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Special Geometry

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Abstract. A *special manifold* is an allowed target manifold for the vector multiplets of $D=4$, $N=2$ supergravity. These manifolds are of interest for string theory because the moduli spaces of Calabi–Yau threefolds and $c=9$, (2, 2) conformal field theories are special. Previous work has given a local, coordinate-dependent characterization of special geometry. A global description of special geometries is given herein, and their properties are studied. A special manifold $\mathscr{M}$ of complex dimension n is characterized by the existence of a holomorphic $Sp(2n+2,R)\otimes GL(1,C)$ vector bundle over $\mathscr{M}$ with a nowhere-vanishing holomorphic section Ω. The Kähler potential on $\mathscr{M}$ is the logarithm of the $Sp(2n+2,R)$ invariant norm of Ω.

I. Introduction

The construction of a supersymmetric field theory proceeds by demanding that the action is invariant under some chosen group of supersymmetry transformations. This places constraints on the particle content and couplings of the theory. In theories with scalars, by viewing the scalar fields as coordinates on a target manifold $\mathscr{M}$, it is often possible to reinterpret these constraints as constraints on the geometry of $\mathscr{M}$. This reinterpretation is not always straightforward, because the constraints arising from supersymmetry are expressed locally and in a particular coordinate system on $\mathscr{M}$.

As examples, it is known that local $N=1$, supersymmetry in four dimensions requires that $\mathscr{M}$ is a Kähler manifold [1] of restricted type[1] [2]. Local $N=2$ supersymmetry with $(0,\frac{1}{2})$ chiral multiplets requires that $\mathscr{M}$ is quaternionic [3].

Oddly enough, the allowed geometry of the target space $\mathscr{M}$ of locally $N=2$ supersymmetric $(0,\frac{1}{2},1)$ vector multiplets [4] – we shall refer to this as *special*

[1] This means that the Kähler form $\mathscr{J}$ is an even element of integral cohomology, or equivalently that there exist a line bundle with first Chern class equal to $[\mathscr{J}]$

geometry[2] – has not heretofore been understood. It is the purpose of this paper to fill this gap.

This gap is especially serious because of the relevance of special geometries to string theory. Compactification of a IIa or IIb string theory with a $c=9$, $(2,2)$ conformal field theory leads to an $N=2$ supersymmetric four dimensional field theory with vector multiplets [5]. The massless scalars of these vector multiplets are coordinates of the conformal field theory moduli space $\mathcal{M}$. Therefore, *the geometry of the moduli space of* $c=9$, $(2,2)$ *conformal field theories is special.* It likewise follows that the moduli space geometry of Calabi–Yau threefolds is special.

Of course, since $N=2$ supersymmetry cannot be broken in four dimensions, such theories cannot describe nature. However, *heterotic* compactification with a $c=9$, $(2,2)$ conformal field theory leads to phenomenologically interesting theories with $N=1$ supersymmetry in four dimensions. As pointed out by Seiberg [5], the tree-level metric on $\mathcal{M}$ which appears in the low energy Lagrangian depends only on the choice of conformal field theory, and is the same for type II or heterotic compactification. It was further shown by Dixon, Kaplunovsky and Louis [6], combined with the results of [4, 7, 8, 9], that in fact all the tree-level parameters in the low energy Lagrangian of a $c=9$, $(2,2)$ heterotic compactification are in this manner indirectly constrained by $N=2$ spacetime supersymmetry.

The net effect is that we can have our cake and eat it too: the low energy Lagrangian of a heterotic compactification is subject to the powerful constraints of $N=2$ supersymmetry, yet it has the phenomenologically desirable features of $N=1$ supersymmetry. It is thus crucial to understand the nature of the $N=2$ constraints.

The local constraints on special geometry implied by $N=2$ supersymmetry were derived in "special" coordinates by deWit, Lauwers and van Proeyen [4]. The entire Lagrangian is locally specified by a holomorphic function $\mathcal{F}(Z)$, where Z is a coordinate on $\mathcal{M}$. The metric $\mathcal{G}$ on $\mathcal{M}$ is Kähler, and the Kähler potential is related to $\mathcal{F}$ by

$$\mathcal{K} = -\ln\left((\partial_N\mathcal{F} - \partial_{\bar{N}}\bar{\mathcal{F}})(Z^{\bar{N}} - Z^N) + 2\mathcal{F} + 2\bar{\mathcal{F}}\right). \tag{1}$$

All other couplings are determined from $\mathcal{F}$. For example, the metric g^v governing the kinetic term of the vector field is

$$g^v_{MN} = \partial_M\partial_N\mathcal{F}. \tag{2}$$

The geometric significance of (1) and (2) is far from obvious. The expressions do not even appear to transform covariantly under diffeomorphisms of $\mathcal{M}$.

In this paper we will unravel the meaning of (1) and the geometry of $\mathcal{M}$ as follows. $N=1$ supersymmetry implies that $\mathcal{M}$ is a Kähler manifold of restricted type [2]. This implies the existence of a line bundle L whose first Chern class equals the Kähler form. Assume the existence of a $2n+2$ dimensional holomorphic $Sp(2n+2, R)$ vector bundle $\mathcal{H}$ over $\mathcal{M}$ with a compatible hermitian metric, and a

[2] These geometries have previously been referred to as *restricted Kähler*. However that phrase has a different and previously established meaning (as in the previous footnote), so we use instead the word *special*

holomorphic section Ω of $\mathscr{H} \otimes L$ whose norm is the exponential of the Kähler potential on $\mathscr{M}$. We will show that this implies the existence of special coordinates in which the geometry locally takes the form of Eq. (1). Conversely we shall show that if the geometry is of the form (1) in every patch, the bundle $\mathscr{H}$ and the section Ω exist globally. Thus special geometry, and in particular the $c=9$, (2,2) moduli space, is geometrically characterized by this section Ω.

It is further shown that Ω and $\mathscr{D}\Omega$, where $\mathscr{D}$ is the hermitian connection, provide, along with their complex conjugates, a basis for the bundle $E \equiv (L \otimes (T^* \otimes L) \otimes (\bar{T}^* \otimes \bar{L}) \otimes \bar{L})$, where T^* is the holomorphic cotangent bundle. Properties of this basis then imply the existence of a flat, non-hermitian connection $\mathbb{D}$ on E. The flatness of $\mathbb{D}$ can then be used to derive strong constraints on the geometry of $\mathscr{M}$.

For the moduli space of complex structures on Calabi–Yau threefolds, $\mathscr{H}$ is the Hodge bundle with fibers $H^3(X, C)$ and the section Ω is simply the cohomology class defined by the holomorphic (3,0) form. For conformal field theories, $\mathscr{H}$ is presumably the bundle whose fibres are the chiral cohomology classes (with respect to $G^{\pm}_{-1/2}$), and Ω the section defined by the top weight chiral primary field, (or its dual). (Evidence in favor of this latter conjecture was given in [10].) The flat connection $\mathbb{D}$ is the hermitian connection associated to the Zamolodchikov metric [11], supplemented by a non-hermitian piece given by the structure constants of the chiral ring [12, 13, 14, 15]. The flatness of $\mathbb{D}$ then gives interesting relationships among the conformal field theory correlation functions.

The striking feature of special geometry is that all geometric quantities are derived from *holomorphic* sections of various bundles. Thus one expects that special geometries are – unlike Kähler geometries – labelled by a finite number of parameters (after specification of boundary data). We hope that this property will be useful in understanding low energy string theory.

This paper is organized as follows. In Sect. II the section Ω and its properties are discussed in the context of the Calabi–Yau moduli space. It is shown to imply the existence of the flat connection $\mathbb{D}$ on the bundle E and various identities relating the Riemann curvature to holomorphic sections. In Sect. III these identities are shown to imply the local existence of special coordinate systems in which the geometry takes the canonical supergravity form (1). The notion of "integral" coordinates, in which some duality symmetries are linearly realized, is also discussed. In Sect. IV two equivalent definitions of a special manifold are given; one based on patching data for a cover of $\mathscr{M}$ and the other based on the global existence of certain geometric structures. In Sect. V we discuss how these geometric structures are identified within a $c=9$, (2,2) conformal field theory. In Sect. VI the generalization of special geometry describing the moduli spaces of conformal field theories with $c>9$ or Calabi–Yau spaces of dimension greater than three is discussed. We conclude in Sect. VII with discussion of some open problems.

Portions of the analysis in this paper follow arguments which have appeared previously in different contexts, as referenced in the text. We mention Bryant and Griffiths [16], Tian [17] and in particular the important work of Candelas [15, 18]. Interesting observations on some local aspects of special geometry are made in an appendix of an article by Cecotti, Ferrara and Girardello [7]. This paper may be viewed as an exercise in the theory of variations of Hodge structures, which is discussed for example in [19, 20].

II. Calabi–Yau Moduli Space

In this section we consider the moduli space $\mathscr{M}$ of complex structures on a three dimensional Calabi–Yau manifold X [16–18]. Consider the Hodge bundle $\mathscr{H}$ [21] over $\mathscr{M}$ whose fibers are $H^3(X,C)$. $H^3(X,C)$ is a $b_3 = 2n+2$ dimensional complex vector space, where n is the complex dimension of $\mathscr{M}$. The intersection matrix on $H^3 \otimes H^3$ provides a hermitian metric on $\mathscr{H}$:

$$\langle A|\bar{B}\rangle \equiv i \int_X d^6x A \wedge \bar{B}. \tag{3}$$

Poincaré duality on X implies that $H^3(X,Z)$ is self-dual. In addition the inner product obeys:

$$\langle A|\bar{B}\rangle = \langle B|\bar{A}\rangle^* = -\langle \bar{B}|A\rangle. \tag{4}$$

This implies there is a basis of real forms α_A, β^A for $H^3(X,C)$ obeying

$$\begin{aligned} |\alpha_A\rangle &= |\alpha_A\rangle^*, \\ |\beta^B\rangle &= |\beta^B\rangle^*, \\ \langle \alpha_A|\alpha_B\rangle &= \langle \beta^A|\beta^B\rangle = 0, \\ \langle \alpha_A|\beta^B\rangle &= i\delta^B_A, \\ \langle \beta^B|\alpha_A\rangle &= -i\delta^B_A, \end{aligned} \tag{5}$$

where $A,B = 0,1,\ldots,n$. These supply a set of local flat sections in every patch of a good cover of $\mathscr{M}$. Such bases are unique up to $Sp(2n+2,R)$ transformations, so $\mathscr{H}$ must be a flat, holomorphic $Sp(2n+2,R)$ vector bundle.

On every Calabi–Yau threefold, i.e. for every point Z in $\mathscr{M}$, there is a holomorphic (3,0) form Ω. This defines a section of the projective bundle associated to $\mathscr{H}$ which has been shown to be holomorphic [22]. The section is only defined projectively since Ω is uniquely defined only up to projective transformations which are constant on X but not on $\mathscr{M}$.

$$\Omega \xrightarrow{f} e^{f(Z)}\Omega. \tag{6}$$

The inner product of Ω with its complex conjugate $\bar{\Omega}$ varies under the projective transformations (6) as

$$\langle \Omega|\bar{\Omega}\rangle \xrightarrow{f} e^{f(Z)+f(\bar{Z})}\langle \Omega|\bar{\Omega}\rangle. \tag{7}$$

Positivity of the inner product (7) follows from the Hodge–Riemann bilinear relation [23]. We may then define, following [17, 18], the quantity

$$\mathscr{K}(Z,\bar{Z}) = -\ln\langle \Omega|\Omega\rangle, \tag{8}$$

which transforms under (6) as

$$\mathscr{K}(Z,\bar{Z}) \xrightarrow{f} \mathscr{K}(Z,\bar{Z}) - f(Z) - \bar{f}(\bar{Z}). \tag{9}$$

Identifying the projective transformations as Kähler transformations, we see that $\mathscr{K}$ defines a Kähler potential on $\mathscr{M}$. The associated Kähler metric is

$$\mathscr{G}_{M\bar{N}} = \partial_M \partial_{\bar{N}} \mathscr{K}. \tag{10}$$

It will be seen momentarily that $\mathscr{G}$ is positive.

In general the Kähler potential of course cannot be globally defined as a scalar on $\mathscr{M}$. Rather Kähler potentials on adjacent patches U_1 and U_2 are related on intersections by Kähler transformations:

$$\mathscr{K}_2 = \mathscr{K}_1 - f_{21} - \bar{f}_{21}. \tag{11}$$

$e^{\mathscr{K}}$ is then a section of a real line bundle. Since the Kähler form $\mathscr{J}$ is an even element of integral cohomology[3] (i.e. the manifold is of restricted type), this line bundle can be holomorphically decomposed as $\bar{L}^{-1} \otimes L^{-1}$. A section V of L on adjacent patches is related by

$$V_2 = e^{f_{21}} V_1. \tag{12}$$

From (6) we see that a choice of Ω at every point Z of $\mathscr{M}$ gives a section of $\mathscr{H} \otimes L$.

Motion on $\mathscr{M}$ corresponds to a deformation of the complex structure on X. Infinitesimally, the holomorphic differentials $d\omega^a$ on X mix linearly with the antiholomorphic differentials:

$$\partial_M d\omega^a = \mu_{M\bar{b}}{}^a d\omega^{\bar{b}} + \nu_{Mb}{}^a d\omega^b, \tag{13}$$

where μ_M is in $H^1(X, T)$. Ω provides an isomorphism $H^1(X,T) \to H^{2,1}(X,C)$:

$$G_{M\bar{b}cd} = \mu_{M\bar{b}}{}^a \Omega_{acd}. \tag{14}$$

The (2, 1) forms G_M are therefore cotangent to $\mathscr{M}$.

It follows from (13) that under infinitesimal motion on $\mathscr{M}$, a closed (p,q) form will mix only with closed $(p \pm 1, q \mp 1)$ forms [21]. In particular Ω will mix only with the (2, 1) forms G_M:

$$\partial_M \Omega = G_M - \mathscr{K}_M \Omega. \tag{15}$$

$\mathscr{K}_M$ can be determined from the fact that a (p,q) and (p',q') form are orthogonal unless $p + p' = q + q' = 3$. Orthogonality of $\bar{\Omega}$ and G_M gives

$$\mathscr{K}_M = -\frac{\langle \partial_M \Omega | \bar{\Omega} \rangle}{\langle \Omega | \bar{\Omega} \rangle} = \partial_M \mathscr{K}. \tag{16}$$

Define a derivative $\mathscr{D}_M = \partial_M + \partial_M \mathscr{K}$ which acts covariantly on sections of L. We may then write

$$G_M = \mathscr{D}_M \Omega \tag{17}$$

from which it is evident that G_M is a section of $T^* \otimes \mathscr{H} \otimes L$, where T^* is the holomorphic cotangent bundle.

The inner product of G_M and $G_{\bar{N}}$ gives minus the Kähler metric on $T \otimes L^{-1}$.

$$\langle G_M | G_{\bar{N}} \rangle = -e^{-\mathscr{K}} \mathscr{G}_{M\bar{N}}. \tag{18}$$

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[3] This was shown for Calabi–Yau moduli space in [17], and more generally for moduli spaces of $D = 4$ supergravity theories in [2]

The Hodge–Riemann bilinear relations for the intersection matrix on $H^{2,1}\otimes H^{1,2}$ then implies positively of $\mathscr{G}$ [23].

The inner products of the three forms $(\Omega, G_M, G_{\bar M}, \bar\Omega)$ with the $2n+2$ local flat sections (5) defines $2n+2$ basis elements for the bundle $E\equiv(L\oplus(T^*\otimes L)\oplus(\bar T^*\otimes\bar L)\oplus L)$. A flat, metric-compatible, non-hermitian connection $\mathbb{D}$ on E can be found by requiring that this basis is covariantly constant. In general such a requirement only gives a connection locally, because the transformations (on the fibers of $\mathscr{H}$) relating bases on adjacent patches would not leave $\mathbb{D}$ invariant. However, in this case the connection so defined will transform globally as a connection on E. This is because the transition functions among the basis elements are *constant* $Sp(2n+2, R)$ transformations in each intersection acting on the fibers of $\mathscr{H}$. They thus leave the connection $\mathbb{D}$ invariant.

To construct $\mathbb{D}$ explicitly, define

$$\mathbb{D}=\mathscr{D}+\mathscr{C},\ \bar{\mathbb{D}}=\bar{\mathscr{D}}+\bar{\mathscr{C}}, \tag{19}$$

where $\mathscr{D}$ is the metric-compatible hermitian connection. The connection $(1,0)$-form $\mathscr{C}$ is then determined from the requirements

$$\begin{aligned}
\mathbb{D}\Omega&=\mathscr{D}\Omega+\mathscr{C}_0{}^0\Omega+\mathscr{C}_0{}^MG_M+\mathscr{C}_0{}^{\bar M}G_{\bar M}+\mathscr{C}_0{}^{\bar 0}\bar\Omega=0,\\
\mathbb{D}G_M&=\mathscr{D}G_M+\mathscr{C}^0_M\Omega+\mathscr{C}^N_MG_N+\mathscr{C}^{\bar N}_MG_{\bar N}+\mathscr{C}^{\bar 0}_M\bar\Omega=0,\\
\mathbb{D}G_{\bar M}&=\mathscr{D}G_{\bar M}+\mathscr{C}^0_{\bar M}\Omega+\mathscr{C}^N_{\bar M}G_N+\mathscr{C}^{\bar N}_{\bar M}G_{\bar N}+\mathscr{C}^{\bar 0}_{\bar M}\bar\Omega=0,\\
\mathbb{D}\bar\Omega&=\mathscr{D}\bar\Omega+\mathscr{C}_{\bar 0}{}^0\Omega+\mathscr{C}_{\bar 0}{}^MG_M+\mathscr{C}_{\bar 0}{}^{\bar M}G_{\bar M}+\mathscr{C}_{\bar 0}{}^{\bar 0}\bar\Omega=0,
\end{aligned} \tag{20}$$

where $\mathscr{D}=dZ^M\mathscr{D}_M$ and subscripts $0(\bar 0)$ refer to $\Omega(\bar\Omega)$. The $(0,1)$ form $\bar{\mathscr{C}}$ is defined by the complex conjugate of (20).

Equations for the individual connection coefficients can now be obtained by taking the inner product of (20) with $\Omega, G_M, G_{\bar M}$ and $\bar\Omega$. Several useful formulas in solving for $\mathscr{C}$ are

$$\langle G_M|G_{\bar N}\rangle=-e^{-\mathscr{K}}\mathscr{G}_{M\bar N}, \tag{21}$$

$$[\mathscr{D}_M,\mathscr{D}_N]=0, \tag{22}$$

$$[\mathscr{D}_M,\mathscr{D}_{\bar N}]\Omega=-\mathscr{G}_{M\bar N}\Omega, \tag{23}$$

$$[\mathscr{D}_M,\mathscr{D}_{\bar N}]\bar\Omega=\mathscr{G}_{M\bar N}\bar\Omega. \tag{24}$$

It is then straightforward to show that the non-vanishing entries of $\mathscr{C}$ are

$$\mathscr{C}_0{}^M=-dZ^M, \tag{25}$$

$$\mathscr{C}^{\bar N}_M=-e^{\mathscr{K}}\mathscr{G}^{\bar NP}\mathscr{F}_{MNP}dZ^N, \tag{26}$$

$$\mathscr{C}^{\bar 0}_{\bar M}=-\mathscr{G}_{\bar MN}dZ^N, \tag{27}$$

where $\mathscr{F}_{MNP}$ is defined as

$$\mathscr{F}_{MNP}=\langle\Omega|\mathscr{D}_M\mathscr{D}_N\mathscr{D}_P\Omega\rangle \tag{28}$$

and is a section of $\mathrm{sym}(T^*)^3\otimes L^2$.[4]

[4] This object is not to be confused with the superpotential on $\mathscr{M}$, which transforms under Kähler transformations as a section of L. According to [6, 24, 14] the matter superpotential of a heterotic compactification is given by $C^MC^NC^P\mathscr{F}_{MNP}$. This would seem to imply that the matter superfields C^M are sections of $T\otimes L^{-1/3}$ and the corresponding topological restriction that $\frac{1}{3}c_1(L)$ is integral

p derivatives of a $(3,0)$ form gives a three form with antiholomorphic rank at most p. The only nonvanishing term in (28) therefore has three derivatives acting on Ω. An alternate formula for $\mathscr{F}_{MNP}$ is thus [18]

$$\mathscr{F}_{MNP} = \langle \Omega | \partial_M \partial_N \partial_P \Omega \rangle. \tag{29}$$

While this formula is not manifestly covariant, differentiating it leads immediately to the conclusion

$$\partial_{\bar{K}} \mathscr{F}_{MNP} = 0, \tag{30}$$

i.e. $\mathscr{F}_{MNP}$ is a holomorphic section.

Since the connection $\mathbb{D}$ is flat, the associated curvature $\mathbb{R}$ must vanish. Further information about the geometry can be learned from the condition $\mathbb{R} = 0$. For example

$$0 = \langle G_P | [\mathbb{D}_M, \mathbb{D}_N] G_Q \rangle = 2\mathscr{D}_{[M} \mathscr{F}_{N]PQ}. \tag{31}$$

It follows that locally [6]

$$\mathscr{F}_{MNP} = \mathscr{D}_M \mathscr{D}_N \mathscr{D}_P \mathscr{S}, \tag{32}$$

where $\mathscr{S}$ is not necessarily holomorphic. The proof of this is essentially an iteration of the Poincaré lemma of Dolbeault cohomology. The values of $\mathscr{S}$ on overlaps of patches may differ by solutions of $\mathscr{D}_M \mathscr{D}_N \mathscr{D}_P \mathscr{S}_0 = 0$, so that $\mathscr{S}$ may not be a global section of L^2.

A formula for $\mathscr{F}_{MNP}$ as the third derivative of a holomorphic section can be obtained by considering the quantity:

$$\mathscr{F}(Z, Z') = \langle \Omega(Z) | \Omega(Z') \rangle. \tag{33}$$

$\mathscr{F}(Z, Z')$ is a holomorphic section of the line bundle $L \otimes L'$ over $\mathscr{M} \times \mathscr{M}'$, where $\mathscr{M}' = \mathscr{M}$. It also obeys

$$\mathscr{F}(Z, Z') = -\mathscr{F}(Z', Z). \tag{34}$$

$\mathscr{F}_{MNP}$ is then given by the manifestly covariant formula

$$\mathscr{F}_{MNP}(Z) = \mathscr{D}_M \mathscr{D}_N \mathscr{D}_P \mathscr{F}(Z', Z)|_{Z' = Z} \tag{35}$$

or the manifestly holomorphic formula

$$\mathscr{F}_{MNP}(Z) = \partial_M \partial_N \partial_P \mathscr{F}(Z', Z)|_{Z' = Z}. \tag{36}$$

Another useful equation is

$$\begin{aligned} 0 &= \langle G_P | [\mathbb{D}_M, \mathbb{D}_{\bar{N}}] G_{\bar{Q}} \rangle e^{\mathscr{K}} \\ &= \mathscr{R}_{M\bar{N}P\bar{Q}} - \mathscr{G}_{M\bar{N}} \mathscr{G}_{P\bar{Q}} - \mathscr{G}_{M\bar{Q}} \mathscr{G}_{P\bar{N}} + e^{2\mathscr{K}} \mathscr{F}_{MPR} \mathscr{F}_{\bar{N}\bar{Q}\bar{S}} \mathscr{G}^{R\bar{S}}, \end{aligned} \tag{37}$$

where $\mathscr{R}$ is the Riemann tensor of $\mathscr{G}$. Coordinate dependent versions of this equation have appeared previously in the supergravity literature [25] and [6]. We see here that it has a geometric interpretation as expressing the flatness of the connection $\mathbb{D}$ on E.

A formula for the Ricci form of $\mathscr{M}$ is obtained by tracing (37) with the complex structure:

$$\mathscr{R}_{M\bar{N}} = -(n+1)\mathscr{G}_{M\bar{N}} + \partial_M \mathscr{V}_{\bar{N}}. \tag{38}$$

The one form $\mathscr{V}_{\bar{N}}$ is given by

$$\mathscr{V}_{\bar{N}} = e^{2\mathscr{K}} \mathscr{F}_{\bar{N}\bar{K}\bar{L}} \mathscr{D}_M \mathscr{D}_N \mathscr{S} \mathscr{G}^{M\bar{K}} \mathscr{G}^{N\bar{L}}. \tag{39}$$

Similar relations may be derived for the complexified moduli space of Kähler forms on the Calabi–Yau space X. In this case the appropriate bundle $\mathscr{H}$ is the bundle whose fibers are the sums of the even rank cohomology classes, and the appropriate Ω is again a section of $\mathscr{H} \otimes L$.

III. Special Coordinates

In the preceding sections the geometry of $\mathscr{M}$ has been described in a coordinate-independent manner. Previous discussions of $\mathscr{M}$ – both in the supergravity and mathematics literature – have been primarily in the context of a particular coordinate frame which we shall refer to as *special* coordinates. These special coordinates are in part characterized by the fact that $\mathscr{F}_{MNP}$ is locally expressible as the third partial derivative of a holomorphic function.[5] Special coordinates do not naturally arise in conformal field theory, and this has made it difficult to apply results on the structure of $\mathscr{M}$ to conformal field theory. In this section we attempt to rectify this by showing how the local existence of special coordinates follows in the general coordinate-independent framework of the previous section.

In the beginning of the previous section, a real basis of local flat sections of $\mathscr{H}$ obeying

$$\begin{aligned} \langle \alpha_A | \alpha_B \rangle &= \langle \beta^A | \beta^B \rangle = 0, \\ \langle \alpha_A | \beta^B \rangle &= i\delta^B_A, \\ \langle \beta^B | \alpha_A \rangle &= -i\delta^B_A, \end{aligned} \tag{40}$$

where $A, B = 0, 1, \ldots, n$, was introduced. Ω may be expanded in terms of this basis:

$$\Omega(Z) = W^A(Z)\alpha_A + i\mathscr{F}_A(Z)\beta^A. \tag{41}$$

The coefficients W^A and $\mathscr{F}_A$ are holomorphic and transform as sections of L since Ω is a holomorphic section of $\mathscr{H} \otimes L$.

We wish to show that W^A (or alternately $\mathscr{F}_A$) are good projective coordinates in the neighborhood of a generic point Z^M_0. Because the variation of Ω is always non-zero under motion along $\mathscr{M}$, we expect the $2n+2$ coefficients of Ω in a fixed basis to provide complex coordinates on $\mathscr{M}$. However, since $\mathscr{M}$ has complex dimension n, only half of these $2n+2$ coefficients are required for complex projective coordinates.

We now demonstrate that, for sufficiently smooth Ω, W^A provide good projective coordinates near Z^M_0. The proof is somewhat cumbersome, those uninterested in the details may skip directly to Eq. (58).

W^A are good projective coordinates in a neighborhood of Z_0 if

$$\sum_A W^{\bar{A}}(Z_0) W^A(Z_0) \neq 0, \tag{42}$$

[5] It is curious that special coordinates are not singled out by this condition alone. In [6] it was shown that $\mathscr{F}_{MNP} = \partial_M \partial_N \partial_P \mathscr{F}'$ in "holonormal" coordinates, but this $\mathscr{F}'$ is not related to $\mathscr{K}$ by (1)

and there are n linearly independent one forms

$$dX^i = X^i_A W^A{}_{,M}(Z_0) dZ^M \quad i = 1, \ldots, n. \tag{43}$$

Under these circumstances n of the W^A's provide ordinary coordinates (the Jacobian is non-zero) and the remaining W^A extends these to projective coordinates.

Equation (42) is a consequence of the non-vanishing norm of Ω. To verify condition (43), consider a Kähler gauge in which

$$\partial_M \mathscr{K}(Z_0, \bar{Z}_0) = 0, \tag{44}$$

$$\mathscr{K}(Z_0, \bar{Z}_0) = \ln 2 \tag{45}$$

and a coordinate system in which

$$\mathscr{G}_{M\bar{N}}(Z_0) = \delta_{M\bar{N}}. \tag{46}$$

With respect to these coordinates, define the basis

$$\begin{aligned} \alpha_0 &= [\Omega + \bar{\Omega}]_{Z_0}, \\ \alpha_M &= [G_M + G_{\bar{M}}]_{Z_0}, \\ \beta^0 &= i[-\Omega + \bar{\Omega}]_{Z_0}, \\ \beta^M &= i[G_M - G_{\bar{M}}]_{Z_0}, \end{aligned} \tag{47}$$

which obeys (40). Using the formulae

$$\begin{aligned} W^A{}_{,M} &= i\langle \beta^A | \partial_M \Omega \rangle, \\ \mathscr{F}_{A,M} &= -\langle \alpha_A | \partial_M \Omega \rangle, \end{aligned} \tag{48}$$

one finds that in this basis

$$\begin{aligned} W^0{}_{,M}(Z_0) &= 0, \\ W^N{}_{,M}(Z_0) &= \tfrac{1}{2}\delta^N_M, \\ \mathscr{F}_{0,M}(Z_0) &= 0, \\ \mathscr{F}_{N,M}(Z_0) &= -\tfrac{1}{2}\delta_{NM}. \end{aligned} \tag{49}$$

From this it is easily seen that there are n linearly independent dX^i's by choosing $X^i_A = \delta^i_A$ in (43). The W^A defined in the particular basis (47) are thus good projective coordinates in a neighborhood of Z_0.

It remains to show that the W^A defined with respect to a general basis are good projective coordinates. A general basis is obtained by acting with an $Sp(2n+2, R)$ matrix S on the basis (47). One then needs to show there are n linearly independent one forms

$$dX^i = X^i_A W'^A{}_{,M}(Z_0) dZ^M \quad i = 1, .., n \tag{50}$$

where

$$W'^A{}_{,M} = i\langle \beta^A | S \partial_M \Omega \rangle. \tag{51}$$

To this end, consider the n dimensional subspace of real vectors $\tilde{X}^i_A$ obeying

$$(S^{-1}\tilde{X}^i)_0 = 0. \tag{52}$$

Linear dependence of the corresponding $d\tilde{X}^i$'s would imply a non-trival solution of

$$\langle S^{-1}\lambda | \partial_M \Omega(Z_0)\rangle = 0, \tag{53}$$

where $\lambda_A = \lambda_i \tilde{X}^i_A$ is subject to the constraint (52). Using the relation

$$\partial_M \Omega(Z_0) = \frac{1}{2}\alpha_M - \frac{i}{2}\beta^M \tag{54}$$

implied by (47), one finds that (53) is equivalent to

$$(S^{-1}\lambda)^M = i(S^{-1}\lambda)_M. \tag{55}$$

The norm of $S^{-1}\lambda$ is then

$$\langle (S^{-1}\lambda)^* | S^{-1}\lambda\rangle = 2\sum_A (S^{-1}\lambda)^*_A (S^{-1}\lambda)_A \geqq 0. \tag{56}$$

On the other hand

$$\langle (S^{-1}\lambda)^* | S^{-1}\lambda\rangle = \langle \lambda^* | \lambda\rangle = 0 \tag{57}$$

since S is real and $\lambda = \lambda_A \beta^A$. We therefore conclude that λ must vanish, and the n $d\tilde{X}^i$'s are linearly independent.

This establishes that the W^A defined with respect to any basis for H^3 of the form (40) are good projective coordinates in the Kähler gauge (44), (45). However a change of Kähler gauge multiplies W^A by a non-vanishing holomorphic function, so the result holds in any gauge.

We therefore conclude that W^A are locally good complex projective coordinates. We will refer to these coordinates as special projective coordinates. Non-projective special coordinates can then be defined in the usual manner.

In special projective coordinates, Ω takes the form [6]

$$\Omega(W) = W^A \alpha_A + i\mathscr{F}_A(W)\beta^A. \tag{58}$$

The Kähler potential is

$$\mathscr{K} = -\ln(W^A \mathscr{F}_{\bar{A}} + \mathscr{F}_A W^{\bar{A}}), \tag{59}$$

where $W^{\bar{A}} \equiv (W^A)^*$ and $\mathscr{F}_{\bar{A}} \equiv (\mathscr{F}_A)^*$. From the relation

$$0 = \langle \Omega | \partial_A \Omega\rangle = -W^B \partial_A \mathscr{F}_B + \mathscr{F}_A \tag{60}$$

one learns that

$$\mathscr{F}_A = \partial_A \mathscr{F} \tag{61}$$

where

$$\mathscr{F} \equiv \tfrac{1}{2} W^A \mathscr{F}_A \tag{62}$$

is a locally holomorphic function of projective weight two [18]. It further follows [18] that the section of $\mathrm{sym}(T^*)^3 \otimes L^2$ defined in the previous section:

$$\mathscr{F}_{ABC} = \langle \Omega | \partial_A \partial_B \partial_C \Omega\rangle \tag{63}$$

is, in special coordinates, the third derivative of the holomorphic function $\mathscr{F}$,

$$\mathscr{F}_{ABC} = \partial_A \partial_B \partial_C \mathscr{F}. \tag{64}$$

Equations (59) and (61) reduce to the supergravity formula (1) after transforming to non-protective special coordinates [9]. This then establishes that the special geometries discussed in Sect. II satisfy the requirements of the target space geometries of $D = 4, N = 2$ vector multiplet sigma models coupled to supergravity. These requirements were previously stated in a local, coordinate dependent form: the action is determined entirely by a locally holomorphic function $\mathscr{F}$, and the Kähler potential for the vector multiplet is related to $\mathscr{F}$ by (61) and (59). In this section we have seen that the special geometries defined geometrically in the previous section always locally admit special coordinates in which this is the case.

How unique are special coordinates? Our starting point in Eq. (40), was a basis for the fibers of $\mathscr{H}$ with an $Sp(2n+2, R)$ invariant inner product. This basis is unique up to $Sp(2n+2, R)$ rotations, which therefore parameterize the "special" coordinate transformations which preserve the special structure [7]. Since these transformations in general mix up α_A and β^B, they will also mix up $\partial_A \mathscr{F}$ and W^B:

$$\begin{pmatrix} i\partial\mathscr{F} \\ W \end{pmatrix}' = M \begin{pmatrix} i\partial\mathscr{F} \\ W \end{pmatrix}, \tag{65}$$

where M is in $Sp(2n+2, R)$ and $i\partial_A \mathscr{F}$ and W^B have been grouped into a $2n+2$ component column vector. Special coordinate transformations are referred to as duality rotations in the supergravity literature [7].

For Calabi–Yau moduli spaces there is a preferred subclass of special coordinates, used in [16] which we shall call "integral coordinates." These are obtained by choosing α_A and β^B to generate integral cohomology, and are integral periods of Ω. (Projective) integral coordinates are unique up to $Sp(2n+2, Z)$ transformations. Two values of Ω for which $(i\mathscr{F}_A, W^A)$ differ by an $Sp(2n+2, Z)$ transformation correspond to the same point in $\mathscr{M}$. Thus the integral periods of Ω give coordinates on a Teichmueller space $\mathscr{T}$. $Sp(2n+2, Z)$ is then a subgroup of the modular group by which $\mathscr{T}$ is divided to obtain $\mathscr{M}$. These are also the duality transformations of string theory. The analog of integral coordinates for conformal field theory moduli space is discussed in Sect. V.

IV. Definitions of a Special Manifold

Special geometries arise in the construction of four dimensional Lagrangians with n $N = 2$ vector super multiplets coupled to supergravity. The complex scalar components of the vector supermultiplet are coordinates on an n complex dimensional manifold $\mathscr{M}$. Their kinetic term is determined by a metric $\mathscr{G}$ on $\mathscr{M}$. Invariance of the Lagrangian under $N = 2$ supersymmetry places local constraints on the geometry of $\mathscr{M}$. Namely, there must exist a homogeneous of degree two holomorphic function $\mathscr{F}$ related to the metric in projective coordinates W by [4]

$$\mathscr{G}_{A\bar{B}} = -\partial_A \partial_{\bar{B}} \ln (W^A \partial_{\bar{A}} \bar{\mathscr{F}} + \partial_A \mathscr{F} W^{\bar{A}}). \tag{66}$$

Equation (66) reduces to (1) in non-projective coordinates. Of course this expression does not make sense globally. One must in addition specify the transformations relating the various quantities on overlaps of patches. In order that $N = 2$ supersymmetry transformations can be defined in each patch, the transformation rules must preserve the relation (66) between $\mathscr{G}_{A\bar{B}}$, $\partial_A \mathscr{F}$ and W^A. This restricts the allowed transformations to the $Sp(2n+2,R)\otimes GL(1,C)$ transformations discussed in Sect. IV.

Guided by this we are led to the following definition of a special geometry:

Definition (1). *A special manifold is an n dimensional Kähler manifold of restricted type such that on each patch U_i of a good cover there exist complex projective coordinates W_i^A and a homogeneous, degree two holomorphic function $\mathscr{F}_i(W)$ related to the Kähler potential $\mathscr{K}_i$ by*

$$\mathscr{K}_i(W,\bar{W}) = -\ln(W_i^A\partial_{\bar{A}}\bar{\mathscr{F}}_i + \partial_A\mathscr{F}_i W_i^{\bar{A}}). \tag{67}$$

On intersections of adjacent patches U_i and U_j, $\partial_A\mathscr{F}$ and W^A are related by special coordinate transformations

$$\begin{pmatrix} i\partial\mathscr{F} \\ W \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} i\partial\mathscr{F} \\ W \end{pmatrix}_j, \tag{68}$$

where the f_{ij} are holomorphic and M_{ij} is a constant element of $Sp(2n+2,R)$.

The transition of functions are subject to the usual consistency conditions on triple overlaps:

$$\begin{aligned} e^{f_{ij}+f_{jk}+f_{ki}} &= 1, \\ M_{ij}M_{jk}M_{ki} &= 1. \end{aligned} \tag{69}$$

Definition (1) of course refers to a particular coordinate system. An alternate, coordinate-independent definition is:

Definition (2). *Let L denote the complex line bundle whose first Chern class equals the Kähler form, $\mathscr{J}$, of an n dimensional Kähler manifold $\mathscr{M}$ of restricted type. Let $\mathscr{H}$ denote a holomorphic $Sp(2n+2,R)$ vector bundle over $\mathscr{M}$ and $\langle , | , \rangle$ the compatible hermitian metric on $\mathscr{H}$. $\mathscr{M}$ is a special manifold if, for some choice of $\mathscr{H}$, there exists a holomorphic section Ω of $\mathscr{H}\otimes L$ with the property*

$$\mathscr{J} = -\frac{i}{2\pi}\partial\bar{\partial}\ln\langle\Omega|\bar{\Omega}\rangle. \tag{70}$$

Note that the transition functions of a holomorphic $Sp(2n+2,R)$ vector bundle are necessarily constant on each overlap.

The equivalence of definitions (1) and (2) follow from the results of Sects. (II) and (III). To see that (2) follows from (1), note that W^A and $i\partial_A\mathscr{F}$ are the components of the section Ω in each patch. To see that (1) follows from (2), recall that the existence of a holomorphic section of $\mathscr{H}\otimes L$ with the property $\mathscr{J} = -\frac{i}{2\pi}\partial\bar{\partial}\ln\langle\Omega|\Omega\rangle$ was our starting point in the beginning of Sect. III. This led, in Sect. IV, to the local existence *of special coordinates* W^A and one form $\partial_A\mathscr{F}$ obeying the relations of definition (1).

It is possible that an alternate definition of a special manifold can be given as a manifold admitting a flat connection on the bundle E with the properties of the connection $\mathbb{D}$. The bundle E is canonically associated with every Kähler manifold of restricted type. Such a definition would have the advantage of not requiring the extra structure associated to the bundle $\mathscr{H}$.

V. $C=9,(2,2)$ Superconformal Field Theory

In the previous sections we have developed the theory of special manifolds using Calabi–Yau moduli space as a concrete example. However the results apply equally to the moduli space of $c=9,(2,2)$ conformal field theories.

To see this, recall [26, 5, 7] that this moduli space has a product structure $\mathscr{M}_1 \times \mathscr{M}_2$, where $\mathscr{M}_1$ is associated with the (c,c) ring of chiral[6] operators, and $\mathscr{M}_2$ with the (a,c) ring.[7] Compactification of a type IIa string leads to $D=4$, $N=2$ vector multiplet with $\mathscr{M}_1$ as a target space. This is possible only if $\mathscr{M}_1$ is a special manifold. Similarly, IIb compactification leads to a $D=4$, $N=2$ vector multiplet with $\mathscr{M}_2$ as a target space, so $\mathscr{M}_2$ must also be a special manifold.

While this argument implies that $\mathscr{M}_1$ and $\mathscr{M}_2$ are special, more work is required to explicitly identify the various geometric objects discussed in this paper with objects in a two dimensional field theory. Such an identification would be extremely useful for understanding the structure of $N=2$ conformal field theories, and might enable one to go beyond the $c=9$ case discussed here. For the case of the Calabi–Yau moduli space, for example the bundle $\mathscr{H}$ was identified as the bundle whose fibers are $H^3(X)$, the projective section Ω was identified as the section defined by the $(3,0)$ form, etc. In Sects. II and III we showed that properties of these objects then implied properties of the $D=4$, $N=2$ Lagrangian, and vice versa.

For the conformal field theory moduli space, the situation is quite different, since there have been few studies of its geometry. However the results of the present paper, along with the work of [12, 13] relating the cohomology ring of a Calabi–Yau manifold to the chiral ring of a conformal field theory, allow us to make educated guesses identifying the various sections and bundles with elements of a conformal field theory. In this section we will describe these educated guesses, but we will not verify them here. Some progress in directly verifying them using purely two dimensional methods was made in [10].

The section Ω is presumably identified with the top weight chiral primary field Ω^+ (or its dual [10]) which is known to exist in every $(2,2)$ conformal field theory. The main problem is to show that Ω^+ defines a holomorphic section of the vector bundle whose fibers are the cohomology classes (with respect to $G^{\pm}_{-1/2}$) of chiral fields, and that an appropriate inner product can be defined on these fibers.

Steps in this direction were taken in [10], where it was shown that Ω^+ is invariant under antiholomorphic deformations of the moduli, and that $\ln\langle\Omega^-\Omega^+\rangle$ is the Kähler potential on $\mathscr{M}$.

In fact the results of [10] required only $(0,2)$ (rather than the full $(2,2)$) supersymmetry. It is an open question how much, if any, of the analysis of this paper might be applicable to the $(0,2)$ moduli space.

The coordinates usually adopted in analyzing deformations of conformal field theories are not special. Since many formulae simplify in special coordinates, it would be useful to understand the transformation to special coordinates in conformal field theory, which is roughly as follows. In the usual presentation of a

[6] We use the word chiral here to denote fields which commute with $G^{\pm}_{-1/2}$, as opposed to left or right moving fields

[7] This has only been shown locally, but seems likely to be valid globally

$c=9,(2,2)$ conformal field theory, one is given the weights, multiplicities and correlation functions of the chiral primary fields. Holomorphic deformations of the conformal field theory are generated by weight $(\frac{1}{2},\frac{1}{2})$ charge $(-1,-1)$ chiral primary fields P_M, and antiholomorphic deformations by their charge $(1,1)$ hermitian conjugates. (Charge $(-1,1)$ fields will be ignored in the following, but similar statements apply.) There is a canonical isomorphism of P_M with charge $(-1,2)$ chiral primary fields G_M

$$G_M=[\Omega^+_{-1/2},P_M], \tag{71}$$

where $\Omega^+_{-1/2}=\int\frac{dz}{2\pi i}\Omega^+(z)$, in precise analogy with (14). Ω^+ will mix with G_M under deformations of the conformal field theory, so the chiral fields with $Q_R-Q_L=3$, where $Q_R(Q_L)$ is the right (left) $U(1)$ charge, should be identified as a basis for the bundle E of Sect. II. The analog of Poincaré duality was proven for (2, 2) conformal field theories in [13]. This implies that a real basis $|\alpha_A\rangle, |\beta^B\rangle$ for the fibers of $\mathscr{H}$ represented as chiral states, with an $Sp(2n+2,R)$ invariant inner product (as in Eq. (40) or (47)) can be constructed in any conformal field theory, i.e. at any point Z in $\mathscr{M}$. This basis may be expressed in terms of the more conventional states of the form $|G_M\rangle$ obtained by acting with chiral operators on the vacuum at Z:

$$|\alpha_A\rangle=\mathscr{E}^0_A|\Omega^+\rangle+\mathscr{E}^M_A|G_M\rangle+\mathscr{E}^{\bar M}_A|G_{\bar M}\rangle+\mathscr{E}^{\bar 0}_A|\bar\Omega^-\rangle, \tag{72}$$

$$|\beta^B\rangle=\mathscr{E}^{B0}|\Omega^+\rangle+\mathscr{E}^{BM}|G_M\rangle+\mathscr{E}^{B\bar M}|G_{\bar M}\rangle+\mathscr{E}^{B\bar 0}|\bar\Omega^-\rangle. \tag{73}$$

Local constant sections are then obtained by parallel transport of this basis. The "vielbeins" $\mathscr{E}$ defining these sections are given as the solution of

$$\mathbb{D}\mathscr{E}=0 \tag{74}$$

with the flat connection $\mathbb{D}$ of Eq. (44). In conformal field theory terms, this connection is essentially the hermitian connection associated to the Zamolodchikov metric supplemented by a non-hermitian piece proportional to the structure constants of the chiral ring.

In the basis (72), special projective coordinates might be defined by

$$W^A=i\langle\beta^A|\Omega\rangle. \tag{75}$$

In these coordinates, $\partial_A\mathscr{F}$ is

$$\partial_A\mathscr{F}(W)=-\langle\alpha_A|\Omega\rangle \tag{76}$$

and $\mathscr{F}$ is

$$\mathscr{F}=-\frac{i}{2}\langle\beta^A|\Omega\rangle\langle\alpha_A|\Omega\rangle. \tag{77}$$

However, the technology for defining the variation of matrix elements such as (76) and (77) on $\mathscr{M}$ is as yet not well understood (at least by us!), so at present these expressions are formal.

It would certainly be of interest to understand the analog of integral coordinates for the moduli space of $c=9$, (2,2) conformal field theories or, equivalently, the analog of integral cohomology. A possibly relevant clue is that, for the Calabi–Yau case, points in Teichmueller space $\mathscr{T}$ for which Ω is itself a generator of integral cohomology are fixed points of a subgroup of the modular group (just as for genus one curves $\tau = i$ is invariant under $\tau \to -\frac{1}{\tau}\Big)$. Thus we might expect that "integral" chiral cohomology classes could be identified by maximal chiral cohomology classes at fixed points of modular (i.e. duality) transformations. These fixed points are often characterized by enlarged symmetries which make them easy to analyze.

VI. General Values of C and D

In the previous sections, the moduli spaces of $c = 9$ conformal field theories and Calabi–Yau threefolds have been analyzed. String or Kaluza–Klein compactification relates these to the moduli space of $N = 2$, $D = 4$ supergravity. For conformal field theories with $c > 9$ or Calabi–Yau spaces of dimension d greater than three, useful information cannot be obtained, at least not in any simple way, by considering string or Kaluza–Klein compactification. However these moduli spaces are still characterized by holomorphic sections Ω of certain vector bundles. Thus we expect that an analysis along the lines of Sect. II could still lead to geometric information about the moduli space. In mathematical terms, we are interested in Hodge structures of weight $\frac{c}{3}$ with the property that the maximal filtration $F^{c/3}$ is one dimensional. In this section we make some preliminary remarks on this problem.

As in the case $d = 3$, a Kähler potential may be defined by

$$\mathscr{K} = -\ln\langle\Omega|\bar{\Omega}\rangle, \tag{78}$$

where Ω is a $(d, 0)$ form and the Hodge metric is

$$\langle A|\bar{B}\rangle = i^{d^2}\int A \wedge \bar{B}. \tag{79}$$

Then

$$G_M = \mathscr{D}_M \Omega \tag{80}$$

is a $(d-1, 1)$ form. Since the metric

$$\langle G_M|G_{\bar{N}}\rangle = -\mathscr{G}_{M\bar{N}}e^{-\mathscr{K}} \tag{81}$$

is non-degenerate, G_M forms a cotangent basis on $\mathscr{M}$.

Now consider

$$G_{MN} = \mathscr{D}_M \mathscr{D}_N \Omega. \tag{82}$$

The inner product of G_{MN} with $G_{\bar{K}}$ is

$$\begin{aligned}\langle G_{\bar{K}}|G_{MN}\rangle &= \mathscr{D}_M e^{-\mathscr{K}}\mathscr{G}_{N\bar{K}} \\ &\quad - \mathscr{G}_{N\bar{K}}\langle\bar{\Omega}|G_M\rangle = 0,\end{aligned} \tag{83}$$

which implies that G_{MN} is a $(d-2, 2)$ form. Its inner product with $(2, d-2)$ forms is

$$\begin{aligned}\langle G_{\bar{P}\bar{Q}}|G_{MN}\rangle &= \mathscr{D}_{\bar{P}}\langle G_{\bar{Q}}|G_{MN}\rangle \\ &\quad + e^{-\mathscr{K}}(R_{M\bar{P}N\bar{Q}} - \mathscr{G}_{M\bar{P}}\mathscr{G}_{N\bar{Q}} - \mathscr{G}_{M\bar{Q}}\mathscr{G}_{N\bar{P}}) \\ &= e^{-\mathscr{K}}(R_{M\bar{P}N\bar{Q}} - \mathscr{G}_{M\bar{P}}\mathscr{G}_{N\bar{Q}} - \mathscr{G}_{M\bar{Q}}\mathscr{G}_{N\bar{P}}).\end{aligned} \tag{84}$$

The Hodge–Riemann bilinear relation then implies

$$R_{M\bar{P}N\bar{Q}} - \mathcal{G}_{M\bar{P}}\mathcal{G}_{N\bar{Q}} - \mathcal{G}_{M\bar{Q}}\mathcal{G}_{N\bar{P}} \leqq 0, \tag{85}$$

i.e. the eigenvalues of the sectional curvature are bounded. This result is valid for all d, and is possibly a consequence of more general theorems [19] on curvatures of Hodge bundles.

The obstacle to further progress at this point is that for $d > 3$ we do not know how much of $H^{d-2,2}(X)$ is spanned by the forms G_{MN}, or, equivalently, the number of zero eigenvalues of the metric (84). In order to derive a flat connection analogous to (20) one needs to know that the $\mathcal{D}_{M_1} \ldots \mathcal{D}_{M_q} \Omega$ for $q \geqq \frac{d}{2}$ can be expressed as linear combinations of $\mathcal{D}_{\bar{M}_1} \ldots \mathcal{D}_{\bar{M}_{d-q}} \bar{\Omega}$. We do not know if this is possible.

Note that for $d > 3$, the coefficients of Ω as a section of the bundle of d forms cannot provide projective coordinates on $\mathcal{M}$, because there are too many of them. They may however provide some highly redundant coordinates which can be reduced to coordinates on $\mathcal{M}$ by some identifications.

VII. Some Open Problems

We conclude this paper by mentioning some open problems in special geometry. While there are many interesting problems in the general context of Hodge theory, the following are mentioned for their relevance to string theory.

1. *Classification of Special Manifolds.* A classification of special manifolds would amount to a description of all possible (2, 2) supersymmetric vacua of string theory (though there is of course no guarantee that every special geometry arises as the moduli space of some string vacua). From the point of view of extracting low-energy physics, this is the best form in which to obtain a description of string vacua (as opposed to e.g. as representations of two dimensional algebras), because the moduli space geometry gives directly the low-energy tree-level couplings. A complete classification is undoubtedly a forbidding problem. A perhaps more reasonable problem is classification of low dimensional special manifolds. This is also the physically interesting case, since the dimension of the special manifold is the number of (anti)families of a superstring compactification. Some important insights on this problem are in [7].
2. *Superpotentials on Special Manifolds.* It is possible, and perhaps even plausible, that non-perturbative string effects will generate a superpotential $\mathcal{W}$ on $\mathcal{M}$. If so, it was shown in [2] that $\mathcal{W}$ must be a section of the line bundle L. The physical ground state of the theory will then lie at the minimum of

$$V = e^{\mathcal{K}}(\mathcal{G}^{M\bar{N}}\mathcal{D}_M\mathcal{W}\mathcal{D}_{\bar{N}}\bar{\mathcal{W}} - 3\mathcal{W}\bar{\mathcal{W}}), \tag{86}$$

and supersymmetry will be unbroken if and only if $\mathcal{D}_M\mathcal{W}$ vanishes at the minimum. On a general Kähler manifold it is difficult to obtain any general information about the nature of the minima, but for the more restricted special manifolds it may be possible. An interesting recent discussion appears in [27].
3. *Compactification of Special Manifolds.* A typical structure of a special manifold seems to be contractible space divided by some group action, as for the moduli space

of curves. It may be possible, as in the case of curves, to compactify special manifolds by the addition of boundary components in a way that preserves at least some, though possibly not all of the special structure. An alternate exciting possibility, suggested in [28, 18], is that special geometries might be "glued" together along their boundaries in such a way as to form a "universal special geometry" which contains the moduli spaces of all supersymmetric string vacua.

Some understanding of this issue is probably essential for any progress on (2) since there are generally many sections on non-compact manifolds.

4. *Examples of Special Manifolds.* Global moduli spaces for $c = 3$ and $c = 6$, $(4,4)$ [29] conformal field theories have been described. For the case $c = 9$, the 6-torus has been discussed, but this has more than $(2,2)$ supersymmetry. No global description of any $c = 9$, $(2,2)$ conformal field theory moduli space has been given to our knowledge until very recently [30]. Some local descriptions are given for example in [7, 24, 15, 31]. $c = 9$, $(2,2)$ is the case of most interesting for string theory; it is also evident from this paper that such moduli spaces have special properties not shared by other values of c. These spaces are deserving of further study.

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