follows:

$$\|u\|_1 = \sum_{\gamma} |u_\gamma|. \tag{1.11}$$

If $\|t_{\gamma_0}\|_1$ denotes the L^1-operator norm of t_{γ_0}, the condition we assume to be fulfilled is

$$\sup_x \sum_{|\gamma_0|_c = n, \gamma_0 \ni x} \|t_{\gamma_0}\|_1 \leqq 1 \tag{1.12}$$

for all $n = 1, 2, \ldots$.

In Sect. 2, we consider the ground state problem for the Hamiltonian H_λ. We use a method based on dressing transformations that is a refinement of the one used in [1]. As remarked in [1] already, it seems to be impossible to construct unitary dressing transformations involving only operators belonging a commutative algebra. On the other hand, the techniques developed in [1] to control the analyticity of the dressing transformation for λ small, are based on such commutativity properties. As a matter of fact, the construction of unitary dressing transformations necessitates three ingredients that are not contained in [1]. First, one can relax a little the condition of commutativity without losing control of the convergence of the cluster expansions involved. Second, the dressing transformation must not be written as the exponential of a skew-symmetric operator, but as the product of an infinite number of operators of such a form. Since we are dealing with a non-commutative operator, this makes a difference. The third new point to understand, is how to solve a problem of an algebraic nature. In fact, in order to control the convergence of cluster expansions, the operators entering into the dressing transformation must belong to an algebra of skew-symmetric operators satisfying a condition of weak non-commutativity. A simple minded comparison of the number of free parameters and the number of constraints characterizing such an algebra, gives a very discouraging result for systems with more than three levels per site. However, thanks to a strike of luck, several algebras having the right properties exist and, in Sect. 2, we construct one of them.

Unlike the (undressed) Green's function $(H_\lambda - z)^{-1}$, the dressed Green's functions have a remarkably simple behavior in that their kernel in the basis of excitations decays exponentially fast with the separation among the supports of the excitations in its two arguments. This feature of dressed Green's functions is one of the basic properties of the dressed representation and it is independent of the property of unitarity. However, it turns out to be of no use for applications if it cannot be combined with estimates in the L^2-operator norm for Green's functions. Typically, one would like to derive such an estimate from information on the gap. However, if the dressed Hamiltonian is not selfadjoint, i.e. if the dressing transformation is not unitary, this cannot be done in any satisfactory way.

The second reason why unitarity is a property of crucial importance, is that there are situations like the one considered in Part II, in which the lack of analyticity forces one to use non-perturbative (i.e. variational) arguments in the dressed representation. However, the variational principle concerns only selfadjoint eigenvalue problems and selfadjointness, in general, is preserved only by unitary transformations.

In the rest of this section, we introduce some notations and give a more precise statement of the results proven in Part I.

We use the following ansatz for the unitary dressing transformation to be constructed:

$$U(\lambda) = \lim_{v \to \infty} e^{R^1(\lambda)} \cdots e^{R^v(\lambda)}. \tag{1.13}$$

$U(\lambda)$ has to solve the following conjugacy problem:

$$U(\lambda)^{-1} H_\lambda U(\lambda)|0\rangle = E_0(\lambda)|0\rangle \tag{1.14}$$

for all λ small enough, where $E_0(\lambda)$ is a constant. The operator $R^v(\lambda)$ has the form

$$R^v(\lambda) = \sum_{n=1}^{\infty} \lambda^n R_n^v \tag{1.15}$$

with

$$R_n^v = \sum_{\gamma:|s(\gamma)|=v} r_{n\gamma} \tau_\gamma. \tag{1.16}$$

τ_γ is a skew symmetric operator with support $s(\gamma)$, i.e. it acts only on the spins in $s(\gamma)$ and it is such that

$$\tau_\gamma|0\rangle = |\gamma\rangle. \tag{1.17}$$

Notation. In this paper we denote with c or c_0 all positive constants independent of Λ. They may depend on $|s|$, though. Of course, $c = 2c < c$. The notation c_0 will be used for constants defined in preceding sections, while c denotes any constant arising in the current section.

The following is the first result proven in this part:

Theorem 1.1. *If the operators τ_γ are chosen as indicated in Sect. 2, the dressing transformation $U(\lambda)$ solving the conjugacy problem (1.14) is uniquely determined and it admits an analytic extension to a disc $\{\lambda \in \mathbb{C} \,|\, |\lambda| < c\}$. Moreover, we have*

$$\sum_{n=1}^{\infty} \lambda^n \sup_{x \in \Lambda} \sum_{\gamma: x \in s(\gamma)} \| r_{n\gamma} \tau_\gamma \|_1 \leqq c. \tag{1.18}$$

To formulate the other results, let us write the dressed Hamiltonian as follows:

$$U(\lambda)^{-1} H_\lambda U(\lambda) = S + V(\lambda) + E_0(\lambda), \tag{1.19}$$

where

$$S = \sum_{x \in \Lambda} S_x, \tag{1.20}$$

$E_0(\lambda)$ is the constant appearing in (1.14) and $V(\lambda)$ is the remainder. Due to (1.14), we have

$$V(\lambda)|0\rangle = 0. \tag{1.21}$$

Equation (1.21) is one of the basic properties of the operator $V(\lambda)$ and is at the origin of several of the following results:

Theorem 1.2. *If $\lambda \leqq c$, we have*

(i) *$V(\lambda)$ has the form*

$$V(\lambda) = \sum_{\gamma_0 \subset \Lambda} v_{\gamma_0}(\lambda), \tag{1.22}$$

where v_{γ_0} is an operator with support γ_0 such that

$$\sup_x \sum_{\substack{\gamma_0 : |\gamma_0|_c \geqq n \\ x \in \gamma_0}} \| v_\gamma(\lambda) \|_1 \leqq (c\lambda)^{n-1}; \tag{1.23}$$

(ii) *$V(\lambda)$ satisfies the following relative form boundedness estimate with respect to S:*

$$|\langle u | V(\lambda) | u \rangle \leqq c\lambda \langle u | S | u \rangle \tag{1.24}$$

for all $u \in \mathscr{H}(\Lambda)$;

(iii) *$V(\lambda)$ is relatively bounded in L^1-norm with respect to S, in the following sense:*

$$\| S^{-1/2} V(\lambda) S^{-1/2} \|_1 \leqq c\lambda; \tag{1.25}$$

(iv) *$V(\lambda)$ is relatively bounded in L^2-norm with respect to S, in the following sense:*

$$\| V(\lambda) | u \rangle \|_2 \leqq (c\lambda) \| Su \|_2, \quad \forall u \in \mathscr{H}; \tag{1.26}$$

(v) *$V(\lambda)|0\rangle$ is the ground state of H_λ for λ small and its energy is separated by a gap $(1 - 0(\lambda))$ from the rest of the spectrum of $\mathscr{H}_\lambda$;*

(vi) *The kernel of the dressed Green's function*

$$G(\gamma, \gamma') \equiv \langle \gamma | (S + V(\lambda))^{-1} | \gamma' \rangle \tag{1.27}$$

for $s(\gamma)$, $s(\gamma') \neq \varnothing$, satisfies the following decay estimate

$$\sup_\gamma \sum_{\gamma' : d_c(s(\lambda), s(\gamma')) \geqq k} |G(\gamma, \gamma')| \leqq (c\lambda)^k, \tag{1.28}$$

where, if $\gamma_0, \gamma'_0 \subset \Lambda$, we define

$$\begin{aligned} d_c(\gamma_0, \gamma'_0) = \min \{ |\Gamma| \ & \text{for } \Gamma \subset \Lambda \text{ such that for all } x \in \gamma \backslash \gamma'_0 \text{ (respectively } \gamma'_0 \backslash \gamma_0) \\ & \text{there is a path in } \Gamma \text{ joining it to } \gamma'_0 \text{ (respectively } \gamma_0) \} \end{aligned} \tag{1.29}$$

The three relative boundedness estimates contained in Theorem 1.2 are all quite important, because they permit us to control the convergence of random walk expansions expressing dressed Green's functions in several different norms. Moreover, they permit to establish the positivity of certain operators entering in the variational inequalities contained in Parts II and III. Let us remark that in this problem there are two different norms arising in a natural way: The L^1 norm defined in (1.11) and the L^2-norm of $\mathscr{H}$. The L^1-norm is useful to control cluster expansions, while the L^2-norm is indispensable to bound the norm of Green's functions and to express variational inequalities. This is the reason why we need all the relative boundedness estimates in Theorem 1.2.

Section II contains also the proof of the following result concerning the decay of correlations in the ground state:

Theorem 1.3. *If $\mathcal{O}_{\gamma_0}$ and $\mathcal{O}_{\gamma'_0}$, are two operators of L^1-operator norm one and with supports γ_0 and γ'_0, respectively, we have*

$$|\langle U(\lambda)0 | \mathcal{O}_{\gamma_0} \mathcal{O}_{\gamma'_0} | U(\lambda)0 \rangle - \langle U(\lambda)0 | \mathcal{O}_{\gamma_0} | U(\lambda)0 \rangle \langle U(\lambda)0 | \mathcal{O}_{\gamma'_0} | U(\lambda)0 \rangle | \leqq (c)^{d(\gamma_0, \gamma'_0)}, \tag{1.30}$$

where

$$d(\gamma_0, \gamma'_0) = \min\{d(x, y), x \in \gamma_0, y \in \gamma'_0\}.$$

2. Construction of a Unitary Dressing Transformation

The aim of this section is to define the operators τ_γ in (1.16) and to give a proof of Theorem 1.1.

To define the operators τ_γ in (4.16), let us look at $\mathbb{C}^{N+1}$ as the subspace of the linear space

$$\mathbb{C}^{\mathbb{Z}} = \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \tag{2.1}$$

of the vectors $(u(k))_{k \in \mathbb{Z}}$ with $u(k) = u(k')$ if $k \sim k'$, where $\sim$ is the following equivalence relation:

$$k \sim k' \quad \text{iff} \quad \exists n \in \mathbb{Z} \text{ such that } k = \pm k' \bmod 2(N+1). \tag{2.2}$$

Let us introduce the following operators acting on $\mathbb{C}^{\mathbb{Z}}$,

$$(T_n u)(k) = \tfrac{1}{2}[u(k+n) + u(k-n)]. \tag{2.3}$$

The subspace $\mathbb{C}^N$ defined above is invariant under T_n. In fact, for all integers $n \in \mathbb{Z}$, if $k \sim k'$ then either $k+n \sim k'+n$ or $k+n \sim k'-n$. Hence, if $u \in \mathbb{C}_1^{\mathbb{Z}}$, $n = \mathbb{Z}$ and $k \sim k'$ we have

$$\tfrac{1}{2}(u(k+n) + u(k-n)) = \tfrac{1}{2}(u(k'+n) + u(k'-n)). \tag{2.4}$$

The operators T_n in (2.3) are symmetric and we have

$$T_n |0\rangle = |n\rangle, \tag{2.5}$$

for all $n \in \{0, 1, \ldots, N\}$ and

$$T_n T_m = \tfrac{1}{2}(T_{n+m} + T_{n-m}), \tag{2.6}$$

$$T_{-n} = T_n, \tag{2.7}$$

for all $n, m \in \mathbb{Z}$. In particular, from (2.6) and (2.7) we see that the operators T_n are mutually commuting, i.e.

$$[T_n, T_m] = 0 \quad \forall n, m \in \{0, 1, \ldots, N\}. \tag{2.8}$$

Let $T_{n,x}$ be the operator T_n defined in (2.3) and acting on the copy of $\mathbb{C}^{N_x+1}$ attached to the site x of Λ. Moreover, let $\tilde{T}_{n,x}$ be the skew-symmetric operator with support $\{x\}$, such that

$$\tilde{T}_{n,x} = |n\rangle_{xx}\langle 0| - |0\rangle_{xx}\langle n|. \tag{2.9}$$

If γ is an excitation, let us define

$$\tau_r = \frac{1}{|s(\gamma)|} \sum_{x_0 \in s(\gamma)} \tilde{T}_{\gamma(x_0), x_0} \cdot \left(\prod_{x \in s(\gamma) \setminus \{x_0\}} T_{\gamma(x), x} \right). \tag{2.10}$$

The operators τ_γ in (2.10) are skew-symmetric as required, and their L^1-operator norm is one. Unlike the operators used in [1] and [2], they do not form a commutative algebra, but since they contain only one center of non-commutativity "diluted" over all the support, the norm of the commutators is small enough to control the cluster expansions defining $R(\lambda)$. Such expansions are the next topic to be discussed.

By expanding both members of (1.10) in powers of λ and equating the coefficients, we find the following recurrence relations for the operators R_n^v:

$$\sum_{v=1}^{n+1} SR_n^v|0\rangle = -\Bigg\{\sum_x \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1+\cdots+i_k=n \\ k>1}} \frac{1}{(v)!}[\cdots[s_x, R_{i_1}^{v_1}]\cdots R_{i_k}^{v_k}]$$

$$+ \sum_{|\gamma_0|_c \leqq n} \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_i+\cdots+i_k=n-|\gamma_0|_c+1}} \frac{1}{(v)!}[\cdots[t_{\gamma_0}, R_{i_1}^{v_1}]\cdots R_{i_k}^{v_k}]\Bigg\}|0\rangle, \quad (2.11)$$

where S is defined as in (1.20) and

$$(v)! = \prod_{s=1}^{\infty} (\#\{v_i = s\})!. \quad (2.12)$$

Let us remark that if $n=1$, $r_{1\gamma}=0$ unless $|s(\gamma)|_c=2$, while at the n^{th} order of perturbation theory, in the operator R_n^v only clusters γ of size

$$|s(\gamma)| \leqq |s(\gamma)|_c \leqq n+1 \quad (2.13)$$

are present.

In order to control the convergence of the expansion defined by (2.11), let us introduce the following numerical sequence

$$r_n^* \equiv \sup_x \sum_{\gamma: x\in s(\gamma)} |s(\gamma)||r_{n\gamma}| = \sup_x \left\| \sum_{\gamma: x\in s(\gamma)} |s(\gamma)| r_{n_\gamma} \tau_\gamma |0\rangle \right\|_1 . \quad (2.14)$$

For $n=1$, we have

$$\begin{aligned} r_1^* &= \sup_x \sum_{\gamma: x\in s(\gamma)} |s(\gamma)||r_{1\gamma}| \\ &\leqq \sup_x \sum_{\substack{\gamma_0: x\in\gamma_0 \\ |\gamma_0|_c=2}} |\gamma_0| \| t_{\gamma_0}|0\rangle \|_1 \leqq 2. \end{aligned} \quad (2.15)$$

If $x\in\Lambda$, let P_x be the orthogonal projection onto the subspace of the states with the spin in x excited. Let us estimate the following norm:

$$\begin{aligned} &\sup_x \left\| P_x \sum_{\substack{|\gamma_0|_c=v_0 \\ |s(\gamma_1)|=v_1}} [t_{\gamma_0}, r_{i_1\gamma_1}\tau_{\gamma_1}]|0\rangle \right\|_1 \\ &\leqq \sup_x \sum_{\substack{|\gamma_0|_c=v_0 \\ |s(\gamma_1)|=v_1, x\in\gamma_0}} \| [t_{\gamma_0}, \tau_{\gamma_1}] \|_1 |r_{i_1\gamma_1}| + \sup_x \sum_{\substack{|\gamma_0|_c=v_0 \\ |s(\gamma_1)|=v_1, x\in s(\gamma_1)}} \| [t_{\gamma_0}, \tau_{\gamma_1}] \|_1 |r_{i_1\gamma_1}| \\ &\leqq 2\Bigg(\sup_x \sum_{\substack{|\gamma_0|_c=v_0 \\ x\in\gamma_0}} \| t_{\gamma_0} \|_1 |\gamma_0| \Bigg)\Bigg(\sup_y \sum_{\substack{|s(\gamma_1)|=v_1 \\ y\in s(\gamma_1)}} |r_{i_1\gamma_1}| \Bigg) \\ &\quad + 2\Bigg(\sup_x \sum_{\substack{|s(\gamma_1)|=v_1 \\ x\in s(\gamma_1)}} |r_{i_1\gamma_1}||s(\gamma_1)| \Bigg)\Bigg(\sup_y \sum_{\substack{|\gamma_0|_c=v_0 \\ y\in\gamma_0}} \| t_{\gamma_0} \|_1 \Bigg) \\ &\leqq 2(v_0+v_1) r_{i_1v_1}^* \leqq 2(v_0+1) r_{i_1v_1}^* v_1, \end{aligned} \quad (2.16)$$

where

$$r^*_{iv} \equiv \sup_x \sum_{\substack{|s(\gamma)|=v \\ x\in s(\gamma)}} |r_{i\gamma}|. \tag{2.17}$$

Let $\mathscr{T}^{(1)}_{\bar{\gamma}}$ be the operator with support $\bar{\gamma}\subset\Lambda$ such that

$$\sum_{\bar{\gamma}\subset\Lambda} \mathscr{T}^{(1)}_{\bar{\gamma}} = \sum_{\substack{|\gamma_0|_c=v_0 \\ |s(\gamma_1)|=v_1}} [t_{\gamma_0}, r_{i_1\gamma_1}\tau_{\gamma_1}]. \tag{2.18}$$

$\mathscr{T}^{(1)}_{\bar{\gamma}}$ contains only clusters of size $\leqq v_0+v_1$. The number of centers of non-commutativity that $\mathscr{T}^{(1)}_{\bar{\gamma}}$ contains is $\leqq v_0+1$ and, due to (2.16), we have

$$\sup_x \sum_{x\in\bar{\gamma}} \|\mathscr{T}^{(1)}_{\bar{\gamma}}\|_1 \leqq 2(v_0+1) r^*_{i_1v_1} v_1. \tag{2.19}$$

Hence, one can bound as follows the double commutator in (2.11):

$$\begin{aligned}
&\sup_x \left\| P_x \sum_{\substack{|\gamma_0|_c=v_0 \\ |s(\gamma_1)|=v_1, |s(\gamma_2)|=v_2}} [[t_{\gamma_0}, r_{i_1\gamma_1}\tau_{\gamma_1}], r_{i_2\gamma_2}\tau_{\gamma_2}]|0\rangle \right\|_1 \\
&= \sup_x \left\| P_x \sum_{|s(\gamma_2)|=v_2} [\mathscr{T}^{(1)}_{\bar{\gamma}}, r_{i_2\gamma_2}\tau_{\gamma_2}]|0\rangle \right\|_1 \\
&\leqq \sup_x \sum_{x\in\bar{\gamma}, |s(\gamma_2)|=v_2} \|[\mathscr{T}^{(1)}_{\bar{\gamma}}, r_{i_2\gamma_2}\tau_{\gamma_2}]\|_1 \\
&\quad + \sup_x \sum_{x\in s(\gamma_2), |s(\gamma_2)|=v_2} \|[\mathscr{T}^{(1)}_{\bar{\gamma}}, r_{i_2\gamma_2}\tau_{\gamma_2}]\|_1 .
\end{aligned} \tag{2.20}$$

The first sum in (2.20) can be split into a sum over γ_2 such that $s(\gamma_2)$ intersects the centers of noncommutativity of $\mathscr{T}^{(1)}_{\bar{\gamma}}$, plus the remaining terms for which $\bar{\gamma}$ intersects the center of non-commutativity diluted in τ_{γ_2}. The first term is

$$\begin{aligned}
&\leqq 2(v_0+1)\left(\sup_x \sum_{x\in\bar{\gamma}} \|\mathscr{T}^{(1)}_{\bar{\gamma}}\|_1\right)\left(\sup_y \sum_{\substack{y\in s(\gamma_2) \\ |s(\gamma_2)|=v_2}} |r_{i_2\gamma_2}|\right) \\
&\leqq 4(v_0+1)^2 (r^*_{i_1v_1} v_1) r^*_{i_2v_2}.
\end{aligned} \tag{2.21}$$

The remaining terms are bounded from above by

$$\begin{aligned}
&2\frac{v_0+v_1}{v_2}\left(\sup_x \sum_{x\in\bar{\gamma}} \|\mathscr{T}^{(1)}_{\bar{\gamma}}\|_1\right)\left(\sup_y \sum_{\substack{y\in s(\gamma_2) \\ |s(\gamma_2)|=v_2}} |r_{i_2\gamma_2}|\right) \\
&\leqq 4\frac{(v_0+v_1)^2}{v_2} (r^*_{i_1v_1} v_1) r^*_{i_2v_2}.
\end{aligned} \tag{2.22}$$

Finally, the second term in (2.20) is

$$\begin{aligned}
&\leqq 2\left(\sup_x \sum_{\substack{x\in s(\gamma_2) \\ |s(\gamma_2)|=v_2}} |r_{i_2\gamma_2}|\right)\left(v_2 \sup_y \sum_{y\in\bar{\gamma}} \|\mathscr{T}^{(1)}_{\bar{\gamma}}\|_1\right) \\
&\leqq 4(v_0+v_1)(r^*_{i_1v_1} v_1)(r^*_{i_2v_2} v_2).
\end{aligned} \tag{2.23}$$

Adding up (2.21), (2.22) and (2.23), we find

$$(2.20) \leqq 4(v_0 + v_1)\left(v_0 + 1 + \frac{v_0 + v_1}{v_2} + 1 \right)(r^*_{i_1 v_1} v_1)(r^*_{i_2 v_2} v_2). \tag{2.24}$$

By iterating the arguments above, one arrives at the following estimate holding for all $k = 2, 3, \ldots$:

$$\sup_x \left\| P_x \sum_{\substack{|\gamma_0| = v_0, \ldots \\ \ldots |s(\gamma_k)| = v_k}} [\cdots [t_{\gamma_0}, r_{i_1\gamma_1}\tau_{\gamma_1}] \cdots r_{i_k\gamma_k}\tau_{\gamma_k}] |0\rangle \right\|_1$$
$$\leqq 2^k(v_0 + 1)\left(v_0 + 1 + \frac{v_0 + v_1}{v_2} + 1 \right) \cdots$$
$$\cdot \left(v_0 + (k-1) + \frac{v_0 + v_1 + \cdots + v_{k-1}}{v_k} + 1 \right)(r^*_{i_1 v_1} v_1) \cdots (r^*_{i_k v_k} v_k). \tag{2.25}$$

Since the volumes are ordered by construction, i.e. $v_1 \leqq \cdots \leqq v_k$, we have

$$(2.29) \leqq 4^k(v_0 + 1)(v_0 + 2) \cdots (v_0 + k)\left(\prod_{j=1}^{k} r^*_{i_j v_j} v_j \right)$$
$$= 4^k k! \binom{v_0 + k}{v_0} \left(\prod_{j=1}^{k} r^*_{i_j v_j} v_j \right) \leqq 8^k 2^{v_0} k! \left(\sum_{j=1}^{k} r^*_{i_j v_j} v_j \right). \tag{2.26}$$

A similar estimate can be derived for the first term in (2.11). Namely, we have

$$\sup_x \left\| P_x \left[\cdots \left[\sum_y s_y, R^{v_1}_{i_1} \right] \cdots, R^{v_k}_{i_k} \right] |0\rangle \right\|_1 \leqq 8^k |s| k! \left(\prod_{j=1}^{k} r^*_{i_j v_j} v_j \right), \tag{2.27}$$

where $|s| \equiv \sup_x \| s_x \|$. On the other hand, the L^1-norm of the left-hand side of (2.11) can be bounded from below as follows:

$$\left\| P_x \sum_{v=1}^{n+1} S R^v_n |0\rangle \right\|_1 = \sum_{x \in s(\gamma)} |r_{n\gamma}| \, \| S \tau_\gamma |0\rangle \|_1 \geqq \sum_{x \in s(\gamma)} |r_{n\gamma}| |s(\gamma)| = r^*_n. \tag{2.28}$$

Thus, for all $n \geqq 2$ the following recurrence inequalities hold:

$$r^*_n \leqq 1 + \sum_{v_0 = 2}^{n+1} \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1 + \cdots + i_k = n - v_0 + 1}} 8^k 2^{v_0} k! \frac{1}{(v)!} \left(\prod_{j=1}^{k} r^*_{i_j v_j} v_j \right)$$
$$+ \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1 + \cdots + i_k = n}} 8^k |s| k! \frac{1}{(v)!} \left(\prod_{j=1}^{k} r^*_{i_j v_j} v_j \right)$$
$$\leqq 1 + \sum_{v_0 = 2}^{n+1} \sum_{i_1 + \cdots + i_k = n - v_0 + 1} 8^k 2^{v_0} r^*_{i_1} \cdots r^*_{i_k} + \sum_{i_1 + \cdots + i_k = n} 8^k |s| r^*_{i_1} \cdots r^*_{i_k}. \tag{2.29}$$

If $r^*(\lambda)$ is the formal power series

$$r^*(\lambda) \sim \sum_{n=1}^{\infty} \lambda^n r^*_n, \tag{2.30}$$

we have

$$r^*(\lambda) \lesssim \sum_{n=1}^{\infty} \lambda^n + \left(\sum_{v_0=2}^{\infty} \lambda^{v_0-1} 2^{v_0} \right)(1 - 8r^*(\lambda))^{-1} + |s| \left[\frac{1}{1 - 8r^*(\lambda)} - 1 - 8r^*(\lambda) \right]$$

$$= \frac{\lambda}{1-\lambda} + \left[\frac{4\lambda}{1-2\lambda} + 64|s|r^*(\lambda)^2 \right](1 - 8r^*(\lambda))^{-1}. \tag{2.31}$$

Since the equation

$$a^*(\lambda) = \frac{\lambda}{1-\lambda} + \left[\frac{4\lambda}{1-2\lambda} + 64|s|a^*(\lambda)^2 \right](1 - 8a^*(\lambda))^{-1}, \tag{2.32}$$

has a solution $a^*(\lambda)$ analytic in a neighborhood of $\lambda = 0$, and since $a^*(\lambda)$ majorates the series (2.30), we conclude that (2.30) converges for λ small. In particular, the series

$$\sum_{n=1}^{\infty} \lambda^n \sup_x \sum_{\gamma: x \in s(\gamma)} |r_{n\gamma}| |s(\gamma)| \tag{2.33}$$

converges for λ small. This implies that $U(\lambda)$ is analytic in λ for λ in a disc around $\lambda = 0$ *independent* of Λ. This completes the proof of Theorem 1.1.

Appendix – Removing the Condition $|s| < \infty$. The condition $|s| < \infty$ is not satisfied in certain models of interest like the one considered in [1] with Hamiltonian

$$H_\lambda = \sum_{x \in \mathbb{Z}^d} \frac{1}{2} n_x(n_x + 1) + \lambda \sum_{\langle xy \rangle \subset \mathbb{Z}^d} (c_x^+ c_y + c_y^+ c_x). \tag{2.34}$$

The Hilbert space is

$$\mathscr{H} = \bigotimes_x l^2(\mathbb{N}), \tag{2.35}$$

n_x is the number operator in x and c_x^+, c_x are the Bose creation and annihilation operators in x. In this case the spectrum of $s_x \equiv \frac{1}{2}(n_x + 1)n_x$ is unbounded. However, if as in [1] we restrict ourselves to the subspace with one particle per site, we can still find a unitary dressing transformation for H_λ. We have to modify as follows the definition of $T_n: l^2(\mathbb{N}) \to l^2(\mathbb{N})$:

$$T_n \sum_{k=0}^{\infty} u(k)|0\rangle = \sum_{k=0}^{\infty} \frac{1}{2}(u(|k+n|) + u(|k-n|)). \tag{2.36}$$

The bound (2.31) can be replaced by

$$\sup_x \left\| P_x \left[\cdots \left[\sum_y s_y, R_{i_1}^{v_1} \right] \cdots R_{i_k}^{v_k} \right] |0\rangle \right\|_1 \leqq \frac{n(n+1)}{2} 8^k k! \left(\sum_{j=1}^{k} r_{i_j v_j}^* v_j \right) \tag{2.37}$$

that is true because at the n^{th} order of perturbation theory there can be at most $n+1$ particles in one site. This forces to change (2.31) as follows:

$$r^*(\lambda) \lesssim \frac{\lambda}{1-\lambda} + \frac{4\lambda}{1-2\lambda}(1 - 8r^*(\lambda))^{-1} + F(r^*(\lambda)), \tag{2.38}$$

where

$$F(z) = \frac{z}{(1-z)^2}(2z - z^2) = \sum_{n=1}^{\infty} \frac{n(n+1)}{z} z^n. \tag{2.39}$$

However, the analyticity of (2.38) follows from an application of the implicit function theorem as above.

3. Relative Boundedness Results and Decay of Dressed Green's Functions

This section is dedicated to the study of the dressed Hamiltonian

$$S + V(\lambda) \equiv U(\lambda)^{-1} H_\lambda U(\lambda) - E_0(\lambda) \tag{3.1}$$

defined in Sect. 1 after (1.19), and contains the proof of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2.

(i) We have

$$\begin{aligned} V(\lambda) = \sum_{m=1}^{\infty} \lambda^m \Bigg\{ & \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1 + \cdots + i_k = m}} \frac{1}{(v)!} [\cdots [S, R_{i_1}^{v_1}] \cdots R_{i_k}^{v_k}] \\ & + \sum_{j=1}^{m} \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1 + \cdots + i_k = m-j}} \frac{1}{(v)!} \left[\cdots \left[\sum_{|\gamma_0|_c = j} t_{\gamma_0}, R_{i_1}^{v_1} \right] \cdots \right] \Bigg\}. \end{aligned} \tag{3.2}$$

Hence, if we express $V(\lambda)$ as a sum

$$V(\lambda) = \sum_{\gamma_0 \subset \Lambda} v_{\gamma_0}(\lambda) \tag{3.3}$$

of operators $v_{\gamma_0}(\lambda)$ with support γ_0, we see that the first non-vanishing term in the expansion for $v_{\gamma_0}(\lambda)$ in powers of λ is of order at least $|\gamma_0|_c - 1$. Hence, due to the bounds in Sect. 2, we have

$$\sup_x \sum_{\substack{\gamma_0 : |\gamma_0|_c \geqq n \\ x \in \gamma_0}} \| v_{\gamma_0}(\lambda) \|_1 \leqq \sum_{m=n-1}^{\infty} \lambda^m (|s| r_m^* + r_m^*) \leqq (|s| + 1) \sum_{m=n-1}^{\infty} a_m^* \lambda^m \leqq (c_0 \lambda)^{n-1}, \tag{3.4}$$

where

$$a^*(\lambda) = \sum_{m=1}^{\infty} a_m^* \lambda^m \tag{3.5}$$

is the function analytic at $\lambda = 0$, implicitly defined by Eq. (2.35). Q.E.D.

(ii) If

$$u = \sum_\gamma u_\gamma \tau_\gamma |0\rangle \in \mathscr{H}(\Lambda) \tag{3.6}$$

is a wave function, we have

$$\begin{aligned} |\langle u | V(\lambda) | u \rangle| &\leqq \sum_\gamma u_\gamma^2 |\langle \gamma | V(\lambda) | \gamma \rangle| + 2 \sum_{|u_\gamma| \leqq |u_\gamma|'} |u_\gamma u_{\gamma'}| |\langle \gamma' | V(\lambda) | \gamma \rangle| \\ &\leqq 2 \sum_\gamma |u_\gamma|^2 \sum_{\gamma'} |\langle \gamma' | V(\lambda) | \gamma \rangle|. \end{aligned} \tag{3.7}$$

It is enough to prove that the bound

$$\sum_{\gamma'} |\langle \gamma' | V(\lambda) | \gamma \rangle| \leqq (c_0 \lambda) |s(\gamma)| \tag{3.8}$$

holds for all excitations γ. Since by construction we have

$$V(\lambda)|0\rangle = 0, \tag{3.9}$$

one can estimate as follows the left-hand side of (3.8):

$$\sum_{\gamma'} |\langle \gamma' | V(\lambda) | \gamma \rangle| = \| V(\lambda) \tau_\gamma |0\rangle \|_1 = \| [V(\lambda), \tau_\gamma] |0\rangle \|_1$$

$$\leqq |s(\gamma)| \sup_x \sum_{\gamma_0 : x \in \gamma_0} \| v_{\gamma_0}(\lambda) \|_1 \leqq (c_0 \lambda) |s(\gamma)|. \quad \text{Q.E.D.} \tag{3.10}$$

(iii) We have

$$\| S^{-1/2} V(\lambda) S^{-1/2} \|_1 \leqq \sup_\gamma \| S^{-1/2} V(\lambda) S^{-1/2} \tau_\gamma |0\rangle \|_1 \tag{3.11}$$

$$\leqq \sup_\gamma \sum_{j=1}^\infty \sum_{\substack{\gamma_0 : |\gamma_0|_c = j+1 \\ s(\gamma) \cap \gamma_0 \neq \emptyset}} \frac{\| v_{\gamma_0}(\lambda) \|_1}{\max(|s(\gamma)| - |\gamma_0|, 1)^{1/2} |s(\gamma)|^{1/2}}$$

$$\leqq \sup_\gamma \sum_{j=1}^\infty \frac{|s(\gamma)| (c_0 \lambda)^j}{\max(|s(\gamma)| - j, 1)^{1/2} |s(\gamma)|^{1/2}} \leqq \sum_{j=1}^\infty e^{1+j} (c_0 \lambda)^j \leqq (c\lambda),$$

where we used the inequality

$$\left(\frac{|s(\gamma)|}{|s(\gamma)| - j} \right)^{1/2} \leqq e^{j+1} \tag{3.12}$$

that holds for all $j = 1, 2, \ldots, (|s(\gamma)| - 1)$. Q.E.D.

(iv) We have

$$|\langle V(\lambda) u | V(\lambda) u \rangle| = \left| \sum_{\gamma, \gamma'} u_\gamma u'_\gamma \langle \gamma' | V(\lambda)^2 | \gamma \rangle \right|$$

$$\leqq \sum_\gamma u_\gamma^2 \langle \gamma | V(\lambda)^2 | \gamma \rangle + 2 \sum_{|u_\gamma| \leqq |u_{\gamma'}|} |u_\gamma u_{\gamma'}| |\langle \gamma' | V(\lambda)^2 | \gamma \rangle|$$

$$\leqq 2 \sum_\gamma u_\gamma^2 \sum_{\gamma'} |\langle \gamma' | V(\lambda)^2 | \gamma \rangle \tag{3.13}$$

Moreover, we have

$$\sum_{\gamma'} |\langle \gamma' | V(\lambda)^2 | \gamma \rangle \leqq \sum_{\gamma' \gamma''} |\langle \gamma' | V(\lambda) | \gamma'' \rangle| |\langle \gamma'' | V(\lambda) | \gamma \rangle|$$

$$\leqq \sum_{\gamma''} (c\lambda) |s(\gamma'')| |\langle \gamma'' | V(\lambda) | \gamma \rangle, \tag{3.14}$$

where we used again (3.8). To conclude the proof, it suffices to verify the bound

$$\sum_{\gamma''} |s(\gamma'')| |\langle \gamma'' | V(\lambda) | \gamma \rangle| \leqq (c\lambda) |s(\gamma)|^2. \tag{3.15}$$

We have

$$\sum_{\gamma''} |s(\gamma'')| |\langle \gamma'' | V(\lambda) | \gamma \rangle| \leqq \sum_{j=0}^\infty \sum_{\gamma'' : ||s(\gamma'')| - |s(\gamma)|| = j} (|s(\gamma)| + j) |\langle \gamma'' | [V(\lambda), \tau_\gamma] |0\rangle|$$

$$\leqq \sum_{j=0}^{\infty} (|s(\gamma)|+j)|s(\gamma)| \sup_x \sum_{\substack{\gamma_0 : x \in \gamma_0 \\ |\gamma_0| \geqq j}} \| v_{\gamma_0}(\lambda) \|_1$$

$$\leqq \sum_{j=0}^{\infty} |s(\gamma)|(|s(\gamma)|+j)(c_0\lambda)^{\bar{j}}, \tag{3.16}$$

where $\bar{j} = \max(1,j)$. Hence, we have

$$(3.15) \leqq |s(\gamma)|^2 (c_0\gamma)(1+(1-c_0\lambda)^{-1}) + |s(\gamma)|(c_0\lambda)(1-c_0\lambda)^{-2} \leqq (c\lambda)|s(\gamma)|^2 \tag{3.17}$$

if λ is small enough. Q.E.D.

(v) As a Corollary of any of the estimates (ii), (iii) and (iv), we have that $U(\lambda)|0\rangle$ is the ground state of H_λ and its energy is separated by a gap $(1-0(\lambda))$ from the rest of the spectrum of H_λ. This is a consequence of quite standard analyticity arguments; see [1] and references therein.

(vi) Let us prove the bound

$$\sum_{\gamma' : d_c(s(\gamma), s(\gamma')) \geqq k} |G(\gamma, \gamma')| \leqq (c\lambda)^k, \tag{3.18}$$

where $G = (S + V(\lambda))^{-1}$, $s(\gamma), s(\gamma') \neq \emptyset$ and $d_c(s(\gamma), s(\gamma'))$ is the distance defined in (1.29). We have

$$\sum_{\gamma' : d_c(s(\gamma), s(\gamma')) = k} |\langle \gamma' | S^{-1/2} V(\lambda) S^{-1/2} | \gamma \rangle| \leqq \left(\frac{|s(\gamma)|}{\max(|s(\gamma)|-k, 1)} \right)^{1/2} (c\lambda)^{\bar{k}} \leqq (c\lambda)^{\bar{k}}, \tag{3.19}$$

where we use (3.12). By expanding the resolvent in geometric series

$$(S + V(\lambda))^{-1} = \sum_{j=0}^{\infty} S^{-1/2} [S^{-1/2} V(\lambda) S^{-1/2}]^j S^{-1/2}, \tag{3.20}$$

we find

$$\sum_{d_c(s(\gamma'), s(\gamma)) = k} |\langle \gamma' | (S + V(\lambda))^{-1} | \gamma \rangle|$$

$$\leqq \frac{1}{\max(|s(\gamma)|-k, 1)} \sum_{n=1}^{\infty} \sum_{\gamma_1, \dots, \gamma_{n-1} \neq \emptyset} |\langle \gamma' | S^{-1/2} V(\lambda) S^{-1/2} | \gamma_{n-1} \rangle|$$

$$\cdot |\langle \gamma_{n-1} | S^{-1/2} V(\lambda) S^{-1/2} | \gamma_{n-2} \rangle| \cdots |\langle \gamma_1 | S^{-1/2} V(\lambda) S^{-1/2} | \gamma \rangle$$

$$\leqq \sum_{n=1}^{\infty} \sum_{k_1 + \dots + k_n \geqq k} (c\lambda)^{\bar{k}_1 + \dots + \bar{k}_n} \leqq \sum_{j=k}^{\infty} b_j \lambda^j, \tag{3.21}$$

where the sequence b_j is such that

$$\sum_{j=1}^{\infty} b_j \lambda^j = (c\lambda)[1 + (1 - c\lambda)^{-1}][1 - (c\lambda)(1 + (1 - c\lambda)^{-1})]^{-1}. \tag{3.22}$$

Q.E.D.

Proof of Theorem 1.2. If $\mathcal{O}_{\gamma_0}$ is an operator with support $\gamma_0 \subset \Lambda$, its vacuum expectation value can be expressed through the following cluster expansion that converges for $\lambda < c$:

$$\langle U(\lambda)0|\mathcal{O}_{\gamma_0}|U(\lambda)0\rangle = \langle 0|U(\lambda)^{-1}\mathcal{O}_{\gamma_0}U(\lambda)|0\rangle$$

$$= \sum_{n=1}^{\infty} \lambda^n \sum_{\substack{v_1 \leqq \dots \leqq v_k \\ i_1 + \dots + i_k = n}} \sum_{\substack{|s(\gamma_1)| = v_1, \dots, \\ |s(\gamma_k)| = v_k}} \langle 0|[\cdots[\mathcal{O}_{\gamma_0}, r_{i_1\gamma_1}\tau_{\gamma_1}]\cdots r_{i_k\gamma_k}\tau_{\gamma_k}]|0\rangle. \tag{3.23}$$

The expectation values appearing here vanish unless

$$s(\gamma_1) \cap \gamma_0 \neq \emptyset, \dots, s(\gamma_k) \cap [\gamma_0 \cup \dots \cup s(\gamma_{k-1})] \neq \emptyset. \tag{3.24}$$

In the case in which we have a product of two operators $\mathcal{O}_{\gamma_0}$ and $\mathcal{O}_{\gamma_0'}$, with supports γ_0, γ_0' respectively, separated by a distance $d(\gamma_0, \gamma_0') = n$, the expansions in powers of λ for the two functions

$$\langle U(\lambda)0|\mathcal{O}_{\gamma_0}\mathcal{O}_{\gamma_0'}|U(\lambda)0\rangle \tag{3.25}$$

and

$$\langle U(\lambda)0|\mathcal{O}_{\gamma_0}|U(\lambda)0\rangle\langle U(\lambda)0|\mathcal{O}_{\gamma_0'}|U(\lambda)0\rangle \tag{3.26}$$

coincide up to order n. In fact, only at orders $\geqq n$ there are clustering touching both γ_0 and γ_0'. Since the l^1-operator norm of $\mathcal{O}_{\gamma_0}$ and $\mathcal{O}_{\gamma_0'}$ is assumed to be normalized to one, the sum of the terms of order $\geqq n$ can be estimated to be less than $(c\lambda)^n$. Q.E.D.

Part II. Exponential Decay of Dressed Eigenfunctions Below Threshold

4. Introduction, Notations and Results

In this part of the paper, we consider a model with Hamiltonian operator of the form

$$H_\lambda = \sum_{x \in \Lambda} s_x + \sum_{\gamma_0 \subset \Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0}. \tag{4.1}$$

Here, the notations have the same meaning as in Part I, except that s_x is no longer assumed to have a gap $\geqq 1$ for x in a finite set $\mathcal{S} \subset \Lambda$. We study the asymptotic behavior away from $\mathcal{S}$ of eigenfunctions of low energy and we are interested in finding bounds that hold for all λ smaller than a constant independent of both $\mathcal{S}$ and Λ. The estimates we establish are valid starting from a distance from $\mathcal{S}$ that depends on $|\mathcal{S}|$ and on the energy of the eigenfunction, but not on the size of Λ.

Since for $x \in \mathcal{S}$ the gap of s_x can be arbitrarily small, the attempt to look for a dressing transformation solving the ground state problem, is bound to fail due to small divisor problems. This is not a mere technical obstruction, but it signs a possible lack of analyticity of the ground state as a function of the perturbation parameter λ. Nonetheless, it is still possible to use perturbative methods to study the asymptotic behavior of the eigenfunction "below threshold", i.e. the eigenfunctions with energy less than the ground state energy plus one. Since an excitonic quasiparticle away from $\mathcal{S}$ has potential energy at least one, in an eigenstate below threshold there cannot be quasiparticles in a scattering state free to move to infinity. Since on $\mathcal{S}$ the potential energy for excitations is every low, one expects the existence of several eigenfunctions below threshold. They are mutually orthogonal and ought to be quite different one from each other, close to $\mathscr{S}$. However, their asymptotic behavior far away from $\mathscr{S}$ is very similar because they are locally very close to the ground state. To state this property more precisely, it is convenient to pass to a unitarily equivalent dressed representation and a few definitions are needed.

Let us introduce the regularized Hamiltonian H_λ^{reg} as follows:

$$H_\lambda^{\text{reg}} = \sum_{x\in\Lambda\backslash\mathscr{S}} s_x + \sum_{x\in\mathscr{S}} (1 - P_{|0\rangle_x}) + \sum_{\gamma_0\subset\Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0}. \tag{4.2}$$

Let us compute as in Part I a unitary dressing transformation $U(\lambda)$ for H_λ^{reg} and let $E_{0,\lambda}^{\text{reg}}$ be the ground state energy of H_λ^{reg}. We propose to work with the dressed Hamiltonian

$$S + V(\lambda) + W(\lambda) \equiv U(\lambda)^{-1} H_\lambda U(\lambda) - E_{0,\lambda}^{\text{reg}}, \tag{4.3}$$

where

$$S = \sum_{x\in\Lambda} s_x, \tag{4.4}$$

$$V(\lambda) = U(\lambda)^{-1} H_\lambda^{\text{reg}} U(\lambda) - E_{0,\lambda}^{\text{reg}} - \sum_{x\in\Lambda\backslash\mathscr{S}} s_x - \sum_{x\in\mathscr{S}'} (1 - P_{|0\rangle_x}) \tag{4.5}$$

and $W(\lambda)$ is the remainder

$$W(\lambda) = U(\lambda)^{-1}\left[\sum_{x\in\mathscr{S}} (s_x - 1 + P_{|0\rangle_x})\right] U(\lambda) - \sum_{x\in\mathscr{S}} (s_x - 1 + P_{|0\rangle_x}). \tag{4.6}$$

We still have

$$V(\lambda)|0\rangle = 0. \tag{4.7}$$

Moreover, if we represent $W(\lambda)$ in the form

$$W(\lambda) = \sum_{\gamma_0\subset\Lambda} w(\gamma_0), \tag{4.8}$$

where $w(\gamma_0)$ is an operator with support γ_0, we have

$$\sum_{\gamma_0 : d_{\mathscr{S}}(\emptyset,\gamma_0)\geqq k} \| w(\gamma_0) \|_1 \leqq |\mathscr{S}| (c_0\lambda)^{\bar{k}}, \tag{4.9}$$

where $\bar{k} = \max(1, k)$. If $A \subset \Lambda$ is a set, $d_A(\gamma_0, \gamma_0')$ denotes the distance between the two sets $\gamma_0, \gamma_0' \subset \Lambda$ defined as follows:

$$d_A(\gamma_0, \gamma_0') = \inf\{|\Gamma|, \text{ where } \Gamma \subset \Lambda \text{ is such that for each point of } \gamma_0\backslash\gamma_0' \text{ (respectively } \gamma_0'\backslash\gamma_0) \text{ there exists a path in } \Gamma \text{ connecting it to } \gamma_0' \text{ (respectively } \gamma_0) \text{ and/or to } A\}. \tag{4.10}$$

Equation (4.9) is a consequence of the estimates in Sect. 2 on the coefficients $r_{n\gamma}$ entering in the dressing transformation $U(\lambda)$.

If n is an integer $\geqq 1$, let us define the neighborhood $\bar{\mathscr{S}}_n$ of $\mathscr{S}$ as follows:

$$\bar{\mathscr{S}}_n = \{x\in\Lambda \text{ such that } d(x, \mathscr{S}) \leqq n\} \tag{4.11}$$

and its boundary

$$\partial\bar{\mathscr{S}}_n = \{x\in\Lambda \text{ such that } n \leqq d(x, \mathscr{S}) \leqq n - 1\}. \tag{4.12}$$

The following is the main result of this part:

Theorem 4.1. *Let $\theta \in [0, 1)$. There are constants independent of θ such that if*

$$n \geqq 2 \frac{\log(c|\mathscr{S}|(1-\theta)^{-1})}{|\log c_0 \lambda|} \tag{4.13}$$

and

$$\lambda \leqq c(1-\theta)^2 \tag{4.14}$$

then the following is true: If we represent an eigenfunction u of $S + V(\lambda) + W(\lambda)$ in the form

$$u = \sum_{s(\gamma) \subset \sim \bar{\mathscr{S}}_n} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle \tag{4.15}$$

with $\phi_\gamma \in \mathscr{H}(\bar{\mathscr{S}}_n)$, and if the energy E of u is such that

$$E \leqq E_{0,\lambda} + \theta, \tag{4.16}$$

$E_{0,\lambda}$ being the ground state energy of $H_{0,\lambda}$, we have

$$\sum_{d_{\bar{\mathscr{S}}_n}(\varnothing, s(\gamma)) \geqq k} \|\phi_\gamma\|_2 \leqq |\partial \bar{\mathscr{S}}_n| (1-\theta)^{-k-1} (c\sqrt{\lambda})^k \tag{4.17}$$

for all $k = 1, 2, \ldots$.

Notations. Let us introduce the $L^{2,1}$-norm $\|\cdot\|_{2,1}$ as follows: if $u \in \mathscr{H}(\Lambda)$ and we write it in the form

$$u = \sum_{s(\gamma) \subset \sim \bar{\mathscr{S}}_n} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle, \tag{4.18}$$

then

$$\|u\|_{2,1} \equiv \sum_{s(\gamma) \subset \sim \bar{\mathscr{S}}_n} \|\phi_\gamma\|_2 . \tag{4.19}$$

Here n is the integer in Theorem 4.1.

For the applications in Part III, we do not need this result in full generality, but only the following Corollary:

Corollary 4.2. *If*

$$u = \sum_\gamma u_\gamma \tau_\gamma | 0 \rangle \tag{4.20}$$

is an eigenfunction of $S + V + W$ with energy

$$E \leqq E_{0\lambda} + \tfrac{1}{2}, \tag{4.21}$$

then there are constants such that if $\lambda \leqq c$ and

$$n \geqq 2 \frac{\log c|\mathscr{S}|}{|\log c_0 \lambda|} \tag{4.22}$$

we have

$$\left(\sum_{d_{\bar{\mathscr{S}}_n}(\varnothing, \gamma) \geqq k} |u_\gamma|^2 \right)^{1/2} \leqq |\partial \bar{\mathscr{S}}_n| (c\sqrt{\lambda})^k . \tag{4.23}$$

This Corollary follows from Theorem 4.1 with $\theta = 1/2$ because the bound (4.17) in $L^{2,1}$-norm is stronger than the bound (4.23) in L^2-norm.

As remarked above, the proof of such results is based on a combination of perturbative and variational arguments. The latter are necessary in order to get some information about the behavior of the eigenfunctions near $\mathcal{S}$. From them one can see that if we go far enough from $\mathcal{S}$, we can control the decay of u with a convergent random walk expansion. In the rest of this introductory section, let us introduce some other notations and state the intermediate result to be proven with a variational method in Sect. 5. The random walk expansion that permits us to conclude the proof of Theorem 4.1, is discussed in Sect. 6.

If $A \subset \Lambda$, let us define V_A, $V_{\partial A}$ and $V_{\sim A}$ as follows. If $U_A(\lambda)$ is the unitary dressing transformation computed as in Part I for the Hamiltonian H_λ^{reg} restricted to A, i.e. for

$$H_\lambda^{\text{reg}}(A) = \sum_{x \in \mathcal{S} \cap A} (1 - P_{|0\rangle_x}) + \sum_{x \in A \setminus \mathcal{S}} s_x + \sum_{\gamma_0 \subset A} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0} \tag{4.24}$$

then

$$V_A(\lambda) = U_A(\lambda)^{-1} H_\lambda^{\text{reg}}(A) U_A(\lambda) - \sum_{x \in \mathcal{S} \cap A} (1 - P_{|0\rangle_x}) - \sum_{x \in A \setminus \mathcal{S}} s_x - E_{0\lambda}^{\text{reg}}(A), \tag{4.25}$$

where $E_{0\lambda}^{\text{reg}}(A)$ is the ground state energy of $H_\lambda^{\text{reg}}(A)$. Analogously I define $V_{\sim A}(\lambda)$. The boundary term $V_{\partial A}(\lambda)$ is the remainder

$$V_{\partial A}(\lambda) \equiv V(\lambda) - V_A(\lambda) - V_{\sim A}(\lambda). \tag{4.26}$$

Let us notice the following basic property of $V_{\partial A}(\lambda)$:

$$V_{\partial A}(\lambda)|0\rangle = 0. \tag{4.27}$$

One can treat the W term in a similar way and define

$$W_A(\lambda) = U_A(\lambda)^{-1} \left[\sum_{x \in \mathcal{S} \cap A} (s_x - 1 + P_{|0\rangle_x}) \right] U_A(\lambda) - \sum_{x \in \mathcal{S}} (s_x - 1 + P_{|0\rangle_x}) \tag{4.28}$$

and

$$W_{\partial A}(\lambda) = W(\lambda) - W_A(\lambda). \tag{4.29}$$

The S term requires a special treatment. In the following, if $n \geqq 0$ is an integer, we decompose S into the sum of three parts

$$S = (S_{\bar{\mathcal{S}}_n} - S_{\partial \mathcal{S}_n}^{\text{in}}) + S_{\partial \bar{\mathcal{S}}_n} + (S_{\sim \bar{\mathcal{S}}_n} - S_{\partial \mathcal{S}_n}^{\text{out}}). \tag{4.30}$$

Here,

$$S_{\bar{\mathcal{S}}_n} \equiv \sum_{x \in \bar{\mathcal{S}}_n} s_x, \quad S_{\sim \bar{\mathcal{S}}_n} \equiv \sum_{x \in \sim \bar{\mathcal{S}}_n} s_x, \tag{4.31}$$

and

$$S_{\partial \bar{\mathcal{S}}_n} \equiv \sum_{x \in \sim \mathcal{S}} (c_0 \lambda)^{(1/2) d(x, \bar{\mathcal{S}}_n) + 1/2} s_x, \tag{4.32}$$

$$S_{\partial \mathcal{S}_n}^{\text{in}} \equiv \sum_{x \in \bar{\mathcal{S}}_n \setminus \mathcal{S}} (c_0 \lambda)^{(1/2) d(x, \bar{\mathcal{S}}_n) + 1/2} s_x, \tag{4.33}$$

$$S_{\partial \mathcal{S}_n}^{\text{out}} = S_{\partial \bar{\mathcal{S}}_n} - S_{\partial \mathcal{S}_n}^{\text{in}}. \tag{4.34}$$

The constant c_0 appearing in (4.32) and (4.33) is fixed in Sect. 5 so that $V_{\partial\bar{\mathscr{S}}_n}$ turns out to be relatively form bounded with respect to $S_{\partial\bar{\mathscr{S}}_n}$ in the following sense:

$$|\langle u|V_{\partial\bar{\mathscr{S}}_n}|u\rangle \leqq \langle u|\mathscr{S}_{\partial\bar{\mathscr{S}}_n}|u\rangle + (c_0\lambda)^{n+1}|\mathscr{S}| \tag{4.35}$$

for all $u \in \mathscr{H}(\Lambda)$ and all integers $n \geqq 1$.

The main result in Sect. 5 is an estimate on the asymptotic behavior for large n of the ground state energy E_n of the operator

$$S_{\bar{\mathscr{S}}_n} + V_{\bar{\mathscr{S}}_n} + W_{\bar{\mathscr{S}}_n} - S^{\text{in}}_{\partial\bar{\mathscr{S}}_n}. \tag{4.36}$$

Theorem 4.3. *If $E_{0\lambda}$ is the ground state energy of $S + V(\lambda) + W(\lambda)$, we have*

$$|E_{0\lambda} - E_n| \leqq (c\lambda)^{1+(1/2)n}|\mathscr{S}| \tag{4.37}$$

for some constant c (independent of $\mathscr{S}$ and Λ).

5. Preliminary Decay Estimates for the Ground State

In this section, we fix the constant c_0 appearing in the definition (4.32) of the operators $S_{\partial\bar{\mathscr{S}}_n}$ and we prove Theorem 4.3.

Let us start with the following lemma.

Lemma 5.1. *If we expand $V_{\partial\bar{\mathscr{S}}_n}(\lambda)$ as follows:*

$$V_{\partial\bar{\mathscr{S}}_n} = \sum_{\gamma_0 \subset \Lambda} v_{\partial\bar{\mathscr{S}}_n}(\gamma_0), \tag{5.1}$$

where $v_\partial(\gamma_0)$ is an operator with support γ_0, we have

$$\sum_{\gamma_0 : x \in \gamma_0} \|v_{\partial\bar{\mathscr{S}}_n}(\gamma_0)\|_1 \leqq (c_0\lambda)^{d(x,\partial\bar{\mathscr{S}}_n)+1} \tag{5.2}$$

for all $n = 0, 1, 2, \ldots$ and all $x \in \Lambda$.

Proof. Although explicit formulas are too long to be worth writing here, it is not difficult to see that $V_{\partial\bar{\mathscr{S}}_n}$ can be expressed as the sum of clusters of operators that can have one of the following two forms: Either they can be written as

$$[\cdots[\lambda^{|\gamma_0|_c - 1} t_{\gamma_0}, r_{i_1\gamma_1}\tau_{\gamma_1}] \cdots r_{i_k\gamma_k}\tau_{\gamma_k}] \tag{5.3}$$

with $\gamma_0 \cap \bar{\mathscr{S}}_n \neq \emptyset$ and $\gamma_0 \cap (\sim \bar{\mathscr{S}}) \neq \emptyset$ and the coefficients r_{i_j,γ_j}, $j = 1, \ldots, k$, are those appearing in the total dressing transformation $U(\lambda)$. Or the clusters in $V_{\partial\bar{\mathscr{S}}_n}$ can have the form (5.3), but with $\gamma_0 \subset \bar{\mathscr{S}}_n$ or $\gamma_0 \subset (\sim \bar{\mathscr{S}}_n)$ and the coefficients $r_{i_j\gamma_j}$ come from $U(\lambda)$, $U_{\bar{\mathscr{S}}_n}(\lambda)$ or $U_{\sim\bar{\mathscr{S}}_n}(\lambda)$. In this case, the sequence $(i_1, \ldots, i_k)$ must be such that

$$i_1 + \cdots + i_k \geqq 1 + d(\gamma_0, \partial\bar{\mathscr{S}}_n). \tag{5.4}$$

In fact, the difference among the operators $U(\lambda)$, $U_{\bar{\mathscr{S}}_n}(\lambda)$ and $U_{\sim\bar{\mathscr{S}}_n}(\lambda)$ are due to those clusters, which among the clusters appearing at lower orders of perturbation theory that generate them, have at least one cluster intersecting both $\bar{\mathscr{S}}_n$ and $\sim \bar{\mathscr{S}}_n$.

If we decompose $V_{\partial\bar{\mathscr{S}}_n}$ as in (5.1) and if x is a site of Λ, then the minimum order in λ at which in the expansion (5.3) there appear clusters containing x, is $\geqq d(x, \partial\bar{\mathscr{S}}_n) + 1$. Hence, (5.2) follows from the estimates in Part I. Q.E.D.

Let us define

$$S_{\partial\bar{\mathcal{S}}_n} = \sum_{x\in\sim\mathcal{S}} (c_0\lambda)^{(1/2)d(x,\partial\bar{\mathcal{S}}_n)+1/2} s_x, \tag{5.5}$$

where c_0 is the minimum constant such that (5.2) is true $\forall n$ and $\forall x\in\Lambda$. Let us suppose that $\lambda \leqq 1/64c_0$, so that

$$S_{\partial\bar{\mathcal{S}}_n} \leqq \tfrac{1}{8}S. \tag{5.6}$$

I also suppose that λ is so small that

$$\tfrac{1}{4}S_A > |V_A| \tag{5.7}$$

for all regions $A\subset\Lambda$ such that $A\cap\mathcal{S}=\emptyset$.

Lemma 5.2. *We have*

$$|\langle u|V_{\partial\bar{\mathcal{S}}_n}|u\rangle \leqq \langle u|S_{\partial\bar{\mathcal{S}}_n}|u\rangle + (c_0\lambda)^{n+1}|\mathcal{S}| \tag{5.8}$$

for all $u\in\mathcal{H}(\Lambda)$ *and all* $n=0,1,2,\dots$.

Proof. By expanding u in the excitation basis

$$u = \sum_\gamma u_\gamma \tau_\gamma|0\rangle, \tag{5.9}$$

we find

$$\begin{aligned}
|\langle u|V_{\partial\bar{\mathcal{S}}_n}|u\rangle| &\leqq \sum_\gamma u_\gamma^2|\langle\gamma|V_{\partial\bar{\mathcal{S}}_n}|\gamma\rangle| + 2\sum_{|u_{\gamma'}|\leqq|u_\gamma|}|u_\gamma u'_\gamma|\,|\langle\gamma'|V_{\partial\bar{\mathcal{S}}_n}|\gamma\rangle| \\
&\leqq 2\sum_\gamma u_\gamma^2\sum_{\gamma'}|\langle\gamma'|V_{\partial\bar{\mathcal{S}}_n}|\gamma\rangle| \\
&\leqq 2\sum_\gamma u_\gamma^2\sum_{x\in s(\gamma)}\sum_{\gamma_0:x\in\gamma_0}|\langle\gamma'|[v_{\partial\bar{\mathcal{S}}_n}(\gamma_0),\tau_\gamma]|0\rangle| \leqq 2\sum_\gamma u_\gamma^2\sum_{x\in s(\gamma)}(c_0\lambda)^{1+d(x,\partial\bar{\mathcal{S}}_n)} \\
&\leqq \langle u|S_{\partial\bar{\mathcal{S}}_n}|u\rangle + 2\sum_\gamma u_\gamma^2\sum_{x\in s(\gamma)\cap\mathcal{S}}(c_0\lambda)^{1+d(x,\partial\bar{\mathcal{S}}_n)} \leqq \langle u|S_{\partial\bar{\mathcal{S}}_n}|u\rangle + 2(c_0\lambda)^{n+1},
\end{aligned} \tag{5.10}$$

from which (5.8) follows. Q.E.D.

It is now possible to pass to the proof of Theorem 4.3. I prove by induction on n the following result, from which Theorem 4.3 follows:

Theorem 5.3. *For all* $n=1,2,\dots$ *and all integers* m *with* $0\leqq m<n$, *we have*

(i) $$|E_n - E_m| \leqq (1+3d_m)(c_0\lambda)^{1+m/2}|\mathcal{S}|, \tag{5.11}$$

(ii) $$\langle u_n|S_{\bar{\mathcal{S}}_n\setminus\bar{\mathcal{S}}_m}|u_n\rangle \leqq 3(d_m+1)(c_0\lambda)^{1+m/2}|\mathcal{S}|, \tag{5.12}$$

(iii) $$\langle u_n|S_{\partial\bar{\mathcal{S}}_n}|u_n\rangle \leqq d_n(c_0\lambda)^{1+n/2}|\mathcal{S}|, \tag{5.13}$$

where

$$d_n = 9\cdot 4^{n-1} - 1. \tag{5.14}$$

Proof. For $n=1$ and $m=0$, we have

$$\begin{aligned}
E_1 &\leqq \langle u_0|S_{\bar{\mathcal{S}}_1} + V_{\bar{\mathcal{S}}_1} + W_{\bar{\mathcal{S}}_1} - S_{\partial\bar{\mathcal{S}}_1}|u_0\rangle \\
&= \langle u_0|S_{\mathcal{S}} + V_{\mathcal{S}} + W_{\mathcal{S}} + V_{\partial\mathcal{S}} + W_{\partial\mathcal{S}}|u_0\rangle \leqq E_0 + 2(c_0\lambda)|\mathcal{S}|,
\end{aligned} \tag{5.15}$$

where the boundary operators $V_{\partial\mathscr{S}}$ and $W_{\partial\mathscr{S}}$ are the ones giving the decompositions

$$V_{\bar{\mathscr{S}}_1} = V_{\mathscr{S}} + V_{\partial\mathscr{S}} + V_{\bar{\mathscr{S}}_1\backslash\mathscr{S}}, \tag{5.16}$$

$$W_{\bar{\mathscr{S}}_1} = W_{\mathscr{S}} + W_{\partial\mathscr{S}}. \tag{5.17}$$

On the other hand, we have

$$\begin{aligned} E_1 &= \langle u_1 | S_{\bar{\mathscr{S}}_1} + V_{\bar{\mathscr{S}}_1} + W_{\bar{\mathscr{S}}_1} - S_{\partial\bar{\mathscr{S}}_1} | u_1 \rangle \\ &= \langle u_1 | S_{\mathscr{S}} + V_{\mathscr{S}} + W_{\mathscr{S}} + V_{\partial\mathscr{S}} + W_{\partial\mathscr{S}} + S_{\bar{\mathscr{S}}_1\backslash\mathscr{S}} + V_{\bar{\mathscr{S}}_1\backslash\mathscr{S}} - S_{\partial\bar{\mathscr{S}}_1} | u_1 \rangle \\ &\geqq E_0 - 2(c_0\lambda)|\mathscr{S}| + \tfrac{1}{2}\langle u_1 | \mathscr{S}_{\bar{\mathscr{S}}_1\backslash\mathscr{S}} | u_1 \rangle. \end{aligned} \tag{5.18}$$

Hence

$$|E_1 - E_0| \leqq 2(c_0\lambda)|\mathscr{S}| \tag{5.19}$$

and

$$\langle u_1 | S_{\bar{\mathscr{S}}_1\backslash\mathscr{S}} | u_1 \rangle \leqq 2[E_1 - E_0 + 2(c_0\lambda)|\mathscr{S}|] \leqq 8(c_0\lambda)|\mathscr{S}|. \tag{5.20}$$

Let now $n > 1$ and let us suppose that (i), (ii) and (iii) have been proven up to $n-1$. We have

$$\begin{aligned} E_n &\leqq \langle u_m | S_{\bar{\mathscr{S}}_n} + V_{\bar{\mathscr{S}}_n} + W_{\bar{\mathscr{S}}_n} - S_{\partial\bar{\mathscr{S}}_n} | u_m \rangle \\ &= \langle u_m | (S_{\bar{\mathscr{S}}_m} + V_{\bar{\mathscr{S}}_m} + W_{\bar{\mathscr{S}}_m} - S_{\bar{\mathscr{S}}_m}) + (S_{\partial\bar{\mathscr{S}}_m} + V_{\partial\bar{\mathscr{S}}_m} - S_{\partial\bar{\mathscr{S}}_n} + W_{\partial\bar{\mathscr{S}}_m} | u_m \rangle \\ &\leqq E_m + (c_0\lambda)^{m+1}|\mathscr{S}| + (2 + (c_0\lambda)^{1/2(n-m)})\langle u_m | S_{\partial\bar{\mathscr{S}}_m} | u_m \rangle \\ &\leqq E_m + (3d_m + 1)(c_0\lambda)^{1+m/2}|\mathscr{S}|. \end{aligned} \tag{5.21}$$

On the other hand, we have

$$\begin{aligned} E_n &= \langle u_n | S_{\bar{\mathscr{S}}_n} + V_{\bar{\mathscr{S}}_n} + W_{\bar{\mathscr{S}}_n} - S_{\partial\bar{\mathscr{S}}_n} | u_n \rangle \\ &= \langle u_n | (S_{\bar{\mathscr{S}}_m} + V_{\bar{\mathscr{S}}_m} + W_{\bar{\mathscr{S}}_m} - S_{\partial\bar{\mathscr{S}}_m}) \\ &\quad + (S_{\partial\bar{\mathscr{S}}_m} + V_{\partial\bar{\mathscr{S}}_m}) + W_{\partial\bar{\mathscr{S}}_m} + (S_{\bar{\mathscr{S}}_n\backslash\bar{\mathscr{S}}_m} - S^{\text{out}}_{\partial\bar{\mathscr{S}}_m} + V_{\bar{\mathscr{S}}_n\backslash\bar{\mathscr{S}}_m} - S_{\partial\bar{\mathscr{S}}_n}) | u_n \rangle \\ &\geqq E_m - 2(c_0\lambda)^{m+1}|\mathscr{S}| + \tfrac{1}{2}\langle u_n | S_{\bar{\mathscr{S}}_n\backslash\bar{\mathscr{S}}_m} | u_n \rangle. \end{aligned} \tag{5.22}$$

Hence, we find

$$|E_n - E_m| \leqq (3d_m + 1)(c_0\lambda)^{1+m/2}|\mathscr{S}| \tag{5.23}$$

and

$$\langle u_n | S_{\bar{\mathscr{S}}_n\backslash\bar{\mathscr{S}}_m} | u_n \rangle \leqq 3(d_m + 1)(c_0\lambda)^{1+m/2}|\mathscr{S}|. \tag{5.24}$$

Finally, we have

$$\begin{aligned} \langle u_n | S_{\partial\bar{\mathscr{S}}_n} | u_n \rangle &\leqq \sum_{j=0}^{n-1} (c_0\lambda)^{1/2(n-j)} \langle u_n | S_{\bar{\mathscr{S}}_{j+1}\backslash\bar{\mathscr{S}}_j} | u_n \rangle \\ &\leqq \sum_{j=0}^{n-1} (c_0\lambda)^{1/2(n-j)} (c_0\lambda)^{1+j/2} |\mathscr{S}| 3(1 + d_j) \\ &= \sum_{j=0}^{n-1} 3(1 + d_j)(c_0\lambda)^{1+n/2}|\mathscr{S}|. \end{aligned} \tag{5.25}$$

Hence (5.13) holds if

$$d_n = \sum_{j=0}^{n-1} 3(1+d_j) = 3 + 4d_{n-1} = 9\cdot 4^{n-1} - 1. \tag{5.26}$$

Q.E.D.

6. Decay Estimates in $L^{2,1}$-Norm Below Threshold

This section contains the proof of Theorem 4.1.

Let n be an integer to be fixed later on and let P be the orthogonal projection onto the space spanned by the states of the form $\phi \otimes |0_{\sim\bar{\mathscr{S}}_n}\rangle$ with $\phi \in \mathscr{H}(\bar{\mathscr{S}}_n)$. Let

$$\Pi = 1 - P. \tag{6.1}$$

Let u be a dressed eigenstate with energy $E < E_0 + \theta$. We have

$$(S + V + W - E)u = 0. \tag{6.2}$$

Hence,

$$(S + V + W - E)\Pi u = (S + V + W - E)Pu, \tag{6.3}$$

i.e.

$$\left(D_0 + \sum_{k=0}^{\infty} V^{(k)}_{\partial\bar{\mathscr{S}}_n} + W^{(k)}_{\partial\bar{\mathscr{S}}_n} + V^{(k)}_{\sim\bar{\mathscr{S}}_n}\right)\Pi u = u_0. \tag{6.4}$$

Here we use the notations

$$u_0 = \Pi(S + V + W - E)Pu = \Pi(V_{\partial\bar{\mathscr{S}}_n} + W_{\partial\bar{\mathscr{S}}_n})Pu \tag{6.5}$$

and

$$D_0 = \Pi(S + V_{\bar{\mathscr{S}}_n} + W_{\bar{\mathscr{S}}_n} - E)\Pi. \tag{6.6}$$

The operator $V^{(k)}_{\partial\bar{\mathscr{S}}_n}$ has the following matrix elements:

$$\langle\gamma'|V^{(k)}_{\partial\bar{\mathscr{S}}_n}|\gamma\rangle = \begin{cases}\langle\gamma'|\Pi V_{\partial\bar{\mathscr{S}}_n}\Pi|\gamma\rangle & \text{if } \; d_{\bar{\mathscr{S}}_n}(s(\gamma), s(\gamma')) = k \\ 0 & \text{otherwise,}\end{cases} \tag{6.7}$$

and $V^{(k)}_{\sim\bar{\mathscr{S}}_n}$ and $W^{(k)}_{\partial\bar{\mathscr{S}}_n}$ are defined in a similar way.

The proof is based on decay estimates for u_0 and on the following representation for Πu:

$$\begin{aligned}\Pi u &= \left(D_0 + \sum_{k=0}^{\infty} V^{(k)}_{\partial\bar{\mathscr{S}}_n} + W^{(k)}_{\partial\bar{\mathscr{S}}_n} + V^{(k)}_{\sim\bar{\mathscr{S}}_n}\right)^{-1} u_0 \\ &= \sum_{j=0}^{\infty} D_0^{-1/2}\left[D_0^{-1/2}\left(\sum_{k=0}^{\infty} V^{(k)}_{\partial\bar{\mathscr{S}}_n} + W^{(k)}_{\partial\bar{\mathscr{S}}_n} + V^{(k)}_{\sim\bar{\mathscr{S}}_n}\right)D_0^{-1/2}\right]^j D_0^{-1/2}u_0.\end{aligned} \tag{6.8}$$

The geometric series expansion (6.8) provides us with a random walk expansion for Πu. To control it, one needs the relative boundedness estimates that are contained in the following three lemmas.

In the following we assume that n satisfies a bound of the form (4.13) so that

$$\| W^{(k)}_{\partial \mathscr{S}_n} \|_{2,1} \leqq (c_0 \lambda)^{n+\bar{k}} |\mathscr{S}| \leqq (c_0\lambda)^{\bar{k}} \tag{6.9}$$

and

$$|E_{0\lambda} - E_n| \leqq 2(27 \cdot 4^{n-1} - 2)(c_0\lambda)^{1+(n/2)} |\mathscr{S}| \leqq \tfrac{1}{4}(1-\theta), \tag{6.10}$$

where E_n is the ground state energy of

$$S_{\bar{\mathscr{S}}_n} + V_{\bar{\mathscr{S}}_n} + W_{\bar{\mathscr{S}}_n} - S^{\text{in}}_{\partial \bar{\mathscr{S}}_n} \tag{6.11}$$

in $\mathscr{H}(\bar{\mathscr{S}}_n)$. $S^{\text{in}}_{\partial \bar{\mathscr{S}}_n}$ and $S_{\partial \bar{\mathscr{S}}_n}$ are defined as in Sect. 5. Under such hypothesis, the following three lemmas are true:

Lemma 6.1. *We have*

$$\| D_0^{-1} \|_{2,1} \leqq \frac{4}{3(1-\theta)} \tag{6.12}$$

and

$$\| D_0^{-1/2} W^{(k)}_{\partial \bar{\mathscr{S}}_n} D_0^{-1/2} \|_{2,1} \leqq \frac{4(c_0\lambda)^{\bar{k}}}{3(1-\theta)}. \tag{6.13}$$

Proof. This is an immediate consequence of (6.9) and (6.10). Q.E.D.

Lemma 6.2. *We have*

$$\| D_0^{-1/2} V^{(k)}_{\sim \bar{\mathscr{S}}_n} D_0^{-1/2} \|_{2,1} \leqq \frac{4}{3} \frac{k+1}{1-\theta} (c_0\lambda)^{\bar{k}}. \tag{6.14}$$

Proof. In fact

$$\| D_0^{-1/2} V^{(k)}_{\sim \bar{\mathscr{S}}_n} D_0^{-1/2} \|_{2,1} = \sup_{\substack{\|\phi\|_2 = 1 \\ \phi \in \mathscr{H}(\bar{\mathscr{S}}_n)}} \ \sup_{\substack{s(\gamma) \subset \sim \bar{\mathscr{S}}_n \\ s(\gamma) \neq 0}} \| D_0^{-1/2} V^{(k)}_{\sim \bar{\mathscr{S}}_n} D_0^{-1/2} (\phi \otimes (\tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle) \|_{2,1} \tag{6.15}$$

and if $\phi \in \mathscr{H}(\bar{\mathscr{S}}_n)$ is such that $\|\phi\|_2 = 1$ and γ is an excitation with $\varnothing \neq S(\gamma) \subset \sim \bar{\mathscr{S}}_n$, we have

$$\begin{aligned} &\| D_0^{-1/2} V^{(k)}_{\sim \bar{\mathscr{S}}_n} D_0^{-1/2} \phi \otimes (\tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle) \|_{2,1} \\ &\leqq \left(|s(\gamma)| - \frac{1+3\theta}{4} \right)^{-1/2} \sum_{\gamma'} \left(|s(\gamma')| - \frac{1+3\theta}{4} \right)^{-1/2} |\langle \gamma' | V^{(k)}_{\sim \bar{\mathscr{S}}_n} | \gamma \rangle| \\ &\leqq \left(|s(\gamma)| - \frac{1+3\theta}{4} \right)^{-1/2} \left(\max(|s(\gamma)| - k, 1) - \frac{1+3\theta}{4} \right)^{-1/2} |s(\gamma)| (c_0\lambda)^{\bar{k}} \\ &\leqq \frac{4}{3} \frac{k+1}{1-\theta} (c_0\lambda)^{\bar{k}} \quad \text{Q.E.D.} \end{aligned} \tag{6.16}$$

Lemma 6.3. *We have*

$$\| D_0^{-1/2} V^{(k)}_{\partial \bar{\mathscr{S}}_n} D_0^{-1/2} \|_{2,1} \leqq \frac{4}{3} \frac{k+1}{1-\theta} (c_0\lambda)^{\bar{k}} + (c_0\lambda)^{\bar{k}/2} \tag{6.17}$$

Proof. If

$$V^{(k)}_{\partial\bar{\mathscr{S}}_n} = \sum_{\gamma_0 \subset \Lambda} v^{(k)}_{\partial}(\gamma_0) \tag{6.18}$$

gives the expansion of $V^{(k)}_{\partial\bar{\mathscr{S}}_n}$ in operators $v^{(k)}_{\partial}(\gamma_0)$ with support γ_0, let us rewrite $V^{(k)}_{\partial\bar{\mathscr{S}}_n}$ as follows:

$$V^{(k)}_{\partial\bar{\mathscr{S}}_n} = \sum_{\gamma_0 \subset \Lambda} \operatorname{ad} v^{(k)}_{\partial}(\gamma_0), \tag{6.19}$$

where $\operatorname{ad} v^{(k)}_{\partial}(\gamma_0)$ is the operator such that

$$\operatorname{ad} v^{(k)}_{\partial}(\gamma_0)|\gamma\rangle = \operatorname{ad} v^{(k)}_{\partial}(\gamma_0)\tau_\gamma|0\rangle \equiv [v^{(k)}_{\partial}(\gamma_0), \tau_\gamma]|0\rangle. \tag{6.20}$$

Let us remark that the support of the operators $\operatorname{ad} v^{(k)}_{\partial}(\gamma_0)$ is the entire set Λ. We have

$$\|D_0^{-1/2} V^{(k)}_{\partial\bar{\mathscr{S}}_n} D_0^{-1/2}\|_{2,1} = \sup_{\substack{s(\gamma) \subset \sim\bar{\mathscr{S}}_n \\ s(\gamma) \neq \emptyset}} \sup_{\substack{\phi \in \mathscr{H}(\bar{S}_n) \\ \|\phi\| = 1}} \|D_0^{-1/2} V^{(k)}_{\partial\bar{\mathscr{S}}_n} D_0^{-1/2} \phi \otimes (\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle)\|_{2,1}. \tag{6.21}$$

Let us fix an excitation γ with $\emptyset \neq s(\gamma) \subset \sim\bar{\mathscr{S}}_n$ and let $\phi \in \mathscr{H}(\bar{\mathscr{S}}_n)$ be a state such that $\|\phi\|_2 = 1$. We have

$$\begin{aligned} &\|D_0^{-1/2} V^{(k)}_{\partial\bar{\mathscr{S}}_n} D_0^{-1/2} \phi \otimes (\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle)\|_{2,1} \\ &\leqq \sum_{\gamma_0 : \gamma_0 \cap s(\gamma) \neq \emptyset} \|D_0^{-1/2} \operatorname{ad} v^{(k)}_{\partial}(\gamma_0) D_0^{-1/2} \phi \otimes (\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle)\|_{2,1} \\ &+ \sum_{\gamma_0 : \gamma_0 \cap s(\gamma) = \emptyset} \|D_0^{-1/2} \operatorname{ad} v^{(k)}_{\partial}(\gamma_0) D_0^{-1/2} \phi \otimes (\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle)\|_{2,1}. \end{aligned} \tag{6.22}$$

The first term can be estimated as in (6.16) and is

$$\leqq \frac{4}{3}\frac{k+1}{1-\theta}(c_0\lambda)^{\bar{k}}. \tag{6.23}$$

To bound the second term, let us introduce the operators $\bar{v}^{(k)}_0(\gamma_0)$ with $\gamma_0 \subset \bar{\mathscr{S}}_n$ such that

$$\bar{v}^{(k)}_{\partial}(\gamma_0) = \sum_{\gamma'_0 \cap \bar{\mathscr{S}}_n = \gamma_0} F_1 \operatorname{ad} v^{(k)}_{\partial}(\gamma'_0) F_2 : \mathscr{H}(\bar{\mathscr{S}}_n) \to \mathscr{H}(\bar{\mathscr{S}}_n), \tag{6.24}$$

where F_2 is the injection: $\mathscr{H}(\bar{\mathscr{S}}_n) \to \mathscr{H}(\Lambda)$ such that

$$F_2\phi = \phi \otimes |0_{\sim\bar{\mathscr{S}}_n}\rangle \tag{6.25}$$

for all $\phi \in \mathscr{H}(\bar{\mathscr{S}}_n)$, and F_1 is its left inverse such that

$$F_1 \phi \otimes \psi = \phi \tag{6.26}$$

for all $\phi \in \mathscr{H}(\bar{\mathscr{S}}_n)$ and $\psi \in \mathscr{H}(\sim\bar{\mathscr{S}}_n)$. Due to the bounds in Sect. 2 on the operators entering into the dressing transformation, we have

$$\sum_{\substack{\gamma_0 : x \in \gamma_0 \\ \gamma_0 \subset \sim\bar{\mathscr{S}}_n}} \|\bar{v}^{(k)}_{\partial}(\gamma_0)\|_1 \leqq (c_0\lambda)^{m(x)} \tag{6.27}$$

for all $x \in \Lambda$, where

$$m(x) \equiv \max(\bar{k}, d(x, \partial\bar{\mathscr{S}}_n) + 1). \tag{6.28}$$

The second term in (6.22) is

$$\leqq \sum_{\gamma_0 \subset \bar{\mathscr{P}}_n} \| (S_{\bar{\mathscr{P}}_n} + V_{\bar{\mathscr{P}}_n} + W_{\bar{\mathscr{P}}_n} - E + 1)^{-1/2} \bar{v}_0^{(k)}(\gamma_0) \cdot (S_{\bar{\mathscr{P}}_n} + V_{\bar{\mathscr{P}}_n} + W_{\bar{\mathscr{P}}_n} - E + 1)^{1/2} \|_2 \leqq (c_0\lambda)^{\bar{k}/2}. \tag{6.29}$$

To prove this, it suffices to establish the following relative boundedness estimate:

$$\sum_{\gamma_0 \subset \bar{\mathscr{P}}_n} |\langle \phi | \bar{v}_\partial^{(k)}(\gamma_0) | \phi \rangle| \leqq (c_0\lambda)^{\bar{k}/2} \langle \phi | (S_{\bar{\mathscr{P}}_n} + V_{\bar{\mathscr{P}}_n} - E + 1) | \phi \rangle. \tag{6.30}$$

By expanding ϕ in the basis of excitations

$$\phi = \sum_{s(\gamma) \subset \bar{\mathscr{P}}_n} \phi_\gamma \tau_\gamma |0\rangle, \tag{6.31}$$

we find

$$\begin{aligned}
\sum_{\gamma_0 \subset \bar{\mathscr{P}}_n} |\langle \phi | \bar{v}_\partial^{(k)}(\gamma_0) | \phi \rangle| &\leqq 2 \sum_{\gamma_0 \subset \bar{\mathscr{P}}_n} \sum_{s(\gamma) \subset \bar{\mathscr{P}}_n} \phi_\gamma^2 \sum_{s(\gamma') \subset \bar{\mathscr{P}}_n} |\langle \gamma' | \bar{v}_\partial^{(k)}(\gamma_0) | \gamma \rangle| \\
&\leqq 2 \sum_{s(\gamma) \subset \bar{\mathscr{P}}_n} \phi_\gamma^2 \sum_{x \in s(\gamma)} \sum_{\gamma_0 : x \in \gamma_0} \sum_{s(\gamma') \subset \bar{\mathscr{P}}_n} |\langle \gamma' | \bar{v}_\partial^{(k)}(\gamma_0) | \gamma \rangle| \\
&\leqq \sum_{s(\gamma) \subset \bar{\mathscr{P}}_n} \phi_\gamma^2 \sum_{x \in s(\gamma)} (c_0\lambda)^{m(x)} \\
&\leqq 2(c_0\lambda)^{\bar{k}/2} \sum_{s(\gamma) \subset \bar{\mathscr{P}}_n} \phi_\gamma^2 \sum_{x \in s(\gamma)} (c_0\lambda)^{(d(x, \partial \bar{S}_n) + 1)/2} \\
&\leqq (c_0\lambda)^{\bar{k}/2} \langle \phi | S_{\partial \bar{\mathscr{P}}_n}^{\text{in}} | \phi \rangle.
\end{aligned} \tag{6.32}$$

Since

$$\begin{aligned}
&\langle \phi | S_{\bar{\mathscr{P}}_n} + V_{\bar{\mathscr{P}}_n} + W_{\bar{\mathscr{P}}_n} - E + 1 - S_{\partial \bar{\mathscr{P}}_n}^{\text{in}} | \phi \rangle \\
&\quad \geqq E_n - E + 1 \geqq E_{0\lambda} - E + 1 - \tfrac{1}{4}(1 - \theta) \geqq \tfrac{3}{4}(1 - \theta) > 0,
\end{aligned} \tag{6.33}$$

we have

$$(6.32) \leqq (c_0\lambda)^{\bar{k}/2} \langle \phi | (S_{\bar{\mathscr{P}}_n} + V_{\bar{\mathscr{P}}_n} + W_{\bar{\mathscr{P}}_n} - E + 1 - S_{\partial \bar{\mathscr{P}}_n}^{\text{in}}) + S_{\partial \bar{\mathscr{P}}_n}^{\text{in}} | \phi \rangle, \tag{6.34}$$

which proves (6.30). Q.E.D.

As a consequence of the three Lemmas above, we find the bound

$$\| D_0^{-1/2} Z^{(k)} D_0^{-1/2} \|_{2,1} \leqq (1 - \theta)^{-1} (c\lambda)^{\bar{k}/2} \tag{6.35}$$

for some constant c and all integers $k \geqq 0$, where

$$Z^{(k)} \equiv V_{\partial \bar{\mathscr{P}}_n}^{(k)} + V_{\sim \bar{\mathscr{P}}_n}^{(k)} + W_{\partial \bar{\mathscr{P}}_n}^{(k)}. \tag{6.36}$$

From (6.35) follows that the geometric series expansion

$$\left(D_0 + \sum_{k=0}^{\infty} Z^{(k)} \right)^{-1} = \sum_{j=0}^{\infty} \sum_{k_1 \cdots k_j = 0}^{\infty} D_0^{-1/2} \prod_{i=1}^{j} [D_0^{-1/2} Z^{(k_i)} D_0^{-1/2}] D_0^{-1/2} \tag{6.37}$$

converges in $L^{2,1}$ operator norm if

$$\sum_{k=0}^{\infty} (c\lambda)^{\bar{k}/2} (1 - \theta)^{-1} \leqq \frac{1}{2}, \tag{6.38}$$

i.e. if

$$\lambda \leqq c(1-\theta)^2 \tag{6.39}$$

for some constant c. Let us remark that the operator $D_0^{-1/2} Z^{(k_i)} D_0^{-1/2}$ has nonvanishing matrix elements only between states $|\gamma\rangle, |\gamma'\rangle$ with

$$d_{\bar{\mathscr{S}}_n}(s(\gamma), s(\gamma')) = k_i. \tag{6.40}$$

This is due to the fact that $D_0^{-1/2}$ does not induce transitions outside $\bar{\mathscr{S}}_n$. We have

Lemma 6.4. *Let n be an integer $\geqq 0$ and let $P_\gamma^{\geqq n}$ be the orthogonal projection onto the space spanned by the states $|\gamma\rangle$ with*

$$d_{\bar{\mathscr{S}}_n}(s(\gamma), s(\gamma')) \geqq n. \tag{6.41}$$

Then, we have

$$\left\| P_\gamma^{\geqq n} \left(D_0 + \sum_{k=0}^{\infty} Z^{(k)} \right)^{-1} |\gamma\rangle \right\|_{2,1} \leqq (1-\theta)^{-1} \left(\frac{c\sqrt{\lambda}}{1-\theta} \right)^{\bar{n}} \tag{6.42}$$

where $\bar{n} = \max(n, 1)$..

Proof. We have

$$\begin{aligned} &\left\| P_\gamma^{\geqq n} \left(D_0 + \sum_{k=0}^{\infty} Z^{(k)} \right)^{-1} |\gamma\rangle \right\|_{2,1} \\ &\leqq \sum_{j=0}^{\infty} \sum_{k_1 + \cdots + k_j \geqq n} \prod_{i=1}^{j} \| D_0^{-1/2} Z^{(k_i)} D_0^{-1/2} \|_{2,1} \| D_0^{-1/2} \|^2 \\ &\leqq \sum_{j=0}^{\infty} \sum_{k_1 + \cdots + k_j \geqq n} c(1-\theta)^{-1} \left(\frac{c\sqrt{2}}{1-\theta} \right)^{\sum \bar{k}_i} \leqq (1-\theta)^{-1} \left(\frac{c\sqrt{\lambda}}{1-\theta} \right)^{\bar{n}}. \end{aligned} \tag{6.43}$$

Q.E.D.

The proof of Theorem 4.1 is now easy to conclude.
The operator

$$T \equiv \Pi(V_{\partial \bar{\mathscr{S}}_n} + W_{\partial \bar{\mathscr{S}}_n}) \tag{6.44}$$

can be decomposed

$$T = \sum_{k=0}^{\infty} T^{(k)} \tag{6.45}$$

in the same way as is done for $V_{\partial \bar{\mathscr{S}}_n}$ in (6.7) and we have

$$\| T^k \|_{2,1} \leqq (c_0 \lambda)^{\bar{k}} (|\partial \bar{\mathscr{S}}_n| + |\mathscr{S}| (c_0 \lambda)^n) \leqq (c_0 \lambda)^{\bar{k}} |\partial \bar{\mathscr{S}}_n|. \tag{6.46}$$

Thus, the same expansion used in the proof of Lemma 6.5 now gives

$$\left\| P_\phi^{\geqq k} \left(D_0 + \sum_{k=0}^{\infty} Z^k \right)^{-1} TPu \right\|_{2,1} \leqq (1-\theta)^{-1} |\partial \bar{\mathscr{S}}_n| \left(\frac{c\sqrt{\lambda}}{1-\theta} \right)^k \tag{6.47}$$

for all $k \geqq 1$. Q.E.D.

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