Ward Identities
for Non-Commutative Geometry★

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Dedicated to Res Jost and Arthur Wightman

Abstract. We interpret the cocycle condition in quantum field theory as a set of integrated Ward identities for non-commutative geometry.

I. Basic Notions

The Wightman functions of a super-symmetric quantum field theory given by a super-trace functional have a geometric or cohomological interpretation. This was shown in joint work of the authors with Lesniewski [JLO1]. This property is summarized by the construction of a cocycle τ for the ∂-complex of entire cyclic cohomology, namely a solution to the equation

$$\partial\tau = 0. \tag{I.1}$$

Here τ is a time average of certain Euclidean Wightman functionals, and ∂ is a standard coboundary operator of non-commutative geometry, see [C, JLO1]. Furthermore, this natural cohomological interpretation can be generalized to the case when the Wightman functionals τ are constructed from a finite-temperature functional satisfying a super-version of the KMS condition of statistical mechanics [K, JLO2, JLWis]. In this case the super-trace (associated with finite volume theories) need not exist – only the super-KMS functional has to be defined.

In this note we investigate how Eq. (I.1) has a set of identities as its consequence, which can be interpreted as a symmetry of the Wightman functions. We call these identities "Ward identities for non-commutative geometry." Let us consider an example. Let γ denote the Z_2 grading in our theory, equal to $(-I)^{N_f}$ in models, and let $A \to A^\gamma = \gamma A \gamma$ denote the action of the grading on field operators. Let $A(t) = e^{-tH} A e^{tH}$ denote the propagation of A to Euclidean time t, and let $\langle A \rangle$ denote the expectation

$$\langle A \rangle = \mathrm{Str}(A e^{-H}), \tag{I.2}$$

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or more generally a super-KMS functional constructed as a limit of functionals of the form (I.2), and which have the properties of [JLO2]. Also let d denote the super derivation generated by a self-adjoint super charge operator Q by the graded commutator,

$$dA = QA - A^\gamma Q, \tag{I.3}$$

thus omitting a factor of i, used in [JLO1]. From this we infer that

$$d^2 A = [H, A]$$

with $H = Q^2$.

Our example of the Ward identity is

$$\langle A_1 A_2 \rangle - \langle A_2^\gamma A_1 \rangle = -\int_0^1 \langle dA_1 dA_2^\gamma(t) \rangle dt. \tag{I.4}$$

If $A_1 = \psi(x, 0)^*$, $A_2 = \psi(y, 0)$, then (I.4) says that

$$\langle \{ \psi(x, 0)^*, \psi(y, 0) \} \rangle = \int_0^1 \langle d\psi(x, 0)^* d\psi(y, t) \rangle dt, \tag{I.5}$$

which in a canonical theory gives a sum rule for the right side of (I.5). In general, $d\psi$ involves the interaction potential, so (I.5) relates a sum of 2-point functions on the left side to vertex operators on the right. For this reason (I.5) can be regarded as a Ward identity.

II. The General Identity

The general identity is formulated in terms of

$$\tau_n(A_1, \ldots, A_n) = (-1)^{n(n-1)/2} \int_{\sigma_{n-1}} \langle A_1 dA_2^\gamma(t_1) \ldots dA_n^{\gamma^{n-1}}(t_{n-1}) \rangle dt, \tag{II.1}$$

where $dt = dt_1 \ldots dt_{n-1}$ and where σ_n denotes the simplex $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq 1$. We adopt the notation that τ_n depends on n, rather than on $n + 1$ variables; this follows Quillen and simplifies our expressions, but it differs from our earlier work, and from Connes' notation as well. For convenience, we use somewhat different conventions than in our earlier work: we omit the factor i in the definition of d, and we replace the constant in τ_n by $(-1)^{n(n-1)/2}$. These changes have the effect of replacing the coboundary operator $b + B$ in our earlier papers by the new coboundary operator

$$\partial = b - B.$$

It is known that τ satisfies the identity

$$(b\tau_n)(A_1, \ldots, A_{n+1}) = (-1)^{n(n-1)/2} \int_{\sigma_n} \langle dA_1 dA_2^\gamma(t_1) \ldots dA_{n+1}^{\gamma^n}(t_n) \rangle dt, \tag{II.2}$$

see [JLO1] and also [EFJL, JLO2]. Here b is the coboundary operator which maps n-cochains to $n + 1$-cochains. Explicitly,

$$(b\tau_n)(A_1, \ldots, A_{n+1}) = \sum_{j=1}^n (-1)^{j+1} \tau_n(A_1, \ldots, A_j A_{j+1}, \ldots, A_{n+1})$$

$$+ (-1)^n \tau_n(A_{n+1}^{\gamma^n} A_1, A_2, \ldots, A_n). \tag{II.3}$$

Writing out (II.2) in the case $n = 1$ we have the Ward identity (I.4). We claim that it is reasonable to pose

Definition II.1. The identities (II.2) are the "Ward identities" of non-commutative geometry.

These identities have an expansion in terms of operator products, generalizing (I.4–5). If $A_1, ..., A_{n+1}$ are $n+1$ operators, the identities can be written

$$\sum_{j=1}^{n} (-1)^{j+1} \int_{\sigma_{n-1}} \langle A_1 dA_2^{\gamma}(t_1)...d(A_j^{\gamma^{j-1}}(t_{j-1})A_{j+1}^{\gamma^{j-1}}(t_{j-1}))...dA_{n+1}^{\gamma^n}(t_{n-1})\rangle dt$$

$$+ (-1)^n \int_{\sigma_{n-1}} \langle A_{n+1}^{\gamma^n} A_1 dA_2^{\gamma}(t_1)...dA_n^{\gamma^{n-1}}(t_{n-1})\rangle dt$$

$$= (-1)^n \int_{\sigma_n} \langle dA_1 dA_2^{\gamma}(t_1)...dA_{n+1}^{\gamma^n}(t_n)\rangle dt . \tag{II.4}$$

In particular, the leading singularities which would arise in an operator product expansion about coinciding points in the terms on the left side (each of which have equal time contributions) must add to exactly equal the leading singularity in the term on the right.

III. Exponential Formalism

In a recent paper, Quillen studied the JLO-cocycle using his formalism of super-connections; see Sect. 8 of [Q]. Here we use a similar method, in order to give a direct derivation of our Ward identities.

Let $\mathfrak{A}$ denote the algebra of field operators. In this paper we assume that this algebra is an algebra of bounded operators; but for considerations such as in Sect. II, $\mathfrak{A}$ should be extended to an algebra including certain unbounded operators. A typical situation would be the case where the operators in $\mathfrak{A}$ are defined on a common, dense, invariant domain $\mathscr{D}$.

For simplicity we assume $\mathfrak{A}$ to be a subalgebra of $\mathscr{L}(\mathscr{H})$, the set of bounded linear operators acting on a Hilbert space $\mathscr{H}$. Following [Q], we define n-cochains as n-linear maps from $\mathfrak{A}$ to $\mathscr{L}(\mathscr{H})$, an algebra of such operators, containing $\mathfrak{A}$, and acting on a Hilbert space $\mathscr{H}$,

$$f_n : A_1 \times A_2 \times A_3 \times ... \times A_n \mapsto f_n(A_1, ..., A_n) \in \mathscr{L}(\mathscr{H})$$

for $A_i \in \mathfrak{A}$. A 0-cochain will be a "constant" map from $\mathfrak{A}$ to $\mathscr{L}(\mathscr{H})$, namely an operator on $\mathscr{L}(\mathscr{H})$. A cochain $\mathbf{f}$ will be a sequence

$$\mathbf{f} = (f_0, f_1, ..., f_n, ...)$$

of n-cochains on $\mathfrak{A}$. On the other hand, if f is an n-cochain, we let $\mathbf{f}$ denote the cochain sequence with

$$(\mathbf{f})_m = \begin{cases} f & \text{if } m = n \\ 0 & \text{otherwise}. \end{cases}$$

Let $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ denote the space of cochains in the above sense, and let $\mathscr{C}_{\mathscr{H}}^n(\mathfrak{A}) \subset \mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ denote the n-cochains. Furthermore, in case we choose for $\mathscr{H}$ the one-dimensional subspace space $\mathbb{C}$ of $\mathscr{H}$, we let $\mathscr{C}_{\mathbb{C}}(\mathfrak{A}) \subset \mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ denote cochains such that each component takes values which are multiples of the identity operator I in

$\mathcal{L}(\mathcal{H})$. Thus $\mathcal{C}_{\mathbb{C}}(\mathfrak{A})$ is the space of cochains in the ordinary sense, namely sequences where f_n is an n-linear functional on $\mathfrak{A}$.

We define a map ω,

$$\omega : \mathcal{C}_{\mathcal{H}}(\mathfrak{A}) \to \mathcal{C}_{\mathbb{C}}(\mathfrak{A})$$

by

$$\omega(\mathbf{f}) = \{\omega(f_0), \omega(f_1), ..., \omega(f_n), ...\},$$

where

$$\omega(f_n)(A_1, ..., A_n) = \mathrm{Tr}(\gamma^n f_n(A_1, ..., A_n)). \tag{III.1}$$

Of course, the map ω may not be defined for all cochains $\mathbf{f}$, or the domain of $\omega(\cdot)$ may not be all of $\mathfrak{A}$. However here we study algebraic aspects of this construction, returning briefly to comment on analytic questions at the end.

There is a natural multiplication on $\mathcal{C}_{\mathcal{H}}(\mathfrak{A})$, so it can be regarded as an algebra with the product of components,

$$(\mathbf{f} \cdot \mathbf{g})_k = \sum_{n+m=k} f_n \cdot g_m, \tag{III.2}$$

where

$$f_n \cdot g_m(A_1, ..., A_{n+m}) = f_n(A_1, ..., A_n) g_m(A_{n+1}, ..., A_{n+m}). \tag{III.3}$$

Remark. Quillen [Q] uses a different sign convention in his product.

The multiplication on the right is multiplication of operators in $\mathcal{L}(\mathcal{H})$. The *identity cochain* θ is the one-cochain

$$\theta : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$$

$$A \mapsto A,$$

where we consider $\mathfrak{A}$ as a subset of $\mathcal{L}(\mathcal{H})$. Also let $\boldsymbol{\theta} = (0, \theta, 0, ...)$ denote the corresponding cochain in $\mathcal{C}_{\mathcal{H}}(\mathfrak{A})$. Let γ denote the operator on $\mathcal{H}$ which defines the $\mathbb{Z}_2$-grading, and such that $A^\gamma = \gamma A \gamma$. Let b' denote the coboundary operator

$$(b'f_n)(A_1, ..., A_{n+1}) = \sum_{j=1}^{n} (-1)^{j+1} f_n(A_1, ..., A_j A_{j+1}, ..., A_{n+1}). \tag{III.4}$$

On 0-cochains, we define b' to be 0.

Let us decompose $\mathbf{f}$ according to the grading γ, namely $\mathbf{f} = \mathbf{f}^+ + \mathbf{f}^-$, where

$$\mathbf{f}^\pm = \tfrac{1}{2}(\mathbf{f} \pm \mathbf{f}_\gamma),$$

and

$$(\mathbf{f}_\gamma)_n(A_1, ..., A_n) = \mathbf{f}_n(A_1^\gamma, ..., A_n^\gamma).$$

Correspondingly we decompose the space $\mathcal{C}_{\mathcal{H}}(\mathfrak{A})$ of cochains according to the grading γ and write

$$\mathcal{C}_{\mathcal{H}}(\mathfrak{A}) = \mathcal{C}_{\mathcal{H}}(\mathfrak{A})^+ \oplus \mathcal{C}_{\mathcal{H}}(\mathfrak{A})^-.$$

Then the operator b of the previous section is generalized to

$$(bf_n)(A_1, ..., A_{n+1}) = (b'f_n)(A_1, ..., A_{n+1})$$
$$+ (-1)^n (f_n^+(A_{n+1}^\gamma A_1, A_2, ..., A_n) + f_n^-(A_{n+1}A_1, A_2, ..., A_n)).$$

(III.5)

We use the convention

$$f_n(A_1^{\Gamma i_1}, A_2^{\Gamma i_2}, ..., A_n^{\Gamma i_n}) = f_n^+(A_1^{\gamma i_1}, A_2^{\gamma i_2}, ..., A_n^{\gamma i_n}) + f_n^-(A_1, A_2, ..., A_n) \quad \text{(III.6)}$$

and

$$f_n(A_1^\Gamma A_2, A_3, ..., A_{n+1}) = f_n^+(A_1^\gamma A_2, A_3, ..., A_{n+1}) + f_n^-(A_1 A_2, A_3, ..., A_{n+1}).$$

Thus

$$(bf_n)(A_1, ..., A_{n+1}) = (b'f_n)(A_1, ..., A_{n+1}) + (-1)^n f_n(A_{n+1}^\Gamma A_1, A_2, ..., A_n). \text{(III.7)}$$

We also introduce a generalization of Connes' coboundary operator B which is defined by

$$(Bf_n)(A_1, ..., A_{n-1}) = \sum_{j=0}^{n-2} (-1)^{nj} \Big((f_n(1, A_{n-j}^\Gamma, ..., A_{n-1}^\Gamma, A_1, ..., A_{n-j-1})$$
$$+ (-1)^n f_n(A_{n-j}^\Gamma, ..., A_{n-1}^\Gamma, A_1, ..., A_{n-j-1}, 1) \Big). \quad \text{(III.8)}$$

Furthermore, we define $Bf_0 = 0$.

It is convenient, given the form of B, to introduce a cyclic permutation operator on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$. Let us define the operator T on $\mathscr{C}_{\mathscr{H}}^n(\mathfrak{A})$ by

$$(Tf_n)(A_1, ..., A_n) = (-1)^{n+1} f_n(A_n^\Gamma, A_1, ..., A_{n-1}). \quad \text{(III.9)}$$

The total symmetrization operator on f_n is given by

$$(\text{Sym} f_n)(A_1, ..., A_n) = \sum_{k=0}^{n-1} T^k f_n. \quad \text{(III.10)}$$

In the case when

$$\mathbf{f} \in \bigoplus_n (\mathscr{C}_{\mathscr{H}}^{2n}(\mathfrak{A}) \cap \mathscr{C}_{\mathscr{H}}(\mathfrak{A})^+) \oplus (\mathscr{C}_{\mathscr{H}}^{2n+1}(\mathfrak{A}) \cap \mathscr{C}_{\mathscr{H}}(\mathfrak{A})^-),$$

we note that

$$(Tf_n)(A_1, ..., A_n) = (-1)^{n+1} f_n(A_n^{\gamma^{n+1}}, A_1, ..., A_{n-1}) \quad \text{(III.11)}$$

and

$$(\text{Sym} f_n)(A_1, ..., A_n) = \sum_{k=0}^{n-1} (-1)^{(n+1)k} f_n(A_{n-k+1}^{\gamma^{n+1}}, ... A_n^{\gamma^{n+1}}, A_1, ..., A_{n-k}).$$

(III.12)

Furthermore, let us define the projection $\mathbf{f}^{(\geq k)}$ of $\mathbf{f}$ by

$$\mathbf{f}^{(\geq k)} = (0, 0, ..., 0, f_k, f_{k+1}, ...).$$

Then we have the corresponding formula of $(\text{Sym} \mathbf{f})$ on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$, namely

$$(\text{Sym} \mathbf{f}) = \sum_{k=1}^{\infty} T^{k-1} \mathbf{f}^{(\geq k)}. \quad \text{(III.13)}$$

This agrees with (III.10) on the n-th component f_n of $\mathbf{f}$.

The superderivation d on $\mathfrak{A}$ can be lifted to a superderivation on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ as follows: define d as a map of $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ into itself by

$$d\mathbf{f} = Q\mathbf{f} - \mathbf{f}'Q.$$

Here Q is interpreted as a constant cochain, namely an element of $\mathscr{C}_{\mathscr{H}}^0(\mathfrak{A})$. We then can write (III.11) in the form

$$T\omega(d\theta\mathbf{f}) = \omega(\mathbf{f}'d\theta). \tag{III.14}$$

In particular, the nth component of the left-hand side is

$$(T\omega(d\theta\mathbf{f}))_n(A_1, \ldots, A_n)$$
$$= (-1)^{n+1}\omega(dA_n^{\gamma^{n+1}} \cdot f_{n-1}(A_1, \ldots, A_n))$$
$$= \omega(\gamma^{n+1}dA_n\gamma^{n+1}f_{n-1}(A_1, \ldots, A_n))$$
$$= \omega(\gamma f_{n-1}(A_1, \ldots, A_{n-1})\gamma dA_n),$$

which is the nth component of the right-hand side.

It is also useful to introduce a superderivation D with respect to a grading of $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ different from γ, namely a grading ε defined by

$$(\mathbf{f}^\varepsilon)_n = (-1)^n\mathbf{f}_n.$$

The superderivation D on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ is given by

$$(D\mathbf{f}) = \gamma Q\mathbf{f} - \mathbf{f}^\varepsilon\gamma Q.$$

When acting on θ, the relation between D and d is that

$$D\theta = -d(\gamma\theta), \qquad d\theta = -D(\gamma\theta). \tag{III.15}$$

The above definitions indicate how we extend maps defined on $\mathscr{C}_{\mathscr{H}}^n$ for all n to maps on $\mathscr{C}_{\mathscr{H}}$, and we use the same symbol to denote this extension. For example,

$$(\gamma\mathbf{f})_n(A_1, \ldots, A_n) = \gamma f_n(A_1, \ldots, A_n),$$
$$(b'\mathbf{f})_n(A_1, \ldots, A_n) = b'f_{n-1}(A_1, \ldots, A_n).$$

Lemma III.1 (Graded Leibnitz Rule). *The operators d, D, and b' are superderivations on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ with respect to the gradings γ, ε, and ε respectively:*

$$d(\mathbf{fg}) = (d\mathbf{f})\mathbf{g} + f^\gamma d\mathbf{g}, \qquad D(\mathbf{fg}) = (D\mathbf{f})\mathbf{g} + f^\varepsilon D\mathbf{g}, \qquad b'(\mathbf{fg}) = (b'\mathbf{f})\mathbf{g} + f^\varepsilon b'\mathbf{g}. \tag{III.16}$$

The supersymmetry of the mapping ω can be expressed in terms of the invariance of ω under the differential D. In addition, the commutativity of the maps b' and ω involves γ, see Lemma III.3.

Lemma III.2 (Supersymmetry of ω). *We have*

$$\omega(D\mathbf{f}) = 0, \qquad \omega((d\theta)\mathbf{f}) = -\omega(\gamma\theta(D\mathbf{f})). \tag{III.17}$$

Proof. We compute using $Q\gamma = -\gamma Q$,

$$\omega(D\mathbf{f})_n = \omega(\gamma Q f_n - f_n^\varepsilon\gamma Q)$$
$$= \text{Tr}(\gamma^{n+1}Qf_n - (-1)^n\gamma^n f_n\gamma Q) = \text{Tr}(\gamma^{n+1}(Qf_n - Qf_n)) = 0.$$

In particular,

$$0 = \omega(D(\gamma\theta\mathbf{f})) = \omega(D(\gamma\theta)\mathbf{f}) - \omega(\gamma\theta D\mathbf{f}) = -\omega((d\theta)\mathbf{f}) - \omega(\gamma\theta D\mathbf{f}).$$

Lemma III.3 (Projection of b, b', and B from $\mathscr{C}_{\mathscr{H}}(A)$ to $\mathscr{C}_{\mathbb{C}}(A)$). We have

$$b'\omega = \omega\gamma b', \qquad b\omega = \omega\gamma b, \qquad B\omega = \omega\gamma B, \tag{III.18}$$

and furthermore

$$b'\theta = \theta^2. \tag{III.19}$$

Proof. We have

$$b'(\omega(f_n(A_1, ..., A_{n+1}))) = b'(\mathrm{Tr}(\gamma^n f_n(A_1, ..., A_{n+1})))$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \mathrm{Tr}(\gamma^n f_n(A_1, ..., A_j A_{j+1}, ..., A_{n+1}))$$

$$= \omega(\gamma(b'f_n))(A_1, ..., A_{n+1}).$$

We use here $\gamma^2 = 1$. The calculations for b and for B are similar. The calculation of $b'\theta$ follows from its definition.

IV. The Entire Cyclic Cocycle

The particular cochain we are going to study in the following is

$$\mathbf{f} = \theta e^{-\mathbf{K}}, \tag{IV.1}$$

with

$$\mathbf{K} = \mathbf{H} - \gamma d\theta. \tag{IV.2}$$

Here $e^{-\mathbf{H}}$ is the cochain $(e^{-H}, 0, ...)$ with $H = Q^2$. The exponential $e^{-\mathbf{K}}$ is defined by the Duhamel expansion

$$e^{-\mathbf{K}} = \sum_{n=0}^{\infty} \int_{\sigma_n} e^{-t_1 H} \gamma d\theta e^{-(t_2-t_1)H} \gamma d\theta ... e^{-(t_n-t_{n-1})H} \gamma d\theta e^{-(1-t_n)} dt.$$

This sequence is the unique solution to the equation

$$e^{-\mathbf{K}} = e^{-\mathbf{H}} + \int_0^1 e^{-s\mathbf{K}} \gamma d\theta e^{-(1-s)\mathbf{H}} ds. \tag{IV.3}$$

Under the map ω, the cochain $\theta e^{-\mathbf{K}}$ on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ yields the cochain τ on $\mathscr{C}_{\mathbb{C}}(\mathfrak{A})$ introduced in (II.1), namely

$$\tau = (\tau_0, \tau_1, ..., \tau_n, ...) = \omega(\theta e^{-\mathbf{K}}). \tag{IV.4}$$

Explicitly,

$$\tau_n(A_1, ..., A_n) = \int_{\sigma_{n-1}} \mathrm{Tr}(\gamma^n A_1 \gamma dA_2(t_1) \gamma dA_3(t_2) ... \gamma dA_n(t_{n-1})) dt$$

$$= (-1)^{n(n-1)/2} \int_{\sigma_{n-1}} \langle A_1 dA_2^{\gamma}(t_1) dA_3^{\gamma^2}(t_2) ... dA_n^{\gamma^{n-1}}(t_{n-1}) \rangle dt.$$

We now give the basic identity of perturbation theory.

Proposition IV.1. *Let σ denote a grading of $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ and let ∂ denote a superderivation on $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ with respect to this grading, such that $\partial e^{-s\mathbf{H}} = 0$ and $(\partial(\gamma d\theta))_0 = 0$. Then for $t \geq 0$,*

$$\partial e^{-t\mathbf{K}} = \int_0^t e^{-s\mathbf{K}^\sigma} \partial(\gamma d\theta) e^{-(t-s)\mathbf{K}} ds. \tag{IV.5}$$

Proof. Applying ∂ to (IV.3), we find

$$\partial e^{-t\mathbf{K}} = \int_0^t \partial(e^{-s\mathbf{K}})\gamma d\theta e^{-(t-s)\mathbf{H}} ds + \int_0^t e^{-s\mathbf{K}^\sigma} \partial(\gamma d\theta) e^{-(t-s)\mathbf{H}} ds. \tag{IV.6}$$

Inspecting the nth component of this equation, we see that $(\partial e^{-t\mathbf{K}})_n$ on the left-hand side is expressed in terms of $(\partial e^{-t\mathbf{K}})_m$, for $m < n$. Thus to prove (IV.5) by induction in n, we may substitute (IV.5) into the right-hand side of (IV.6) – as the induction hypothesis. This gives

$$\partial e^{-t\mathbf{K}} = \int_{0 \leq s_1 \leq s_2 \leq t} e^{-(s_2-s_1)\mathbf{K}^\sigma} \partial(\gamma d\theta) e^{-s_1\mathbf{K}} \gamma d\theta e^{-(t-s_2)\mathbf{H}} ds$$

$$+ \int_0^t e^{-s\mathbf{K}^\sigma} \partial(\gamma d\theta) e^{-(t-s)\mathbf{H}} ds.$$

In the first integral we substitute $s = s_2 - s_1$ and $s' = s_1$, to find

$$\partial e^{-t\mathbf{K}} = \int_0^t e^{-s\mathbf{K}^\sigma} \partial(\gamma d\theta) \left(e^{-(t-s)\mathbf{H}} + \int_0^{t-s} e^{-s'\mathbf{K}} \gamma d\theta e^{-(t-s-s')\mathbf{H}} ds' \right) ds$$

$$= \int_0^t e^{-s\mathbf{K}^\sigma} \partial(\gamma d\theta) e^{-(t-s)\mathbf{K}} ds,$$

where we have again used (IV.3). Hence (IV.5) holds for all components.

Our main results are the following, from which the Ward identity and the cocycle condition are immediate consequences:

Proposition IV.2 (Ward Identity). *The coboundary operator b on $\mathscr{C}_{\mathbb{C}}(\mathfrak{A})$ satisfies*

$$b\tau = b\omega(\theta e^{-\mathbf{K}}) = \omega(d\theta e^{-\mathbf{K}}). \tag{IV.7}$$

Proposition IV.3 (Cocycle Condition). *The coboundary operator B on $\mathscr{C}_{\mathbb{C}}(\mathfrak{A})$ satisfies*

$$B\tau = B\omega(\theta e^{-\mathbf{K}}) = \omega(d\theta e^{-\mathbf{K}}). \tag{IV.8}$$

Therefore, Eq. (IV.8) can be written

$$b\tau = B\tau, \quad \text{or} \quad \partial\tau = 0. \tag{IV.9}$$

Remark. As a consequence of Lemma III.3, we have

$$b'\tau = \omega(\gamma b'\theta e^{-\mathbf{K}}).$$

Since (III.7) can be written

$$b\tau = b'\tau - \omega(\gamma\theta e^{-\mathbf{K}^\varepsilon}\theta),$$

we also rewrite b as

$$b\tau = \omega(\gamma b'\theta e^{-\mathbf{K}} - \gamma\theta e^{-\mathbf{K}^\varepsilon}\theta). \tag{IV.10}$$

Lemma IV.4. *The coboundary operator b' satisfies*

$$b'e^{-\mathbf{K}} = \theta e^{-\mathbf{K}} - e^{-\mathbf{K}^\varepsilon}\theta + De^{-\mathbf{K}}. \qquad (IV.11)$$

Furthermore,

$$De^{-\mathbf{K}} = \int_0^1 e^{-(1-s)\mathbf{K}^\varepsilon}[\theta, \mathbf{H}]e^{-s\mathbf{K}}ds, \qquad (IV.12)$$

and

$$b'e^{-\mathbf{K}} = \int_0^1 e^{-(1-s)\mathbf{K}^\varepsilon}\gamma d\theta^2 e^{-s\mathbf{K}}ds. \qquad (IV.13)$$

Proof. The identities (IV.12–13) follow from Proposition IV.1,

$$D(\gamma d\theta) = \gamma Q\gamma d\theta - (\gamma d\theta)^\varepsilon\gamma Q = d^2\theta = [\mathbf{H}, \theta], \qquad (IV.14)$$

and

$$b'(\gamma d\theta) = \gamma d\theta^2.$$

Finally we note that

$$\theta e^{-\mathbf{K}} - e^{-\mathbf{K}^\varepsilon}\theta = \int_0^1 \frac{d}{ds}(e^{-(1-s)\mathbf{K}^\varepsilon}\theta e^{-s\mathbf{K}})ds$$

$$= \int_0^1 (e^{-(1-s)\mathbf{K}^\varepsilon}(-\theta\mathbf{K} + \mathbf{K}^\varepsilon\theta)e^{-s\mathbf{K}})ds$$

$$= \int_0^1 (e^{-(1-s)\mathbf{K}^\varepsilon}(-[\theta, \mathbf{H}] + \gamma d\theta^2)e^{-s\mathbf{K}})ds = (-D + b')e^{-\mathbf{K}},$$

thus proving (IV.11).

Proof of Proposition IV.2. The identity (IV.10) is a translation of (III.5). Using (III.19), (IV.11), and the superderivation property of b',

$$b'\theta e^{-\mathbf{K}} = \theta^2 e^{-\mathbf{K}} - \theta b'e^{-\mathbf{K}} = \theta e^{-\mathbf{K}^\varepsilon}\theta - \theta De^{-\mathbf{K}}. \qquad (IV.15)$$

Multiplying by γ, and using Lemma III.2, we obtain

$$b'\omega(\theta e^{-\mathbf{K}}) = \omega(\gamma b'\theta e^{-\mathbf{K}}) = \omega(\gamma\theta e^{-\mathbf{K}^\varepsilon}\theta) - \omega(\gamma\theta De^{-\mathbf{K}})$$

$$= \omega(\gamma\theta e^{-\mathbf{K}^\varepsilon}\theta) + \omega((d\theta)e^{-\mathbf{K}}). \qquad (IV.16)$$

Thus by (IV.10), we obtain (IV.4).

Proof of Proposition IV.3. We calculate using Proposition IV.1, $dI = 0$, and $Bf_1 = 0$ that

$$B\omega(\theta e^{-\mathbf{K}}) = \omega(\gamma B\theta e^{-\mathbf{K}}) = \omega(\gamma B\theta(e^{-\mathbf{K}} - e^{-\mathbf{H}})).$$

Here $e^{-\mathbf{H}}$ is the zero-order term in $e^{-\mathbf{K}}$. Thus using the definition (III.8) of B we have

$$B\omega(\theta e^{-\mathbf{K}}) = \mathrm{Sym}\,\omega(\gamma(e^{-\mathbf{K}} - e^{-\mathbf{H}})).$$

Note that

$$\omega(\gamma(e^{-\mathbf{K}} - e^{-\mathbf{H}})) \in \bigoplus_n (\mathscr{C}_{\mathscr{H}}^{2n}(\mathfrak{A}) \cap \mathscr{C}_{\mathscr{H}}(\mathfrak{A})^+) \oplus (\mathscr{C}_{\mathscr{H}}^{2n+1}(\mathfrak{A}) \cap \mathscr{C}_{\mathscr{H}}(\mathfrak{A})^-),$$

so that (III.12–13) apply. We are thus reduced to proving

Proposition IV.5. *With the choice*

$$\mathbf{f}=\omega(\gamma(e^{-\mathbf{K}}-e^{-\mathbf{H}})),$$

we have

$$\mathrm{Sym}\,\mathbf{f}=\sum_{k=1}^{\infty}T^{k-1}\mathbf{f}^{(\leqq k)}=\omega(d\theta e^{-\mathbf{K}}).$$

Proof. We repeatedly apply the perturbation identity (IV.3) to the definition of **f** to obtain

$$\mathbf{f}^{(\geqq k)}=\int_{\sigma_k}\omega(\gamma e^{-s_1\mathbf{H}}\gamma d\theta e^{-(s_2-s_1)\mathbf{H}}\gamma d\theta\dots e^{-(s_k-s_{k-1})\mathbf{H}}\gamma d\theta e^{-(1-s_k)\mathbf{K}})ds$$

$$=\int_{\sigma_k}\omega(e^{-s_1\mathbf{H}}d\theta e^{-(s_2-s_1)\mathbf{H}}\gamma d\theta\dots e^{-(s_k-s_{k-1})\mathbf{H}}\gamma d\theta e^{-(1-s_k)\mathbf{K}})ds.$$

Apply the identity (III.14) to this expression $(k-1)$ times and replace s_k by t. Thus we obtain,

$$T^{k-1}\mathbf{f}^{(\geqq k)}$$

$$=\int_{0}^{1}\left(\int_{0\leqq s_1\leqq\dots\leqq s_{k-1}\leqq t}\omega(d\theta e^{-(1-t)\mathbf{K}}e^{-s_1\mathbf{H}}\gamma d\theta e^{-(s_2-s_1)\mathbf{H}}\dots\gamma d\theta e^{-(t-s_{k-1})\mathbf{H}})ds\right)dt.$$

Summing over k we obtain,

$$\mathrm{Sym}\,\mathbf{f}=\sum_{k=1}^{\infty}T^{k-1}\mathbf{f}^{(\geqq k)}=\int_{0}^{1}\omega(d\theta e^{-(1-t)\mathbf{K}}e^{-t\mathbf{K}})dt=\int_{0}^{1}\omega(d\theta e^{-\mathbf{K}})dt=\omega(d\theta e^{-\mathbf{K}}),$$

as claimed.

Let us finish this section with a few remarks on topology of the space of cochains. We put a norm on the space of cochains as follows. Let us define a norm on the space $\mathscr{C}_{\mathscr{H}}^{n}$ by

$$\|f_n\|_n=\mathrm{Sup}_{A_1,\dots,A_n\in\mathfrak{A}}\frac{\|f_n(A_1,\dots,A_n)\|_{\mathscr{L}(\mathscr{H})}}{\|A_1\|_*\dots\|A_n\|_*},$$

where $\|A\|_*=\|A\|+\|dA\|$. With this topology, we can define a norm $\|\|\cdot\|\|$ on $\mathscr{C}_{\mathscr{H}}$ by

$$\|\|\mathbf{f}\|\|=\mathrm{Sup}_n(n+1)!^{1/2}\|f_n\|_n.$$

This norm restricts in a natural way to $\mathscr{C}_{\mathbb{C}}(\mathfrak{A})$, by replacing the $\mathscr{L}(\mathscr{H})$ norm with the absolute value. This is the norm on ordinary cochains. We define the space of entire cochains $\mathscr{C}_{\mathrm{entire}}(\mathfrak{A})$ as the subspace of $\mathscr{C}_{\mathbb{C}}(\mathfrak{A})$ with finite $\|\|\cdot\|\|$ norm. Connes has established that the operators b and B act on the space $\mathscr{C}_{\mathrm{entire}}(\mathfrak{A})$ of entire cochains [C].

Proposition IV.6. *The cocycle defined in (IV.4) is entire,* $\tau\in\mathscr{C}_{\mathrm{entire}}(\mathfrak{A})$.

Proof. Since $\tau_0=0$, the estimates of [JLO1] show that $\|\tau_n\|_n\leqq\mathrm{Tr}\,e^{-H}(n-1)!^{-1}$ for $n>0$. Hence we have $\|\|\tau\|\|\leqq\mathrm{Sup}_n\mathrm{Tr}\,e^{-H}(n+1)!^{1/2}(n-1)!^{-1}\leqq3\,\mathrm{Tr}\,e^{-H}$.

V. Exponential Form of the Super-KMS Functional

It is possible to define the cocycle τ under more general conditions, namely without the assumption of the existence of a Fredholm operator Q and without the existence of a trace class heat kernel e^{-Q^2}. Instead, we substitute a super-derivation and unitary action as explained in [JLO2] and use a super-KMS functional to define a generalized trace on $\mathfrak{A}$.

We assume that $\mathfrak{A}$ has the structure of a quantum algebra. In particular, $\mathfrak{A}$ is represented as an algebra of operators on a Hilbert space $\mathscr{H}$. There is a $\mathbb{Z}_2$-grading on $\mathfrak{A}$ which acts as a self adjoint operator γ on $\mathscr{H}$. There is a continuous, one-parameter group α_t of *-automorphisms of $\mathfrak{A}$, whose generator is the square of a densely defined super-derivation d on $\mathfrak{A}$. Furthermore, we then know that there is a dense subalgebra $\mathfrak{A}_\alpha$ of $\mathfrak{A}$ defined by those elements such that $\alpha_t(A)$ extends to an entire function of t. We assume that $\mathfrak{A}_\alpha$ is an invariant domain for d.

Under these assumptions, a super-KMS functional on $\mathfrak{A}$ is a continuous linear functional ω such that when restricted to $\mathfrak{A}_\alpha$ the following two relations hold:

$$\omega(dA)=0, \quad \text{and} \quad \omega(AB)=\omega(B^\gamma\alpha_i(A)). \tag{V.1}$$

We extend the definition of ω to $\mathscr{C}_{\mathscr{H}}(\mathfrak{A})$ by defining

$$\omega(\mathbf{f})_n = \omega(\gamma^{n+1}f_n).$$

Remark that if in addition we have the structures used in Sects. I–IV, then an example of a super-KMS functional is $\omega(\cdot)=\mathrm{Tr}(\gamma\cdot e^{-Q^2})$.

Theorem V.1. *If ω is a super-KMS functional on $\mathfrak{A}$, and $s\in\mathbb{R}$ then there exists a family of entire cochains on $\mathfrak{A}$ defined by*

$$\mathbf{g}(s)= \sum_{n=0}^{\infty} \int_{\sigma_n} \omega(\theta\alpha_{st_1}(\gamma d\theta)\alpha_{st_2}(\gamma d\theta)\ldots\alpha_{st_n}(\gamma d\theta))dt. \tag{V.2}$$

This family of cochains is the boundary value of a cochain-valued function $\mathbf{g}(s)$, which is analytic in the strip $0<\mathrm{Im}\,s<1$ and satisfies

$$\|\|\mathbf{g}(s)\|\| \leqq 3\|\omega\|,$$

uniformly in the strip. Furthermore, $\mathbf{g}(s)$ has a continuous boundary value as $\mathrm{Im}\,s\to 1$.

Remark. The cochain function $\mathbf{g}(s)$ can be interpreted as a trace of a wave operator, for in case that Q and H exist, then $\mathbf{g}(s)=\omega(\theta e^{isH+\gamma d\theta}e^{-isH})$.

Theorem V.2. *If ω is a super-KMS functional on $\mathfrak{A}$, then there exists an entire cyclic cocycle τ on $\mathfrak{A}$ defined by*

$$\tau = \mathbf{g}(i). \tag{V.3}$$

In other words,

$$\tau \in \mathscr{C}_{\mathrm{entire}}(\mathfrak{A}), \quad \text{and} \quad \partial\tau=0. \tag{V.4}$$

The cohomology classes of τ are stable under bounded perturbations of d and the corresponding perturbations of α and of ω.

The proofs of Theorems V.1–2 have an analytic and an algebraic part. The analytic estimates follow closely the arguments of [JLO2, JLWis], so we do not repeat them. The algebraic calculations parallel those in Sects. III and IV.

References

[C]   Connes, A.: Entire cyclic cohomology of banach algebras and characters of θ-summable Fredholm modules. K-Theory **1**, 519–548 (1988)
[EFJL]   Ernst, K., Feng, P., Jaffe, A., Lesniewski, A.: Quantum K-theory. II. Homotopy invariance of the Chern character. J. Funct. Anal. (to appear)
[JLO1]   Jaffe, A., Lesniewski, A., Osterwalder, K.: Quantum K-theory. I. The Chern character. Commun. Math. Phys. **118**, 1–14 (1988)
[JLO2]   Jaffe, A., Lesniewski, A., Osterwalder, K.: On super-KMS functionals and entire cyclic cohomology. K-theory **2**, 675–682 (1989)
[JLWis]   Jaffe, A., Lesniewski, A., Wisniowski, M.: Deformation of super-KMS functionals. Commun. Math. Phys. **121**, 527–540 (1989)
[K]   Kastler, D.: Cyclic cocycles from graded KMS functionals. Commun. Math. Phys. **121**, 345–350 (1989)
[Q]   Quillen, D.: Algebra cochains and cyclic cohomology. Publications Mathématiques I.H.E.S. **68**, 139–174 (1989)