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On the Diffusive Nature of Entropy Flow in Infinite Systems: Remarks to a Paper by Guo-Papanicolau-Varadhan*

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Abstract. The hydrodynamic behaviour of interacting diffusion processes is investigated by means of entropy (free energy) arguments. The methods of [13] are simplified and extended to infinite systems including a case of anharmonic oscillators in a degenerate thermal noise. Following [14, 15] and [3–5] we derive a priori bounds for the rate of entropy production in finite volumes as the size of the whole system is infinitely extended. The flow of entropy through the boundary is controlled in much the same way as energy flow in diffusive systems [4].

0. Introduction

In a recent paper Guo-Papanicolau-Varadhan [13] proposed a new, fairly general approach to the hydrodynamic description of microscopically reversible spin systems in finite volumes. Using the free energy (relative entropy) of the model as a Liapunov function, they found that space-time averages of the evolved state approach a canonical local equilibrium, cf. Holley [14]. Although the parameter of this canonical state, that is the mean spin, has not been identified yet at this stage, a beautiful second entropy argument shows that the mean spin happens to be stable at the macroscopic level, therefore it is controlled by the conservation law. This means that the evolution equation of this conserved quantity closes up in the hydrodynamic limit, and a non-linear diffusion equation is obtained. From a probabilistic point of view, this result is a sophisticated law of large numbers formulated in a functional space; a more advanced technology yields also the related theory of large deviations [2]. The main purpose of this paper is to extend the entropy arguments of [13] to infinite systems, see Fritz [6, 7] and Funaki [10] for some previous results based on a different method. We are interested also in the

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hydrodynamic behaviour of certain anharmonic systems in a degenerate thermal noise, cf. Fritz-Maes [9]. Our main tool is an a priori bound revealing the diffusive nature of the flow of free energy, see Holley-Stroock [15] and Fritz [3, 5] for some previous results. Results of this kind are sufficient to derive the law of large numbers in the hydrodynamic limit for some symmetric, and weakly asymmetric infinite systems. For the associated large deviation theory [2] one also needs an additional a priori bound controlling the local entropy of the space-time process, cf. Lemma 6.1 of [13]. In a preliminary version [8] of this paper we did some calculations on the basis of the Maruyama-Girsanov formula. Although our bound seems to be sharp in some cases, it is not sufficient for the study of large deviations, so we do not discuss this question here.

1. Problems and Main Results

In this section we follow a possibly simple presentation of some ideas and results; generalizations and technical details will be added later. First we consider interacting diffusion processes ω_k indexed by the set of integers, $\mathscr{Z}$, thus the configurations of the system are real sequences $\omega=(\omega_k)_{k\in\mathscr{Z}}$. The evolution law is given by an infinite system of stochastic differential equations:

$$d\omega_k=\tfrac{1}{2}[V'(\omega_{k+1})-2V'(\omega_k)+V'(\omega_{k-1})]dt+dw_{k-1}-dw_k\,,\qquad k\in\mathscr{Z}\,,\tag{1.1}$$

where $V:\mathbb{R}\to\mathbb{R}$ is a one-body potential, $V'=dV/dx$, and w_k, $k\in\mathscr{Z}$ is a family of independent, standard Wiener processes. Notice that (1.1) is just a conservation law for the spin ω, it is in fact a Ginzburg-Landau lattice model with (free) energy $H(\omega)=\sum V(\omega_k)$, thus we have a family of reversible states λ_z, $z\in\mathbb{R}$,

$$\begin{aligned}\lambda_z(\mathrm{d}\omega)&=\prod_{k\in\mathscr{Z}}\exp[z\omega_k-V(\omega_k)-\log\Sigma(z)]d\omega_k\,,\\ \Sigma(z)&=\int\exp[zx-V(x)]dx\,.\end{aligned}\tag{1.2}$$

For convenience we assume that V has two continuous derivatives, V'' is bounded, and $\liminf_{|x|\to\infty}V''(x)>0$. The second condition implies $\Sigma(z)<+\infty$ for all $z\in\mathbb{R}$, while the first one yields the existence of unique strong solutions to (1.1) in a configuration space, Q, defined as

$$\Omega=\left[\omega:\lim_{|k|\to\infty}|\omega_k|e^{-\delta|k|}=0\text{ for }\delta>0\right],\tag{1.3}$$

see [6, 7] for some further references. Equip Ω with its relative product topology and the associated Borel structure, of course, $\lambda_z(\Omega)=1$ for all z. The generator of the diffusion defined by (1.1) in Ω is actually an extension of an elliptic operator, $\mathbb{G}$:

$$\mathbb{G}\varphi=\tfrac{1}{2}\sum_{k\in\mathscr{Z}}[(\partial_{k+1}-\partial_k)^2\varphi-(V'(\omega_{k+1})-V'(\omega_k))(\partial_{k+1}\varphi-\partial_k\varphi)]\,,\tag{1.4}$$

where $\partial_k\varphi=\partial\varphi/\partial\omega_k$, while $\varphi:\Omega\to\mathbb{R}$ is a smooth cylinder function. Although the proof is immediate, we shall need such a statement for finite dimensional diffusions only.

We are interested in the asymptotic behaviour of the rescaled spin field $S^\varepsilon=S_t^\varepsilon(\varphi)$ as $\varepsilon\to 0$,

$$S_t^\varepsilon(\varphi)=\varepsilon\sum_{k\in\mathscr{Z}}\varphi(\varepsilon k)\omega_k(t/\varepsilon^2)\,,\qquad \varepsilon>0,\ \varphi\in\mathbb{C}_0^2(\mathbb{R})\,,\tag{1.5}$$

where $\mathbb{C}_0^2(\mathbb{R})$ denotes the space of twice continuously differentiable $\varphi:\mathbb{R}\to\mathbb{R}$ with compact support. Since S^ε is a normalized sum like that we have in the law of large numbers, a deterministic limit is expected:

$$\lim_{\varepsilon\to 0} S_t^\varepsilon(\varphi)=\int \varphi(x)m_t(x)dx \quad \text{in probability} \tag{1.6}$$

with some asymptotic density m_t for each $t>0$, at least if the very same statement holds true at $t=0$. This means that we are given a family of initial distributions μ^ε, $\varepsilon>0$, on Ω, and an initial density $\varrho:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{\varepsilon\to 0} S_0^\varepsilon(\varphi)=\int \varphi(x)\varrho(x)dx \quad \text{in } \mu^\varepsilon \text{ for } \varphi\in\mathbb{C}_0^2(\Omega). \tag{1.7}$$

A formal derivation of the limiting equation is quite easy. From (1.2) we see that

$$\int V'(\omega_k)\lambda_z(d\omega)=z \quad \text{and} \quad \int \omega_k\lambda_z(d\omega)=\Sigma'(z)/\Sigma(z) \tag{1.8}$$

for all $k\in\mathscr{Z}$, thus $z=J'(\varrho)$ whenever $\varrho=\int\omega_k\lambda_z(d\omega)$, where J denotes the convex conjugate (Legendre transform) of $\log\Sigma$,

$$J(\varrho)=\sup_z\,[z\varrho-\log\Sigma(z)]\,. \tag{1.9}$$

This means that if the system approaches a local equilibrium as $\varepsilon\to 0$, that is $\omega(t/\varepsilon^2)$ is distributed in an asymptotic sense by a measure of type (1.2) with some spatially inhomogeneous profile $z_t=z_t(x)$ of the chemical potential, then $m_t=\Sigma'(z_t)/\Sigma(z_t)$ must satisfy a nonlinear diffusion equation, namely

$$\frac{\partial m_t(x)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x^2}J'(m_t(x))\,,\qquad m_0=\varrho\,. \tag{1.10}$$

There are two rigorous methods to derive (1.10) from a microscopic model like (1.1). If V is strictly convex, then (1.1) behaves as a parabolic equation of divergence form, thus we have some very effective a priori bounds; in fact, these bounds do not depend on ε or the actual realization of the process. In this case the initial distribution is almost arbitrary, besides (1.7) it is sufficient to assume that μ^ε, $\varepsilon>0$ is a tight family with respect to a certain weak topology of Ω, see [6, 7, 9, 10] for a precise formulation and further results. The challenging problem of a non-convex V has been solved by Guo-Papanicolau-Varadhan [13], they consider (1.1) with periodic boundary conditions, i.e. on a circle. In this case it is natural to assume that the relative entropy of the initial distribution with respect to some equilibrium state λ_z is bounded by a multiple of the number of active sites; thus entropy can be used as a Liapunov function for the evolved measure. They show that this condition together with (1.6) imply (1.10) as the hydrodynamic limit of (1.1), both equations should be considered with periodic boundary conditions. Further developments based on the same method are presented in [2, 11, 16, 18]. For an early application of entropy as a Liapunov function for symmetric diffusion see Nash [17].

In the case of an infinite system we have to consider local quantities, thus an additional difficulty appears: boundary effects should also be controlled. Let $\mathscr{R}_n$ denote the σ-field of Ω generated by the variables $\omega_{-n},\omega_{-n+1},\dots,\omega_n$, we define a reference measure, Q_n on $\mathscr{R}_n$ by its Lebesque density q_n,

$$q_n(\omega)=\exp\left[-\sum_{k=-n}^{n}V(\omega_k)\right], \tag{1.11}$$

and $p_n = d\mu_n/dQ_n$ whenever μ_n, the restriction of a Borel probability μ to $\mathscr{R}_n$, is absolutely continuous. Then the (non-equilibrium) free energy of μ in the box $[-n, n]$ is defined as

$$F_n(\mu) = \int \log p_n d\mu \quad \text{if} \quad \mu_n \ll Q_n, \qquad F_n(\mu) = +\infty \text{ otherwise}. \tag{1.12}$$

In a smooth case the temporal derivative of $F_n(\mu)$ along the evolution (1.1) can be decomposed as

$$\begin{aligned} \dot{F}_n(\mu) &= \int \mathbb{G} \log p_n d\mu = \int p_{n+1} \mathbb{G} \log p_n dQ_{n+1} \\ &= -\frac{1}{2} \sum_{k \in \mathscr{Z}} \int \frac{1}{p_n} (\partial_{k+1} p_n - \partial_k p_n)(\partial_{k+1} p_{n+1} - \partial_k p_{n+1}) dQ_{n+1} \\ &= -D_n(\mu) + B_n(\mu), \end{aligned} \tag{1.13}$$

where

$$D_n(\mu) = \frac{1}{2} \sum_{k=-n}^{n-1} \int \frac{1}{p_n} (\partial_{k+1} p_n - \partial_k p_n)^2 dQ_n, \tag{1.14}$$

while $B_n(\mu)$ is the rest from the second line of (1.13). It is well known that D_n admits a variational characterization [1], but B_n and $\dot{F}_n$ may not be defined for arbitrary μ.

Consider first $B_n(\mu)$, the boundary term of entropy (free energy) production; using the variational characterization of entropy, and the Schwarz inequality as in [3], we obtain two bounds on B_n:

$$B_n(\mu) \leqq K[K + F_{n+1}(\mu) - F_n(\mu)], \tag{1.15}$$

$$B_n(\mu) \leqq [K(K + F_{n+1}(\mu) - F_n(\mu))]^{1/2} [D_{n+1}(\mu) - D_n(\mu)]^{1/2}, \tag{1.16}$$

where K is a constant depending only on V. Notice that D_n is increasing by convexity, while the increment of F_n is bounded from below. On the other hand, if $\mu_t = \mu \mathbb{P}^t$ denotes the evolved measure, then at least at a formal level we have

$$F_n(\mu_t) + \int_0^t D_n(\mu_s) ds \leqq F_n(\mu_0) + \int_0^t B_n(\mu_s) ds. \tag{1.17}$$

The above set of inequalities can be solved in various situations; difficulties concerning the smoothness of μ_t are postponed to the next section.

In a stationary regime $D_n = B_n$, and we expect that $D_n(\mu) = 0$ for all n, i.e. μ is a canonical Gibbs state in the sense that μ and λ_z have identical conditional distributions given the total spin inside any box, and the configuration outside of it. Comparing (1.15) and (1.16) we obtain that

$$D_{n+1}(\mu) \leqq [K(2K + F_{n+2}(\mu) - F_n(\mu))]^{1/2} [D_{n+1}(\mu) - D_n(\mu)]^{1/2}, \tag{1.18}$$

therefore, if $D_m > 0$ for some m, then

$$1 \leqq [K(2K + F_{n+2}(\mu) - F_n(\mu))]^{1/2} \left[\frac{1}{D_n(\mu)} - \frac{1}{D_{n+1}(\mu)} \right]^{1/2} \tag{1.19}$$

for $n \geqq m$, which results in a contradiction in two cases.

Theorem 1. *If μ is a stationary state such that $F_n(\mu) < +\infty$ for each n, then any of the conditions $F_n(\mu) = o(n^2)$, or $F_{n+1}(\mu) - F_n(\mu) = \mathcal{O}(n)$ implies that μ is a canonical Gibbs state.* □

The formal part of the proof is immediate from (1.19). If $F_n = o(n^2)$, then $n-m \leqq \delta n(D_m)^{-1/2}$ follows for each $\delta > 0$ and $n > n_\delta$ by the Schwarz inequality, thus $D_m(\mu) \leqq \delta^2$. In the second case we get $c/n \leqq D_n^{-1} - D_{n+1}^{-1}$ for $n \geqq m$ with some $c > 0$, and the contradiction follows by summing over n. Technical details of the proof are to be added in the next section, but let us remark here that there are many other stationary states. Indeed, if $z_k = a + bk$, $z = (z_k)_{k \in \mathscr{Z}}$, and λ_z denotes the product measure defined by (1.2) with this linear profile of the chemical potential z, then each measure of this type is stationary, and also reversible with respect to (1.1). Notice that $F_n(\lambda_z) = \mathcal{O}(n^3)$ if $b \neq 0$.

In a time dependent situation we use (1.16) and (1.17) to estimate F_n and D_n. It might be interesting to see that energy flow in parabolic systems is controlled by the very same set of inequalities. For a simple example, let $V: \mathbb{R} \to \mathbb{R}$ be strictly convex, for $\omega \in \Omega$ define,

$$u_n = \sum_{k=-n}^{n} \omega_k, \qquad v_n = \sum_{k=-n}^{n-1} (\omega_{k+1} - \omega_k)[V'(\omega_{k+1}) - V'(\omega_k)],$$

and assume that ω evolves according to the deterministic (drift) part of (1.1). Differentiating u_n we obtain immediately that

$$\dot{u}_n + v_n \leqq [K(u_{n+1} - u_n)]^{1/2} (v_{n+1} - v_n)^{1/2}, \tag{1.20}$$

which is satisfied also by $u_n = Kn + F_n$ and $v_n = D_n$. This differential inequality can be solved by means of a trick of [4], if

$$F_n(\mu_0) \leqq Cn \quad \text{for} \quad n \geqq 1, \tag{1.21}$$

then we obtain that

$$F_n(\mu_t) + \int_0^t D_n(\mu_s) ds \leqq C\bar{K}[n^2 + t]^{1/2} \tag{1.22}$$

for all $n \geqq 1$ and $t \geqq 0$, where $\bar{K}$ is a new constant depending only on K. This a priori bound allows us to apply the method of [13] to infinite systems with a minor change. Since we are not able to derive an effective bound for the local space-time entropy of the infinitely extended system, cf. Lemma 6.1 of [13] and Proposition 4 of [8], we can not refer to tightness of the rescaled distribution on the space of measure valued trajectories. Nevertheless, we can manage by means of an $\mathbb{H}^{-2}$ topology, thus we do not need any further information on the dynamics. On the other hand, (1.22) is a microscopic bound, thus we need not introduce space averages of the evolved measure. The crucial Theorem 4.7 of [13] (two-block estimate) will also be simplified. In Sect. 4 we prove

Theorem 2. *Suppose* (1.7) *and* $F_n(\mu^\varepsilon) \leqq Cn$ *for all* $n \geqq 1$ *and* $\varepsilon > 0$. *Then* (1.6) *holds true for all* $t > 0$, *and the limiting density* m_t *is specified as the unique weak solution to* (1.10) *such that* $m_0 = \varrho$ *and*

$$\int_0^T \int [m_t^2(x) + |\nabla m_t(x)|^2] e^{-|x|} dx\, dt < +\infty$$

for all $T > 0$; $\nabla m_t = \partial m_t / \partial x$. $\square$

The dimension of the space, and the concrete form of the interaction is not relevant for the proof, once we have Theorem 4.2 of [13] (equivalence of ensembles), we can extend the law of large numbers to infinite volumes, cf. Rezakhanlou [18].

Our second question concerns the limits of the free energy (entropy) method, its basic condition is certainly the reversibility of the microscopic system. Weakly asymmetric problems can be treated as lower order perturbations of reversible models, see [2, 9, 16]. In such cases the reversible component dominates the asymmetric part in such a brutal way that the latter has no influence on the structure of local equilibrium. Hamiltonian systems in a thermal noise are more delicate. Indeed, the Hamiltonian part preserves entropy, thus we can weaken the strength of the noise. Let us consider an anharmonic chain on $\mathscr{Z}$ with Hamiltonian

$$H(p,q)=\sum_{k\in\mathscr{Z}}[p_k^2/2+V(q_{k+1}-q_k)], \tag{1.23}$$

where $p=(p_k)_{k\in\mathscr{Z}}$, $q_k=(q_k)_{k\in\mathscr{Z}}$, $p_k,q_k\in\mathbb{R}$, and $V:\mathbb{R}\to\mathbb{R}$ is a symmetric potential with the same regularity properties we had before. The equations of motion can be written as

$$dp_k=[V'(r_k)-V'(r_{k-1})]dt, \qquad dq_k=p_k dt, \tag{1.24}$$

where $r_k=q_{k+1}-q_k$ denotes the deformation. This model admits stationary states in the (p,r) variables, $r=(r_k)_{k\in\mathscr{Z}}$, but they are not reversible in the usual sense. For the entropy method we need some stochasticity of the evolution, usually the noise and damping are added to the equations for p_k, see [5, 9]. However, this kind of random perturbation is not regular enough, even the weak uniqueness of solutions to the corresponding limiting equation is problematic. That is why we investigate the following system:

$$\begin{aligned} dp_k&=[V'(r_k)-V'(r_{k-1})]dt, \\ dr_k&=(p_{k+1}-p_k)dt+\frac{\alpha}{2}[V'(r_{k+1})-2V'(r_k)+V'(r_{k-1})]dt \\ &\quad+\sqrt{\alpha}(dw_{k-1}-dw_k), \end{aligned} \tag{1.25}$$

where $\alpha>0$, and w_k, $k\in\mathscr{Z}$ is a family of independent standard Wiener processes. The configuration space of (1.25) is chosen as $\Omega^2=\Omega\times\Omega$, see (1.3), thus $\omega=(p,r)=(p_k,r_k)_{k\in\mathscr{Z}}$ if $\omega\in\Omega^2$. This law admits two additive integrals: $\sum p_k$ and $\sum r_k$, and all canonical Gibbs states with energy $H(\omega)=\sum[p_k^2/2+V(r_k)]$ at unit temperature are stationary measures of (1.25) for all $\alpha>0$.

There is another, a little bit more convincing motivation of (1.25). Let us consider a Hamiltonian particle system interacting by a symmetric pair potential V. We have then a stochastic evolution such that both the momentum and the particle number are conserved, namely

$$dp_k=-\partial_k H dt, \qquad dq_k=p_k dt-\frac{\alpha}{2}\partial_k H dt+\sqrt{\alpha}dw_k, \tag{1.26}$$

where $\alpha>0$, H denotes the total energy, $\partial_k H=\partial H/\partial q_k$, and w_k are independent Wiener processes, but we do not know any similar, momentum preserving perturbation to $dp_k=-\partial_k H dt$. Now, if the dimension of the space is just one, and V has such a big hard core that only neighboring particles can interact, then (1.26) reduces to (1.25) for p_k and the interparticle distance $r_k=q_{k+1}-q_k$; notice that, due to the hard core of V, (1.26) preserves the order of particles on $\mathbb{R}$.

From the point of view of hydrodynamics, there are two different limiting procedures for (1.25). The hyperbolic scaling is the most natural one, then the conserved fields are rescaled as

$$P_t^\varepsilon(\varphi)=\sum_{k\in\mathscr{X}}\varepsilon\varphi(\varepsilon k)p_k(t/\varepsilon), \qquad R_t^\varepsilon(\varphi)=\sum_{k\in\mathscr{X}}\varepsilon\varphi(\varepsilon k)r_k(t/\varepsilon), \tag{1.27}$$

and their limiting densities, π and ϱ are to be defined by

$$\lim_{\varepsilon\to 0}P_t^\varepsilon(\varphi)=\int\varphi(x)\pi_t(x)dx, \qquad \lim_{\varepsilon\to 0}R_t^\varepsilon(\varphi)=\int\varphi(x)\varrho_t(x)dx \tag{1.28}$$

in probability for $\varphi\in\mathbb{C}_0^2(\mathbb{R})$. Since (1.25) describes a thermal equilibrium, a nonlinear wave equation is expected:

$$\partial_t\pi=\nabla J'(\varrho), \qquad \partial_t\varrho=\nabla\pi, \tag{1.29}$$

where $\partial_t=\partial/\partial t$, $\nabla=\partial/\partial x$, and J is the same as in (1.10). In the case (1.26) of particle systems a formal calculation yields $\partial_t\varrho+\nabla(\varrho u)=0$, $\partial_t(\varrho u)+\nabla[\varrho u^2+p(\varrho)]=0$ as limiting equations, where ϱ is the density, u is the velocity field, and p denotes the dynamic pressure; they are the hydrodynamic equations of an isentropic gas. Unfortunately, the entropy argument yields only a very weak form of local equilibrium in these cases, we do not have the so-called two-block estimate. The problem we are able to solve belongs to the weakly asymmetric category.

Suppose that $\alpha=\bar{\alpha}/\varepsilon$, and $\bar{\alpha}>0$ is fixed. This massive noise changes even the macroscopic equation, we expect

$$\partial_t\pi=\nabla J'(\varrho), \qquad \partial_t\varrho=\nabla\pi+\frac{\bar{\alpha}}{2}\Delta J'(\varrho) \tag{1.30}$$

as the hydrodynamic limit of (1.25) with this scaling, where $\Delta=\partial/\partial x^2$. The conditions of this statement are similar to those of Theorem 2. In this case $\mathscr{R}_n$ is defined as the σ-field of Ω^2 generated by the variables $p_{-n+1}, p_{-n+2}, \ldots, p_n, r_{-n}, r_{-n+1}, \ldots, r_n$,

$$g_n(\omega)=\exp\left[-\sum_{k=-n+1}^{n}p_k^2/2-\sum_{k=-n}^{n}V(r_k)\right], \tag{1.31}$$

Q_n is a finite measure on $\mathscr{R}_n$, and $f_n=d\mu_n/dQ_n$ whenever μ_n is the restriction of a Borel probability μ to $\mathscr{R}_n$. We define F_n by $F_n(\mu)=\int\log f_n d\mu$ if $\mu_n\ll Q_n$, and $D_n(\mu)$ by (1.14) with $\partial_k=\partial/\partial r_k$ and f_n in the place of p_n, at least if f_n is smooth. Our inequality controlling the flow of free energy now becomes

$$\begin{aligned}\partial_tF_n(\mu_t)+\alpha D_n(\mu_t)&\leqq K[K+F_{n+1}(\mu_t)-F_n(\mu_t)]\\&+\alpha[K(K+F_{n+1}(\mu_t)-F_n(\mu_t))]^{1/2}[D_{n+1}(\mu_t)-D_n(\mu_t)]^{1/2},\end{aligned}$$

whence, if $F_n(\mu_0)\leqq Cn$ for $n\geqq 1$, we obtain that

$$F_n(\mu_t)+\alpha\int_0^t D_n(\mu_s)ds\leqq C\bar{K}[t+(n^2+\alpha t)^{1/2}] \tag{1.33}$$

for all $n\geqq 1$ and $t\geqq 0$; $\bar{K}$ is a new constant depending only on V. Although D_n controls only the distribution of the deformation r, the right-hand side of our system (1.25) of evolution equations depends in a linear way on the momentum p, thus we do not need any further information on the distribution of p. In the last section we prove

Theorem 3. *Let $\alpha=\bar{\alpha}/\varepsilon$ and suppose that μ^ε, the initial distribution satisfies $F_n(\mu^\varepsilon)\leqq Cn$ for $n\geqq 1$, and (1.28) for $t=0$. Then (1.25) implies (1.28) for all $t>0$ with limiting* *densities* π *and* ϱ *specified as a unique weak solution to* (1.30), *such that*

$$\int_0^T \int [\pi_t^2(x)+\varrho_t^2(x)+|\nabla \varrho_t(x)|^2]e^{-|x|}dx\,dt<+\infty \quad \text{for} \quad T>0. \quad \square$$

The proof is based on our a priori bound (1.33). To understand its structure, notice that the generator of the Markov process defined by (1.25) can be written as a sum: $\mathbb{G}=\mathbb{L}+\alpha\mathbb{G}_r$, where

$$\mathbb{L}\varphi=\sum_{k\in\mathscr{Z}}[\partial_k'\varphi(\partial_k H-\partial_{k-1}H)+\partial_k\varphi(\partial_{k+1}'H-\partial_k'H)], \tag{1.34}$$

$\partial_k'=\partial/\partial p_k$; while $\mathbb{G}_r$ is of type (1.4), it is acting only on functions of the r variables. The Hamiltonian part of the evolution preserves the free energy, the contribution of $\mathbb{L}$ to the temporal derivative of F_n consists of boundary terms only, namely

$$B_n'(\mu)=\int\frac{\partial_{-n}f_n}{f_n}\partial_{-n}'f_{n+1}dQ_{n+1}-\int\frac{\partial_n f_n}{f_n}\partial_{n+1}'f_{n+1}dQ_{n+1}. \tag{1.35}$$

It is a very fortunate situation that B_n' can be controlled by means of other boundary terms of $\dot{F}_n$, thus we get (1.32).

2. A Priori Bound for the Flow of Free Energy in Reversible Systems

The problem admits a fairly general formulation for systems of real valued spins on a countable, connected graph $\mathscr{S}$. The configuration space is defined as

$$\Omega=\left[\omega\in\mathbb{R}^{\mathscr{S}}:\lim_{|k|\to\infty}|\omega_k|e^{-\delta|k|}=0 \text{ for } \delta>0\right], \tag{2.1}$$

where $|k|$ denotes the length of the shortest path connecting k with a distinguished site $\theta\in\mathscr{S}$. Let S_k denote the set of neighbors of $k\in\mathscr{S}$ including k itself, $\Lambda_n=[k\in\mathscr{S}:|k|\leqq n]$; the graph structure of $\mathscr{S}$ is characterized by the requirements that $\sup_{k\in\mathscr{S}}\operatorname{card}S_k<+\infty$, and $\lim\operatorname{card}\Lambda_{n+1}/\operatorname{card}\Lambda_n=1$ as $n\to\infty$. The Borel field of Ω with respect to its product topology will be denoted by $\mathscr{R},\mathscr{R}_n$ is the σ-field generated by the variables ω_k such that $k\in\Lambda_{2n}$, $\mathbb{C}_b(\varrho)$ is the space of continuous and bounded $\varphi:\Omega\to\mathbb{R}$, while $\mathbb{C}_0^2(\Omega)$ denotes the space of twice continuously differentiable cylinder functions $\varphi:\Omega\to\mathbb{R}$ with bounded second derivatives.

The interaction of the system is given by a family of potentials $V_B\in\mathbb{C}_0^2(\Omega)$, where $B\subset\mathscr{S}$ is finite, and V_B may depend on ω_k only if $k\in B$. For convenience we assume that V is finite in the sense that $V_B\neq 0$ implies $B\subset S_k$ for $k\in B$, $V_B(\omega)\geqq 0$, and all second derivatives of any V_B are bounded by the very same constant, while $\sup|\partial_k V_B(\omega^0)|<+\infty$ with $\omega_j^0=0$ for $j\in\mathscr{S}$. To ensure the existence of Gibbs states with energy $H(\omega)=\Sigma V_B(\omega)$ we need a condition of superstability: we have some $a_0>0$ and $a_1\geqq 0$ such that $V_B(\omega)\geqq a_0\omega_k^2-a_1$ whenever B consists of a single site, k. Let λ denote a Gibbs state with energy H at temperature 1. We consider interacting diffusion processes in Ω generated by $\mathbb{G}$,

$$\mathbb{G}=\sum_{b\in\mathscr{S}^+}\mathbb{G}_b,\quad \mathbb{G}_b\varphi=\tfrac{1}{2}[\partial_b(c_b\partial_b\varphi)-c_b(\partial_b H)\partial_b\varphi],\quad \varphi\in\mathbb{C}_0^2(\Omega), \tag{2.2}$$

where $\mathscr{S}^+\subset\Omega$ such that $|b_k|\leqq 1$ for each $k\in\mathscr{S}$, $b_k\neq 0$ implies $b_j=0$ for $j\notin S_k$, and $\partial_b\varphi=\sum_{k\in\mathscr{S}}b_k\partial\varphi/\partial\omega_k$. The coefficients $c_b\in\mathbb{C}_0^2(\Omega)$ are assumed to be uniformly bounded together with their first and second derivatives, $c_b(\omega) \geqq 1$ for all $\omega \in \Omega$ unless it is identically zero, and c_b may depend on ω_j only if $j \in S_k$ for some $k \in \mathscr{S}$ such that $b_k \neq 0$. Let S_k^+ denote the set of $b \in \mathscr{S}^+$ such that $b_k \neq 0$; we assume also that S_k^+ is never empty, and $\sup_{k \in \mathscr{S}} \operatorname{card} S_k^+ < +\infty$. The stochastic equations associated with $\mathbb{G}$ are

$$d\omega_k + \tfrac{1}{2} \sum_{b \in S_k^+} b_k [c_b \partial_b H dt - \partial_b c_b dt + 2\sqrt{c_b}\, dw_b] = 0, \tag{2.3}$$

where w_b, $b \in \mathscr{S}^+$ is a family of independent, standard Wiener processes.

There are two basic examples of such processes. If $\mathscr{S}^+ = \mathscr{S}$ and $\partial_b = \partial/\partial\omega_b$, then we obtain a class of stochastic gradient systems, see e.g. [3]. Ginzburg-Landau lattice models with conservation law are defined on an oriented graph $\mathscr{S}$, $\mathscr{S}^+$ is the set of positively oriented bonds, while $\partial_b = \partial_j - \partial_k$ if b is the bond directed from k to j. If $\partial_b c_b = 0$ for all b, then c_b can be interpreted as the conductivity of bond b.

Under the above set of conditions, which will be assumed in this section, it is easy to solve (2.3) in Ω in a unique way, see [7] for a brief explanation and further references. In fact, it is possible to construct a strongly continuous Markov semigroup, $\mathbb{P}^t$, in $\mathbb{C}_b(\Omega)$ such that $\mathbb{C}_0^2(\Omega)$ is a core of its generator denoted also by $\mathbb{G}$. Finite-dimensional approximations to (2.3) can be obtained by letting $c_b = 0$ outside of a finite subset of $\mathscr{S}$.

Consider now the free energy in a finite box and its rate of production. Let Q_n denote a finite measure on $\mathscr{R}_n$ with Lebesque density $q_n(\omega) = \exp[-H_n(\omega)]$,

$$H_n(\omega) = \sum_{B \subset \Lambda_{2n}} V_B(\omega). \tag{2.4}$$

If μ is a Borel probability on Ω, and $b \in \Lambda_n^+$, then

$$F_n(\mu) = \sup[\textstyle\int \varphi d\mu - \log \int e^{\varphi} dQ_n : \varphi \in \mathbb{C}_b(\Omega) \cap \mathscr{R}_n], \tag{2.5}$$

$$D_n^b(\mu) = 4 \sup\left[-\int \frac{1}{\varphi} \mathbb{G}_b \varphi d\mu : \inf \varphi > 0,\ \varphi \in \mathbb{C}_0^2(\Omega) \cap \mathscr{R}_n \right], \tag{2.6}$$

$$D_n(\mu) = \sum_{b \in \Lambda_n^+} D_n^b(\mu), \qquad \Lambda_n^+ = [b \in \mathscr{S}^+ : b_k = 0 \text{ unless } k \in \Lambda_{2n-1}], \tag{2.7}$$

where $\varphi \in \mathscr{R}_n$ indicates that φ is $\mathscr{R}_n$-measurable. Since Λ_{-1} is empty, $D_0(\mu) = 0$. In the smooth case we have $F_n(\mu) = \int \log p_n d\mu$,

$$D_n^b(\mu) = \frac{1}{2} \int \frac{c_b(\omega)}{p_n(\omega)} (\partial_b p_n(\omega))^2 Q_n(d\omega), \tag{2.8}$$

where μ_n is the restriction of μ to $\mathscr{R}_n$, and $p_n = d\mu_n/dQ_n$, see [1]. In view of (2.5) and (2.6), both F_n and D_n^b are convex and lower semi-continuous functions of μ with respect to the weak topology of probability measures, and $D_n^b(\mu) \leqq D_{n+1}^b(\mu)$ whenever $b \in \Lambda_n^+$. Since Q_{n+1} is not an extension of Q_n, and they are not normed, F_n is not necessarily an increasing sequence. Nevertheless $F_{n+1} - F_n$, a conditional free energy, can also be used to estimate expectations in the spirit of (2.5).

Lemma 1. *There exists a constant K depending only on S and V such that if $\Lambda_n^\partial = \Lambda_{2n+2} \setminus \Lambda_{2n}$, then*

$$\sum_{k \in \Lambda_n^\partial} \int \omega_k^2 d\mu \leqq K[K \operatorname{card} \Lambda_n^\partial + F_{n+1}(\mu) - F_n(\mu)]. \quad \square$$

Proof. Let $0<\alpha<a_0$ and $g_n(\omega)=\exp\left[-\alpha \sum_{k\in\Lambda_n^\partial}\omega_k^2\right]$, by convexity of $x\log x$, and by the properties of the interaction

$$\int p_{n+1}\log\left[\frac{p_n q_n}{p_{n+1}q_{n+1}}g_n\right]dQ_{n+1}\leqq\log\int\frac{p_n q_n}{q_{n+1}}g_n dQ_{n+1}\leqq\tilde{K}\operatorname{card}\Lambda_n^\partial,$$

which completes the proof by a direct calculation. □

The flux of free energy can be handled as follows. Let f_n denote the Lebesgue density of μ_n, and set $h_n^b=\partial_b p_n/p_n$, then

$$h_{n+1}^b=\partial_b f_{n+1}/f_{n+1}-\partial_b q_n/q_n+\partial_b H_{n+1}-\partial_b H_n,$$

thus we have

$$\begin{gathered}\dot{F}_n(\mu)=\int \mathbb{G}\log p_n d\mu=-\tfrac{1}{2}\sum_{b\in\mathscr{S}^+}\int c_b h_n^b\partial_b p_{n+1}dQ_{n+1}=-D_n(\mu)+B_n(\mu),\\ B_n(\mu)=-\tfrac{1}{2}\sum_{k\notin\Lambda_n^+}\int c_b h_n^b h_{n+1}^b d\mu.\end{gathered}\tag{2.9}$$

Since $D_n^b\leqq D_{n+1}^b$ if $b\in\Lambda_n^+$ by convexity of D, we get

$$\begin{gathered}B_n(\mu)\leqq[X_n(\mu)[D_{n+1}(\mu)-D_n(\mu)]]^{1/2},\\ X_n(\mu)=\tfrac{1}{2}\sum_{b\notin\Lambda_n^+}\int c_b(h_n^b)^2 d\mu.\end{gathered}\tag{2.10}$$

On the other hand, integrating by parts we obtain that

$$\begin{aligned}\int c_b h_n^b\partial_b f_{n+1}d\omega&=-\int(\partial_b c_b)h_n^b d\mu-\int\mu(c_b|\mathscr{R}_n)(\partial_b h_n^b)f_n d\omega\\&=\int[\partial_b\mu(c_b|\mathscr{R}_n)-\partial_b c_b]h_n^b d\mu+\int\mu(c_b|\mathscr{R}_n)h_n^b\partial_b f_n d\omega,\end{aligned}$$

where $\mu(\varphi|\mathscr{R}_n)=\int\varphi(\omega)\mu(d\omega|\mathscr{R}_n)$, thus by the Schwarz inequality we obtain a second bound for B_n, namely

$$B_n(\mu)=-X_n(\mu)+[X_n(\mu)R_n(\mu)]^{1/2}\leqq\tfrac{1}{4}R_n(\mu),\tag{2.11}$$

$$\begin{aligned}R_n(\mu)=&\sum_{b\notin\Lambda^+}\int\mu(c_b|\mathscr{R}_n)\left[\frac{\mu(c_b\partial_b H_{n+1}-c_b\partial_b H_n|\mathscr{R}_n)}{\mu(c_b|\mathscr{R}_n)}\right]^2 d\mu\\&+\sum_{b\notin\Lambda_n^+}\int\mu(c_b|\mathscr{R}_n)\left[\frac{\partial_b\mu(c_b|\mathscr{R}_n)-\mu(\partial_b c_b|\mathscr{R}_n)}{\mu(c_b|\mathscr{R}_n)}\right]^2 d\mu;\end{aligned}\tag{2.12}$$

if $c_b=0$ then the corresponding term vanishes by convention.

Lemma 2. *Suppose that $F_{n+1}(\mu)<+\infty$ and $p_{n+1}\in\mathbb{C}_0^2(\Omega)$ is positive, then we have the following bounds for $B_n(\mu)$:*

$$B_n(\mu)\leqq\min[\tfrac{1}{4}R_n(\mu),[R_n(\mu)(D_{n+1}(\mu)-D_n(\mu))]^{1/2}],\tag{2.13}$$

and $R_n(\mu)$ is controllable if $b_k\neq 0$ implies $\partial c_b/\partial\omega_j=0$ whenever $|j|>|k|$; in such situations we have a constant K depending only on $\mathscr{S}$ and V such that

$$R_n(\mu)\leqq K[K\operatorname{card}\Lambda_n^\partial+F_{n+1}(\mu)-F_n(\mu)].\quad\square\tag{2.14}$$

Proof. The first statement of (2.13) is just (2.10), but $B_n\geqq 0$ and (2.11) yield $X_n\leqq R_n$, thus (2.11) results in (2.13). In view of Lemma 1, (2.14) reduces to

$$R_n(\mu)\leqq K_1\left[\operatorname{card}\Lambda_n^\partial+\int\sum_{k\in\Lambda_n^\partial\cup\Lambda_{n-1}^\partial}\omega_k^2 d\mu\right],\tag{2.15}$$

which can easily be verified as $\partial_b \mu(c_b|\mathscr{R}_n) = \mu(\partial_b c_b|\mathscr{R}_n)$ by assumption. Indeed, the first partial derivatives of V_B satisfy a uniform bound of type $a+b\sum_{k\in S}|\omega_k|$, while c_b, $\operatorname{card} S_k$, $\operatorname{card} S_k^+$ are also uniformly bounded, thus (2.15) follows by a direct calculation.

Remark. The assumption of Lemma 2 concerning the dependence of c_b on the configuration will also be assumed in the rest of the paper. A natural version is $c_b(\omega) = c_k(\omega_k)$ if $b_j = 0$ for $j \neq k$; then c_b is a constant unless b is sitting on a single site of $\mathscr{S}$. This condition seems to be a very technical one, but I do not see any way to remove it. $\square$

The second set of inequalities of Lemma 2 can be solved by means of a sophisticated weight function of [4], the following lemma covers also the case of anharmonic systems, see (1.32).

Lemma 3. *Suppose that* $u_{n+1}(t) \geqq u_n(t) \geqq 0$ *and* $v_{n+1}(t) \geqq v_n(t) \geqq 0$, $t \geqq 0$, $n = 0, 1, \dots$ *satisfy* $u_n(0) \leqq C \operatorname{card} \Lambda_{2n}$, *and*

$$du_n/dt + \alpha v_n \leqq a(u_{n+1} - u_{n-1}) + \alpha[K(u_{n+1} - u_{n-1})(v_{n+1} - v_n)]^{1/2},$$

where $\alpha > 0$, $a \geqq 0$, $K > 0$, $u_{-1} = 0$. *Then we have a constant M depending only on K such that for all* $t \geqq 0$ *and* $n \geqq 1$,

$$u_n(t) + \alpha \int_0^t v_n(s)ds \leqq MC \sum_{m=0}^{\infty} \frac{1}{r} \exp\left(-\frac{m}{r}\right) \operatorname{card} \Lambda_{2m},$$

where $r = aMt + [n^2 + \alpha t]^{1/2}$. $\square$

Proof. Let $\theta_n(r) = \int \theta(x/r)\theta(n-x)dx$ for $r > 0$, where $\theta : \mathbb{R} \to (0, 1]$ is defined by $\theta(0) = 1$ and

$$-\frac{\theta'(x)}{\theta(x)} = \begin{cases} 0 & \text{if } |x| \leqq 1 \\ \operatorname{sign} x & \text{if } |x| \geqq 3 \\ \frac{1}{2}(x\text{-}\operatorname{sign} x) & \text{if } 1 \leqq |x| \leqq 3, \end{cases}$$

and consider

$$u(t,r) = \sum_{n=0}^{\infty} (\theta_n - \theta_{n+1})u_n(t), \qquad v(t,r) = \sum_{n=0}^{\infty} (\theta_n - \theta_{n+1})v_n(t).$$

We may assume that $r \geqq 1$, then $\theta_n \leqq e^{1/r}\theta_{n+1}$ implies $\theta_n - \theta_{n+1} \leqq (2/r)\theta_{n+1}$ and $\theta_n - \theta_{n+1} \leqq 2\min[\theta'_n, \theta'_{n+1}]$ by an easy calculation, see [4]. On the other hand,

$$\sum_{n=0}^{\infty} (\theta_n - \theta_{n+1})v_n = \theta_0 v_0 + \sum_{n=0}^{\infty} \theta_{n+1}(v_{n+1} - v_n),$$

whence by means of $xy - y^2/2 \leqq x^2/2$ we obtain that

$$\frac{\partial u}{\partial t} + \frac{\alpha}{2} v \leqq \left(4a + \frac{3\alpha K}{r}\right)\frac{\partial u}{\partial r},$$

which can be solved by the method of characteristics. Let $r(t) = n$ and $dr/ds + 4a + 3\alpha K/r = 0$ for $0 \leqq s \leqq t$, then

$$u(t,n) + \frac{\alpha}{2}\int_0^t v(s, r(s))ds \leqq u(0, r(0)),$$

and $dr^2/dt+8ar+6\alpha K=0$, consequently

$$r^2(0)\leqq n^2+6\alpha Kt+8a\int_0^t r(s)ds\leqq n^2+6\alpha Kt+8atr(0),$$

that is $r(0)\leqq 8at+[n^2+6\alpha Kt]^{1/2}$. Since $e^{-n/r}\leqq\theta_n(r)\leqq M'e^{-n/r}$, the statement follows from the initial condition. $\square$

Now we are in a position to prove the main tool of this paper.

Proposition 1. *Let $\mu_t=\mu\mathscr{P}^t$ and $F_n(\mu)\leqq C\,\mathrm{card}\,\Lambda_{2n}$ for all n, then we have a constant M depending only on $\mathscr{S}$ and V such that*

$$F_n(\mu_t)+\int_0^t D_n(\mu_s)ds\leqq C\sum_{m=0}^{\infty}\frac{M}{r}\exp\left(-\frac{m}{r}\right)\mathrm{card}\,\Lambda_{2m}$$

for $n\geqq 1$ and $t\geqq 0$, where $r=M[n^2+t]^{1/2}$. $\square$

Proof. Let $u_n(t)=F_n(\mu_t)+K\,\mathrm{card}\,\Lambda_{2n}$, $v_n(t)=D_n(\mu_t)$, and suppose first that $c_b=0$ outside of a finite set of $\mathscr{S}^+$, and $\mu\ll\lambda$ with a nice density such that Lemma 2 is applicable, then Lemma 3 implies the statement immediately. Since the infinite system can be approximated by such a partial dynamics, Proposition 1 extends also to this case by lower semicontinuity of F_n and D_n. To complete the proof, we have to find an approximation μ^δ to the initial state μ in such a way that Lemma 2 applies to each $\mu^\delta\mathbb{P}^t$, $F_m(\mu^\delta)\leqq(C+1)\,\mathrm{card}\,\Lambda_{2m}$, and $\lim_{\delta\to 0}F_m(\mu^\delta)=F_m(\mu)$ for each m, which is not difficult. Indeed, let g_m^δ denote the joint density of some independent normal variables ω_k, $k\in\Lambda_{2m}$ of mean zero and variance δ, and define μ^δ by $\mu^\delta=g_\infty^\delta*\mu$, i.e.

$$\int\varphi d\mu^\delta=\iint\varphi(\omega)g_m^\delta(\omega-\bar\omega)\mu(d\bar\omega)\quad\text{if}\quad\varphi\in\mathscr{R}_m.$$

Let $f_m^\delta=g_m^\delta*f_m$ and $q_m^\delta=g_m^\delta*q_m$, where f_m is the Lebesgue density of μ on $\mathscr{R}_m$, and $g*f$ denotes the convolution of g and f. Notice that f_m^δ is just the Lebesque density of μ^δ on $\mathscr{R}_m$, thus by Jensen's inequality we obtain that

$$F_m(\mu)\geqq\int f_m^\delta\log\frac{f_m^\delta}{q_m^\delta}d\omega=F_m(\mu_m^\delta)+\int f_m^\delta\log\frac{q_m}{q_m^\delta}d\omega.$$

The last term here can be estimated by means of the quadratic upper bound of H_m, thus $\mu^\delta\to\mu$ in the above sense. $\square$

The free energy argument yields the following technical result on the stationary states of reversible diffusion processes.

Proposition 2. *Suppose that $\mu=\mu\mathbb{P}^t$, then any of the following two sets of conditions implies $D_n(\mu)=0$ for each n,*
(i) $\mathrm{card}\,\Lambda_{2n}=o(n^2)$ *and* $F_n(\mu)=o(n^2)$,
(ii) $\mathrm{card}\,\Lambda_n^0=\mathcal{O}(n)$, $F_0(\mu)<+\infty$, *and* $F_{n+1}(\mu)-F_n(\mu)=\mathcal{O}(n)$. $\square$

Proof. We need both inequalities of Lemma 2. In view of the argument outlined in Sect. 1, the only problem is that of the smoothness of μ. In the frames of Lemma 2 from $D_{n+1}=D_n+D_{n+1}-D_n$ we obtain by a direct calculation that

$$D_{n+1}^t\leqq|F_n(\mu_0)-F_n(\mu_t)|_++[C_n^t(D_{n+1}^t-D_n^t)]^{1/2},\tag{2.17}$$

where the following abbreviations are used: $|u|_+ = \max[0,u]$,

$$D_n^t = \int_0^t D_n(\mu_s)ds,$$

$$C_n^t = F_{n+1}(\mu_0) + K't \operatorname{card} \Lambda_n^\partial \cup \Lambda_{n+1}^\partial + K' \int_0^t [R_n(\mu_s) + R_{n+1}(\mu_s)]ds.$$

There are two ways to derive a priori bounds for D. Suppose that $D_m^t > 0$, in the case of (ii) we follow [15] and derive

$$\sum_{k=m}^{n-1} (C_k^t)^{-1} \leqq 2(D_m^t)^{-2} \sum_{k=m}^{n-1} |F_k(\mu_0) - F_k(\mu_t)|_+^2 (C_k^t)^{-1} + 2(D_m^t)^{-1}, \tag{2.18}$$

in the case of (i) we use the Schwarz inequality to conclude

$$n - m \leqq (D_m^t)^{-1} \sum_{k=m}^{n-1} |F_k(\mu_0) - F_k(\mu_t)|_+ + \left[\sum_{k=m}^{n-1} C_k^t\right]^{1/2} (D_m^t)^{-1/2}. \tag{2.19}$$

Now we are in a position to exploit the lower semi-continuity of F and D. Let $\mu^\delta \to \mu$ as in the previous proof, then $F_k(\mu^\delta) \to F_k(\mu)$ and $F_k(\mu_t) = F_k(\mu) \leqq \liminf F_k(\mu_t^\delta)$, thus the first sums on the right-hand sides of (2.18) and (2.19) vanish in both cases. On the other hand, the dynamics depend in a continuous way on initial data, thus C_k^t also converge to their values corresponding to the stationary state, consequently both (i) and (ii) yield bounds for C_k^t. This means that the proof can be completed by means of our elementary calculations given after Theorem 1. □

If the entropy is not locally finite then a regularization trick of [3] is still available; the conditions $\operatorname{card} \Lambda_n^\partial = O(n)$ seem to be sufficient to conclude that every stationary state satisfies $D_n(\mu) = 0$ for each n, which implies that μ is a reversible measure of any other evolution specified by such coefficients $\bar{c}_b$ that $\bar{c}_b \neq 0$ implies $c_b \neq 0$. Further consequences depend on the structure of $\mathcal{S}^+$, if $\mathcal{S} \neq \mathcal{S}^+$ then reversibility may be a much weaker property of μ than $D_n(\mu) = 0$ for each n. The variational characterization (2.6) of the Donsker-Varadhan rate function is not really convenient for concrete calculations, we prefer its following consequence, see [3].

Lemma 4. *Let $f \in \mathbb{C}_0^2(\Omega)$ be bounded, $b \in \Lambda_n^+$ and $c_b \geqq 1$ for each $\omega \in \Omega$. If μ is a Borel probability, and $F_n(\mu) < +\infty$, then*

$$\int f \partial_b H d\mu \leqq \int \partial_b f d\mu + \left[\int f^2 d\mu\right]^{1/2} [D_n^b(\mu)]^{1/2}. \quad \square$$

Proof. Our condition implies that all expectations are finite, and μ has a density p_n with respect to Q_n on $\mathcal{R}_n$. If p_n is smooth then we can integrate by parts, thus

$$\int f \partial_b H d\mu = \int f(\partial_b H) p_n dQ_n = \int \partial_b f d\mu + \int f \partial_b p_n dQ_n,$$

which implies the statement by the Schwarz inequality. The proof of the general case is based on Sect. 4 of [1], we may, and do assume that $c_b = 1$ and $D_n^b(\mu) < +\infty$. Observe first that

$$\int f \mathbb{G}_b h dQ_n = -\tfrac{1}{2} \int (\partial_b f) \partial_b h dQ_n,$$

thus $(-\mathbb{G}_b)^{1/2}$ is the closure of ∂_b in $\mathbb{L}^2(Q_n)$. The corresponding stochastic equations are obtained by letting $c_{b'} = 0$ in (2.3) whenever $b' \neq b$, thus we see that $\mathbb{G}_b$ generates a strongly continuous semigroup of self-adjoint contractions $\mathbb{P}_b^t$ in

$\mathbb{L}^2(Q_n)$. Although $\mathbb{P}_b^t$ does not admit a transition density in general, following the proof of Theorem 5 in [1] we obtain that

$$\int f(\omega)[f(\omega)-\mathbb{P}_b^t f(\omega)]Q_n(d\omega) \leqq tD_n^b(\mu),$$

where $f=\sqrt{p_n}$, consequently f belongs to the domain of $(-\mathbb{G}_b)^{1/2}$, and the $\mathbb{L}^2$-norm of $(-\mathbb{G}_b)^{1/2}f$ is not greater than $[D_n^b(\mu)]^{1/2}$, which completes the proof. □

The following consequence of Lemma 4 results in a simplified proof of the famous two-blocks estimate of [13]. We are going to show that the gradient of energy varies slowly in space if the production of free energy is small. Let $\partial_k H = \partial H/\partial\omega_k$ and $f(\omega)=\sum_{k\in\mathcal{S}} \varphi_k \partial_k H$, where $\varphi:\mathcal{S}\to\mathbb{R}$ vanishes outside of a finite set. We can estimate $\int f^2 d\mu$ by means of Lemma 4 as soon as we find some $g:\mathcal{S}^+\to\mathbb{R}$ such that $\varphi_k=\mathcal{B}_k g$, where $\mathcal{B}_k g=\sum_{b\in\mathcal{S}^+} b_k g_b$; then we have an identity

$$f(\omega)=\sum_{k\in\mathcal{S}} \varphi_k \partial_k H(\omega)=\sum_{b\in\mathcal{S}^+} g_b \partial_b H(\omega), \qquad \varphi_k=\mathcal{B}_k g. \tag{2.20}$$

Let us remark that if $\mathcal{S}^+$ is the set of positively oriented bonds, then bond variables like g play the role of vector fields, thus $\mathcal{B}$ can be interpreted as a discrete version of div.

Lemma 5. *Let f,φ,g as in (2.20) and suppose that $g_b=0$ if $b\notin\Lambda_n^+$, $c_b\geqq 1$ if $b\in\Lambda_n^+$. We have some K depending only on $\mathcal{S}$ and V in such a way that $F_n(\mu)<+\infty$ implies*

$$\int f^2 d\mu \leqq K\sum_{k\in\mathcal{S}} \varphi_k^2 + 2\Big[\sum_{b\in\mathcal{S}^+} |g_b|\,[D_n^b(\mu)]^{1/2}\Big]^2. \ \square$$

Proof. From Lemma 4 we get

$$\int f^2 d\mu = \sum_{b\in\mathcal{S}^+} \int f g_b \partial_b H d\mu \leqq \sum_{b\in\mathcal{S}^+} \int g_b \partial_b f d\mu + [\textstyle\int f^2 d\mu]^{1/2} \sum_{b\in\mathcal{S}^+} |g_b|\,[D_n^b(\mu)]^{1/2}.$$

Observe now that $\sum g_b\partial_b f=\sum\sum \varphi_k\varphi_j\partial_k\partial_j H$ in view of (2.20), thus the first sum on the right-hand side can be estimated by a multiple of $\sum\varphi_k^2$. On the other hand, $X^2\leqq a+bX$ implies $X^2\leqq 2a+2b^2$, which completes the proof. □

There are several ways to use this lemma. For example, choosing $\varphi_k=1$ or -1 on two different domains, while $\varphi_k=0$ otherwise, we obtain a bound for the mean square deviation of the corresponding averages of $\partial_k H$. In view of the one-block estimate, this yields an asymptotic bound of the same kind also for the block spins. Since Proposition 1 controls only macroscopic space-time averages of $\omega_k^2(t)$, it will be very useful to compare microscopic and macroscopic averages in this way allowing us to investigate local equilibrium in fixed microscopic domains.

3. Anharmonic Systems in a Thermal Noise

Here we investigate the flow of free energy in systems like (1.26), the following generalization is immediate. Let $\mathcal{S}$ and $\mathcal{S}^+$ be as in the previous section, configurations are couples (p,r) such that $p=(p_k)_{k\in\mathcal{S}}\in\Omega$, $r=(r_b)_{b\in\mathcal{S}^+}\in\Omega^+$, where

$$\Omega^+=\Big[r\in\mathbb{R}^{\mathcal{S}^+}: \lim_{|b|\to\infty} |r_b|e^{-\delta|b|}=0 \text{ for } \delta>0\Big], \tag{3.1}$$

and $|b|$ denotes the smallest value of $|k|$, $k \in \mathscr{S}$ such that $b_k \neq 0$. Let S_b^+ denote the set of $b' \in \mathscr{S}^+$ such that $b_k b'_k \neq 0$ for some $k \in \mathscr{S}$, then $\sup_{b \in \mathscr{S}^+} \operatorname{card} S_b^+ < +\infty$, thus the corresponding graph structure of $\mathscr{S}^+$ is essentially the same as that of $\mathscr{S}$. Assume also that $b_k = b_j$ for all $b \in \mathscr{S}^+$ implies $j = k$, then $\mathscr{S}$ can be identified with a subset of Ω^+, thus the relation of $\mathscr{S}$ and $\mathscr{S}^+$ is symmetric. We consider a Hamiltonian of type $H = H(p, r)$,

$$H(p,r) = \sum_{k \in \mathscr{S}} \tfrac{1}{2} p_k^2 + \sum_{B \subset \mathscr{S}^+} V_B(r), \tag{3.2}$$

where $V_B \in \mathbb{C}_0^2(\Omega^+)$ depends on r_b only if $b \in B$; regularity and stability properties of V are the same as in Sect. 2 with $\mathscr{S}$ in the place of the present $\mathscr{S}^+$. In the original setup we had p_k and q_k as the canonical coordinates, and the potential energy happened to be a function of some new variables $r_b = \sum b_k q_k$. Let us introduce the corresponding differential operators: $\partial_k = \partial/\partial p_k$, $\partial_b^+ = \partial/\partial r_b$, $\partial_b = \sum b_k \partial_k$, $\partial_k^+ = \sum b_k \partial_b^+$, then $\partial H/\partial q_k = \partial_k^+ H$, thus the stochastic equations of motion should be as

$$\begin{aligned} dp_k &= -\partial_k^+ H dt, \\ dr_b &= \partial_b H dt - \frac{\alpha}{2} \sum_{k \in \mathscr{S}} b_k \partial_k^+ H dt + \sqrt{\alpha} \sum_{k \in \mathscr{S}} b_k dw_k, \qquad k \in \mathscr{S},\, b \in \mathscr{S}^+, \end{aligned} \tag{3.3}$$

where $\alpha > 0$ and w_k, $k \in \mathscr{S}$ is an independent family of standard Wiener processes. Observe that (1.26) is obtained as a particular case of (3.3) if $b = (k \to k+1)$, $b_i = 1$ for $i = k+1$, $b_i = -1$ for $i = k$, $b_i = 0$ otherwise, while k runs over $\mathscr{S} = \mathscr{Z}$. A similar construction is possible even if $\mathscr{S} = \mathscr{Z}^d$ with $d > 1$, and $\mathscr{S}^+$ is the set of positively oriented bonds. However, if we insist on the original interpretation, then $\sum r_b = 0$ should be postulated for any circle in $\mathscr{Z}^d$, and the constrained equilibrium states are not really understood.

There is no additional difficulty concerning the Markov semigroup, $\mathbb{P}^t$ defined by (3.3), it is strongly continuous in $\mathbb{C}_b(\Omega \times \Omega^+)$, and $\mathbb{C}_0^2(\Omega \times \Omega^+)$ is a core of its generator, $\mathbb{G}$. From the Ito formula we obtain that $\mathbb{G} = \mathbb{L} + \alpha \mathbb{G}^+$, $\mathbb{L} = \sum_{k \in \mathscr{S}} \mathbb{L}_k$, $\mathbb{G}^+ = \sum_{k \in \mathscr{S}} \mathbb{G}_k^+$, and

$$\mathbb{L}_k \varphi = (\partial_k H)\partial_k^+ \varphi - (\partial_k^+ H)\partial_k \varphi, \qquad \mathbb{G}_k^+ \varphi = \tfrac{1}{2}[(\partial_k^+)^2 \varphi - (\partial_k^+ H)\partial_k^+ \varphi] \tag{3.4}$$

for $\varphi \in \mathbb{C}_0^2(\Omega \times \Omega^+)$. Therefore a direct calculation shows that every Gibbs state with energy H and temperature one is a stationary state of $\mathbb{P}^t$, and it is even reversible with respect to the evolution generated by any of $\mathbb{G}_k^+$, $k \in \mathscr{S}$.

In order to define the local free energy and its rate of production, let $\mathscr{R}_n$ denote the σ-field of $\Omega \times \Omega^+$ generated by the variables p_k, $k \in \Lambda_{2n-1}$ and r_b, $b \in \Lambda_{2n}^+$, where Λ_{2n}^+ is the set of $b \in \mathscr{S}^+$ such that $b_k \neq 0$ for some $k \in \Lambda_{2n}$. A finite reference measure Q_n is defined on each $\mathscr{R}_n$ by its Lebesque density $g_n = e^{-H_n}$,

$$H_n(p,r) = \sum_{k \in \Lambda_{2n-1}} \tfrac{1}{2} p_k^2 + \sum_{B \subset \Lambda_{2n}^+} V_B(r). \tag{3.5}$$

If μ is a Borel probability on $\Omega \times \Omega^+$ then μ_n denotes its restriction to $\mathscr{R}_n$, and $F_n(\mu) = \int \log f_n d\mu$ if $d\mu_n = f_n dQ_n$, and $F_n(\mu) = +\infty$ otherwise. A formal differentiation results in

$$\int \mathbb{G} \log f_n d\mu + \alpha D_n(\mu) = \alpha B_n^+(\mu) + B_n(\mu), \tag{3.6}$$

$$D_n^k(\mu) = 4 \sup \left[-\int \frac{1}{\varphi} \mathbb{G}_k^+ \varphi d\mu : 0 < \varphi \in \mathbb{C}_0^2(\Omega \times \Omega^+) \cap \mathscr{R}_n \right], \tag{3.7}$$

$$D_n(\mu) = \sum_{k \in \Lambda_{2n-1}} D_n^k(\mu), \qquad B_n(\mu) = \int \mathbb{L} \log f_n d\mu, \tag{3.8}$$

$$B_n^+(\mu) = \frac{1}{2} \sum_{k \notin \Lambda_{2n-1}} \int \partial_k^+ f_{n+1} \frac{\partial_k^+ f_n}{f_n} dQ_{n+1}. \tag{3.9}$$

Of course, if f_{n+1} is smooth enough, and $k \in \Lambda_{2n-1}$, then

$$D_n^k(\mu) = \frac{1}{2} \int \frac{1}{f_n} (\partial_k^+ f_n)^2 dQ_n, \tag{3.10}$$

while, again by integrating by parts we obtain that

$$B_n(\mu) = \sum_{k \in \mathscr{S}} \int \frac{1}{f_n} [(\partial_k f_{n+1}) \partial_k^+ f_n - (\partial_k^+ f_{n+1}) \partial_k f_n] dQ_{n+1}. \tag{3.11}$$

We have two further identities, for each k and n

$$\mu[\partial_k f_{n+1}/f_{n+1} | \mathscr{R}_n] = \partial_k f_n / f_n + \mu[\partial_k H_{n+1} - \partial_k H_n | \mathscr{R}_n], \tag{3.12}$$

$$\mu[\partial_k^+ f_{n+1}/f_{n+1} | \mathscr{R}_n] = \partial_k^+ f_n / f_n + \mu[\partial_k^+ H_{n+1} - \partial_k^+ H_n | \mathscr{R}_n]. \tag{3.13}$$

Like in Sect. 2, from (3.13) we obtain

$$B_n^+(\mu) \leqq -X_n^+(\mu) + [X_n^+(\mu) R_n^+(\mu)]^{1/2} \leqq \frac{1}{4} R_n^+(\mu), \tag{3.14}$$

$$\begin{aligned} X_n^+(\mu) &= \frac{1}{2} \sum_{k \notin \Lambda_{2n-1}} \int \frac{1}{f_n} (\partial_k^+ f_n)^2 dQ_n, \\ R_n^+(\mu) &= \sum_{k \notin \Lambda_{2n-1}} \int [\mu(\partial_k^+ H_{n+1} - \partial_k^+ H_n | \mathscr{R}_n)]^2 d\mu \\ &\quad + \sum_{k \notin \Lambda_{2n-1}} \int [\mu(\partial_k H_{n+1} - \partial_k H_n | \mathscr{R}_n)]^2 d\mu, \end{aligned} \tag{3.15}$$

while from (3.9) by monotonicity of D_n and by the Schwarz inequality

$$B_n^+(\mu) \leqq [X_n^+(\mu)(D_{n+1}(\mu) - D_n(\mu))]^{1/2}. \tag{3.16}$$

The Hamiltonian contribution, $B_n(\mu)$ can be treated in a similar manner, but its structure is different. From (3.12) and (3.13) we see that all terms with $k \in \Lambda_{2n-1}$ vanish by asymmetry, while $\partial_k f_n = 0$ otherwise, thus by the Schwarz inequality

$$B_n(\mu) \leqq [X_n^+(\mu) R_n^+(\mu)]^{1/2}. \tag{3.17}$$

Comparing the estimates above, we obtain

Proposition 3. *There exists some constant M depending only on S, S^+, and V such that $\alpha \geqq 1$ and $F_n(\mu) \leqq C \operatorname{card} \Lambda_{2n}$ imply*

$$F_n(\mu_t) + \alpha \int_0^t D_n(\mu_s) ds \leqq C \sum_{m=0}^{\infty} \frac{M}{R} \exp\left(-\frac{m}{R}\right) \operatorname{card} \Lambda_{2m}$$

for all $n \geqq 1$ *and* $t \geqq 0$, *where* $R = Mt + M(n^2 + \alpha t)^{1/2}$, *and* $\mu_t = \mu \mathbb{P}^t$ *is the time evolved measure via* (3.3). □

Proof. From (3.17) and (3.14) we have

$$B_n(\mu) + \alpha B_n^+(\mu) \leqq -\alpha X_n^+(\mu) + (1+\alpha)\left[X_n^+(\mu) R_n^+(\mu)\right]^{1/2};$$

$$B_n(\mu) + \alpha B_n^+(\mu) \leqq \left[X_n^+(\mu) R_n^+(\mu)\right]^{1/2} + \alpha\left[X_n^+(\mu)\left(D_{n+1}(\mu) - D_n(\mu)\right)\right]^{1/2}$$

follows from (3.16) and (3.17). If the common left-hand side exceeds $[X_n^+ R_n^+]^{1/2}$, then $X_n^+(\mu) \leqq R_n^+(\mu)$, consequently

$$B_n(\mu) + \alpha B_n^+(\mu) \leqq R_n^+(\mu) + \alpha\left[R_n^+(\mu)\left(D_{n+1}(\mu) - D_n(\mu)\right)\right]^{1/2}, \tag{3.18}$$

while the first inequality yields simply R_n^+ as an upper bound in the opposite case, which proves (3.18). Now we are in a position to estimate R_n^+ by means of the increment of F_n, cf. Lemma 1, thus we obtain in the smooth case that

$$R_n^+(\mu) \leqq K\left[K \operatorname{card} \Lambda_{2n+2} + F_{n+1}(\mu) - K \operatorname{card} \Lambda_{2n-2} - F_{n-1}(\mu)\right],$$

where K is a universal constant, thus the final inequality follows by Lemma 3. The treatment of the general case is the very same as in the proof of Proposition 1. □

Let us remark that if α is small, then we obtain R_n^+/α as the first term on the right-hand side of (3.18), thus R would grow as fast as $n + t/\alpha$. This means that α must be bounded away from zero, but an upper bound is not needed. Our inequalities are sufficient to derive such statements that every stationary measure is a (canonical) Gibbs state in translation invariant situations only. We are not going to discuss this problem here, see [5] for a particular case.

4. Passage to the Hydrodynamic Limit

We follow the argument of Guo-Papanicolau-Varadhan [13]. In view of Proposition 1, the most crucial steps of the proofs remain unchanged, some of them are even simplified a little bit. Since we do not have effective a priori bounds for the local free energy of the process in space and time, we can not extend Proposition 6.1 of [13] to infinite systems, thus some modifications of the original proof can not be avoided. We prove first Theorem 2.

We start with a proof of the tightness of the family of time averages of the evolved measures with respect to the relative product topology of $\mathbb{R}^{\mathscr{X}}$. In view of Proposition 1, the time average of $F_n(\mu_t)$ remains bounded by a multiple of n only if $t = \mathcal{O}(n^2)$, thus the following averages will be frequently used in this last section. If v is a real sequence indexed by $\mathscr{X}$, then

$$\Sigma_n(k, v) = \frac{1}{2n+1} \sum_{j=k-n}^{k+n} v_j, \qquad \bar{\mu}_t = \frac{1}{t} \int_0^t \mu \mathbb{P}^s ds, \tag{4.1}$$

and the set of Borel probabilities μ on $\mathbb{R}^{\mathscr{X}}$ such that $F_n(\mu) \leqq Cn$ for $n > 0$ will be denoted by $P(C)$. If $\mu \in P(C)$ then Proposition 1 yields the following a priori bounds:

$$D_n(\bar{\mu}_t) \leqq M \frac{C}{t}(n^2 + t)^{1/2}, \qquad F_n(\bar{\mu}_t) \leqq M \frac{C}{\sqrt{t}}(n^2 + t) \tag{4.2}$$

for all $n \geqq 1$ and $t > 0$, where $C \geqq 1$ by assumption, and M is a constant depending only on V. The variational formula (2.5) implies that if $\mu \in P(C)$ and $C \geqq 1$, then

$$\int \Sigma_n(0, \omega^2) \bar{\mu}_t(d\omega) \leqq MC\left(\frac{n}{\sqrt{t}} + \frac{\sqrt{t}}{n}\right), \quad t > 0,\ n \geqq 1, \tag{4.3}$$

where $\omega^2 = (\omega_k^2)_{k \in \mathscr{Z}}$, and M depends only on V. Since the a priori bound of D_n vanishes even in macroscopic domains as t goes to infinity, Lemma 5 yields a microscopic a priori bound.

Lemma 6. *If $\mu \in P(C)$, $C > 1$, then $\int \omega_k^2 d\bar{\mu}_t \leqq KC(1 + |k|)$ with the same constant for all μ, $t > 0$, and $k \in \mathscr{Z}$.* □

Proof. Let us apply Lemma 5 with $\varphi : \mathscr{Z} \to \mathbb{R}$ such that $\varphi_0 = 1 - (1 + 2n)^{-1}$, $\varphi_k = -(1 + 2n)^{-1}$ if $|k| \leqq n$ but $k \neq 0$, otherwise $\varphi_k = 0$. Remember that if $b = (k, k+1)$ then $\partial_b = \partial_{k+1} - \partial_k$, thus g_b of Lemma 5 is just the sum of φ_j for $j \leqq k$, consequently $\sum g_b^2 \leqq 1 + 2n$, while $\sum \varphi_k^2 \leqq 2$, whence by (4.2)

$$\int [V'(\omega_0) - \Sigma_n(0, V')]^2 d\bar{\mu}_t \leqq K_1 \left[1 + \frac{n}{t}(n^2 + t)^{1/2}\right],$$

where $V' = (V'(\omega_k))_{k \in \mathscr{Z}}$. On the other hand, from (4.3)

$$\int [\Sigma_n(0, V')]^2 d\bar{\mu}_t \leqq \int \Sigma_n(0, V'^2) d\bar{\mu}_t \leqq MC\left(\frac{n}{\sqrt{t}} + \frac{\sqrt{t}}{n}\right),$$

where n is still free. Choose n as the integer part of $\sqrt{t}$, and compare the inequalities above; the case of $k = 0$ follows by an easy calculation as $|V'(x)| \geqq a_1|x| - a_2$ with some $a_1 > 0$. If $k \neq 0$ then it is enough to notice that the free energy of μ in a box $[k - n, k + n]$ is bounded by $C(n + |k|) \leqq Cn(1 + |k|)$ if $n \geqq 1$, which completes the proof by repeating the same argument. □

Equal blocks can also be compared by means of Lemma 5, we get:

Lemma 7. *Let $\mu \in P(C)$, $C > 1$, $N \geqq M \geqq 2L + 1 > 1$, then*

$$\sum_{k=-N}^{N} \int [\Sigma_L(k, V') - \Sigma_L(k + M, V')]^2 d\bar{\mu}_t \leqq K\left[\frac{N}{L} + \frac{C}{t}(M + L)^2 (N^2 + t)^{1/2}\right],$$

where K does not depend on C, N, L, μ, and t. □

Proof. We define a function φ for each $k : \varphi_j^{(k)} = (1 + 2L)^{-1}$ if $|j - k - M| \leqq L$, $\varphi_j^{(k)} = -(1 + 2L)^{-1}$ if $|j - k| \leqq L$, and it is zero otherwise, thus $\sum_{i \in \mathscr{Z}} [\varphi_j^{(k)}]^2 = (1 + 2L)^{-1}$. The associated $g^{(k)}$ vanishes outside of $[k - L, k + M + L]$, and $\sum_{b \in \mathscr{Z}^+} [g_b^{(k)}]^2 \leqq 1 + M + 2L$ for each k. Thus applying Lemma 5 to each term of the left-hand side, and counting the frequency of each D_{2N}^b on the right-hand side of the inequality, we obtain the statement. □

In view of Lemma 6 the family $[\bar{\mu}_t : \mu \in P(C), t > 0]$ is tight in $\mathbb{R}^{\mathscr{Z}}$, and (4.2) implies $D_n(\bar{\mu}) = 0$ for each n and for any limit point $\bar{\mu}$ of a subsequence along which $t \to \infty$, consequently $\bar{\mu}$ is a canonical Gibbs state. Moreover, from the results of Sects. 3 and 6 of [13] we also know that Lemma 6 and (4.2) imply

$$\lim_{L \to \infty} \limsup_{t \to \infty} \sup_{\mu \in P(C)} \int |\Sigma_L(k, V') - J'(\Sigma_L(k, \omega))| \bar{\mu}_t(d\omega) = 0 \tag{4.4}$$

for each $k \in \mathscr{Z}$; a DeFinetti type extremal decomposition theorem can also be used to derive such a statement, see [11, 12]. Observe now that

$$|J'(\Sigma_L(k,\omega)) - J'(\Sigma_L(k+M,\omega))| \leqq |J'(\Sigma_L(k,\omega)) - \Sigma_L(k,V')| \\ + |J'(\Sigma_L(k+M,\omega)) - \Sigma_L(k+M,V')| + |\Sigma_L(k,V') - \Sigma_L(k+M,V')|, \quad (4.5)$$

thus combining (4.4), the one-block estimate, and Lemma 7, we obtain the celebrated two-block estimate of [13].

Proposition 4. *Let $\mu \in P(C)$, $C>1$, $0<\delta<1$, $N>\sqrt{t}$, and $\delta\sqrt{t} \geqq M \geqq 2L+1>1$, then*

$$\frac{1}{N}\sum_{k=-N}^{N} \int |\Sigma_L(k,\omega) - \Sigma_L(k+M,\omega)|\bar{\mu}_t(d\omega) \leqq M\left[\frac{C}{L} + C\delta^2\right]^{1/2} + R_L(t,C),$$

where M and R depend only on V, and $\lim_{L\to\infty} \limsup_{t\to\infty} R_L(t,C) = 0$. □

Proof. The first term on the right-hand side comes from Lemma 7, the second one is the contribution of (4.4). Since $N \geqq \sqrt{t}$, we can use (4.3) to show uniformity of the convergence of R_L. In the present formulation it is relevant that J is strictly convex, cf. [18]. □

Now we are in a position to turn to the macroscopic picture, the configurations of the system will be interpreted as step functions on $\mathbb{R}$. Let $\varepsilon>0$ denote the scaling parameter, $\mu^\varepsilon \in P(C)$ is the initial distribution, and $\mathfrak{P}_\varepsilon$ is the law of the rescaled process $\omega^\varepsilon = \omega_t^\varepsilon(x) = \omega_{[x/\varepsilon]}(t/\varepsilon^2)$. We consider $\mathfrak{P}_\varepsilon$ as a Borel probability on a trajectory space $\mathbb{C}_w([0,\infty), \mathbb{H}^{-2})$ defined as follows: If $\varphi: \mathbb{R} \to \mathbb{R}$ is locally integrable then $\|\varphi\|$ is defined by $\|\varphi\|^2 = \int \varphi^2(x)\exp(|x|)dx$, $\|\varphi\|_1^2 = \|\varphi\|^2 + \|\varphi'\|^2$, $\|\varphi\|_2^2 = \|\varphi\|^2 + \|\varphi'\|_1^2$, and $\mathbb{H}$, $\mathbb{H}^1$, $\mathbb{H}^2$ denote the associated Hilbert spaces. Their dual spaces with respect to the usual scalar product $\langle\varphi,\sigma\rangle = \int \varphi(x)\sigma(x)dx$ are $\mathbb{H}^*$, $\mathbb{H}^{-1}$, $\mathbb{H}^{-2}$, respectively; $\|\sigma\|_*^2 = \int \sigma^2(x)\exp(-|x|)dx$ and

$$\|\sigma\|_i^* = \sup[\langle\varphi,\sigma\rangle : \|\varphi\|_i \leqq 1], \quad i=1,2 \quad (4.6)$$

define the corresponding dual norms. The original configuration space Ω is embedded into $\mathbb{H}^*$ for each $\varepsilon>0$ in a natural way. From the Ito formula we obtain immediately that if $\varphi \in \mathbb{H}^2$ then

$$\langle\varphi,\omega_t^\varepsilon\rangle = \langle\varphi,\omega_0^\varepsilon\rangle + \tfrac{1}{2}\int_0^t \langle\varphi'', V'(\omega_s^\varepsilon)\rangle ds + M_\varepsilon(t,\varphi), \quad (4.7)$$

where $V'(\omega_s^\varepsilon) = V'(\omega_s^\varepsilon(x))$, and M_ε is the sum of the martingale part and a remainder coming from the difference of φ'' and its lattice approximation. By means of (4.3) it follows easily that

$$\int \sup_{t\leqq T}[M_\varepsilon(t,\varphi)]^2 d\mathfrak{P}_\varepsilon \leqq KCT\varepsilon\|\varphi\|_2^2, \quad (4.8)$$

where K depends only on V, and the very same argument yields

$$\int_0^T \int \|\omega_t^\varepsilon\|_*^2 d\mathfrak{P}_\varepsilon dt \leqq KCT, \quad (4.9)$$

$$\int \sup_{t\leqq T}[\|\omega_t^\varepsilon\|_2^*]^2 d\mathfrak{P}_\varepsilon \leqq KCT, \quad (4.10)$$

$$\lim_{\delta\to 0}\sup_{\varepsilon>0} \mathfrak{P}_\varepsilon\left[\sup_{t<T}\sup_{0<s<\delta} |\langle\varphi,\omega_{t+s}^\varepsilon\rangle - \langle\varphi,\omega_t^\varepsilon\rangle| > a\right] = 0 \quad (4.11)$$

for all $T>0$, $a>0$, and $\varphi \in \mathbb{H}^2$, provided that K is large enough.

Consider now $\mathbb{C}_w = \mathbb{C}_w([0, \infty), \mathbb{H}^{-2})$, the space of weakly continuous trajectories in $\mathbb{H}^{-2}$. In view of (4.10), (4.11) and the Arzela-Ascoli theorem the family $\mathfrak{P}_\varepsilon$, $\varepsilon > 0$ is tight on $\mathbb{C}_w$, let $\mathfrak{P}$ denote any of its limit points as $\varepsilon \to 0$. From (4.9)

$$\int_0^T \int \|\sigma_t\|_*^2 \mathfrak{P}(d\sigma) dt \leqq CKT, \tag{4.12}$$

which means that $\mathfrak{P}$ is concentrated on a set of locally square integrable trajectories $\sigma = \sigma_t = \sigma_t(x)$, see Lemma 6.3 of [13]. Therefore we can define a functional I for each $t \geqq 0$, $\varphi \in \mathbb{C}_0^2(\mathbb{R})$ and P-a.e. $\sigma \in \mathbb{C}_w([0, \infty), \mathbb{H}^{-2})$ by

$$I(t, \varphi, \sigma, \mathbb{K}) = \left| \langle \varphi, \sigma_t \rangle - \langle \varphi, \sigma_0 \rangle - \tfrac{1}{2} \int_0^t \langle \varphi, \mathbb{K}(\sigma_s) \rangle ds \right|, \tag{4.13}$$

where $\mathbb{K}: \mathbb{H}^* \to \mathbb{H}^*$ is a uniformly Lipschitz continuous map to be specified later in several different ways. The main part of the proof is to show that $\int I(t, \varphi, \sigma, \mathbb{K}_J) \mathfrak{P}(d\sigma) = 0$ for each t and φ with $\mathbb{K}_J(u) = J'(u(\cdot))$; then the final statement follows by the weak uniqueness of the limiting equation. From (4.7) and (4.8) we know that $\lim_{\varepsilon \to 0} \int I(t, \varphi, \sigma, \mathbb{K}_V) \mathfrak{P}_\varepsilon(d\sigma) = 0$, while

$$\int I(t, \varphi, \sigma, \mathbb{K}_J) \mathfrak{P}(d\sigma) \leqq \liminf_{\delta \to 0} \int I(t, \varphi, \sigma, \mathbb{K}_J^\delta) \mathfrak{P}(d\sigma), \tag{4.14}$$

$$\int I(t, \varphi, \sigma, \mathbb{K}_J^\delta) \leqq \liminf_{\varepsilon \to 0} \int I(t, \varphi, \sigma, \mathbb{K}_J^\delta) \mathfrak{P}_\varepsilon(d\sigma) \tag{4.15}$$

follow by the Fatou lemma and by the lower semi-continuity of I, where $\mathbb{K}_J^\delta(u) = J'(g_\delta * u)$, $g_\delta(x) = g(x/\delta)/\delta$, and $g \in \mathbb{C}_0^2(\mathbb{R})$ is a probability density vanishing outside of the interval $(-1, 1)$.

On the other hand, let

$$\mathbb{K}_{J,L}^{(\varepsilon)}(u) = J'(\Sigma_L([x/\varepsilon], u^\varepsilon)), \quad \bar{\mathbb{K}}_{V,L}^{(\varepsilon)}(u) = \Sigma_L([x/\varepsilon], V'(u^\varepsilon))$$

whenever $u: \mathbb{R} \to \mathbb{R}$ is a step function of step size ε, and $u^\varepsilon = (u(\varepsilon k))_{k \in \mathfrak{Z}}$. The one-block estimate (4.4) and the a priori bound (4.8) imply that

$$\lim_{L \to \infty} \limsup_{\varepsilon \to 0} \int |I(t, \varphi, \sigma, \bar{\mathbb{K}}_{V,L}^{(\varepsilon)}) - I(t, \varphi, \sigma, \mathbb{K}_{J,L}^{(\varepsilon)})| \mathfrak{P}_\varepsilon(d\sigma) = 0,$$

and $\mathbb{K}_{V,L}^{(\varepsilon)}$ is almost the same as $\mathbb{K}_V$ if L is fixed while ε goes to zero, consequently

$$\lim_{L \to \infty} \limsup_{\varepsilon \to 0} \int I(t, \varphi, \sigma, \mathbb{K}_{J,L}^{(\varepsilon)}) P_\varepsilon(d\sigma) = 0. \tag{4.16}$$

The crucial step of the proof is to fill in the gap between the large microscopic average Σ_L, $L \to \infty$ and the small macroscopic average $g_{\delta *}$, $\delta \to 0$. This is exactly the task of the two-block estimate Proposition 4. Indeed, as $g_\delta(x + \varepsilon L) - g_\delta(x) = \mathcal{O}(\varepsilon L / \delta^2)$,

$$\limsup_{\varepsilon \to 0} \int |I(t, \varphi, \sigma, \mathbb{K}_J^\delta) - I(t, \varphi, \sigma, \mathbb{K}_{J,L}^{(\varepsilon)})| \mathfrak{P}_\varepsilon(d\sigma) \leqq CM \left(\delta + \frac{1}{L} \right), \tag{4.17}$$

which completes the proof by weak uniqueness of (1.10). Like in [13], the two-block estimate shows that $\mathfrak{P}$ is concentrated on weakly differentiable trajectories, moreover

$$\int_0^T \int \| \nabla J'(\sigma_t) \|_*^2 \mathfrak{P}(d\sigma) dt \leqq CKT, \tag{4.18}$$

which is a sufficient regularity condition of weak uniqueness, see the proof of the same property of the more general equation (1.30) below.

The proof of Theorem 3 is almost the same as that of Theorem 2, so we discuss only the differences. Proposition 3 yields a fundamental a priori bound: for $\mu \in P(C)$, $C>1$ we have

$$D_n(\bar{\mu}_t) \leqq M\frac{C}{t}(\sqrt{\varepsilon t}+\varepsilon t+\varepsilon n), \qquad F_n(\bar{\mu}_t) \leqq MC\sqrt{\frac{\varepsilon}{t}}\left(n^2+\frac{t}{\varepsilon}\right), \tag{4.19}$$

which can be used in the same way as (4.2). Of course, here μ is a Borel probability on $\Omega \times \Omega^+$, and D_n does not control the distribution of momenta, but this is even not necessary. Indeed, the weak form of the microscopic evolution law can be written as

$$\begin{aligned} \langle \varphi, p_t^\varepsilon \rangle &= \langle \varphi, p_0^\varepsilon \rangle - \int_0^t \langle \varphi', V'(r_s^\varepsilon) \rangle ds + Y_\varepsilon(t, \varphi), \\ \langle \varphi, r_t^\varepsilon \rangle &= \langle \varphi, r_0^\varepsilon \rangle - \int_0^t \langle \varphi', p_s^\varepsilon \rangle ds + \frac{\bar{\alpha}}{2}\int_0^t \langle \varphi'', V'(r_s^\varepsilon) \rangle ds + M_\varepsilon(t, \varphi), \end{aligned} \tag{4.20}$$

where $p_t^\varepsilon(x) = p_{[x/\varepsilon]}(t/\varepsilon)$, $r_t^\varepsilon(x) = r_{[x/\varepsilon]}(t/\varepsilon)$ are the rescaled trajectories, $\bar{\alpha} = \varepsilon\alpha$, and Y and M vanish as $\varepsilon \to 0$. Since the right-hand side of (4.20) depends only in a linear way on p^ε, its distribution is really irrelevant. In the same way as before, we obtain that any limit distribution of the rescaled process is concentrated on weak solutions satisfying the regularity condition of Theorem 3. Suppose now that (p, r) and $(\bar{p}, \bar{r})$ are weak solutions, and define (u, v) by $\partial_t u = J'(r_t) - J'(\bar{r}_t)$, $u_0 = 0$; $\partial_t v = p_t - \bar{p}_t + \nabla J'(r_t) - \nabla J'(\bar{r}_t)$, $v_0 = 0$. Let $X(t) = \int \theta(x)(u_t^2 + v_t^2)dx$, where θ is the same as in the proof of Lemma 3. Integrating by parts, using the Schwarz inequality and $J'' \geqq 2a > 0$ we obtain that

$$\begin{aligned} \partial_t X &= 2\int \theta u_t(v_t + J'(r_t) - J'(\bar{r}_t))dx + \bar{\alpha}\int \theta v_t(\nabla J'(r_t) - \nabla J'(\bar{r}_t))dx \\ &\leqq KX(t) - a\int \theta(x)[r_t(x) - \bar{r}_t(x)]^2 dx. \end{aligned} \tag{4.21}$$

Since $X(0) = 0$, the Grönwall lemma yields $r_t = \bar{r}_t$ a.s., whence $p_t = \bar{p}_t$ a.s., which completes the proof of Theorem 3.

The problem of isentropic gas dynamics is to be discussed in a forthcoming paper.

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