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Space-Time Fields and Exchange Fields

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Abstract. We derive discrete symmetries of braid group statistics related to charge conjugation and outer automorphisms of the local algebra. The structure of the latter (which are abelian superselection charges) is analyzed in some detail. We use the results to study in great generality a phenomenon recently observed in conformal quantum field theories: the existence of two-dimensional space-time fields with conventional (local, fermionic, dual) commutation relations, expressible as bilinear sums over light-cone fields with exchange algebra commutation relations.

1. Introduction

Braid group statistics is the natural statistics in low space-time dimensions. A fundamental reason why it has for a long time escaped our attention, except for a class of "anyon" models with abelian representations of the braid group, is the intimate relation between statistics and internal symmetries: the ordinary (Lie group) symmetries go along with permutation group statistics only [1], while braid group statistics signals a new type of "quantum symmetry" [2] which we are only recently beginning to understand.

While braid group statistics must be expected in up to $2+1$ space-time dimensions (at least for "gauge charges" localized in narrow tubes extending to space-like infinity), the only explicit occurrence of non-abelian braid group statistics so far is in $1+1$ dimensional conformal quantum field theories. Even there, it is not the statistics associated with the local two-dimensional observables that has been read off the Wightman functions (conformal block functions) completely determined by Ward identities, but in fact the statistics of its "chiral" local light-cone fields, i.e. an effectively one-dimensional quantum field theory with particularly simple kinematics.

By virtue of conformal covariance, space-time fields factorize as bilinear expressions in light-cone fields. The monodromy (analyticity) properties of conformal block functions [3], interpreted as vacuum expectation values of light-cone fields interpolating different superselection sectors, turn into commutation relations (exchange algebra) of the latter [4, 5]. By a non-trivial interplay of the structure constants of the two exchange algebras (on either light-cone), this decomposition is compatible with local commutativity, or more general "conventional" commutation relations of the space-time fields [6, 7].

We use the term "conventional" for commutation relations determined only by the charges carried by the fields, in contradistinction to "exchange" commutation relations (see below) the structure constants of which depend also on the charged sectors among which the fields interpolate. Space-time fields with bosonic or fermionic commutation relations have been studied in terms of boundary conditions imposed on modular invariant partition functions [8]. Yet the general systematics remain somewhat obscure.

In this article we shall address the reverse question. Given two (isomorphic) exchange quantum field theories $\mathscr{F}$ over the oriented real axis. Their tensor product $\mathscr{F}\otimes\mathscr{F}$ is interpreted as a quantum field theory over Minkowski space-time $\mathbb{M}^2=\mathbb{R}^+\times\mathbb{R}^-$ with the identification $t+x=x_+$, $t-x=x_-$ of coordinates. The metric in $\mathbb{M}^2$ is given by $(t,x)^2=t^2-x^2=x_+x_-$. The ordering [1] $>$ on the real axes is such that the quadrant $x_+>0$, $x_->0$ becomes the forward light-cone.

Then we ask for subalgebras $\hat{\mathscr{F}}$ of $\mathscr{F}\otimes\mathscr{F}$ consisting of space-time fields

$$\Phi=\sum F_+\otimes F_- \tag{1.1}$$

with conventional commutation relations. We shall identify several such subalgebras. The underlying systematics are traced back to the outer localized automorphisms (abelian superselection charges) of the algebra of light-cone observables, playing a prominent role in the construction.

The first subalgebra consists of mutually local space-time operators which are left-right symmetric (or rather conjugate): the sum (1.1) contains only terms for which the charge carried by F_- is conjugate to the charge carried by F_+. A second subalgebra consists of "unbalanced" fields with an abelian "excess charge" carried by one of its chiral factors. The commutation relations and operator products of these fields, in general, still depend on the excess charge of the sectors they act on. This dependence cannot always be removed due to an intrinsic obstruction (comparable with pseudo-reality of Lie group representations) preventing the possibility to represent multiplication of equivalence classes of automorphisms in terms of individual multiplication of a unique choice of representatives. If one restricts oneself to excess charges without this obstruction (we give criteria for the obstruction to be absent or uneffective), one obtains a third subalgebra of space-time fields with soliton-like (dual) commutation relations. Finally, among these we identify subalgebras of mutually local space-time operators with excess charge, and discuss extended algebras of local light-cone operators.

We address this question in the framework of algebraic quantum field theory [9–12], based on "first principles" to the largest possible extent (especially locality and spectral properties). In that approach it is apparent that braid group statistics and the exchange algebra, which were at first sight ascribed to the peculiarities of

[1] The choice of the ordering is not immaterial, as soon as issues related to the spectrum condition are discussed

conformal invariance, are in fact basic structural features of general low-dimensional quantum field theories. For our present task it is best suited to derive symmetries of the relevant structure constants, and to elucidate the prominent role of localized automorphisms for the existence of unbalanced space-time fields [2].

The detailed study of braid group statistics leads to a natural (though not canonical) construction of a charged field algebra, the "reduced field bundle" or "abstract exchange algebra" $\mathscr{F}$ [13] from the observable content, i.e. the algebra of observables $\mathscr{A}$, of the theory. The abstract derivation, being completely model independent and in particular never assuming conformal invariance, precisely predicts all the remarkable structural observations, made in large classes of two-dimensional models of conformal field theory, but is not limited to these. Thus the results of the first sections of this article about the structure of superselection rules with braid group statistics are of general validity in low dimensions. Moreover, since in the sequel we do not assume the conformal light-cone theory to be governed by some symmetry group, we allow for the possibility of more general conformal field theories than those given by current algebras based on Lie groups, cosets, or orbifolds. The interesting conjecture that there might be always such a symmetry, is beyond the intentions of the present article.

As a matter of fact, the known conformal models appear quite exhaustive for the admissible braid group statistics. One might speculate whether the conformal models provide some "complete sample collection of prototypes" of braid group statistics, in this respect comparable to the free theories scanning permutation group parastatistics. The study of non-trivial braid group statistics in these simple models may be helpful to devise particle theories with braid group statistics ("plektons"), the scattering matrix of which is strongly constrained by the relevant braid group representations.

In Sects. 2 and 3, we introduce the algebraic framework, as far as it is needed for the present study. In Sect. 4, we prepare by a number of lemmata of general validity the construction of algebras of conventional space-time fields from exchange light-cone fields, which is given in Sect. 5 almost as a corollary of the preceding work. A reader familiar with algebraic quantum field theory may almost directly proceed to Sect. 5, provided he or she accepts the Lemmata 4.4–7. Yet we think that the results of Sects. 2–4, having implications about selection rules for fractional spins, charge conjugation and TPC symmetry, and multiplicative properties of abelian charges in very general situations, are of their own interest.

It is only in Sect. 5, when we specify the algebra of observables $\mathscr{A}$ to be a local net over the real axis (light-cone), that we implicitly assume (though not really use) conformal covariance. The conformal light-cone unites the space-like property of supporting local fields (such as current algebras), and the time-like property of boundedness of the spectrum of the generator of translations. Both these properties are among the first principles of algebraic quantum field theory; the latter may be replaced by the weaker (and technical) assumption of Borchers' "Property B" [10, Theorem 3.3]. It is hardly conceivable how these two axioms

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[2] In fact, one deals with bounded operators instead of Wightman fields. Since point-like limits exist at least in field theories with conformal covariance, leaving the relevant structures unaffected, we shall not always make a clear verbal distinction between fields and operators

can be realized over a one-dimensional "space-time" without conformal covariance.

Let us list here the remaining axioms and assumptions made in Sects. 2–4. Besides the physically immediate axioms of locality and isotony of the C^* algebra (local net) of observables $\mathscr{A}$, we require, as usual, Haag duality as a maximality criterion for $\mathscr{A}$, and restrict ourselves to representations of $\mathscr{A}$ describing arbitrarily narrowly localizable superselection charges. (Both these assumptions can, in fact, be derived from conformal covariance in the vacuum sector.) Furthermore we postulate the existence of conjugate charges and – tied to it – finiteness of statistics. (These properties are reasonable but not self-evident in conformal theories, while they can be derived in massive theories.)

Instead of repeating the complete system of definitions of the framework, we content ourselves – but still with some care about stating the results reliably – with an instructive exposition with emphasis on the physical motivation of the basic objects of our concern (localized morphisms, intertwiners, and the reduced field bundle in Sect. 2, statistics operators and the exchange algebra in Sect. 3, automorphisms in Sect. 4). The reader not familiar with the powerful C^* algebra algorithm is recommended to consult in cases of doubt the original papers [11, 12], the adaptation to low dimensions [13, 14], and [15] containing a pedagogigal introduction, of which we selected and reorganized the results relevant for the present analysis according to our needs.

2. Superselection Charges, Localized Morphisms, and the Reduced Field Bundle

A superselection charge is an equivalence class $[\pi]$ of C^* representations $\pi:\mathscr{A}\to\mathscr{B}(\mathscr{H}_\pi)$ of the local net $\mathscr{A}$ of observables. The latter is (the norm closure of) the union of all its C^* subalgebras (local algebras) $\mathscr{A}(\mathscr{O})$ of observables localized in the finite space-time region [3] $\mathscr{O}$. We are interested in representations locally equivalent to the vacuum representation π_0. This property is usually motivated by the physical picture of localizable charges (particles) undetectable in the causal complement $\mathscr{O}'$ of any finite space-time region $\mathscr{O}$. Though the particle picture is meaningless in a conformal theory, local equivalence of representations of $\mathscr{A}$ is automatically guaranteed in the case of a conformal theory on the (compactified) light-cone [16].

Localizable charges are described in terms of localized and transportable C^* morphisms $\varrho:\mathscr{A}\to\mathscr{A}$ such that $[\pi]\ni\pi_0\circ\varrho:\mathscr{A}\to\mathscr{B}(\mathscr{H}_0)$ are all realized on the Hilbert space $\mathscr{H}_0$ of the vacuum representation. One may assume π_0 faithful and use it is an identification of $\mathscr{A}$ with its image $\pi_0(\mathscr{A})\subset\mathscr{B}(\mathscr{H}_0)$; then the representation π is given by the action of $\mathscr{A}$ on $\mathscr{H}_0$ via the morphism ϱ:

$$A:\Psi\mapsto\varrho(A)\Psi$$

or, better,

$$A:(\varrho,\Psi)\mapsto(\varrho,\varrho(A)\Psi). \tag{2.1}$$

[3] It is sufficient to consider only space-time regions $\mathscr{O}$ which are double-cones (intersections of a forward and a backward light-cone) in $d\geqq 2$, respectively intervals of the real axis in a light-cone theory

The less ambiguous notation $(\varrho, \Psi) \in (\varrho, \mathscr{H}_0) \equiv \mathscr{H}_\varrho$ is employed in order to indicate the nontrivial action of $\mathscr{A}$.

2.1. Definition. (i) A morphism ϱ is said to be localized in the space-time region $\mathcal{O}$, if

$$\varrho(A) = A \ \forall A \in \mathscr{A}(\mathcal{O}'). \tag{2.2}$$

(ii) It is called transportable, if for every $\hat{\mathcal{O}}$ there is an equivalent morphism $\hat{\varrho} \in [\varrho]$ localized in $\hat{\mathcal{O}}$. (See also the remark after Proposition 2.5.)

The product $\varrho_1 \circ \varrho_2 = \varrho_1 \varrho_2$ has the natural interpretation of describing a physical situation containing both charges ϱ_i.

2.2. Proposition [11, Sect. 2]. (i) *The composition induces an abelian class multiplication of superselection charges.*
(ii) *If ϱ_i are localized in $\mathcal{O}_i$ then $\varrho_1\varrho_2$ is localized in $\mathcal{O}_1 \vee \mathcal{O}_2$, the smallest double-cone (interval) containing both $\mathcal{O}_i$.*
(iii) *If $\mathcal{O}_i$ are at space-like distance (causally disconnected), then $\varrho_1\varrho_2 = \varrho_2\varrho_1$.*

Even if ϱ_i are irreducible, $\varrho_1\varrho_2$ needs not to be irreducible (non-additive superselection quantum numbers like irreducible representations of non-abelian Lie groups). But, by Borchers' property B, for every (irreducible) subrepresentation π on $\mathscr{H}_\pi = E\mathscr{H}_0$ of $\varrho_1\varrho_2$, $E \in \varrho_1\varrho_2(\mathscr{A})'$ a projection, there is an isometry T in some local algebra $\mathscr{A}(\mathcal{O})$, $TT^* = E$, $T^*T = \mathbf{1}$, mapping $E\mathscr{H}_0$ unitarily onto $\mathscr{H}_0$. Then the map

$$\varrho(A) := T^* \varrho_1\varrho_2(A) T \tag{2.3}$$

is again a localized morphism, equivalent to the subrepresentation π, and transportable if ϱ_i are. T is an intertwiner from ϱ to $\varrho_1\varrho_2$.

2.3. Definition. (i) T is an intertwiner from ϱ to ϱ' (or: $T \in (\varrho'|\varrho)$), if

$$T\varrho(A) = \varrho'(A) T \ \forall A \in \mathscr{A}. \tag{2.4}$$

(ii) T is an isometry if $T^*T = \mathbf{1}$.

In other words, $T \in (\varrho'|\varrho)$ considered as a map from $\mathscr{H}_\varrho$ to $\mathscr{H}_{\varrho'}$

$$T: (\varrho, \Psi) \mapsto (\varrho', T\Psi) \tag{2.5}$$

intertwines the respective actions (2.1) of $\mathscr{A}$.

2.4. Lemma [11]. (i) *If ϱ_i are localized in $\mathcal{O}_i$, $i = 1, 2$, then $(\varrho_2|\varrho_1) \subset \mathscr{A}(\mathcal{O}_1 \vee \mathcal{O}_2)$.*
(ii) *If ϱ_i have no common (equivalent) subrepresentation, then $(\varrho_2|\varrho_1) = \{0\}$.*
(iii) *If and only if ϱ is irreducible, then $(\varrho|\varrho) \equiv \varrho(\mathscr{A})'$ equals $\mathbb{C}$ (Schur's Lemma).*
(iv) *If $T \in (\varrho_2|\varrho_1)$, $S \in (\varrho_3|\varrho_2)$, then $T^* \in (\varrho_1|\varrho_2)$ and $ST \in (\varrho_3|\varrho_1)$. Moreover, $T \in (\varrho_2\varrho|\varrho_1\varrho)$ and $\varrho(T) \in (\varrho\varrho_2|\varrho\varrho_1)$.*
(v) *If ϱ_1 is irreducible, then $T \in (\varrho_2|\varrho_1)$ is a multiple of an isometry.*

Now we can physically motivate and understand the abstract action of charged fields interpolating among different superselection sectors.

Let $\varrho, \varrho_\alpha, \varrho_\beta$ be irreducible transportable morphisms such that ϱ_β is equivalent to some subrepresentation of $\varrho_\alpha\varrho$. Let $T_e \in (\varrho_\alpha\varrho|\varrho_\beta)$ be an isometry. For $A \in \mathscr{A}$ define the linear operator $(e, A): \mathscr{H}_{\varrho_\alpha} \to \mathscr{H}_{\varrho_\beta}$ by

$$(e, A)(\varrho_\alpha, \Psi) := (\varrho_\beta, T_e^* \varrho_\alpha(A) \Psi). \tag{2.6}$$

This corresponds to the action of A in the background charge ϱ_α, addition of the charge ϱ, and subsequent projection and unitary transport by means of $T_e^* \in (\varrho_\beta|\varrho_\alpha\varrho)$ of the state $(\varrho_\alpha\varrho, \varrho_\alpha(A)\Psi) \in \mathscr{H}_{\varrho_\alpha\varrho}$ to a state in $\mathscr{H}_{\varrho_\beta}$. The collective label e ("super-selection channel") stands for the three irreducible morphisms involved as well as for the specific intertwiner T_e chosen, see below. We shall call $s(e) = \varrho_\alpha$, $r(e) = \varrho_\beta$ the "source" and the "range" of e (referring to the interpolation of the map (e, A)), and $c(e) = \varrho$ the "charge" of e (referring to the charge added by the operator (e, A)), and write $e = (\varrho_\alpha, \varrho, \varrho_\beta)$ if we want to specify only its source, charge, and range.

We shall now restrict ourselves to the set Δ_0 of transportable morphisms, possessing conjugates and having finite statistics (see below). Δ_0 is closed under composition and taking subrepresentations [11, 12]. Moreover, if ϱ_α, $\varrho \in \Delta_0$ are irreducible, then $\varrho_\alpha\varrho$ contains only finitely many inequivalent irreducible subrepresentations ϱ_β, each occurring with finite multiplicity

$$(N_\varrho)_\alpha^\beta \equiv \dim(\varrho_\alpha\varrho|\varrho_\beta) < \infty\,. \tag{2.7}$$

Two irreducible localized morphisms $\varrho, \bar\varrho$ are called conjugate to each other, if $[\varrho\bar\varrho]$ contains the vacuum representation as a subrepresentation, i.e. if there are isometries

$$R \in (\bar\varrho\varrho|id)\,, \qquad \bar R \in (\varrho\bar\varrho|id)\,. \tag{2.8}$$

Finiteness of statistics assures that conjugates are unique (up to equivalence).

After these preliminaries, we define the reduced field bundle.

2.5. Definition. Let $V_0 \subset \Delta_0$ be a countable[4] collection of "reference morphisms," one per equivalence class of irreducible morphisms in Δ_0 (or in some subset Δ closed under composition and taking subrepresentations and conjugates), $id \in V_0$. For every triple $\varrho, \varrho_\alpha, \varrho_\beta \in V_0$ let $N = (N_\varrho)_\alpha^\beta$, and if $N \neq 0$ fix an orthonormal basis of intertwiners $T_e = T^i \in (\varrho_\alpha\varrho|\varrho_\beta)$:

$$T^{i*}T^j = \delta_{ij}\,, \qquad i, j = 1, \ldots, N \tag{2.9}$$

(i.e. here and from now on the collective label e consists apart from its charge, source, and range also of a multiplicity index $i = 1, \ldots, N$. We shall never display the multiplicity indices, and adopt an implicit summation convention for i whenever T_e and T_e^* occur in the same formula). Then

$$\sum_e T_e T_e^* = \mathbf{1}\,, \tag{2.10}$$

where the summation extends over $r(e)$, while $s(e)$, $c(e)$ are kept fixed. If ϱ_α or $\varrho = id$, we choose $T_e = \mathbf{1}$. If $\varrho_\beta = id$ (hence $\varrho_\alpha = \bar\varrho$), we call $T_e =: R_\varrho$.

The reduced space bundle is the sum of Hilbert spaces

$$\mathscr{H} = \bigoplus_{\varrho \in V_0} \mathscr{H}_\varrho \tag{2.11}$$

equipped with the scalar product $\langle(\varrho_1, \Psi_1), (\varrho_2, \Psi_2)\rangle = \delta_{\varrho_1\varrho_2}\langle\Psi_1, \Psi_2\rangle$ induced from the scalar product of $\mathscr{H}_0$.

[4] Countability seems not to be a severe restriction

The reduced field bundle is the sum of vector spaces (extending over all superselection channels e of V_0)

$$\mathscr{F} = \bigoplus_e (e, \mathscr{A}) \tag{2.12}$$

with operators $(e, A) \in \mathscr{F}$ acting on states $(\varrho_\alpha, \Psi) \in \mathscr{H}$ by (cf. (2.6))

$$(e, A)(\varrho_\alpha, \Psi) = \delta_{\varrho_\alpha s(e)}(r(e), T_e^* \varrho_\alpha(A)\Psi). \tag{2.13}$$

2.6. Proposition [13, 14]. *$\mathscr{F}$ is a Banach subalgebra of $\mathscr{B}(\mathscr{H})$. More specifically we have:*

(i) $\|(e, A)\| \leqq \|A\|$.
(ii) *The product of $(e_i, A_i) \in \mathscr{F}$ with charges $\varrho_i = c(e_i)$ is again in $\mathscr{F}$:*

$$(e_2, A_2)(e_1, A_1) = \delta_{s(e_2) r(e_1)} \sum_{e,f} D_{e_1 \circ e_2; f, e}(e, A_f) \in \mathscr{F}, \tag{2.14}$$

where the finite sum extends over all f with $s(f) = \varrho_1$, $c(f) = \varrho_2$ and all e with $s(e) = s(e_1)$, $r(e) = r(e_2)$ and $c(e) = r(f)$ (that is, (e, A_f) do the same interpolation as the product on the left-hand side, and carry charges contained in $\varrho_1 \varrho_2$). The notation $e_1 \circ e_2$ indicates the condition $s(e_2) = r(e_1)$. In (2.14), *with $\varrho_\alpha = s(e_1)$:*

$$A_f = T_f^* \varrho_1(A_2) A_1 \in \mathscr{A}, \tag{2.15}$$

$$D_{e_1 \circ e_2; f, e} = T_{e_2}^* T_{e_1}^* \varrho_\alpha(T_f) T_e \in (r(e)|r(e)) = \mathbb{C}. \tag{2.16}$$

(iii) *$\mathscr{A}$ is contained in $\mathscr{F}$ by the identification* (cf. (2.1))

$$A = \sum_{e, c(e) = id} (e, A). \tag{2.17}$$

(iv) *The following two definitions for (e, A) to be localized in $\mathcal{O}$ (or: $(e, A) \in \mathscr{F}(\mathcal{O})$) are equivalent:*

1. *(e, A) commutes with $\mathscr{A}(\mathcal{O})$ acting on $\mathscr{H}$.*
2. *There are $\hat{\varrho}$ equivalent to $\varrho = c(e)$, $\hat{\varrho}$ localized in $\mathcal{O}$, and $U \in (\hat{\varrho}|\varrho)$ unitary, such that $UA \in \mathscr{A}(\mathcal{O})$.*

(v) *If $(e, A) \in \mathscr{F}(\mathcal{O})$, then $(e, A)^* \in \mathscr{F}(\mathcal{O})$.*

Remarks. 1. By 2.6(iv(2)), the localization is independent of $s(e)$, $r(e)$. Every $(e, A) \in \mathscr{F}$ with A a local operator is contained in some $\mathscr{F}(\mathcal{O})$.
2. It might be reasonable in certain physical contexts to consider a weaker localizability reflecting some "minimal volume" occupied by charges; namely, instead of 2.1(ii) assume only that, if ϱ is localized in $\mathcal{O}$, then for every *translate* $\mathcal{O} + x$ there is $\hat{\varrho} \in [\varrho]$ localized in $\mathcal{O} + x$. The results of this and the following sections are not affected except that 2.6(iv) is then only meaningful for double cones $\mathcal{O}$ sufficiently large to contain the localization region of an equivalent of ϱ, i.e. charged operators cannot be better localized than the charges they carry.

3. Statistics and Exchange Algebra

The statistics of a localized morphism ϱ is a unitary operator $\varepsilon_\varrho = \varepsilon(\varrho, \varrho) \in (\varrho^2|\varrho^2) = \varrho^2(\mathscr{A})' \subset \mathscr{A}$. It induces a collection of unitary representations of the braid groups

B_n (in sufficiently high dimensions: of the permutation groups S_n), related to wave function permutations [17], or commutation relations [12] for fields carrying charge $[\varrho]$. More generally, for any two localized morphisms ϱ_1, ϱ_2 there are unitary operators

$$\varepsilon(\varrho_1,\varrho_2)\in(\varrho_2\varrho_1|\varrho_1\varrho_2), \quad \varepsilon(\varrho_1,\varrho_2)\varepsilon(\varrho_1,\varrho_2)^*=\varepsilon(\varrho_1,\varrho_2)^*\varepsilon(\varrho_1,\varrho_2)=\mathbf{1} \tag{3.1}$$

with properties generalizing representations of the braid groups (see 3.3(i) below).

3.1. Proposition [11, 13]. *Let ϱ_i be localized in $\mathcal{O}_i$. Let $\hat{\varrho}_i\in[\varrho_i]$ be localized in $\hat{\mathcal{O}}_i$ such that $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ are at space-like distance, and $U_i\in(\hat{\varrho}_i|\varrho_i)$ unitary. Then the unitary statistics operator $\varrho_2(U_1^*)U_2^*U_1\varrho_1(U_2)$ is independent of U_i and does not change if $\hat{\mathcal{O}}_i$ are continuously changed within the space-like complements of each other. Thus, in dimension $d\leqq 1+1$, where $\mathcal{O}'$ has two connected components, it can take only two values:*

$$\varrho_2(U_1^*)U_2^*U_1\varrho_1(U_2)=:\begin{cases}\varepsilon(\varrho_1,\varrho_2) & \text{if } \hat{\mathcal{O}}_1>\hat{\mathcal{O}}_2\\ \varepsilon(\varrho_2,\varrho_1)^* & \text{if } \hat{\mathcal{O}}_2>\hat{\mathcal{O}}_1\end{cases}, \tag{3.2}$$

where a global ordering $>$ for space-like separated double-cones (disjoint intervals) has been chosen. In dimension $d\geqq 2+1$, where $\mathcal{O}'$ is connected, these two values coincide:

$$\varepsilon(\varrho_1,\varrho_2)\varepsilon(\varrho_2,\varrho_1)=\mathbf{1}. \tag{3.3}$$

Remark. If charges can only be localized in narrow space-like cones extending to infinity ("gauge charges") [18], then (3.3) holds only for $d\geqq 3+1$.

3.2. Proposition. (i) *The statistics operators of product morphisms are*

$$\begin{aligned}\varepsilon(\varrho_1\varrho_2,\varrho_3)&=\varepsilon(\varrho_1,\varrho_3)\varrho_1(\varepsilon(\varrho_2,\varrho_3)),\\ \varepsilon(\varrho_3,\varrho_1\varrho_2)&=\varrho_1(\varepsilon(\varrho_3,\varrho_2))\varepsilon(\varrho_3,\varrho_1).\end{aligned} \tag{3.4}$$

(ii) *If $T\in(\varrho_2|\varrho_1)$, then*

$$\begin{aligned}\varrho_3(T)\varepsilon(\varrho_1,\varrho_3)&=\varepsilon(\varrho_2,\varrho_3)T,\\ \varrho_3(T)\varepsilon(\varrho_3,\varrho_1)^*&=\varepsilon(\varrho_3,\varrho_2)^*T.\end{aligned} \tag{3.5}$$

Proof. For the first equations, one may choose for simplicity $\hat{\mathcal{O}}_3<\mathcal{O}_1$, $\mathcal{O}_2$, $\hat{\varrho}_1=\varrho_1$, $\hat{\varrho}_2=\varrho_2$, $U_1=U_2=\mathbf{1}$. Then

$$\begin{aligned}\varepsilon(\varrho_1,\varrho_3)\varrho_1(\varepsilon(\varrho_2,\varrho_3))&=U_3^*\varrho_1(U_3)\varrho_1(U_3^*\varrho_2(U_3))=U_3^*\varrho_1\varrho_2(U_3)=\varepsilon(\varrho_1\varrho_2,\varrho_3),\\ \varrho_3(T)\varepsilon(\varrho_1,\varrho_3)&=\varrho_3(T)U_3^*\varrho_1(U_3)=U_3^*\hat{\varrho}_3(T)\varrho_1(U_3)\\ &=U_3^*T\varrho_1(U_3)=U_3^*\varrho_2(U_3)T=\varepsilon(\varrho_2,\varrho_3)T,\end{aligned}$$

where we used Lemma 2.4(i) and (2.2). The second equations are obtained by choosing $\hat{\mathcal{O}}_3>\mathcal{O}_1$, $\mathcal{O}_2$ instead.

3.3. Corollary.
(i) $\varrho_3(\varepsilon(\varrho_1,\varrho_2))\varepsilon(\varrho_1,\varrho_3)\varrho_1(\varepsilon(\varrho_2,\varrho_3))=\varepsilon(\varrho_2,\varrho_3)\varrho_2(\varepsilon(\varrho_1,\varrho_3))\varepsilon(\varrho_1,\varrho_2)$.
(ii) $\varrho_3(R)=\varepsilon(\bar{\varrho}\varrho,\varrho_3)R=\varepsilon(\varrho_3,\bar{\varrho}\varrho)^*R$ *for* $R\in(\bar{\varrho}\varrho|id)$.

Proof. By the substitutions $\varrho_2\to\varrho_2\varrho_1$, $\varrho_1\to\varrho_1\varrho_2$ and $T=\varepsilon(\varrho_1,\varrho_2)$ respectively $\varrho_2\to\bar{\varrho}\varrho$, $\varrho_1\to id$ and $T=R$ in 3.2(ii), using $\varepsilon(\varrho_1,\varrho_2)=\mathbf{1}$ if ϱ_1 or $\varrho_2=id$.

The Corollary 3.3(i) and (3.1) are the relevant equations for the braid group property of the statistics operators. In particular, for $\varrho_i=\varrho$ the unitary operators $\varrho^{k-1}(\varepsilon_\varrho)$ satisfy the defining relations of the generators σ_k of the braid group, reducing to the permutation group if (in high dimensions) also (3.3) holds.

3.4. Lemma. *Let* $(e_i, A_i)\in\mathscr{F}(\mathcal{O}_i)$, $\varrho_i=c(e_i)$. *Then*

$$\varrho_1(A_2)A_1=\begin{Bmatrix}\varepsilon(\varrho_2,\varrho_1)\\ \varepsilon(\varrho_1,\varrho_2)^*\end{Bmatrix}\varrho_2(A_1)A_2 \quad \textit{if} \quad \begin{Bmatrix}\mathcal{O}_2>\mathcal{O}_1\\ \mathcal{O}_1>\mathcal{O}_2\end{Bmatrix}. \tag{3.6}$$

Proof. By Proposition 2.6(v(2)) let $C_i=U_iA_i\in\mathscr{A}(\mathcal{O}_i)$, $U_i\in(\hat\varrho_i|\varrho_i)$, $\hat\varrho_i$ localized in $\mathcal{O}_i$. Then for $\mathcal{O}_i$ at space-like distance one has with (2.2)

$$\varrho_i(A_j)A_i=\varrho_i(U_j^*C_j)U_i^*C_i=\varrho_i(U_j^*)U_i^*\hat\varrho_i(C_j)C_i=\varrho_i(U_j^*)U_i^*C_jC_i,$$

which implies the claim since C_i commute.

3.5. Lemma. (i) *The operators* $\varrho_\alpha(\varepsilon(\varrho_1,\varrho_2))$ *and* $\varrho_\alpha(\varepsilon(\varrho_2,\varrho_1)^*)$ *unitarily map* $(\varrho_\alpha\varrho_1\varrho_2|\varrho_\gamma)$ *into* $(\varrho_\alpha\varrho_2\varrho_1|\varrho_\gamma)$.

(ii) *If* $c(e_i)=\varrho_i$, $s(e_1)=\varrho_\alpha$, $r(e_1)=s(e_2)$, $r(e_2)=\varrho_\gamma$ *all in* V_0, *then* $T_{e_1}T_{e_2}\in(\varrho_\alpha\varrho_1\varrho_2|\varrho_\gamma)$, *and*

$$\varrho_\alpha\begin{pmatrix}\varepsilon(\varrho_1,\varrho_2)\\ \varepsilon(\varrho_2,\varrho_1)^*\end{pmatrix}T_{e_1}T_{e_2}=\sum_{e_2'\circ e_1'}R^{(\pm)}_{e_2'\circ e_1';e_1\circ e_2}T_{e_2'}T_{e_1'}, \tag{3.7}$$

where $c(e_i')=\varrho_i$, $s(e_2')=\varrho_\alpha$, $r(e_2')=s(e_1')$, $r(e_1')=\varrho_\gamma$, *and*

$$R^{(\pm)}_{e_2'\circ e_1';e_1\circ e_2}:=T_{e_1'}^*T_{e_2'}^*\varrho_\alpha\begin{pmatrix}\varepsilon(\varrho_1,\varrho_2)\\ \varepsilon(\varrho_2,\varrho_1)^*\end{pmatrix}T_{e_1}T_{e_2}\in\mathbb{C}. \tag{3.8}$$

(iii) *For* $c(e_i)$, $s(e_1)$, $r(e_2)$ *fixed, the finite square matrices* $D_{e_1\circ e_2;f,e}$ *(see* (2.16)*) and* $R^{(\pm)}_{e_1\circ e_2;e_2'\circ e_1'}$ *are unitary, and* $R^{(-)}=R^{(+)\dagger}$:

$$\sum_{e_2'\circ e_1'}R^{(\pm)}_{e_1\circ e_2;e_2'\circ e_1'}R^{(\mp)}_{e_2'\circ e_1';e_1''\circ e_2''}=\delta_{e_1\circ e_2,e_1''\circ e_2''}, \tag{3.9}$$

$$\sum_{e,f}D_{e_1\circ e_2;f,e}D^*_{e_1'\circ e_2';f,e}=\delta_{e_1\circ e_2,e_1'\circ e_2'},\qquad \sum_{e_1\circ e_2}D_{e_1\circ e_2;f,e}D^*_{e_1\circ e_2;f',e'}=\delta_{ee'}\delta_{ff'} \tag{3.10}$$

Proof. The first statement follows from Lemma 2.4(iv). By 2.4(iii) the operator in (3.8) vanishes for $r(e_1')\neq r(e_2)$ and is a scalar otherwise. Then (3.7), (3.9), and (3.10) follow from the orthonormality (2.9) and completeness (2.10) of intertwiners.

3.6. Proposition (Exchange Algebra [13]). *In the reduced field bundle 2.5, 2.6, one has the following commutation relations: Let* $(e_i, A_i)\in\mathscr{F}(\mathcal{O}_i)$, $r(e_1)=s(e_2)=\varrho_\beta$. *Then* [5] *(with the remaining specifications as in Lemma 3.5),*

$$(e_2,A_2)(e_1,A_1)=\sum_{e_2'\circ e_1'}R^{(\pm)}_{e_1\circ e_2;e_2'\circ e_1'}(e_1',A_1)(e_2',A_2)\quad \textit{if}\quad \begin{Bmatrix}\mathcal{O}_2>\mathcal{O}_1\\ \mathcal{O}_1>\mathcal{O}_2\end{Bmatrix}. \tag{3.11}$$

[5] Note that the structure constants R (3.8) of the exchange algebra and D (2.16) of the operator product expansion (known as "braid matrices" [5–7, 19] and as "duality matrices" [7, 19] respectively in conformal field theory, and satisfying the braid and "pentagon" equations by virtue of Proposition 3.2 and Corollary 3.3) depend on the reference morphisms V_0 and the intertwiners T_e chosen in Definition 2.5. While their transformation behaviour is manifest from the definitions, their actual values are of limited relevance. The *intrinsic* quantities are, e.g., their eigenvalues which can be typically expressed in terms of the invariant statistics parameters introduced below, and the Markov traces associated to the statistics [13]

Proof. Acting on some state (ϱ_α, Ψ), we obtain for the left-hand side

$$(\varrho_\gamma, T^*_{e_2}\varrho_\beta(A_2)T^*_{e_1}\varrho_\alpha(A_1)\Psi) = (\varrho_\gamma, T^*_{e_2}T^*_{e_1}\varrho_\alpha(\varrho_1(A_2)A_1)\Psi),$$

and a similar expression for every operator product on the right-hand side. Using the previous lemmata, the claim follows easily.

4. Automorphisms and Symmetries of the Structure Constants

4.1. Definition. For $\varrho \in \Delta_0$ irreducible, $\bar{\varrho}$ a conjugate, $R \in (\bar{\varrho}\varrho|id)$ an isometry, the statistics parameter of the sector $[\varrho]$ is

$$\lambda_\varrho := R^*\bar{\varrho}(\varepsilon_\varrho)R \in \mathbb{C}. \tag{4.1}$$

The statistics parameter, as an element of $(\varrho|\varrho) = \varrho(\mathscr{A})'$, is a scalar, and does not vanish by definition of Δ_0 (finite statistics). It is independent of the choice of the isometry R and depends only on the equivalence class of ϱ. The statistics parameters of conjugate morphisms coincide. We denote by

$$\frac{\omega(\varrho)}{d(\varrho)} := \lambda(\varrho) = \lambda(\bar{\varrho}) \tag{4.2}$$

the decomposition into a phase $\omega(\varrho)$ (statistics phase) generalizing the distinction between bosons and fermions, and the inverse modulus $d(\varrho) \geqq 1$ (statistical dimension) generalizing the order of (permutation group) para-statistics.

Remarks. 1. For conformally covariant theories on the light-cone, a spin-statistics theorem [14]

$$\omega(\varrho) = \exp 2\pi i h_\varrho, \tag{4.3}$$

relates the statistics phase $\omega(\varrho)$ of a covariant sector to the conformal scaling dimensions $h_\varrho (\mathrm{mod}\, \mathbb{Z})$ of fields carrying charge $[\varrho]$. Analogues are expected to hold also for the Poincaré spin of more general low-dimensional exchange fields. So far, however, the reason for the validity of a spin-statistics theorem is understood only for covariance groups that can geometrically change the sign of a space-like separation by real transformations: the conformal group acting on the compactified light-cone, and the Poincaré group in $2+1$ dimensions. In fact, this action of the covariance group does not imply that the two statistics operators (3.2) coincide, since in these situations the relevant ordering is defined with respect to some reference frame (a "point at infinity" [14] respectively, a space-like direction [20]), but rather relates their difference (the "monodromy" operator $\varepsilon(\varrho_1, \varrho_2)\varepsilon(\varrho_2, \varrho_1)$) to the covariance quantum numbers (spin).
2. The statistics parameter $\lambda(\varrho)$ is a convex sum of the eigenvalues of ε_ϱ. Namely, if $\varepsilon_\varrho = \sum \mu_i E_i$ is the spectral decomposition of the statistics operator, then $\lambda(\varrho) = \sum \mu_i R^*\bar{\varrho}(E_i)R$, where $R^*\bar{\varrho}(E_i)R \in (\varrho|\varrho)$ are positive scalars summing up to 1. In particular, $d(\varrho) \geqq 1$. In conformal models, the statistical dimensions $d(\varrho)$ are known as the normalized entries $S_{0\varrho}/S_{00}$ of the modular matrix, measuring the relative dimensions of representations of the chiral algebra [21]. More generally, the square of the statistical dimension is the von Neumann index of the inclusion of factors $\varrho(\mathscr{A}(\mathscr{O})) \subset \mathscr{A}(\mathscr{O})$ [22].

4.2. Lemma [11, Proposition 2.7]. (i) *The following four definitions for $\tau \in \Delta_0$ irreducible to be an automorphism are equivalent:*

1. *τ possesses an inverse $\tau^{-1} \in \Delta_0$.*
2. *τ^2 is irreducible.*
3. *ε_τ is a scalar (hence $\varepsilon_\tau = \lambda(\tau) = \omega(\tau)$).*
4. *$d(\tau) = 1$.*

(ii) *For $\tau \in \Delta_0$ an automorphism and $\varrho \in \Delta_0$ irreducible, $\varrho\tau \simeq \tau\varrho$ are again irreducible, $d(\varrho\tau) = d(\tau\varrho) = d(\varrho)$, and*

$$\varepsilon(\varrho, \tau)\varepsilon(\tau, \varrho) = \varepsilon(\tau, \varrho)\varepsilon(\varrho, \tau) = \frac{\omega(\tau\varrho)}{\omega(\tau)\omega(\varrho)} =: \Omega_\tau(\varrho). \tag{4.4}$$

(iii) *The equivalence classes of automorphisms in Δ_0 define an abelian group Γ_0 by class multiplication: $[\tau_1][\tau_2] = [\tau_1\tau_2]$, and $[\tau]^{-1} = [\tau^{-1}] = [\bar{\tau}]$.*

Proof. If τ has an inverse, the irreducibility of $\tau\varrho$ and $\varrho\tau$ follows immediately from that of ϱ:

$$\varrho(\tau(\mathscr{A}))' = \varrho(\mathscr{A})' = \mathbb{C}, \qquad \tau(\varrho(\mathscr{A}))' = \tau(\varrho(\mathscr{A})') = \tau(\mathbb{C}) = \mathbb{C}.$$

In particular, τ^2 is irreducible, and $\varepsilon_\tau \in \tau^2(\mathscr{A})$ is a scalar, which coincides with the statistics parameter by definition of the latter, and with the statistics phase by unitarity. Hence (1) implies (2) implies (3) implies (4). Since a nontrivial convex sum of complex phases has modulus < 1, (4) implies that ε_τ has only one eigenvalue, hence (3). To see that (3) implies (1) first note that τ is injective since localized morphisms are norm-preserving maps. If $\varepsilon(\tau)$ is a scalar, then τ is also surjective: let $A \in \mathscr{A}(\mathscr{O}_1)$ and τ be localized in $\mathscr{O}$. Choose $\hat{\tau}$ equivalent to τ and localized in $\hat{\mathscr{O}} > (\mathscr{O}_1 \vee \mathscr{O})$, and $U \in (\hat{\tau}|\tau)$ unitary. Then $U = \varepsilon_\tau \tau(U)$, and

$$A = \hat{\tau}(A) = U\tau(A)U^* = \tau(U)\tau(A)\tau(U^*) \in \tau(\mathscr{A}).$$

Hence τ has an inverse. Since τ is trivial on $\mathscr{A}(\mathscr{O}')$, so is its inverse, hence τ^{-1} is localized in $\mathscr{O}$. It is easy to see that τ^{-1} is transportable if τ is. Of course, the inverse is a conjugate and has finite statistics, hence is in Δ_0. To prove (4.4) remark that $\bar{\varrho}\tau^{-1}$ is conjugate to $\tau\varrho$ with $(\bar{\varrho}\tau^{-1} \circ \tau\varrho|id) = (\bar{\varrho}\varrho|id)$. Let R be an isometry in $(\bar{\varrho}\varrho|id)$. By Proposition 3.2 and Corollary 3.3, one has

$$\begin{aligned}\lambda(\tau\varrho) &= R^*\bar{\varrho}\tau^{-1}[\varepsilon(\tau\varrho, \tau\varrho)]R = R^*\bar{\varrho}\tau^{-1}[\tau(\varepsilon(\tau, \varrho))\varepsilon_\tau\tau^2(\varepsilon_\varrho)\tau(\varepsilon(\varrho, \tau))]R \\ &= \omega(\tau)R^*\bar{\varrho}[\varepsilon(\tau, \varrho)\tau(\varepsilon_\varrho)\varepsilon(\varrho, \tau)]R = \omega(\tau)R^*\bar{\varrho}[\varrho(\varepsilon(\varrho, \tau))\varepsilon_\varrho\varrho(\varepsilon(\tau, \varrho))]R \\ &= \omega(\tau)\varepsilon(\varrho, \tau)R^*\bar{\varrho}(\varepsilon_\varrho)R\varepsilon(\tau, \varrho) = \omega(\tau)\lambda(\varrho)\varepsilon(\varrho, \tau)\varepsilon(\tau, \varrho).\end{aligned}$$

Comparing phases and moduli yields the claim. (iii) is obvious.

4.3. Lemma. *Let $\varrho, \varrho_i \in \Delta_0$ be irreducible, τ, $\tau_i \in \Delta_0$ automorphisms. Then*

(i) $\Omega_{\tau_1\tau_2}(\varrho) = \Omega_{\tau_1}(\varrho)\Omega_{\tau_2}(\varrho)$.
(ii) *If ϱ is equivalent to a subrepresentation of $\varrho_1\varrho_2$, then $\Omega_\tau(\varrho) = \Omega_\tau(\varrho_1)\Omega_\tau(\varrho_2)$.*

Proof. (i) follows from Proposition 3.2(i):

$$\varepsilon(\tau_1\tau_2, \varrho)\varepsilon(\varrho, \tau_1\tau_2) = \varepsilon(\tau_1, \varrho)\tau_1(\varepsilon(\tau_2, \varrho))\tau_1(\varepsilon(\varrho, \tau_2))\varepsilon(\varrho, \tau_1).$$

For (ii) choose an isometry $T \in (\varrho_1\varrho_2|\varrho)$. Then, again using Proposition 3.2:

$$\begin{aligned}\varepsilon(\tau,\varrho)\varepsilon(\varrho,\tau) &= \varepsilon(\tau,\varrho)\tau(T^*T)\varepsilon(\varrho,\tau) = T^*\varepsilon(\tau,\varrho_1\varrho_2)\varepsilon(\varrho_1\varrho_2,\tau)T \\ &= T^*\varrho_1(\varepsilon(\tau,\varrho_2))\varepsilon(\tau,\varrho_1)\varepsilon(\varrho_1,\tau)\varrho_1(\varepsilon(\varrho_2,\tau))T \\ &= \Omega_\tau(\varrho_1)T^*\varrho_1(\varepsilon(\tau,\varrho_2)\varepsilon(\varrho_2,\tau))T\end{aligned}$$

implies the claim.

In view of the spin-statistics theorem (4.3), this lemma provides non-trivial selection rules for scaling dimensions (respectively fractional spins).

Let us return to the group Γ_0 of equivalence classes of automorphisms. In general it is impossible to choose representatives $\tau \in [\tau] \in \Gamma_0$, forming a group isomorphic to Γ_0 by their individual multiplication. We now give criteria, when such an intrinsic obstruction is present or absent.

4.4. Lemma. (i) *Let $\tau \in \Delta_0$ be an automorphism such that $[\tau^\nu]=[id]$. If and only if $\omega(\tau)^\nu=1$, there is $\tilde{\tau}\in[\tau]$ satisfying $\tilde{\tau}^\nu = id$.*

(ii) *Let a subgroup $\Gamma = \bigotimes_i \mathbb{Z}_{\nu_i}$ ($\mathbb{Z}_0 \equiv \mathbb{Z}$) of Γ_0 be generated by τ_i with $[\tau_i^{\nu_i}]=[id]$. If and only if $\omega(\tau_i)^{\nu_i}=1$, one may choose $\tilde{\tau}_i \in [\tau_i]$ generating a subgroup of Δ_0 isomorphic to Γ by individual multiplication.*

Proof. For the "if" statement of (i) we refer to [23, Lemma 2.4], where by virtue of permutation group statistics $\omega(\tau)^\nu=1$ is automatically guaranteed. For the reverse statement it is sufficient to compute

$$1=\varepsilon(id,\tilde{\tau})=\varepsilon(\tilde{\tau}^\nu,\tilde{\tau})=\varepsilon_{\tilde{\tau}}\tilde{\tau}(\varepsilon_{\tilde{\tau}})\dots\tilde{\tau}^{\nu-1}(\varepsilon_{\tilde{\tau}})=\omega(\tilde{\tau})^\nu=\omega(\tau)^\nu.$$

The statement (ii) is an immediate consequence of (i) for every factor $\mathbb{Z}_{\nu_i}$ of Γ'. But the generators $\tilde{\tau}_i$ of different factors may be chosen to have space-like separated localization; then they commute individually.

4.5. Lemma. *Let $\tau \in \Delta_0$ be an automorphism.*

(i) $\omega(\tau^m)=\omega(\tau)^{m^2}$.
(ii) *Suppose $[\tau^\nu]=[id]$. Then $\omega(\tau)^{\nu^2}=\omega(\tau)^{2\nu}=1$. If ν is odd, then $\omega(\tau)^\nu=1$. If for some odd μ there is a fixpoint equivalence class of τ^μ: $[\tau^\mu\varrho]=[\varrho]$, then $\omega(\tau)^\nu=1$. If τ has permutation group statistics, then $\omega(\tau)^\nu=1$.*

Proof. (i) By Proposition 3.2 one can compute $\varepsilon(\tau^m,\tau^m)$ as a product of m^2 factors $\tau^k(\varepsilon_\tau)=\omega(\tau)$, if $m \geqq 0$. For m negative, the claim follows from $\omega(\bar{\varrho})=\omega(\varrho)$.
(ii) The first statement follows from (i) and $\omega(\tau^{\nu+1})=\omega(\tau)$. If $\nu=2n+1$, then $1=\omega(\tau)^{\nu^2}=\omega(\tau)^{2\nu n+\nu}=\omega(\tau)^\nu$. If μ is odd and $[\tau^\mu\varrho]=[\varrho]$, then (by Lemma 4.3) $1=(\Omega_{\tau^\mu}(\varrho))^{-\nu}=\omega(\tau^\mu)^\nu=\omega(\tau)^{\mu^2\nu}=\omega(\tau)^\nu$, since μ^2 is also odd. If τ has permutation group statistics, then $1=\varepsilon_\tau^2=\omega(\tau)^2$ implying $\omega(\tau)^\nu=1$ also for even ν.

Remarks. 1. In conformal field theories, automorphisms with the obstruction, i.e. $\omega(\tau)^\nu=-1$, are encountered. In $SU(2)$ WZW models [24] of level k, the self-conjugate automorphisms have scaling dimensions $\frac{k}{4}$, thus by the spin-statistics theorem $\omega(\tau)^2=(-1)^k$. More generally, in $SU(N)$ WZW models [24] at odd level k there are automorphisms of order $\nu=N$, which have $\omega(\tau)^\nu=-1$. In contrast, in all

unitary minimal models [3] and coset models [25, 26] of $SU(N)$ as well as in the WZW models of even level, the fixpoint condition of the lemma applies, hence $\omega(\tau)^{\nu}=1$ for all automorphisms of order ν, and the obstruction is absent.
2. Whether the obstruction occurs or not is an instance of the intrinsic characterization of sectors given by Longo [22, II. Sect. 6]. In the case $\nu=2$ (τ self-conjugate) it amounts to the property of reality or pseudo-reality [12].

The following two lemmata display some symmetry properties of the structure constants D and R of the operator product expansion and the commutation relations of the reduced field bundle (2.14), (3.11). Besides their particular interest for our task in Sect. 5, they are of general relevance for the TPC symmetry of the exchange algebra [14] and for the study of extended algebras [26].

From now on, we fix a choice $V_0 \subset \Delta$ of irreducible reference morphisms, as in 2.5. It is not always possible to choose V_0 such that with $\tau \in V_0$ an automorphism and $\varrho \in V_0$, also $\varrho\tau$ is in V_0. But there are unique equivalent reference morphisms in V_0 denoted by

$$\varrho^{\tau} \in [\varrho\tau] \cap V_0, \qquad \tau_1+\tau_2 \in [\tau_1\tau_2] \cap V_0. \tag{4.5}$$

One should always be careful to distinguish the individual multiplication $\varrho_1 \circ \varrho_2$, class multiplication $[\varrho_1][\varrho_2]$, and multiplication within V_0 which is only defined if at least one factor is an automorphism: ϱ^{τ} respectively $\tau_1+\tau_2 \equiv \tau_1^{\tau_2}$.

For $\varrho, \tau \in V_0$ there is precisely one superselection channel with source ϱ and charge τ. The corresponding unitary basis intertwiner T_e is called $U(\varrho,\tau) \in (\varrho\tau|\varrho^{\tau})$.

For e a superselection channel of V_0, we denote by $\bar{e}$ the charge conjugate superselection channel with $s(\bar{e})=\overline{s(e)}$, $c(\bar{e})=\overline{c(e)}$, $r(\bar{e})=\overline{r(e)}$, (considered as an independent multiplet of the same size N).

For t a superselection channel of V_0 consisting of automorphisms only, in particular $r(t)=s(t)+c(t)$ and $N=1$, we denote by (et) the superselection channel with $s(et)=s(e)^{s(t)}$, $c(et)=c(e)^{c(t)}$, $r(et)=r(e)^{r(t)}$, (also considered as an independent multiplet of the same size as e).

We define coefficients

$$\zeta(e) := R_{\bar{\varrho}}^{*} R_{\bar{\varrho}_\alpha}^{*} \varrho_\alpha(\varepsilon(\bar{\varrho}_\alpha, \varrho)^{*}) T_e \varrho_\beta(T_{\bar{e}}) R_{\bar{\varrho}_\beta} \in (id|id) = \mathbb{C} \tag{4.6}$$

and

$$\begin{aligned}\mu(e,t) := {} & U^{*}(\varrho_\beta, \tau_0+\tau)\varrho_\beta(T_t^{*}) T_e^{*} \varrho_\alpha(\varepsilon(\tau_0,\varrho))\varrho_\alpha\tau_0(U(\varrho,\tau))U(\varrho_\alpha,\tau_0) \\ & \times T_{(et)} \in (r(et)|r(et)) = \mathbb{C},\end{aligned} \tag{4.7}$$

where $e=(\varrho_\alpha, \varrho, \varrho_\beta)$, $t=(\tau_0, \tau, \tau_0+\tau)$.

Remark that both ζ and μ are in fact $N \times N$ tensors in the suppressed multiplicity indices of e, $\bar{e}$ and e, (te) respectively. One may verify, that as such $\mu(e,t)$ is a unitary matrix, while ζ is unitary up to a factor:

$$\zeta^{\dagger}\zeta = \zeta\zeta^{\dagger} = \frac{d(\varrho_\beta)}{d(\varrho_\alpha)d(\varrho)}\mathbb{1}, \qquad \mu^{\dagger}\mu = \mu\mu^{\dagger} = \mathbb{1}. \tag{4.8}$$

4.6. Lemma.

$$\zeta(e_1)\zeta(e_2)R^{(\pm)}_{\bar{e}_1\circ\bar{e}_2;\bar{e}'_2\circ\bar{e}'_1} = R^{(\mp)*}_{e_1\circ e_2;e'_2\circ e'_1}\zeta(e'_2)\zeta(e'_1), \tag{4.9}$$

$$\zeta(e_1)\zeta(e_2)D_{\bar{e}_1\circ\bar{e}_2;\bar{f},\bar{e}} = D^{*}_{e_1\circ e_2;f,e}\zeta(f)\zeta(e). \tag{4.10}$$

4.7. Lemma.

$$\mu(e_1,t_1)\mu(e_2,t_2)R^{(\pm)}_{(e_1t_1)\circ(e_2t_2);(e'_2t'_2)\circ(e'_1t'_1)}$$
$$=\left\{\begin{matrix}\Omega_{\tau_1}(\varrho_2)\\ \Omega_{\tau_2}(\varrho_1)^*\end{matrix}\right\} R^{(\pm)}_{e_1\circ e_2;e'_2\circ e'_1}R^{(\pm)}_{t_1\circ t_2;t'_2\circ t'_1}\mu(e'_2,t'_2)\mu(e'_1,t'_1)\,, \tag{4.11}$$

$$\mu(e_1,t_1)\mu(e_2,t_2)D_{(e_1t_1)\circ(e_2t_2);(f,s),(e,t)}=D_{e_1\circ e_2;f,e}D_{t_1\circ t_2;s,t}\mu(s,f)\mu(e,t)\,. \tag{4.12}$$

Remarks. 1. Lemma 4.6 is an essential consistency condition for the TPC symmetry [6] of exchange fields. For the derivation in the case of conformal light-cone theories, see [14]. Yet, the general validity of the lemma raises the hope that one might derive TPC symmetry for general exchange field theories.
2. The relevance of Lemma 4.7 for the possibility to extend the local algebra $\mathscr{A}$ to some larger local algebra $\mathscr{A}_{\text{ext}}\subset\mathscr{F}$ will become apparent in the discussion in Sect. 6.

The rest of this section is devoted to the proof of the lemmata. The calculations make extensive use of the intertwiner calculus given by Definition 2.3, Lemma 2.4, Definition 2.5, Proposition 3.2, and Corollary 3.3. Since the "linear" notation used here is quite untransparent, we advise the reader to follow it in a "diagrammatical" notation [27], in which the intertwining properties of the operators involved are easily kept track of, and the numerous equivalence relations are most efficiently visualized. Let us describe this diagrammatical notation in a few words.

Operators are represented by graphs, and irreducible morphisms $\varrho\in\mathcal{V}_0$ are represented by vertical lines drawn to the right of the operator to which ϱ is applied. (No line is drawn for $\varrho=id$.) $A\cdot B$ is represented by the graph of A drawn on top of the graph of B, the graph of A^* is the upside-down mirror image of the graph of A. The graph of an intertwiner $T\in(\varrho_1\ldots\varrho_n|\varrho'_1\ldots\varrho'_m)$ has lines $\varrho'_m,\ldots,\varrho'_1$ "coming in from below," and lines $\varrho_n,\ldots,\varrho_1$ "going out above." E.g., the basic intertwiners are given by

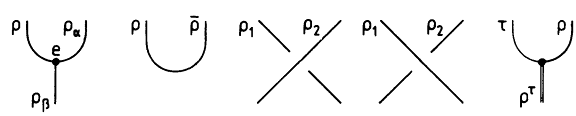

Fig. 1 T_e R_ρ $\varepsilon(\rho_1,\rho_2)$ $\varepsilon(\rho_2,\rho_1)^*$ $U(\rho,\tau)$

By Lemma 2.4(iv), the product ST of two intertwiners is again an intertwiner provided the lower (ingoing) lines of S match with the upper (outgoing) lines of T. Here, if necessary for the matching, vertical lines may be added to the left of the graph of an intertwiner without changing the operator.

With these rules, the formulae (2.4), (3.1), (3.4), (3.5), and Corollary 3.3 turn into apparently "obvious" diagrammatical identities, and the definitions (4.6), (4.7) lose much of their unwieldiness; $\zeta(e)$ describes a "turn-over" that takes e into its conjugate $\bar{e}$, while $\mu(e,t)$ describes the separation of the superselection channel (et)

[6] The role of the factors ζ is partially to account for the dependence on the bases of intertwiners chosen in 2.5, and partially to cancel the factors collected by $*$-conjugation in $\mathscr{F}$ and complex transformations of the covariance group taking x into $-x$

into e and t. In the diagrammatical notation, the manipulations below become quite natural and easily reproducible. In particular, it will be sufficient to demonstrate only the proofs of (4.9), (4.11) for the upper sign; for the lower sign the calculations are quite similar, and (4.10), (4.12) are proven in the same spirit guided by the diagrammatical intuition, but with less difficulty.

Proof of (4.9). Let us start writing the l.h.s. (left-hand side) of (4.9), by inserting the definitions, in the following form ($e_1=(\varrho_\alpha,\varrho_1,\varrho_\beta)$, $e_2=(\varrho_\beta,\varrho_2,\varrho_\gamma)$):

$$\begin{aligned}\text{l.h.s.} &= R_{\bar{\varrho}_1}^* R_{\bar{\varrho}_\alpha}^* \varrho_\alpha(\varepsilon(\bar{\varrho}_\alpha,\varrho_1)^*)\, T_{e_1}\varrho_\beta(T_{\bar{e}_1}) R_{\bar{\varrho}_\beta} \\ &\quad\times R_{\bar{\varrho}_2}^* R_{\bar{\varrho}_\beta}^* \varrho_\beta(\varepsilon(\bar{\varrho}_\beta,\varrho_2)^*)\, T_{e_2}\varrho_\gamma(T_{\bar{e}_2}\cdot T_{\bar{e}_2'}^* T_{\bar{e}_1'}^* \bar{\varrho}_\alpha(\varepsilon(\bar{\varrho}_2,\bar{\varrho}_1))\, T_{\bar{e}_2'} T_{\bar{e}_1'}) R_{\bar{\varrho}_\gamma}\,.\end{aligned}$$

We replace $T_{\bar{e}_2}T_{\bar{e}_2}^*$ by $\mathbf{1}=\sum_{\bar{e}} T_{\bar{e}}T_{\bar{e}}^*$, where the sum extends over $r(\bar{e})$ while $s(\bar{e})=\bar{\varrho}_\beta$, $c(\bar{e})=\bar{\varrho}_2$ are kept fixed, see (2.10). This does not change the expression, since the intertwiner $\varrho_\gamma(T_{\bar{e}}^*\ldots)R_{\bar{\varrho}_\gamma}\in(\varrho_\gamma r(\bar{e})|id)$ vanishes unless $r(\bar{e})=\bar{\varrho}_\gamma$, hence $\bar{e}=\bar{e}_2$. Similarly, after by virtue of (2.4) commuting $R_{\bar{\varrho}_2}^* R_{\bar{\varrho}_\beta}^* = R_{\bar{\varrho}_\beta}^*\varrho_\beta\bar{\varrho}_\beta(R_{\bar{\varrho}_2}^*)$, we may replace $R_{\bar{\varrho}_\beta}R_{\bar{\varrho}_\beta}^*$ by $\mathbf{1}=\sum_e T_eT_e^*$, where the sum extends over $r(e)$ with $s(e)=\varrho_\beta$, $c(e)=\bar{\varrho}_\beta$ fixed. Again this does not change the expression, since the intertwiner $R_{\bar{\varrho}_1}^*\ldots T_e\in(id|r(e))$ vanishes unless $r(e)=id$, hence $T_e=R_{\bar{\varrho}_\beta}$. We thus obtain

$$\begin{aligned}\text{l.h.s.} &= R_{\bar{\varrho}_1}^* R_{\bar{\varrho}_\alpha}^* \varrho_\alpha(\varepsilon(\bar{\varrho}_\alpha,\varrho_1)^*)\, T_{e_1}\varrho_\beta(T_{\bar{e}_1}\bar{\varrho}_\beta(R_{\bar{\varrho}_2}^*)\varepsilon(\bar{\varrho}_\beta,\varrho_2)^*)\, T_{e_2} \\ &\quad\times \varrho_\gamma(T_{\bar{e}_1'}^*\bar{\varrho}_\alpha(\varepsilon(\bar{\varrho}_2,\bar{\varrho}_1))\, T_{\bar{e}_2'}T_{\bar{e}_1'}) R_{\bar{\varrho}_\gamma}\,,\end{aligned}$$

which turns into

$$\begin{aligned}\text{l.h.s.} &= R_{\bar{\varrho}_1}^* R_{\bar{\varrho}_\alpha}^* \varrho_\alpha(\varepsilon(\bar{\varrho}_\alpha,\varrho_1)^*)\, T_{e_1}\varrho_\beta(T_{\bar{e}_1}T_{\bar{e}_1}^*\bar{\varrho}_\alpha\varrho_1(R_{\bar{\varrho}_2}^*)\varepsilon(\bar{\varrho}_\alpha\varrho_1,\varrho_2)^*)\, T_{e_2} \\ &\quad\times \varrho_\gamma(\bar{\varrho}_\alpha(\varepsilon(\bar{\varrho}_2,\bar{\varrho}_1))\, T_{\bar{e}_2'}T_{\bar{e}_1}') R_{\bar{\varrho}_\gamma}\,,\end{aligned}$$

where $T_{\bar{e}_1}$ has been commuted to the left using (2.4) and (3.5). Now, again $T_{\bar{e}_1}T_{\bar{e}_1}^*$ may be replaced by $\mathbf{1}=\sum_{\bar{e}} T_{\bar{e}}T_{\bar{e}}^*$, since the intertwiner $R_{\bar{\varrho}_1}^*\ldots\varrho_\beta(T_{\bar{e}})\in(id|\varrho_\beta r(\bar{e}))$ vanishes unless $r(\bar{e})=\bar{\varrho}_\beta$, hence $\bar{e}=\bar{e}_1$. Finally, commuting the various R^* to the left and the various T_e to the right, we end up with

$$\begin{aligned}\text{l.h.s.} &= R_{\bar{\varrho}_2}^* R_{\bar{\varrho}_1}^* R_{\bar{\varrho}_\alpha}^* \varrho_\alpha[\varepsilon(\bar{\varrho}_\alpha,\varrho_1)^*\varrho_1(\varepsilon(\bar{\varrho}_\alpha\varrho_1,\varrho_2)^*)\varrho_1\varrho_2\bar{\varrho}_\alpha(\varepsilon(\bar{\varrho}_2,\bar{\varrho}_1))] \\ &\quad\times T_{e_1}T_{e_2}\varrho_\gamma(T_{\bar{e}_2'}T_{\bar{e}_1'})R_{\bar{\varrho}_\gamma}\,.\end{aligned}$$

To compute the r.h.s. (right-hand side) of (4.9) we start from ($\varrho_\delta=r(e_2')=s(e_1')$)

$$\begin{aligned}\text{r.h.s.} &= R_{\bar{\varrho}_2}^* R_{\bar{\varrho}_\alpha}^* \varrho_\alpha(\varepsilon(\bar{\varrho}_\alpha,\varrho_2)^*)\, T_{e_2'}\varrho_\delta(T_{\bar{e}_2'}) R_{\bar{\varrho}_\delta} \\ &\quad\times R_{\bar{\varrho}_1}^* R_{\bar{\varrho}_\delta}^* \varrho_\delta(\varepsilon(\bar{\varrho}_\delta,\varrho_1)^*)\, T_{e_1'}\varrho_\gamma(T_{\bar{e}_1'})\cdot T_{e_1'}^* T_{e_2'}^*\varrho_\alpha(\varepsilon(\varrho_1,\varrho_2))\, T_{e_1}T_{e_2}\cdot R_{\bar{\varrho}_\gamma}\,.\end{aligned}$$

Replacing $T_{e_1'}\varrho_\gamma(T_{\bar{e}_1'})=\varrho_\delta\varrho_1(T_{\bar{e}_1'})T_{e_1'}$ and $R_{\bar{\varrho}_1}^*R_{\bar{\varrho}_\delta}^*=R_{\bar{\varrho}_\delta}^*\varrho_\delta\bar{\varrho}_\delta(R_{\bar{\varrho}_1}^*)$, eliminating $T_{e_1'}T_{e_1'}^*$ and $R_{\bar{\varrho}_\delta}R_{\bar{\varrho}_\delta}^*$ by similar arguments as before, then commuting $T_{e_2'}^*$ to the left and eliminating $T_{e_2'}T_{e_2'}^*$ as before, and finally commuting R^* to the left and T_e to the right, we end up with

$$\begin{aligned}\text{r.h.s.} &= R_{\bar{\varrho}_1}^* R_{\bar{\varrho}_2}^* R_{\bar{\varrho}_\alpha}^* \varrho_\alpha[\varepsilon(\bar{\varrho}_\alpha,\varrho_2)^*)\varrho_2(\varepsilon(\bar{\varrho}_\alpha\bar{\varrho}_2,\varrho_1)^*)\varepsilon(\varrho_1,\varrho_2)] \\ &\quad\times T_{e_1}T_{e_2}\varrho_\gamma(T_{\bar{e}_2'}T_{\bar{e}_1'})R_{\bar{\varrho}_\gamma}\,.\end{aligned}$$

Now, to arrive at the desired conclusion, we insert into the expression for the l.h.s. the following formula which is an application of Corollary 3.3(ii):

$$R^*_{\bar{\varrho}_1}=\varrho_2(R^*_{\bar{\varrho}_1})\varepsilon(\varrho_1\bar{\varrho}_1,\varrho_2)=\varrho_2(\bar{\varrho}_2(R^*_{\bar{\varrho}_1})\varepsilon(\bar{\varrho}_2,\varrho_1\bar{\varrho}_1)^*)\varepsilon(\varrho_1\bar{\varrho}_1,\varrho_2),$$

commute $R^*_{\bar{\varrho}_1}$ to the left of $R^*_{\bar{\varrho}_2}$ and $\varrho_2(\varepsilon(\bar{\varrho}_2,\varrho_1\bar{\varrho}_1)^*)\varepsilon(\varrho_1\bar{\varrho}_1,\varrho_2)$ to the right of $R^*_{\bar{\varrho}_\alpha}$. The coincidence of the statistics operators thus collected in the argument of ϱ_α is then due to the braid equivalences (3.1), Proposition 3.2(i), and Corollary 3.3(i).

Proof of (4.11). Let us first prove the unitarity (4.8) of $\mu(e,t)$. Let $e=(\varrho_\alpha,\varrho,\varrho_\beta)$, $t=(\tau_0,\tau,\tau+\tau_0)$. Then

$$\begin{aligned}\mu(e,t)^*\mu(e,t)&=T^*_{(et)}U(\varrho_\alpha,\tau_0)^*\varrho_\alpha\tau_0(U(\varrho,\tau)^*)\varrho_\alpha(\varepsilon(\tau_0,\varrho)^*)T_e\varrho_\beta(T_t)U(\varrho_\beta,\tau_0+\tau)\\&\quad\times U(\varrho_\beta,\tau_0+\tau)^*\varrho_\beta(T_t^*)T_e^*\varrho_\alpha(\varepsilon(\tau_0,\varrho))\varrho_\alpha\tau_0(U(\varrho,\tau))U(\varrho_\alpha,\tau_0)T_{(et)}\\&=T^*_{(et)}U(\varrho_\alpha,\tau_0)^*\varrho_\alpha\tau_0(U(\varrho,\tau)^*)\varrho_\alpha(\varepsilon(\tau_0,\varrho)^*)T_eT_e^*\\&\quad\times\varrho_\alpha(\varepsilon(\tau_0,\varrho))\varrho_\alpha\tau_0(U(\varrho,\tau))U(\varrho_\alpha,\tau_0)T_{(et)},\end{aligned}$$

since U and T_t are unitary. Now, $T_eT_e^*$ may be replaced by $\mathbf{1}=\sum_{e'}T_{e'}T_{e'}^*$, since the intertwiner $T^*_{(et)}\dots T_{e'}\in(\varrho_\beta^{\tau_0+\tau}|r(e')\circ\tau_0\tau)$ vanishes unless $r(e')=\varrho_\beta$, hence $e'=e$. The remaining unitaries ε and U cancel, and we are left with

$$\dots=T^*_{(et)}T_{(et)}=\mathbb{1},$$

if μ is considered as a matrix, hence (et) carrying independent multiplicity labels.

Now, using the definitions and (by similar arguments as before):

$$\mu(e,t)T^*_{(et)}=U(\varrho_\beta,\tau_0+\tau)^*\varrho_\beta(T_t^*)T_e^*\varrho_\alpha(\varepsilon(\tau_0,\varrho))\varrho_\alpha\tau_0(U(\varrho,\tau))U(\varrho_\alpha,\tau_0),$$

we compute (abbreviating $U(\varrho_\gamma,\tau_0+\tau_1+\tau_2)=U_\gamma$ etc.)

$$\begin{aligned}&\mu(e_1,t_1)\mu(e_2,t_2)R^{(+)}_{(e_1t_1)\circ(e_2t_2);(e_2't_2')\circ(e_1't_1')}\mu(e_1',t_1')^*\mu(e_2',t_2')^*\\&=U_\gamma^*\varrho_\gamma(T_{t_2}^*)T_{e_2}^*\varrho_\beta(\varepsilon(\tau_0^{\tau_1},\varrho_2)\tau_0^{\tau_1}(U_2))U_\beta\cdot U_\beta^*\varrho_\beta(T_{t_1}^*)T_{e_1}^*\varrho_\alpha(\varepsilon(\tau_0,\varrho_1)\tau_0(U_1))U_\alpha\\&\quad\times U_\alpha^*\varrho_\alpha(\tau_0(U_2^*)\varepsilon(\tau_0,\varrho_2)^*)T_{e_2'}\varrho_\delta(T_{t_2'})U_\delta\cdot U_\delta^*\varrho_\delta(\tau_0^{\tau_2}(U_1^*)\varepsilon(\tau_0^{\tau_2},\varrho_1)^*)T_{e_1'}\varrho_\gamma(T_{t_1'})U_\gamma.\end{aligned}$$

Since the l.h.s. is a scalar, we may omit U_γ on both sides of this expression. U_β and U_δ cancel trivially, while U_α cancels after commuting it through the statistics operator in the middle. Next, by virtue of Proposition 3.2(ii) we commute U_2^* to the left and U_1 to the right of this statistics operator, and obtain

$$\begin{aligned}\dots&=\varrho_\gamma(T_{t_2}^*)T_{e_2}^*\varrho_\beta(\varepsilon(\tau_0^{\tau_1},\varrho_2)\tau_0^{\tau_1}(U_2)T_{t_1}^*)T_{e_1}^*\varrho_\alpha(\varepsilon(\tau_0,\varrho_1))\\&\quad\times\varrho_\alpha\tau_0[\varrho_1\tau_1(U_2^*)\varepsilon(\varrho_2\tau_2,\varrho_1\tau_1)\varrho_2\tau_2(U_1)]\\&\quad\times\varrho_\alpha(\varepsilon(\tau_0,\varrho_2)^*)T_{e_2'}\varrho_\delta(T_{t_2'}\tau_0^{\tau_2}(U_1^*)\varepsilon(\tau_0^{\tau_2},\varrho_1)^*)T_{e_1'}\varrho_\gamma(T_{t_1'})\\&=\varrho_\gamma(T_{t_2}^*T_{t_1}^*)T_{e_2}^*T_{e_1}^*\\&\quad\times\varrho_\alpha[\varrho_1(\varepsilon(\tau_0\tau_1,\varrho_2))\varepsilon(\tau_0,\varrho_1)\tau_0(\varepsilon(\varrho_2\tau_2,\varrho_1\tau_1))\varepsilon(\tau_0,\varrho_2)^*)\varrho_2(\varepsilon(\tau_0\tau_2,\varrho_1)^*)]\\&\quad\times T_{e_1'}T_{e_2'}\varrho_\gamma(T_{t_2'}T_{t_1'}),\end{aligned}$$

where in the second step we have commuted all T^* to the left and all T to the right, and subsequently cancelled U_i by commuting them through the statistics operators. Evaluating the statistics operators collected in the argument of ϱ_α by virtue of Proposition 3.2(i) and Lemma 4.2(ii), we find

$$[\ldots]=\Omega_{\tau_1}(\varrho_2)\cdot\varepsilon(\varrho_2,\varrho_1)\varrho_2\varrho_1\tau_0(\varepsilon(\tau_2,\tau_1)).$$

Inserting this into the preceding expression, we get

$$\begin{aligned}\ldots&=\Omega_{\tau_1}(\varrho_2)\cdot\varrho_\gamma(T^*_{t_2}T^*_{t_1})\cdot T^*_{e_2}T^*_{e_1}\varrho_\alpha(\varepsilon(\varrho_2,\varrho_1))T_{e'_1}T_{e'_2}\cdot\varrho_\gamma(\tau_0(\varepsilon(\tau_2,\tau_1))T_{t'_2}T_{t'_1})\\&=\Omega_{\tau_1}(\varrho_2)\cdot R^{(+)}_{e_1\circ e_2;e'_2\circ e'_1}\cdot R^{(+)}_{t_1\circ t_2;t'_2\circ t'_1}.\end{aligned}$$

This is, in view of the unitarity of μ, the desired relation.

5. Space-Time Fields from Light-Cone Fields

We specialize the situation of the preceding sections to a local net $\mathscr{A}$ over the oriented light-cone. We choose two identical replica of $\mathscr{A}$ and identify the product of its supports $\mathbb{R}\times\mathbb{R}$ with two-dimensional Minkowski space-time $\mathbb{M}^2$, as in the Introduction. We shall find space-time operators

$$\Phi\in\mathscr{F}(I)\otimes\mathscr{F}(J) \tag{5.1}$$

($I, J\subset\mathbb{R}$ intervals) acting on (subspaces of) $\mathscr{H}\otimes\mathscr{H}$ with "conventional" (local, fermionic, or dual, as opposed to exchange type) commutation relations among each other provided their space-time localizations $\mathcal{O}=I\times J$, which are double cones in the usual sense, are at space-like distance. For $\mathcal{O}_i$ at space-like distance we write $\mathcal{O}_1<\mathcal{O}_2$ if $I_1<I_2$, i.e. $\mathcal{O}_1$ lies in the left causal complement of $\mathcal{O}_2$.

We also derive very simple operator product expansions.

5.1. Proposition. *Let*
$$\hat{\mathscr{H}}_0:=\bigoplus_{\varrho\in V_0}\hat{\mathscr{H}}_{\varrho,\bar\varrho}, \tag{5.2}$$

where $\hat{\mathscr{H}}_{\varrho,\varrho'}:=\mathscr{H}_\varrho\otimes\mathscr{H}_{\varrho'}$. *The space-time operators* [7]

$$\Phi_0^{(\varrho,\bar\varrho)}(A,B):=\sum_{e,\,c(e)=\varrho}\zeta(e)\,(e,A)\otimes(\bar e,B) \tag{5.3}$$

acting on $\mathscr{H}_0$ *commute with each other if they are localized at space-like distance in* $\mathbb{M}^2$. *With the notations as in Proposition 2.6 one has the operator product expansion (for arbitrary localizations)*

$$\Phi_0^{(\varrho_2,\bar\varrho_2)}(A_2,B_2)\Phi_0^{(\varrho_1,\bar\varrho_1)}(A_1,B_1)=\sum_f\zeta(f)\Phi_0^{(\varrho,\bar\varrho)}(A_f,B_{\bar f}). \tag{5.4}$$

Proof. Space-like distance between $\mathcal{O}_i$ in $\mathbb{M}^2$ means $I_1<I_2$, $J_1>J_2$ or $I_1>I_2$, $J_1<J_2$. Hence with the exchange algebra (3.11),

$$\begin{aligned}&\Phi_0^{(\varrho_2,\bar\varrho_2)}(A_2,B_2)\Phi_0^{(\varrho_1,\bar\varrho_1)}(A_1,B_1)\\&\quad=\sum_{e_1,e_2}\zeta(e_2)\zeta(e_1)(e_2,A_2)(e_1,A_1)\otimes(\bar e_2,B_2)(\bar e_1,B_1)\\&\quad=\sum_{e_i,e'_i,e''_i}\zeta(e_2)\zeta(e_1)R^{(\pm)}_{e_1\circ e_2;e'_2\circ e'_1}R^{(\mp)}_{\bar e_1\circ\bar e_2;e''_2\circ\bar e''_1}(e'_1,A_1)(e'_2,A_2)\otimes(\bar e''_1,B_1)(\bar e''_2,B_2).\end{aligned}$$

[7] Here as well as in the subsequent Propositions 5.2 and 5.4 one may verify that the dependence on the choice of the generic intertwiners T_e originated in Definition 2.5 is completely cancelled by the factors ζ and μ, with a remaining dependence only on R_ϱ. Since $\dim(\bar\varrho\varrho|id)=1$, this freedom is just one overall phase factor per charge sector

With Lemma 4.6 and Lemma 3.5(iii) the coefficients summed over e_i yield $\delta_{e_i e_i'} \cdot \zeta(e_1')\zeta(e_2')$:

$$\begin{aligned}\ldots &= \sum_{e_i'} \zeta(e_1')\zeta(e_2')(e_1', A_1)(e_2', A_2) \otimes (\bar{e}_1', B_1)(\bar{e}_2', B_2)\\ &= \Phi_0^{(\varrho_1,\bar{\varrho}_1)}(A_1, B_1)\Phi_0^{(\varrho_2,\bar{\varrho}_2)}(A_2, B_2).\end{aligned}$$

To prove (5.4), we use the operator product expansion (2.14):

$$\begin{aligned}&\Phi_0^{(\varrho_2,\bar{\varrho}_2)}(A_2, B_2)\Phi_0^{(\varrho_1,\bar{\varrho}_1)}(A_1, B_1)\\ &= \sum_{e_i} \zeta(e_2)\zeta(e_1) \sum_{e,f,e',f'} D_{e_1\circ e_2;f,e} D_{\bar{e}_1\circ\bar{e}_2;\bar{f}',\bar{e}'}\\ &= \sum_{e,f} \zeta(f)\zeta(e)(e, A_f) \otimes (\bar{e}, B_{\bar{f}}) = \sum_f \zeta(f)\Phi_0^{(\varrho,\bar{\varrho})}(A_f, B_{\bar{f}}),\end{aligned}$$

where again Lemma 4.6 and Lemma 3.5(iii) have been used.

Let us now discuss space-time fields with excess charge.

5.2. Proposition. *For $\tau \in V_0$ an automorphism let*

$$\hat{\mathscr{H}}_\tau := \bigoplus_{\varrho \in V_0} \hat{\mathscr{H}}_{\varrho^\tau,\bar{\varrho}}. \tag{5.5}$$

Define space-time operators $\Phi_{\tau_0}^{(\varrho^\tau,\bar{\varrho})} : \hat{\mathscr{H}}_{\tau_0} \to \hat{\mathscr{H}}_{\tau_0+\tau}$ by

$$\Phi_{\tau_0}^{(\varrho^\tau,\bar{\varrho})}(A, B) := \sum_{e, c(e)=\varrho} \zeta(e)\mu(e,t)(et, A) \otimes (\bar{e}, B), \tag{5.6}$$

where $t = (\tau_0, \tau, \tau_0 + \tau)$. Then one has

$$\Phi_{\tau_0+\tau_1}^{(\varrho_2^{\tau_2},\bar{\varrho}_2)}\Phi_{\tau_0}^{(\varrho_1^{\tau_1},\bar{\varrho}_1)} = \begin{Bmatrix}\Omega_{\tau_1}(\varrho_2)\\ \Omega_{\tau_2}(\varrho_1)^*\end{Bmatrix} R^{(\pm)}_{t_1\circ t_2; t_2'\circ t_1'} \Phi_{\tau_0+\tau_2}^{(\varrho_1^{\tau_1},\bar{\varrho}_1)}\Phi_{\tau_0}^{(\varrho_2^{\tau_2},\bar{\varrho}_2)} \quad \text{if} \quad \begin{Bmatrix}\mathcal{O}_1 < \mathcal{O}_2\\ \mathcal{O}_2 < \mathcal{O}_1\end{Bmatrix}, \tag{5.7}$$

and

$$\Phi_{\tau_0+\tau_1}^{(\varrho_2^{\tau_2},\bar{\varrho}_2)}(A_2, B_2)\Phi_{\tau_0}^{(\varrho_1^{\tau_1},\bar{\varrho}_1)}(A_1, B_1) = D_{t_1\circ t_2;s,t} \cdot \sum_f \mu(f,s)\zeta(f)\Phi_{\tau_0}^{(\varrho^{\tau_1+\tau_2},\bar{\varrho})}(A_{(fs)}, B_{\bar{f}}). \tag{5.8}$$

The proof is completely analogous to that of the preceding proposition, this time also using Lemma 4.7.

The structure constants in (5.7) and (5.8) in general depend on the excess charge τ_0 of the sector $\hat{\mathscr{H}}_{\tau_0}$ they act on. The following lemma shows that this is related to the obstruction discussed in Sect. 4.

5.3 Lemma. *Let V_0 and a subgroup of automorphisms Γ be chosen such that within Γ one has $\tau_1 + \tau_2 = \tau_1\tau_2$ and $T_t \equiv U(\tau_1, \tau_2) = \mathbf{1}$ (cf. Lemma 4.4). For superselection channels within Γ one has*

$$R^{(\pm)}_{t_1\circ t_2; t_2'\circ t_1'} = \begin{Bmatrix}\varepsilon(\tau_2, \tau_1)\\ \varepsilon(\tau_1, \tau_2)^*\end{Bmatrix} \in \mathbb{C}, \qquad D_{t_1\circ t_2;s,t} = 1. \tag{5.9}$$

Proof. It is sufficient to note that $\varepsilon(\tau_2, \tau_1) \in (\tau_1\tau_2|\tau_2\tau_1) = (\tau_1 + \tau_2|\tau_1 + \tau_2)$ is a scalar, while all basis intertwiners T_t involved in the definitions are 1.

Remark. Equation (5.9) can be expressed in terms of statistics phases alone. Namely, if $\Gamma = \bigotimes_k \mathbb{Z}_{\nu_k}$ as in Lemma 4.4 is generated by $\tau^{(k)}$, $\tau^{(k)\nu_k} = id$, with disjoint localizations I_k, we may assume $I_k > I_l$ if $k > l$, hence

$$\varepsilon(\tau^{(k)}, \tau^{(l)}) = \begin{cases} 1 & \text{if } k > l \\ \Omega_{\tau^{(k)}}(\tau^{(l)}) & \text{if } k < l. \end{cases}$$

Then, for $\tau_i = \sum_k n_i^{(k)} \tau^{(k)} \in \Gamma$ one has

$$\varepsilon(\tau_2, \tau_1) = \prod_{k<l} (\Omega_{\tau^{(k)}}(\tau^{(l)}))^{n_2^{(k)} n_1^{(l)}}.$$

5.4. Proposition. *For $\Gamma \subset V_0$ a subgroup of automorphisms as in 5.3 let*

$$\hat{\mathscr{H}}_\Gamma := \bigoplus_{\tau\in\Gamma} \hat{\mathscr{H}}_\tau. \tag{5.10}$$

Define space-time operators $\Phi^{(\varrho^\tau, \bar\varrho)} \in \mathscr{B}(\hat{\mathscr{H}}_\Gamma)$ by

$$\Phi^{(\varrho^\tau, \bar\varrho)}(A, B) = \sum_{\tau_0\in\Gamma} \Phi_{\tau_0}^{(\varrho^\tau, \bar\varrho)}(A, B). \tag{5.11}$$

Then one has

$$\Phi^{(\varrho_2^{\tau_2}, \bar\varrho_2)} \Phi^{(\varrho_1^{\tau_1}, \bar\varrho_1)} = \begin{Bmatrix} \Omega_{\tau_1}(\varrho_2)\varepsilon(\tau_2, \tau_1) \\ (\Omega_{\tau_2}(\varrho_1)\varepsilon(\tau_1, \tau_2))^* \end{Bmatrix} \Phi^{(\varrho_1^{\tau_1}, \bar\varrho_1)} \Phi^{(\varrho_2^{\tau_2}, \bar\varrho_2)} \quad \text{if} \quad \begin{Bmatrix} \mathcal{O}_1 < \mathcal{O}_2 \\ \mathcal{O}_2 < \mathcal{O}_1 \end{Bmatrix} \tag{5.12}$$

and

$$\Phi^{(\varrho_2^{\tau_2}, \bar\varrho_2)}(A_2, B_2) \Phi^{(\varrho_1^{\tau_1}, \bar\varrho_1)}(A_1, B_1) = \sum_f \mu(f, s) \zeta(f) \Phi^{(\varrho^{\tau_1+\tau_2}, \bar\varrho)}(A_{(fs)}, B_{\bar f}). \tag{5.13}$$

Remark. The commutation relation (5.12) holds also if one chooses only one self-conjugate automorphism $\tau \in V_0$ which may be obstructed: $\omega(\tau)^2 = -1$, i.e. $\tau^2 \neq id$. Namely, the first equation of (5.9) is also valid if either τ_1 or $\tau_2 = id$ or if $\tau_1 = \tau_2$. However, the operator products $\Phi^{(\varrho_2^{\tau}, \bar\varrho_2)} \Phi^{(\varrho_1^{\tau}, \bar\varrho_1)}$ fail to be of the form $\Phi^{(\varrho, \bar\varrho)}$.

Equations (5.12) are in general soliton-like (dual) commutation relations.

Example. Consider the Ising model. There is besides the vacuum sector $id \equiv 0$ only one automorphism $\tau = \bar\tau$ with $h_\tau = \frac{1}{2}$, and one more sector $\varrho = \bar\varrho \in V_0$ with $h_\varrho = \frac{1}{16}$. Identifying respectively

$$\Phi^{(0,0)}, \ \Phi^{(\varrho,\varrho)}, \ \Phi^{(\tau,\tau)}, \ \Phi^{(\tau,0)}, \ \Phi^{(\varrho^\tau,\varrho)}, \ \Phi^{(0,\tau)}$$

with the operators $1, \sigma, m, \psi, \mu, \bar\psi$, all the well-known bosonic, fermionic, and dual commutation relations [28] are read off (5.12), e.g.

$$\sigma\mu = \begin{Bmatrix} \varepsilon(id, \tau)\Omega_\tau(\varrho) \\ (\varepsilon(\tau, id)\Omega_{id}(\varrho))^* \end{Bmatrix} \mu\sigma = (\mp)\mu\sigma,$$

$$\psi\psi = \begin{Bmatrix} \varepsilon(\tau, \tau)\Omega_\tau(0) \\ (\varepsilon(\tau, \tau)\Omega_\tau(0))^* \end{Bmatrix} \psi\psi = -\psi\psi.$$

Observe that (since $\varrho^\tau = \varrho$) σ and μ map the vacuum sector $\hat{\mathscr{H}}_{id,id}$ into two orthogonal copies of the same sector $\hat{\mathscr{H}}_{\varrho,\varrho}$, one contained in $\hat{\mathscr{H}}_0$, the other in $\hat{\mathscr{H}}_\tau$. The fact that these two are equivalent representation sectors of the light-cone observables $\mathscr{A} \otimes \mathscr{A}$ expresses the indistinguishability of the order and disorder operators, although they cannot be identified because of their dual commutation relations. This phenomenon obviously occurs whenever there are fixpoints of the action of some automorphism τ acting in V_0.

5.5. Proposition. *Let $V \subset V_0$ be a subset of reference morphisms representing a subset of irreducible superselection sectors closed under conjugation and composition with subsequent reduction. Let $\Gamma \subset V_0$ be, as in 5.3, a subgroup of automorphisms not subject to the obstruction* [8]. *Then the operators* (5.11) *with $\varrho \in V$, $\tau \in \Gamma$ are mutually local, if and only if*

$$\omega(\tau)=1\,, \qquad \omega(\varrho^\tau)=\omega(\varrho)\ \forall \varrho \in V,\ \tau \in \Gamma\,. \tag{5.14}$$

Proof. $\Phi^{(\varrho^\tau, \bar{\varrho})}$ are mutually local if and only if the coefficients in (5.12) are all unity. Choosing $\tau_1 = id$ one gets $\Omega_\tau(\varrho)=1$. Choosing $\varrho_1 = id$, $\tau_1 = \tau_2$ one gets $\varepsilon_\tau = 1$, i.e. (5.14) as necessary conditions. That these are also sufficient conditions can be seen from the remark after Lemma 5.3.

6. Discussion

We have constructed various subalgebras of the tensor product of two reduced field bundles, consisting of space-time fields with conventional commutation relations. In particular we have identified subalgebras of mutually local space-time fields in Proposition 5.5. Let us consider the extremal cases of this proposition. Choosing $\Gamma = \{id\}$ trivial and $V = V_0$ maximal, we arrive back at the subalgebra constructed in Proposition 5.1. Choosing instead Γ maximal and $V = \{id\}$, the operators $\Phi^{(\tau, id)}(A, \mathbf{1}) \equiv \phi^\tau(A) \otimes \mathbf{1}$ define an extended local net on one light-cone $\mathcal{A}_{\text{ext}}$ containing $\mathcal{A}$ (namely $\Phi^{(id, id)}(A, \mathbf{1}) = A \otimes \mathbf{1}$) with vacuum representation $\mathcal{H}_0^{\text{ext}} = \bigoplus_{\tau \in \Gamma} \mathcal{H}_\tau \otimes \Omega \subset \hat{\mathcal{H}}_\Gamma$. The construction of Sect. 5 allows to study the superselection structure of the extended algebra. Namely, one has [9] for $\varrho \in V_0$

$$\mathcal{A}_{\text{ext}} : \mathcal{H}_\varrho^{\text{ext}} \equiv \bigoplus_{\tau \in \Gamma} \mathcal{H}_{\varrho^\tau} \to \mathcal{H}_\varrho^{\text{ext}}\,.$$

Let us discuss three possibilities. (1) Γ acts freely on its orbit in V_0 through ϱ (e.g., $\varrho = id$). Then the corresponding superselection sector $\mathcal{H}_\varrho^{\text{ext}}$ of $\mathcal{A}_{\text{ext}}$ (the sum over $|\Gamma|$ inequivalent sectors of $\mathcal{A}$) is irreducible. (2) ϱ is a fixpoint for all $\tau \in \Gamma$. Then $\mathcal{H}_\varrho^{\text{ext}}$ is the sum over $|\Gamma|$ identical sectors $\mathcal{H}_\varrho$ of $\mathcal{A}$, which is reducible as a sector of $\mathcal{A}_{\text{ext}}$. [Of course, the generic situation will lie between (1) and (2).] (3) $\varrho \notin V$ is such that $\omega(\varrho^\tau) \neq \omega(\varrho)$ for some $\tau \in \Gamma$. Then by the spin-statistics theorem (4.3), $\mathcal{H}_\varrho^{\text{ext}}$ contains states with non-integer-spaced scaling dimensions, and consequently cannot be a *covariant* sector of $\mathcal{A}_{\text{ext}}$. The following example exhibits all three possibilities in the same model.

Example [8, 26]. Consider the algebra $\mathcal{A}$ generated by the stress-energy tensor with $c = \frac{4}{5}$. The ten covariant superselection sectors V_0 of $\mathcal{A}$ are [29, 3]: the vacuum sector *id*; a self-conjugate automorphism τ with $h_\tau = 3$; two self-conjugate

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[8] Γ needs not to be contained in V. But in view of the condition (5.14) and Lemma 4.3 (ii) no generality is lost (and left-right symmetry restored) if one enlarges V to contain Γ

[9] Since the action on the second tensor factor is trivial, we write only the first factor

sectors ϱ with $h_\varrho = \frac{1}{15}$ respectively $\frac{2}{3}$, which are fixpoints of τ; a pair of self-conjugate sectors $\sigma \neq \sigma^\tau$ with $h_\sigma = \frac{2}{5}$ and $h_{\sigma^\tau} = \frac{7}{5}$; and two pairs of self-conjugate sectors $\pi \neq \pi^\tau$ with $h_\pi = \frac{1}{8}$ respectively $\frac{1}{40}$, $h_{\pi^\tau} = \frac{13}{8}$ respectively $\frac{21}{40}$.

Choosing $\Gamma = \{id, \tau\}$ maximal, the orbits through id, σ, and π have no fixpoints (1), while ϱ stabilize Γ (2). The diagonal and antidiagonal subspaces of $\mathscr{H}_\varrho \oplus \mathscr{H}_{\varrho^\tau}$ turn out to carry inequivalent conjugate representations of $\mathscr{A}_{\mathrm{ext}}$. The representations $\mathscr{H}_\pi \oplus \mathscr{H}_{\pi^\tau}$ are not covariant (3). Thus the six covariant superselection sectors of $\mathscr{A}_{\mathrm{ext}}$ are: the vacuum sector $\mathscr{H}_0^{\mathrm{ext}} = \mathscr{H}_{id} \oplus \mathscr{H}_\tau$; two pairs of conjugate sectors $\mathscr{H}_{\varrho^+}^{\mathrm{ext}}, \mathscr{H}_{\varrho^-}^{\mathrm{ext}} \subset \mathscr{H}_\varrho \oplus \mathscr{H}_\varrho$; and one self-conjugate sector $\mathscr{H}_\sigma^{\mathrm{ext}} = \mathscr{H}_\sigma \oplus \mathscr{H}_{\sigma^\tau}$.

In fact, there are two space-time theories associated with $c = \frac{4}{5}$, namely the minimal ($SU(2)$ coset) models of type A and D. The A type model contains only diagonal operators $\Phi_0^{(\varrho,\varrho)}$ acting in the diagonal Hilbert space $\hat{\mathscr{H}}_0$. The D type model contains non-diagonal operators (with charges different from π, π^τ above) acting in $\hat{\mathscr{H}}_\Gamma$. The A type model is given by our Proposition 5.1, the D type model is given by our Proposition 5.5 with $V = \{id, \tau, \varrho = \varrho^\tau, \sigma, \sigma^\tau\}$. The double occurrence of $\hat{\mathscr{H}}_{\varrho,\varrho} = \hat{\mathscr{H}}_{\varrho^\tau,\varrho}$ in $\hat{\mathscr{H}}_\Gamma$ is well known from the analysis of the modular invariant partition function.

The *same* D type model has also the alternative interpretation as an A type $SU(3)$ coset model [26]. In our language: it is nothing but the construction of Proposition 5.1 with respect to the extended algebra $\mathscr{A}_{\mathrm{ext}}$.

The discussion of the example (and in particular the last remark) sheds also some new light on a result obtained from modular properties required to hold for the partition function [21, Sect. 5]: With respect to the maximally extended light-cone algebras, the local space-time field theory contains every superselection sector precisely once, paired with its conjugate.

Let us discuss the phenomenon of fields which are individually indistinguishable, but "dual" (in general: "v-al") with respect to each other (like the order/disorder fields of the critical Ising model) in the generality offered by Proposition 5.4. Thus let τ be an automorphism such that $\tau^v = id$, ϱ a generic morphism such that $[\varrho\tau] = [\varrho]$, and denote by Φ_k the space-time fields with charge ϱ and excess charge τ^k, $k = 0, \ldots, v-1$:

$$\Phi_k = \Phi^{(\varrho^{(\tau^k)}, \bar{\varrho})} = \Phi^{(\varrho, \bar{\varrho})}.$$

While Φ_k are formally all the same, they interpolate different copies of identical Hilbert spaces. We find the commutation relations

$$\Phi_k \Phi_j = \begin{Bmatrix} \omega(\tau)^{j(k-j)} \\ \omega(\tau)^{k(k-j)} \end{Bmatrix} \Phi_j \Phi_k .$$

In particular, these fields are always local with respect to themselves ($k = j$) and have monodromy determined by the excess charge difference $\omega(\tau)^{-(k-j)^2} = \omega(\tau^{k-j})^{-1}$.

Some comments are in order comparing the present analysis of the space-time field content with the study of modular invariant partition functions [8]. Although there is much overlap in the answers, the questions that have been asked are not the same.

We have constructed subspaces which are generated by local space-time fields acting on the vacuum. Apparently, if these subspaces are chosen maximal, then the trace of $\exp[2\pi i(\tau L_0 - \bar{\tau}\bar{L}_0)]$ in this subspace yields a modular invariant, but in our approach we don't control "maximality." On the other hand, we find also fermionic and dual space-time fields in a uniform setting, while in the modular approach one has to choose different boundary conditions to make visible different types of space-time fields.

The characters contributing to the partition functions measure only the conformal energy (L_0). They cannot distinguish a sector from its conjugate. It is impossible to decide from the modular invariant whether the theory contains a primary state $|h\rangle \otimes |h\rangle$ or rather $|h\rangle \otimes |\bar{h}\rangle$, while our analysis indicates that the former in general cannot be created by a conventional field. (There are many examples, however, where $\varrho = \bar{\varrho}^\tau$ for some automorphism τ, thus $|h\rangle \otimes |h\rangle$ is created by an excess charge field $\Phi^{(\bar{\varrho}^\tau, \varrho)}$. I owe this observation to A. N. Schellekens.)

Finally, while the study of modular invariance is limited to "rational" models, the present analysis applies also for theories with an infinite number of sectors.

The present results fit nicely into the picture of the algebra of local observables being the invariant subalgebra of some algebra of charged fields on which a "symmetry" acts. The symmetry associated with braid group statistics [27] is not a usual symmetry group, but is still expected to be equipped with a coproduct determining the action on a product of fields in terms of the actions on the factors and an antipode playing the role of an inverse [2]. Although the structure of the generalized symmetry is not yet completely clear and maybe not unique: if one assumes two copies of an algebra of multiplet fields carrying representations of the symmetry in 1 : 1 correspondence with their superselection charges, our space-time local fields $\Phi^{(\varrho, \bar{\varrho})}$ appear just as contractions over the symmetry degrees of freedom, i.e. scalars contained in the product of a representation with its conjugate. Fields with excess charge transform like one-dimensional representations of the symmetry.

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