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Localization for a Class of One Dimensional Quasi-Periodic Schrödinger Operators

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Dedicated to Res Jost and Arthur Wightman with respect and affection

Abstract. We prove for small ε and α satisfying a certain Diophantine condition the operator

$$H = -\varepsilon^2 \Delta + \frac{1}{2\pi} \cos 2\pi (j\alpha + \theta) \qquad j \in \mathbb{Z}$$

has pure point spectrum for almost all θ. A similar result is established at low energy for $H = -\frac{d^2}{dx^2} - K^2(\cos 2\pi x + \cos 2\pi(\alpha x + \theta))$ provided K is sufficiently large.

1. Introduction

In this paper we shall study some of the spectral properties of the operator

$$H_c(\theta) = -\frac{d^2}{dx^2} + K^2 v(x, \theta) \tag{1.1}$$

acting on $L_2(R)$, where

$$v(x, \theta) = -\cos 2\pi x - \cos 2\pi(\alpha x + \theta)\,. \tag{1.2}$$

We shall also study its finite difference approximation on $l_2(\mathbb{Z})$ given by

$$H(\theta) = -\varepsilon^2 \Delta + v(j, \theta) = -\varepsilon^2 \Delta + \frac{1}{2\pi} \cos 2\pi(\alpha j + \theta)\,. \tag{1.3}$$

In one dimension the finite difference Laplacian has matrix elements $\Delta_{ij} = 1$ if $|i - j| = 1$ and $\Delta_{ij} = 0$ otherwise. When α is rational the spectra of H and H_c

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are known to be purely absolutely continuous and their generalized eigenstates are Bloch waves $\psi(x) = e^{ikx}p(x)$, where p is a periodic function of x. We shall consider the case where α is irrational and K is large or ε is small. In addition, α is assumed to satisfy the following Diophantine condition:

$$|n\alpha|_1 \geq C_1^2/n^2 \qquad n \neq 0\,. \tag{1.4}$$

Here $|\theta|_1$ is the absolute value of θ modulo 1 defined so that $0 \leq |\theta|_1 \leq \frac{1}{2}$. Weaker conditions on α would suffice, but we shall restrict α as above for convenience. We first discuss the finite difference operator.

Theorem 1.1. *Let α be as in* (1.4). *For ε sufficiently small and $\theta \in \mathbb{R}/\mathbb{Z}$ belonging to a set of measure one, H has pure point spectrum, with eigenfunctions which decay exponentially fast.*

This theorem has also been independently established by Sinai [1]. Our proof also applies if the cosine is replaced by any C^2 periodic function f which is even and has exactly two critical points which are nondegenerate. Returning to the special case (1.3), if α is irrational, then for $2\pi\varepsilon^2 < 1/2$ it is known that H has no absolutely continuous spectrum [2, 3, 4] and when $2\pi\varepsilon^2 > 1/2$, H has no point spectrum [2, 5]. If $2\pi\varepsilon^2 < 1/2$ and α irrational but very well aproximated by rationals, i.e. if α is Liouville, then the spectrum of H is purely singular continuous [3, 6]. The existence of some point spectrum for ε small and α sufficiently irrational has been established in [7].

In the continuum it is a classic theorem of Dinaburg and Sinai [8] that, for any K, H_c has many Bloch wave eigenstates $\psi(x) = \mathrm{qp}(x)e^{ikx}$, where qp is quasi-periodic in x. These eigenstates correspond with the presence of absolutely continuous spectrum. Our next theorem shows that if K is sufficiently large H_c has pure point spectrum at low energy.

Theorem 1.2. *Let α be as in* (1.4). *If K is sufficiently large, then for almost all θ, the spectrum of H_c in the interval $[-2K^2, -2K^2 + 10K\sqrt{1+\alpha^2}]$ is pure point. The eigenfunctions decay exponentially fast.*

This theorem provides the first existence proof for eigenstates of H_c. From general considerations it is known that the spectra of H and H_c are essential. Although many of the techniques and results described above work only for the lattice we shall see that our analysis of H_c is nearly identical to that of H. When K is small, the operator "dual" to H_c has been shown to have pure point spectrum [9] by methods closely related to those described here. Moreover the estimates of [9] together with the ideas in [2] or [5] easily imply that H_c has no point spectrum for small K. We conjecture that H_c has purely absolutely continuous spectrum in this case. For large ε, H is also believed to have only absolutely continuous spectrum.

Determining the nature of the spectrum of H can be viewed as a small divisor problem. Small divisors appear in our analysis of the Green's function $G(E_*) = (H - E_*)^{-1}$ because there are eigenvalues E^i of H which come arbitrarily close to E_*. To overcome this problem we develop a multiscale perturbation scheme in which we keep track of the "location" of the small divisors [10, 12]. For a fixed energ E_* and scale n, the location of the small divisors is given by a family of disjoint singular intervals $S_n(E*, \theta*) = \{I_n^i\}$ defined below. These are certain intervals of length $l_n \geq l_{n-1}^2$, such that $H(\theta*)$ restricted to I_n^i with Dirichlet boundary conditions [denoted $H(I_n^i, \theta*)$] has an eigenvalue $E_n^i(\theta*)$ such that for $n \geq 1$ and some fixed $\beta > 0$,

$$|E_n^i - E*| \leq \delta_n \equiv \exp(-\beta l_n^{2/3}) . \tag{1.5}$$

The uniqueness of E_n^i is proved in Sect. 4. See Sect. 2 for the precise definition of S_n. Many of the ideas appearing here were first used in the case where $v(j)$, $j \in \mathbb{Z}^d$, are independent random variables [10, 11]. Here however, the randomness only appears in the parameter θ, and the $v(j)$ are highly dependent.

If an interval Λ does not meet any member of $S_n(E*, \theta*)$ then the Green's function for $H(\Lambda)$

$$G_\Lambda(E*, x, y) = [H(\Lambda) - E*]^{-1}(x, y)$$

decays exponentially fast, provided $|x - y| \geq l_n$ and $|\theta - \theta*|_1 \leq \delta_n/3$. See Theorem 2.2 for a precise statement, and Appendix A for the proof.

The key observation, which uses the fact that there is only one incommensurate frequency α, is that the centers $c_n^i \in \mathbb{Z}$ of the intervals I_n^i belonging to $S_n(E*, \theta*)$ satisfy the following relation for each $\theta*$:

$$m(c_n^i, c_n^j) \equiv \min\{|(c_n^i - c_n^j)\alpha|_1,\ |(c_n^i + c_n^j)\alpha + 2\theta*|_1\} \leq 4\delta_n^{1/2} . \tag{1.6}$$

See Theorem 2.1. The reason why $m(c_n^i, c_n^j)$ is small is the following. If $H(I_n^i, \theta*)$ and $H(I_n^j, \theta*)$ have eigenvalues which nearly coincide, (as in (1.5)) we will show that the potential restricted to I_n^i or I_n^j must either be nearly translates of each other, in which case $|(c_n^i - c_n^j)\alpha|_1$, is small or they are nearly reflections of each other, so that $|(c_n^i + c_n^j)\alpha + 2\theta*|_1$ is small. This will be proved inductively using perturbation theory. In Sect. 3 we show that the decay estimate on G and (1.6) imply Theorems 1.1 and 1.2.

The above inequality imposes a special geometric pattern on the intervals $I_n^i \in S_n$ as can be seen from the following lemma:

Lemma 1.3. *Let $a, b, c \in \mathbb{Z}$, be distinct points. If $m(a,b) \leq \delta$ and $m(b,c) \leq \delta$, then either $|a-b| \geq \delta^{-1/2}C_1/3$ or $|b-c| \geq \delta^{-1/2}C_1/3$. Also if $|a-b| \leq \delta^{-1/2}C_1/5$, then there is a unique $\tilde{c}$ such that $|a-b| = |c-\tilde{c}|$ and $m(c,\tilde{c}) = |(c+\tilde{c})\alpha + 2\theta*|_1 \leq 3\delta$.*

Proof. If $|(i-j)\alpha|_1 \leq \delta$, then by (1.4), $|i-j| \geq \delta^{-1/2}C_1$. Thus we need only consider the case where $|(a+b)\alpha + 2\theta*|_1 \leq \delta$ and $|(b+c)\alpha + 2\theta*|_1 \leq \delta$. From these inequalities we deduce $|(a-c)\alpha|_1 \leq 2\delta$ so $|a-c| \geq \delta^{-1/2}C_1/\sqrt{2}$, and our first assertion follows easily. To prove the next assertion we can assume $|(a+b)\alpha + 2\theta*|_1 \leq \delta$; otherwise $|(a-b)\alpha|_1 \leq \delta$, which by (1.4) contradicts our distance assumption. If $|(b-c)\alpha|_1 \leq \delta$, then $m(\tilde{c}, c) \leq |[(c+a-b)\alpha + c\alpha] + 2\theta*|_1 \leq |(a+b)\alpha + 2\theta*|_1 + 2|(b-c)\alpha|_1 \leq 3\delta$, where $\tilde{c} = c + (a-b)$. If $|(b+c)\alpha + 2\theta*|_1 \leq \delta$, then we set $\tilde{c} = c - (a-b)$ and a similar inequality holds. The uniqueness of $\tilde{c}$ follows from the first part of the lemma. □

We call $\tilde{c}$ the "mirror image" of c, because, for all j, $v(-j+\tilde{c}, \theta*) = v(j+c, \theta*) + \mathcal{O}(\delta)$. This lemma together with (1.6) shows that each interval I_n of width l_n can contain no more than two intervals I_{n-1}. A third interval is excluded, since Const. $l_n \leq C_1\delta_{n-1}^{-1/4}$.

For the lattice, S_0 has a very simple description. Intervals consist of single sites, $E_0^i = v(i, \theta_*)$ and

$$S_0(E_*, \theta_*) = \{j : |v(j, \theta_*) - E_*| \leq \delta_0\},$$

where $\delta_0 = \varepsilon$. Since $|v(a, \theta_*) - v(b, \theta_*)| \geq m(a, b)^2/2$, (1.6) holds for $n = 0$. Also note that if Λ is an interval such that $\Lambda \cap S_0 = \emptyset$, then since $\|\Delta\| = 2$, and Δ has only nearest neighbor matrix elements,

$$\begin{aligned} |G_\Lambda(E_*; x, y)| &= \sum_{n=0}^{\infty} [|v(j) - E_*|^{-1} \varepsilon^2 \Delta]^n (x, y)\, |v(y) - E_*|^{-1} \\ &\leq \frac{1}{2} \sum_{n \geq |x-y|} \varepsilon^{n-1} \leq \varepsilon^{|x-y|-1}. \end{aligned}$$

In the case of H_c we define I_0^i to be intervals of width 1 centered at $c_0^i \in \mathbb{Z}$. Let $|E_* + 2K^2| \leq 10K$ and define

$$S_0 = \left\{ I_0^i : \inf_{x \in I_0^i} [K^2 v(x) - E_*] \leq 2K \right\}.$$

It is easy to se that these intervals are centered at near minima of $v(x)$ and that they are widely separated $\cong K^{1/4}$. Notice that if $\Lambda \cap S_0 = \emptyset$ then $|G_\Lambda(E_*, x, y)| \leq \exp[-\gamma_0 |x - y|]$, where $\gamma_0 = (2K)^{1/2}$. This follows from the maximum principle. (The constant γ_0 can be improved to K/const by using a simple W.K.B. estimate.) For $n = 0$, (1.6) holds with $\delta_0 = K^{-1}$. See Appendix C for further discussion of the $n = 0$ case.

It is important to note that since I_m^i and I_m^j are integral translates of each other the spectrum of H, σH, satisfies

$$\sigma H[I_n^i, \theta_*] = \sigma H[I_n^j, \theta_* + (c_n^i - c_n^j)\alpha] = \sigma H[I_n^j, \theta_* - (2\theta_* + (c_n^i + c_n^j)\alpha)]. \quad (1.7)$$

The same identity holds for H_c. In the second equality we have used the evenness of the cosine: $v(x + c_n^j, \theta_*) = v(-x + c_n^j, -\theta_* - 2c_n^j\alpha)$. Since we are in one dimension, the eigenvalues as functions of θ never cross and are smooth in θ. Hence the evenness implies

$$E_m^j(\theta) = E_m^j(-\theta - 2c_m^j\alpha). \quad (1.8)$$

To prove (1.6), we shall show that (1.7) holds for the particular eigenvalues $E_{n+1}^i(\theta)$ i.e. $E_{n+1}^i(\theta_*) = E_{n+1}^j(\theta_* \pm m(c_{n+1}^i, c_{n+1}^j))$. Our key estimate gives a bound from below, $|d^2 E_{n+1}^i(\theta)/d\theta^2| \geq 2/3$ when $|dE_{n+1}^i(\theta)/d\theta|$ is small. Hence if $E_{n+1}^i(\theta)$, $E_{n+1}^j(\theta)$ satisfy (1.5) then we obtain (1.6) in the following form

$$2\delta_{n+1} \geq |E_{n+1}^i(\theta_*) - E_{n+1}^j(\theta_*)| = |E_{n+1}^i(\theta_*) - E_{n+1}^i(\theta_* + m)| \geq \frac{m^2}{4}, \quad (1.9)$$

where $m = \pm m(c_{n+1}^i, c_{n+1}^j)$. The last inequality of (1.9) follows from the lower bound on $|d^2 E/d\theta^2|$ and an elementary lemma, Lemma 5.7. See Sect. 5, for details.

The outline of this paper is as follows. Most of the paper is devoted to discussing the finite difference operator H with occasional references to what needs to be modified for the analysis of H_c. In the next section we give a precise definition of S_n and present key identities we shall need for our analysis. The proof of Theorems 1.1 and 1.2 are given in Sect. 3, assuming that (1.6) holds for all n, and that the Green's function decays. In Sect. 4 we assume (1.6) for fixed n and establish properties of the eigenvalues E_{n+1} and the corresponding eigenfunctions. In Sect. 5 we also assume (1.6) and establish a lower bound on $|d^2E^i_{n+1}(\theta)/d\theta^2|$ when $|dE^i_{n+1}(\theta)/d\theta|$ is small. This enables us to prove (1.6) at the next scale, $n+1$, as in (1.9). Some technical comments about the continuum case appear in Appendix C.

2. Definitions and Formulas

Let E_*, θ_* be fixed. For $n = 0$, S_0 is defined in Sect. 1 and (1.6) holds. Let $n \geq 0$. Now assume that S_n is defined and that we have proved by induction that (1.6) holds. We define $S_{n+1} = S_{n+1}(E_*, \theta_*)$ as follows. Let

$$s_n = \min\{|c^i_n - c^j_n| : I^i_n, I^j_n \in S_n\}. \tag{2.1}$$

The c^i_n denote the centers of I^i_n. For $n \geq 1$, if $s_n \geq 4l^2_n$, let I^i_{n+1} have length $l_{n+1} \cong l^2_n$ and center $c^i_{n+1} = c^i_n$. In this case I^i_{n+1} contains a single interval of S_n.

If $s_n \leq 4l^2_n$ we note that each interval $I^i_n \in S_n$ has a "mirror" image $\tilde{I}^i_n$ whose center satisfies $|c^i_n - \tilde{c}^i_n| = s_n$ and

$$m(c^i_n, \tilde{c}^i_n) = |2\theta_* + (c^i_n + \tilde{c}^i_n)\alpha|_1 \leq 12\delta_n^{1/2}. \tag{2.2}$$

This follows from (1.6) and Lemma 1.3. These considerations also imply that the center of any other interval $I^j_n \in S_n$ satisfies

$$|c^j_n - c^i_n| \geq C_1\delta_n^{-1/4}/12 \geq l_{n+2} \geq l_{n+1}. \tag{2.3}$$

The intervals I^i_{n+1} are now defined to be centered at c^i_{n+1} and of width l_{n+1}, where

$$c^i_{n+1} = \tfrac{1}{2}\,[c^i_n + \tilde{c}^i_n], \qquad l_{n+1} \cong l^4_n. \tag{2.4}$$

The collection of intervals consisting of S_n and of its mirror image is called $\bar{S}_n$. Note that in the present case I^i_{n+1} contains precisely 2 elements of $\bar{S}_n$. By (2.3) I^i_{n+1} do not overlap (see Fig. 1).

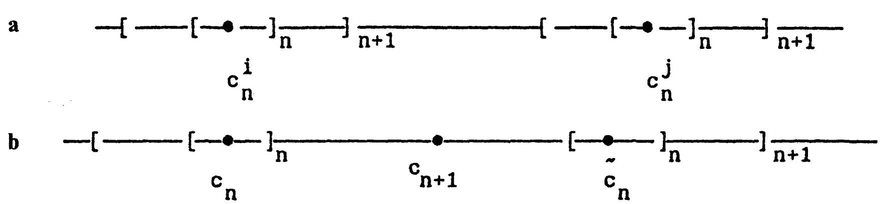

Fig. 1. a The intervals I^i_n and I^j_n are separated by more than $4l^2_n$. The • denote the centers of the intervals I^i_n, I^j_n. In this case $c^i_{n+1} = c^i_n$ and $c^j_{n+1} = c^j_n$. **b** The interval I_n and $\tilde{I}_n$ are separated by less than $4l^2_n$. The interval I_{n+1} is much larger than indicated in the figure

For $n = 0$, we define S_1 as above except that l^2_0 and l^4_0 above are replaced by L^2, L^4 where $L = |\ln \varepsilon|$. This is because l_0 is small and we want l_1 to be large. This special convention will be adopted in the remainder of this paper.

Definition. An interval I_{n+1}^i belongs to the family of *singular* intervals $S_{n+1}(E*, \theta*)$ if $H(I_{n+1}^i)$ has an eigenvalue $E_{n+1}^i(\theta*)$ such that

$$|E* - E_{n+1}^i| \leq \delta_{n+1} \equiv \exp(-\beta l_{n+1}^{2/3}), \tag{2.5}$$

where $\beta = |\ln \varepsilon|$. The nonsingular intervals I_{n+1} are called $(n+1)$-*regular*. The length l_{n+1} of the intervals I_{n+1} is chosen so that its boundary, ∂I_{n+1}, does not meet I_m^i for $m \leq n$. This can be done using Lemma 2.4 below. An arbitrary interval Λ is said to be *n-regular* if every point (or interval) of $\Lambda \cap S_0$ is contained in an m regular interval $I_m \subset \Lambda$ for some $m \leq n$.

In the continuum, the definitions are as above except $\beta = K/2$, $\delta_0 = K^{-1}$. Later we shall use the notation $\delta_{-1} = \varepsilon^{1/4}$, $K^{1/4}$ for the discrete and continuum cases respectively.

Next we formulate *condition* $\mathscr{C}_n$ on the centers of the singular intervals.

Condition $\mathscr{C}_n$. For all $m \leq n$ whenever c_m^i, c_m^j are centers of intervals I_m^i, $I_m^j \in S_m$ then

$$m(c_m^i, c_m^j)^2 \leq 8|E_m^i - E_m^j| \leq 16\delta_m\,. \tag{2.6}$$

Note that (2.6) is a stronger version of (1.6).

Theorem 2.1. *If ε is small $\mathscr{C}_n$ holds for all n, $\theta*$ and $E*$.*

The following remarks hold for all $m \leq n$, assuming $\mathscr{C}_n$ holds.

Remarks. a) The last inequality in (2.6) follows from the fact that both E_m^i and E_m^j belong to the interval $|E - E*| \leq \delta_m$.
b) If $s_m \geq 4l_m^2$, then I_{m+1}^j contains precisely one interval I_m^j in S_m and $I_{m+1}^j \backslash I_m^j$ is the union of m-regular intervals.
c) If $s_m \leq 4l_m^2$, then I_{m+1}^i contains exactly two intervals in $\bar{S}_m$, I_m, $\tilde{I}_m$. By (1.7) and (2.2) the spectrum of $H(I_m)$ and $H(\tilde{I}_m)$ differ by $\mathcal{O}(\delta_m^{1/2})$.
d) If $s_{m-1} \leq 4l_{m-1}^2$ then, by (2.3) the pairs of intervals in $\bar{S}_{m-1}$ are so far separated that $s_m \geq 4l_m^2$ and $s_{m+1} \geq 4l_{m+1}^2$.

Theorem 2.2. *For ε^2 sufficiently small (independent of n) if $\mathscr{C}_{n-1}$ holds and Λ is n-regular then*

$$|G_\Lambda(E, x, y)| \leq \exp -\gamma_n|x - y| \tag{2.7}$$

for all $|E - E*| \leq \delta_n/3$, $|\theta - \theta*| \leq \delta_n/3$ provided $|x - y| \geq l_n^{5/6}$. Moreover $\gamma_n \geq \gamma \geq \frac{1}{2}|\ln \varepsilon$.

For notational compactness we shall generally omit the θ dependence of G.

Analogous theorems hold in the continuum:

Theorem 2.3. *If K is sufficiently large and $E* \in [-2K^2, -2K^2 + 10K\sqrt{1+\alpha^2}]$, then $\mathscr{C}_n$ holds for all n and $\theta*$. For large K* (2.7) *holds as above if we set $\gamma_n \geq \gamma \geq K/\text{const}$ and require $|\theta - \theta*| \leq \delta_n/16\pi K^2$. Derivatives of G in $x, y \in R$ also satisfy* (2.7). *See* (2.9) *below.*

The proof of Theorem 2.2 is given in [10, 12]. See Appendix A where the key steps are explained. Theorem 2.1 will be proved by induction on n.

In the proof of the above theorems we shall frequently express $G_\Lambda = [H(\Lambda) - E]^{-1}$ in terms of G_I where I is a subinterval of Λ. Let us define

$$G_{\Lambda,I} = G_{\Lambda\setminus I} \oplus G_I\,.$$

On the lattice, we can write

$$H(\Lambda) = H(I) + H(\Lambda\setminus I) - \varepsilon^2\Gamma\,,$$

so by the resolvent identity

$$G_\Lambda = G_{\Lambda,I} + \varepsilon^2 G_{\Lambda,I}G_\Lambda\,. \tag{2.8}$$

Here Γ denotes the symmetric matrix corresponding to the boundary of I in Λ

$$\begin{aligned}\Gamma_{x,y} &= 1 \quad \text{if} \quad (x,y) \in \partial I \quad \text{and} \quad (x,y) \notin \partial\Lambda \\ &= 0 \quad \text{otherwise}.\end{aligned}$$

On the lattice we have represent ∂I as a set of unordered nearest neighbor pairs (a,a') such that $a \in I$, $a' \notin I$. We shall frequently identify the boundary of a set with the corresponding matrix, as above. The continuum analogue of (2.8) is

$$G_\Lambda(x,y) = G_{\Lambda,I}(x,y) + \sum_{z\in\partial I\setminus\partial\Lambda} \dot{G}_{\Lambda,I}(x,z)G_\Lambda(z,y)\,, \tag{2.9}$$

where $\dot{G}$ denotes the normal derivative of G with respect to z.

The key estimates involve derivatives of $E_n^i(\theta)$ with respect to θ. Note that, since we are in one dimension, there is no level crossing, and $E_n^i(\theta)$ is well defined and smooth in θ. The first and second derivatives of an eigenvalue $E(\theta)$ of $H(I)$ are given by first and second order perturbation theory:

$$\frac{d}{d\theta}E(\theta) = \langle\psi, v'\psi\rangle \qquad v' = \frac{dv}{d\theta}\,, \tag{2.10}$$

$$\frac{d^2}{d\theta^2}E(\theta) = \langle\psi, v''\psi\rangle - 2\langle\psi, v'G_I^\perp(E)v'\psi\rangle\,, \tag{2.11}$$

where ψ is the normalized eigenfunction corresponding to $E(\theta)$, and $G_I^\perp(E)$ is the Green's function projected onto the orthogonal complement of ψ. If $\tilde{E}$ denotes the eigenvalue of $H(I)$ closest to E, and $\tilde{\psi}$ its eigenfunction, then after projecting out $\tilde{\psi}$ we have

$$\langle\psi, v'G_I^\perp(E)v'\psi\rangle = \frac{\langle\psi, v'\tilde{\psi}\rangle^2}{\tilde{E}-E} + \langle\psi, v'G_I^{\perp\perp}(E)v'\psi\rangle\,. \tag{2.12}$$

The same formulas hold for H_c, except v is replaced by K^2v. In Sect. 5 we show that when $dE/d\theta$ is small, $|\langle\psi, v'\tilde{\psi}\rangle|$ is bounded below and $|E-\tilde{E}|$ is small: hence by (2.10) and (2.11), $|E''(\theta)|$ is large. This is the key step in our proof which enables us to justify (1.9) and (2.6).

Using (2.10) it is easy to see that for any eigenvalue $E(\theta)$ of $H(I,\theta)$,

$$|dE(\theta)/d\theta| \leq 1\,, \qquad |dE(\theta)/d\theta| \leq 4\pi K^2 \tag{2.13}$$

for the discrete and continuum cases respectively. This will be used to obtain narrow intervals in θ near θ_* where our theorems and lemmas apply.

We conclude this section with a technical lemma, assuming $\mathscr{C}_n$ holds.

Lemma 2.4. *If* $\Lambda = [a,b]$ *is an interval then there is a deformed interval* $\Lambda' = [a',b']$ *such that* $a',b' \notin I^i_m$, *for all* $m \leq n$, *and* $|a-a'| + |b-b'| \leq 3l_n$.

Proof. By construction the intervals I^i_n are separated by a distance of at least $2l_n$. The endpoint b can be moved to b_1 by a distance of less than l_n so that $\text{dist}(b_1, I^i_n) \geq l_n/2$ for all i. Similarly b_1 can be moved to b_2, by a distance of less than l_{n-1}, so that $\text{dist}(b_2, I^i_{n-1} \geq l_{n-1}/2$, etc. By using the fact that $l_n/2 \geq \sum_{m=0}^{n-1} l_m$ we see that $b' \equiv b_n$ belongs to no I^i_m, $m \leq n$, and $|b-b'| \leq 3l_n/2$. The other endpoint, a, is treated similarly. □

This lemma enables us to adjust the approximate length of I^i_{n+1}, which is l^2_n or l^4_n, by $\mathcal{O}(l_n)$ so that the endpoints do not meet $I^j_m \in S_m$, for all $m \leq n$. See Appendix D of [10]. This is a requirement of our definition of S_{n+1}. Furthermore, using the special "selfsimilar" structure of S_m or $\bar{S}_m$ described in Lemma 1.3, we can choose l_{n+1} independent of i. This is convenient because otherwise equalities relating E^i_n to E^j_n have exponentially small corrections.

3. Pure Point Spectrum

We shall follow the strategy of [11] to prove that H has pure point spectrum. By a theorem of Berezanskii [13, 14] it is sufficient to prove that every generalized eigenfunction decays exponentially fast. By a generalized eigenfunction we mean any nonzero solution to the equation $H(\theta)\psi = E(\theta)\psi$ such that $|\psi(j)| \leq \text{const.}\, |j|^2$.

Let $E(\theta)$ be a generalized eigenvalue and let Λ_n be an interval of length $2l_n$ centered at 0.

Lemma 3.1. *There is an* $N = N(\theta, E(\theta))$ *such that for all* $n \geq N$, $\Lambda_n \cap I^i_n \neq \emptyset$ *for some* $I^i_n \in S_n(E(\theta), \theta)$.

Proof. If not, there is a sequence $s = n_i \to \infty$ such that $\Lambda_s \cap I^j_s = \emptyset$ for all $I^j_s \in S_s(E)$. Now using Lemma 2.4, choose Λ'_s to be s-regular intervals such that $0 \in \Lambda'_s \subset \Lambda_s$ and $\text{dist}(0, \partial\Lambda'_s) \geq l_s/2$. If $H(\theta)\psi = E(\theta)\psi$, then $\psi(j)$ can be determined from its values on $\partial\Lambda'_s$. For $j \in \Lambda'_s$,

$$|\psi(j)| = \varepsilon^2 \left| \sum_{k,l} G'(E(\theta), j, k)\Gamma'_{kl}\psi(l) \right| . \tag{3.1}$$

Here G' is the Green's function of $H(\Lambda'_s)$ and $\Gamma' = \partial\Lambda'_s$. For $|j| \leq l_s/4$ we see that $|j-k| \geq l_s/4$, so that by Theorem 2.2 and the subquadratic growth of ψ,

$$|\psi(j)| \leq 2e^{-\gamma l_s/4} l^2_2 \varepsilon^2 .$$

Since $\gamma \geq 1$ and l_s can be arbitrarily large $\psi(j) \equiv 0$. This contradiction proves the lemma. □

Lemma 3.2. *For almost all θ_* and any generalized eigenvalue $E(\theta_*)$, there is a finite $N = N(\theta_*, E(\theta_*))$ such that for all $n \geq N$, $\bar{S}_n(E(\theta_*), \theta_*)$ has a unique interval $I_n^* \subset \Lambda_{n+2}$. These intervals have a common center c_N^* and $|c_N^*| \leq 3l_N/2$.*

Proof. The existence of the interval I_n^* follows from Lemma 3.1. If there are two intervals I_n^*, $I_n' \subset \Lambda_{n+2}$, with $I_n' \in \bar{S}_n$, then by Theorem 2.1 and (2.2), the corresponding centers satisfy $m(c_n^*, c_n') \leq 12\delta_n^{1/2}$. Since $|c_n^* - c_n'| \leq cl_n^8$ the Diophantine condition (1.4) implies that

$$m(c_n^*, c_n') = |(c_n^* + c_n')\alpha + 2\theta_*|_1 \leq 12\delta_n^{1/2}. \tag{3.2}$$

The set of θ_* for which (3.2) holds has measure less than $6\delta_n^{1/2}$ and is *independent* of $E(\theta_*)$. Since the number of possible pairs of centers in Λ_{n+2} is less than $4l_{n+2}^2$ the probability that Λ_{n+2} contains 2 singular intervals is less than Const. $\delta_n^{1/2} l_{n+2}^2$. Since the sum over n is finite, by the Borel-Cantelli lemma, with probability one with respect to θ_*, there is an $N < \infty$ so that I_n^* is unique for all $n \geq N$ and so $c_n^* = c_N^*$ all $n \geq N$ by construction. □

Proof of Theorem 1.1. Let $E(\theta_*)$ be a generalized eigenvalue and $N(\theta_*)$ as in the previous lemma. Let $|x| \geq l_{N+1}$. We claim that there is an interval Λ containing x such that for some n,

a) Λ is n regular,

b) $\mathrm{dist}(\partial\Lambda, x) \geq |x|/3 \geq l_n/2$.

Therefore by Theorem 2.2 and the polynomial bound on ψ

$$|\psi(x)| = \left| \sum_{z,z'} G_\Lambda(E; x, z)\Gamma_{zz'}\psi(z') \right| \varepsilon^2 \leq \exp\left(-\frac{\gamma|x|}{3} \right),$$

where $\Gamma = \partial\Lambda$.

To establish our claim, let $n \geq N+1$ be defined so that $l_{n+1} \geq |x| \geq l_n$. The interval $\Lambda' = \left[\frac{x}{2}, \frac{3x}{2}\right]$ does not meet I_{n-1}^* because $|c_{n-1}^*| = |c_N^*| \leq 2l_N$. By Lemma 3.2, since $\Lambda' \subset \Lambda_{n+1}$, we see that Λ' meets no member of S_{n-1}. Using Lemma 2.4, we can deform the interval Λ' to Λ so that it is n-regular and b) holds.

The proof of Theorem 1.2 is exactly the same, except that we use Theorem 2.3 and the formula

$$\psi(x) = \sum_{z \in \partial\Lambda} \dot{G}_\Lambda(E, x, z)\psi(z), \tag{3.3}$$

to recover ψ from its values on $\partial\Lambda$. Recall $\dot{G}$ is the normal derivative of G.

4. Eigenfunctions of $H(I_{n+1})$

In this section we establish some basic facts about the eigenvalues and eigenfunctions of $H(I_{n+1})$ for $I_{n+1} \in S_{n+1}$. We shall assume that $\mathscr{C}_n$ holds [see (2.6)]. If I_{n+1} contains a single element $I_n \in S_n$ (i.e. $s_n \geq 4l_n^2$) then we show that there is only one relevant eigenvalue E_{n+1} near E_* and the eigenfunction is localized well inside I_n. If I_{n+1} contains two singular intervals (i.e. $s_n \leq 4l_n^2$) I_n^+ and $I_n^- \in \bar{S}_n$, then there are two relevant eigenvalues E_{n+1} and $\tilde{E}_{n+1}$. The eigenfunctions are well localized in $I_n^+ \cup I_n^-$. We use the one dimensionality of space to ensure that there are no level crossings, so $E_n(\theta)$ is a smooth function of θ. The following lemma can be used to bound differences of eigenvalues. In more than one dimension these differences can be obtained by induction (see [9]).

Lemma 4.1. *Let ψ_1 and ψ_2 be eigenfunctions of $H(\Lambda)$, where Λ is an interval. If there is an $a \in \mathbb{Z}$ such that*

$$\|[1 - \chi(|x-a| \leq l)]\psi_i\| \leq \tfrac{1}{2}\|\psi_i\|, \qquad i = 1,2, \tag{4.1}$$

then the corresponding eigenvalues satisfy

$$\begin{aligned} |E_1 - E_2| &\geq \left(\frac{\varepsilon}{2}\right)^{4(l+1)} \quad (\textit{finite difference}), \\ |E_1 - E_2| &\geq e^{-(4K+1)l} \quad (\textit{continuum}). \end{aligned} \tag{4.2}$$

The proof of this proposition is standard (see [15]). A proof is given in Appendix B.

Proposition 4.2. *If $I_{n+1} \in \bar{S}_{n+1}$ and $s_n \geq 4l_n^2$, then $H(I_{n+1})$ has a unique eigenvalue E_{n+1} in the interval $|E - E_*| \leq \delta_n/3$ for $|\theta - \theta_*| \leq \delta_n/3$ and the corresponding eigenfunction satisfies*

$$|\psi(x)| \leq e^{-\gamma|c_n - x|/3} \tag{4.3}$$

for $|x - c_n| \geq 4l_n^{1/2}$. Moreover $\|G_{n+1}^{\perp}(E_{n+1})\| \leq 3\delta_n^{-1}$, where $G_{n+1}^{\perp}$ denotes the Green's function for I_{n+1} on the orthogonal complement of ψ_{n+1}.

Proof. We first suppose that $s_n \geq 4l_n^2$ and $s_{n-1} \geq 4l_{n-1}^2$. Then $I_{n+1}\backslash I_n$ and $I_n\backslash I_{n-1}$ are respectively n and $(n-1)$-regular, hence $\Lambda \equiv I_{n+1}\backslash I_{n-1}$ is n regular. For $n = 0$ let $\Lambda \equiv I_1\backslash\{c_0\}$. Let $E \in \sigma H(I_{n+1})$ be such that $|E - E_*| \leq \delta_n/3$. Let $|x - c_n| \geq l_n/2$ and $\Gamma = \partial I_{n-1}$. We determine the value of $\psi(x)$ from its values on the boundary of I_{n-1}:

$$\psi(x) = \varepsilon^2 \sum_{y,y'} G_\Lambda(E,x,y)\Gamma_{yy'}\psi(y'). \tag{4.4}$$

The decay of ψ, (4.3), now follows from Theorem 2.2 and the inequality

$$|x-y| \geq |x - c_n| - |c_n - y| \geq |x - c_n| - l_{n-1} \geq \tfrac{3}{4}|x - c_n| \geq l_n^{5/6}.$$

To obtain decay for $l_n/2 \geq |x - c_{n-1}| \geq 2l_{n-1} = 2l_n^{1/2}$ we set $\Lambda = I_n\backslash I_{n-1}$ which is $(n-1)$-regular and apply (4.4) and Theorem 2.2 as before. Note that $c_n = c_{n-1}$.

Next we suppose $s_{n-1} \leq 4l_{n-1}^2$. Then $I_n \subset I_{n+1}$ contains precisely two intervals I_{n-1}^+, I_{n-1}^- in $\bar{S}_{n-1}$. Set $\Lambda = I_{n+1}\backslash(I_{n-1}^+ \cup I_{n-1}^-)$. Λ is $(n-1)$-regular, because (2.3) shows that I_{n+1} cannot contain three $(n-1)$-singular intervals. Let $|E - E_*| \leq \delta_{n-1}/3$ and set $\Gamma = \partial I_{n-1}^+ \cup \partial I_{n-1}^-$. Then again applying Theorem 2.2 to (4.4) one obtains

$$\begin{aligned} |\psi(x)| &\leq \exp\left(\frac{-3\gamma}{4}|c_{n-1}^+ - x|\right) + \exp\left(\frac{-3\gamma}{4}|c_{n-1}^- - x|\right) \\ &\leq e^{-\gamma|c_n - x|/3} \end{aligned} \tag{4.5}$$

for $|c_n - x| \geq 4l_n^{1/2} \cong 4l_{n-1}^2$. In the last equality we used $c_n = (c_{n-1}^+ + c_{n-1}^-)/2$ and $|c_{n-1}^+ - c_{n-1}^-| \leq 4l_{n-1}^2$.

Thus any eigenvalue in the interval $|E - E_*| \leq \delta_n/3$ corresponds to an eigenfunction localized in an interval of width less than $4l_n^{1/2}$. If $I_{n+1} \in S_{n+1}$, then by definition there is an eigenvalue E_{n+1} such that $|E_{n+1} - E_*| \leq \delta_{n+1}$. By Lemma 4.1 it is unique since any other eigenvalue $\hat{E}$ satisfies

$$|\hat{E} - E_{n+1}| \geq \varepsilon^{4(l_n^{1/2}+1)} \geq \delta_n = e^{-\beta l_n^{2/3}}.$$

Our bound on $G^\perp(E_{n+1})$ follows easily from this estimate. If $\tilde{I}_{n+1}$ is the "mirror" image of $I_{n+1} \in S_{n+1}$, then by (2.2), $m(\tilde{c}_{n+1}, c_{n+1}) = \mathcal{O}(\delta_{n+1}^{1/2})$, and so by (1.7) and (2.13) the spectrum of $H(I_{n+1}^+$ and $H(I_{n+1}^-)$ differ only $\mathcal{O}(\delta_{n+1}^{1/2})$. Therefore the same argument applies for $I_{n+1} \in \bar{S}_{n+1}$. $\square$

Next we consider the case where $s_n \leq 4l_n^2$. Then each $I_{n+1} \in S_{n+1}$ contains two intervals I_n^+, $I_n^- \in \bar{S}_n$ and $s_{n-1} \geq 4l_{n-1}^2$ and $s_{n-2} \geq 4l_{n-2}^2$. See the remarks following Theorem 2.1.

Proposition 4.3. *If $I_{n+1} \in S_{n+1}$ and $s_n \leq 4l_n^2$, then $H(I_{n+1})$ has exactly two eigenvalues E_{n+1}, $\tilde{E}_{n+1}$ in the interval $|E - E_*| \leq \delta_{n-1}/3$ for $|\theta - \theta_*| \leq \delta_n^{1/2}$. The eigenfunctions satisfy*

$$\begin{aligned} \psi_{n+1} &= A\psi_n^+ + B\psi_n^- + \mathcal{O}(\delta_n^3), \\ \tilde{\psi}_{n+1} &= B\psi_n^+ - A\psi_n^- + \mathcal{O}(\delta_n^3), \end{aligned} \tag{4.6}$$

where $A^2 + B^2 = 1$ and ψ_n^+ and ψ_n^- are the normalized eigenfunctions of $H(I_n^+)$ and $H(I_n^-)$ respectively whose eigenvalues E_n^+, E_n^- are closest to E_*. Also $\|G_{n+1}^{\perp\perp}(E_{n+1})\| \leq \mathcal{O}(\delta_{n-1}^{-1})$.

Proof. For $n \geq 1$, the intervals I_n^+ and I_n^- contain just one singular interval each, I_{n-1}^+, I_{n-1}^-. As explained in the previous lemma $\Lambda = I_{n+1} \backslash (I_{n-1}^+ \cup I_{n-1}^-)$ is $(n-1)$-regular and for $E \in H(I_{n+1})$, $|E - E_*| \leq \delta_{n-1}/3$, and $|\theta - \theta_*| \leq \delta_{n-1}/3$, we can conclude that the corresponding eigenfunction ψ decays exponentially fast for x away from c_{n-1}^+, c_{n-1}^- as in (4.5).

To establish (4.6), first note that for $j \in I_n = I_n^+$,

$$(H(I_n) - E)\psi(j) = \varepsilon^2(\Gamma\psi)(j),$$

where $\Gamma = \partial I_n$. Therefore in I_n^+,

$$\psi = a\psi_n^+ + \varepsilon^2 G_n^\perp(E)\Gamma\psi = a\psi_n^+ + \mathcal{O}(\delta_n^3). \tag{4.7}$$

The last term above is obtained using $\frac{1}{2}\|\Gamma\psi\| \cong |\psi(c_n^+ \pm l_n/2)| \leq \exp -\gamma l_n/2$ and $|G_n^\perp(E_n)| \leq 2\delta_{n-1}^{-1}$ from Proposition 4.2. Similar estimates hold for I_n^-. These estimates together with the orthonormality of ψ_{n+1} and $\tilde{\psi}_{n+1}$ yield (4.6). Now since any eigenfunction corresponding to E in the interval $|E - E_*| \leq \delta_{n-1}/3$ has the form (4.6), there can only be two such eigenvalues, E_{n+1}, $\tilde{E}_{n+1}$. The existence of these eigenvalues follows by using $\psi_n^\pm$ as trial wavefunctions,

$$\|(H - E_*)\psi_n^\pm\| \leq \|\Gamma\psi_n^\pm\| + |E_n^\pm - E_*| \leq \delta_n^3 + \mathcal{O}(\delta_n^{1/2}) \ll \delta_{n-1}.$$

See Remark c) following Theorem 2.1.

The bound on $G_{n+1}^{\perp\perp}(E_*) \cong G_{n+1}^{\perp\perp}(E_{n+1})$ follows from the fact that E_{n+1} and $\tilde{E}_{n+1}$ are the only two eigenvalues in the interval $|E - E_*| \leq \delta_{n-1}/3$. $\square$

Corollary 4.4. *There is a unique eigenvalue E_{n+1} for $H(I_{n+1})$, $I_{n+1} \in S_{n+1}$ such that $|E_{n+1} - E_*| \leq \delta_{n+1}^{1/3}$. Hence E_{n+1} is well defined.*

Proof. If $s_n \geq 4l_n^2$ the result follows from Proposition 4.2. If $s_n \leq 4l_n^2$ then Proposition 4.3 shows that ψ_{n+1} and $\tilde{\psi}_{n+1}$ are supported in $I_n^+ \cup I_n^-$ which has width $l_{n+1}^{1/2} \cong l_n^2$. Therefore by Lemma 4.1,

$$|E_{n+1} - \tilde{E}_{n+1}| \geq \exp -\beta l_{n+1}^{1/2} \geq \delta_{n+1}^{1/3} \equiv \exp\left(\frac{-\beta l_{n+1}^{2/3}}{3}\right). \quad \square$$

Lemma 4.5. *If $s_n \geq 4l_n^2$ then for $|\theta - \theta_*| \leq \delta_n/3$*

$$\left|\frac{d^r}{d\theta^r}(E_{n+1}(\theta) - E_n(\theta))\right| \leq \delta_n^{3-\frac{r}{2}} \qquad r = 0, 1, 2. \tag{4.8}$$

Proof. By Proposition 4.2, $|\psi_{n+1}(j)| \leq \exp -\gamma l_n/3$ for j outside I_n. For j inside I_n we have as in the derivation of (4.7),

$$\psi_{n+1} = a\psi_n + \varepsilon^2 G_n^{\perp}(E_{n+1})\Gamma_n \psi_{n+1}\,, \tag{4.9}$$

where $\Gamma_n = \partial I_n$. By Corollary 4.4 $\|G_n^{\perp}(E_{n+1})\| \leq \delta_n^{-1}$ since $|E_{n+1} - E_*| \leq \delta_{n+1}$. Thus the last term of (4.9) is bounded by $\delta_n^{-1} e^{-\gamma l_n/2} \leq \delta_n^4$ hence

$$\|\psi_{n+1} - \psi_n\| \leq \delta_n^4\,. \tag{4.10}$$

This bound and (2.10) yield (4.8) for $r = 0, 1$.

For $r = 2$ we shall use (2.11),

$$E'' = \langle\psi, v''\psi\rangle + 2\langle\psi, v' G^{\perp}(E) v'\psi\rangle\,.$$

We must estimate

$$\begin{aligned} G_{n+1}^{\perp}(E_{n+1}) - G_n^{\perp}(E_n) = &G_{n+1}^{\perp}(E_{n+1})\,[\Gamma_n + (E_{n+1} - E_n)]G_n^{\perp}(E_n) \\ &- G_{n+1}^{\perp}(E_{n+1})P_{n+1}^{\perp}P_n + G_n^{\perp}(E_n)P_n^{\perp}P_{n+1}\,, \end{aligned} \tag{4.11}$$

restricted to I_n. This equation follows from the resolvent identity. We have used the orthogonal projections P_n and P_{n+1} onto ψ_n and ψ_{n+1} respectively and the relation $P_n + P_n^{\perp} = I = P_{n+1} + P_{n+1}^{\perp}$. The last two terms of (4.11) are bounded using (4.10).

$$\|P_{n+1}^{\perp}P_n\| = \|P_n - P_{n+1}P_n\| = \|\psi_n - P_{n+1}\psi_n\| = \mathcal{O}(\delta_n^4)\,,$$

and

$$\|G_n^{\perp}(E_n)\| = \mathcal{O}(\delta_n^{-1}) = \|G_{n+1}^{\perp}(E_{n+1})\|\,,$$

which follows from Corollary 4.4 and Proposition 4.2. The second term on the right side of (4.11) is bounded similarly since $|E_n - E_{n+1}| \leq \mathcal{O}(\delta_n^4)$, by (4.10). Therefore the case $r = 2$ follows if we prove

$$\|\Gamma_n G_n^{\perp}(E_n) v'\psi_n\| \leq \delta_n^3\,.$$

Let χ_n be the characteristic function of the interval

$$\{j : |c_n - j| \le l_n/4\} \subset I_n\,.$$

By the decay of ψ given in Propositions 4.2 and 4.3 we have

$$\|(1-\chi_n)v'\psi_n\| \le \mathcal{O}(\delta_n^4)\,.$$

To prove $\|\Gamma_n G_n^\perp \chi_n\|$ is small let $A = I_n \backslash I_{n-1}$ or $I_n \backslash (I_{n-1}^+ \cup I_{n-1}^-)$ and $\Gamma = \partial I_{n-1}$ or $\partial(I_{n-1}^+ \cup I_{n-1}^-)$ depending on whether I_n contains one or two singular intervals. Then

$$\Gamma_n G_n^\perp \chi_n = \Gamma_n G_A \chi_n + \Gamma_n G_A \Gamma\, G_n^\perp \chi_n - \Gamma_n G_A P_n \chi_n\,. \tag{4.12}$$

Since A is $n-1$ regular $G_A(E)$ decays exponentially fast for $|E - E*| \le \delta_{n-1}/3$. Hence $\Gamma_n G_A \chi_n$ and $\Gamma_n G_A \Gamma$ are exponentially small and $\|G_A(E_n)\| \le \mathcal{O}(\delta_{n-1}^{-1})$. The first two terms on the right side of (4.12) are now clearly less than $\mathcal{O}(\delta^3)$. To estimate the final term we use $\|\Gamma_n G_n \chi_n\| \le \delta_n^3$ and $\|G_A(E_n)\| \cdot \|(1-\chi_n)\psi_n\| \le \delta_n^3$. $\square$

In the continuum the results of this section hold provided $|\theta - \theta*| \le \delta_n/4$ is replaced by $|\theta - \theta*| \le \delta_n/16\pi$. See Theorem (2.3) and (2.13). The proofs are almost exactly the same if we use (2.9), (3.3) and Theorem 2.3. For the case $n = 0$ we use the results of Appendix C. These same remarks apply to our final section.

5. Proof of Theorem 2.1

In this section we shall assume condition $\mathscr{C}_n$ holds [see (2.6)] and then establish $\mathscr{C}_{n+1}$, thereby proving Theorem 2.1. To do this we shall express $E_{n+1}(\theta)$ in terms of $E_n(\theta)$. The resulting estimates on $\frac{d}{d\theta} E_{n+1}(\theta)$ and $\frac{d^2}{d\theta^2} E_{n+1}(\theta)$ enable us to justify (1.9) and hence $\mathscr{C}_{n+1}$.

First let us consider the case where $s_n \ge 4l_n^2$. Recall that Lemma 4.5 implies that for $r = 0, 1, 2,$

$$\left|\frac{d^r}{d\theta^r}\left(E_{n+1}(\theta) - E_n(\theta)\right)\right| \le \delta_n^{3-\frac{r}{2}} \quad \text{for} \quad |\theta - \theta*| \le \delta_n/3\,. \tag{5.1}$$

This is the key estimate for this case.

If $s_n \le 4l_n^2$ we shall need a few lemmas. Let I_n^+, $I_n^- \in \bar{S}_n$ denote the two subintervals of $I_{n+1} \in S_{n+1}$ which are centered at c_n^+, c_n^-, respectively. When $\theta = \theta_s \equiv -c_{n+1}\alpha = -(c_n^+ + c_n^-)\alpha/2$, $H(I_n^\pm)$ are mirror images of each other. In fact

$$v(j + c_n^+, \theta_s + \psi) = v(-j + c_n^-, \theta_s - \psi)\,,$$

and so

$$\frac{d}{d\theta}\left(E_n^+ + E_n^-\right)(\theta_s) = 0\,. \tag{5.2}$$

Note that by $\mathscr{C}_n$ and (2.2)

$$2|\theta_* - \theta_s| \equiv |(c_n^+ + c_n^-)\alpha + 2\theta*| \le 12\delta_n^{1/2}\,.$$

Also by the remark following Theorem 2.1, since $s_n \le 4l_n^2$ we have $s_{n-1} \ge 4l_{n-1}^2$ so that $I_n^\pm$ contains a single interval $I_{n-1}^\pm$ of S_{n-1}. Now we measure the deviation of (5.2) away from $\theta = \theta_s$.

Lemma 5.1. *For $|\theta - \theta_*| \le \delta_n/3$ and $n \ge 1$,*

$$\left|\frac{d}{d\theta}(E_n^+ + E_n^-)\right| \le \mathcal{O}(\delta_n^{1/2}/\delta_{n-1}). \tag{5.3}$$

Proof. At $\theta = \theta_s$ the left side of (5.3) is 0. Since $|\theta_s - \theta_*| \le 6\delta_n^{1/2}$ we have $|\theta - \theta_s| \le 7\delta_n^{1/2}$. By Proposition 4.2 and (2.11) for $|\theta - \theta_*| \le \delta_{n-1}/3$,

$$\left|\frac{d^2}{d\theta^2} E_n^\pm(\theta)\right| = |\langle\psi_n^\pm, v''\psi_n^\pm\rangle \sim 2\langle\psi_n^\pm, v' G_n^\perp(E_n^\pm)v'\psi_n^\pm\rangle| \le 2\pi + 6\delta_{n-1}^{-1}.$$

The lemma now follows by expressing the left side of (5.3) as an integral of the second derivative. □

The following estimates, $\mathcal{D}_n$, will be established by induction on n:
$\mathcal{D}_n$: For $0 \le m \le n$ and $|\theta - \theta_*| \le \delta_m/4$,

$$\left|\frac{d}{d\theta} E_m^i(\theta)\right| \ge \min[\delta_{m-1}^2, \tfrac{1}{2}|\theta + c_m^i\alpha|_1]. \tag{5.4}$$

Recall that $\delta_0 = 2\varepsilon$ and $\delta_{-1} = \varepsilon^{1/4}$.

Note that $dE_m^i(\theta)/d\theta$ vanishes at $\theta = -c_m^i\alpha$ by using the symmetry (1.8). $\mathcal{D}_n$ says that E_m' is small only near a symmetry point.

Lemma 5.2. *If $s_m \le 4l_m^2$ and $\mathcal{D}_m$ holds then, for all θ such that $|\theta - \theta_*| \le \delta_m/4$,*

$$\left|\frac{d}{d\theta} E_m^\pm(\theta)\right| \ge \delta_{m-1}^2. \tag{5.5}$$

Proof. If (5.5) fails then $\mathcal{D}_m$ implies that for some θ in the interval $|\theta - \theta_*| \le \delta_m/4$ and some choice of $+$ or $-$,

$$2\delta_{m-1}^2 \ge |\theta + c_m^\pm\alpha|_1.$$

On the other hand, by (2.2)

$$|(c_m^+ + c_m^-)\alpha + 2\theta_*|_1 \le 12\delta_m^{1/2}.$$

The above two estimates imply that $|(c_m^+ - c_m^-)\alpha|_1 \le 3\delta_{m-1}^2$ which violates the Diophantine condition (1.4), because $|c_m^+ - c_m^-| \le 4l_m^2$. □

Let $E'(\theta)$, $E''(\theta)$ denote the first and second derivatives of E with respect to θ. Recall that $\tilde{E}_{n+1}$ is the eigenvalue closest to E_{n+1}.

Lemma 5.3. *Suppose $s_n \le 4l_n^2$ and $\mathcal{D}_n$ holds. If $|E_{n+1}'(\theta)| \le \delta_n$ for some $|\theta - \theta_*| \le \delta_n/4$, then*

$$E_{n+1}''(\theta) = \frac{[dE_n^+(\theta)/d\theta]^2}{E_{n+1}(\theta) - \tilde{E}_{n+1}(\theta)}(1 + \mathcal{O}(\delta_n)), \tag{5.6}$$

and

$$|E_{n+1}''(\theta)| \ge \tfrac{1}{3}\delta_n^{-2}\delta_{n-1}^4 \ge 1, \tag{5.7}$$

and

$$|\theta + c_{n+1}\alpha|_1 \leq \text{const}\, \delta_n^{3/2}\,. \tag{5.8}$$

The same results hold with E_{n+1} and $\tilde{E}_{n+1}$ interchanged.

Proof. We shall use expressions (2.11) and (2.12) to estimate E''_{n+1}. First note that by Proposition 4.3 and Lemma 5.1,

$$\begin{aligned} E'_{n+1} &= \langle \psi_{n+1}, v'\psi_{n+1}\rangle = A^2\langle \psi_n^+, v'\psi_n^+\rangle + B^2\langle \psi_n^-, v'\psi_n^-\rangle + \mathcal{O}(\delta_n^2) \\ &= A^2 \frac{d}{d\theta} E_n^+(\theta) + B^2 \frac{d}{d\theta} E_n^-(\theta) + \mathcal{O}(\delta_n^2) \\ &= (A^2 - B^2)\frac{d}{d\theta} E_n^+(\theta) + \mathcal{O}(\delta_n^{1/2}/\delta_{n-1})\,. \end{aligned}$$

Now if $|E'_{n+1}(\theta)| \leq \delta_n$, then Lemma 5.2 and the above equation show that $A^2 \cong B^2 \cong 1/2$. Similarly we have

$$|\langle \psi_{n+1}, v'\tilde{\psi}_{n+1}\rangle|^2 \cong |dE_n^+/d\theta|^2 4A^2B^2 \geq \tfrac{1}{2}\,\delta_{n-1}^4\,. \tag{5.9}$$

Next we claim that $|E_n^{\pm}(\theta) - E_{n+1}(\theta)| \leq \mathcal{O}(\delta_n^3)$ whenever $|A| \cong |B|$. This result follows by taking the scalar product of $\psi_n^{\pm}$ with both sides of the identity

$$(H(I_n^{\pm}) - E_{n+1})\psi_{n+1} = \varepsilon^2 \Gamma_n^{\pm}\psi_{n+1} = \mathcal{O}(\delta_n^3)\,,$$

hence

$$\langle \psi_n^{\pm}, \psi_{n+1}\rangle\,(E_n^{\pm} - E_{n+1}) = A^2(E_n^{\pm} - E_{n+1}) = \mathcal{O}(\delta_n^3)\,.$$

The last estimate follows from the decay of ψ_{n+1} given by Proposition 4.3. A similar argument using $\tilde{\psi}_{n+1}$, $\tilde{E}_{n+1}$ shows that $|E_n^{\pm} - \tilde{E}_{n+1}| \leq \mathcal{O}(\delta_n^3) \leq \delta_n^2$. Hence

$$|E_{n+1}(\theta) - \tilde{E}_{n+1}(\theta)| \leq \mathcal{O}(\delta_n^3) \leq \delta_n^2\,. \tag{5.10}$$

To obtain (5.8) note that since $|E_n^+(\theta) - E_n^-(\theta)| \leq \mathcal{O}(\delta_n^3)$, $\mathscr{C}_n$ implies that

$$2|c_{n+1}\alpha + \theta|_1 = |(c_n^+ + c_n^-)\alpha + 2\theta|_1 = \mathcal{O}(\delta_n^{3/2})\,.$$

The proof of (5.6) and (5.7) is now complete using (5.9) and (5.10) to bound the first term on the right-hand side of (2.12). The last term of (2.12) is bounded using the bound $\|G_n^{\perp\perp}\| = \mathcal{O}(\delta_{n-1}^{-1})$ of Proposition 4.3. $\square$

Remarks. If $E_{n+1}(\theta_*) \geq \tilde{E}_{n+1}(\theta_*)$, then this relation holds for all θ, since no level crossing can occur. Hence (5.6) shows that $E'_{n+1}(\theta)$ is increasing and $\tilde{E}'_{n+1}(\theta)$ is decreasing whenever $|E'_{n+1}(\theta)| \leq \delta_n$ and $|\theta - \theta_*| \leq \delta_n/4$. Also both E_{n+1} and $\tilde{E}_{n+1}$ are symmetric about θ_s (see Fig. 2). It is important to note that E'_{n+1} cannot re-enter the band $|E'_{n+1}| \leq \delta_n$, since it is increasing there. Hence the set of θ's where $|\theta - \theta_*| \leq \delta_n/4$ and $|E'_{n+1}| \leq \delta_n$ is an *interval.* If $E_{n+1} \leq \tilde{E}_{n+1}$ then the same arguments apply with E, $\tilde{E}$ interchanged.

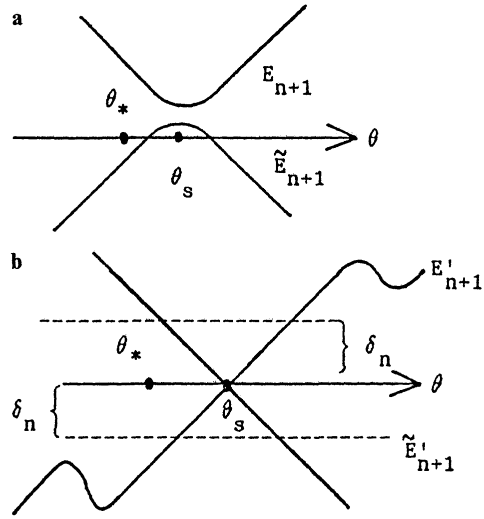

Fig. 2a and b. Graphs of E_{n+1}, $\tilde{E}_{n+1}$ and of E'_{n+1}, $\tilde{E}'_{n+1}$ restricted to $|\theta-\theta_*|\leq\delta_n/4$, assuming $s_n\leq 4l_n^2$. Note that $\theta_s=-c_{n+1}\alpha$

Lemma 5.4. *Suppose $s_n\geq 4l_n^2$ and that $\mathscr{D}_n$ holds. If $|E'_{n+1}(\theta)|\leq\delta_n$ for some θ, $|\theta-\theta_*|\leq\delta_n/4$, then $|E''_{n+1}(\theta)|\geq 2/3$ and $E''_{n+1}(\theta)$ has the same sign for all such θ.*

Proof. Let p be the largest value of $m\leq n$ such that $s_m\leq 4l_m^2$. Then by (5.1), $|E'_{n+1}(\theta)|\cong|E'_{p+1}(\theta)|\leq\delta_{p+1}$, hence (5.7) implies that $|E''_{p+1}(\theta)|$ is large. Applying (5.1) again, we conclude $|E''_{n+1}(\theta)|$ is large. If $s_m\geq 4l_m^2$ for all m we use the fact that $|E''_0(\theta)|\geq 5/6$ and apply (5.1). □

We now establish $\mathscr{D}_n$ [see (5.4)] by induction.

Lemma 5.5. *$\mathscr{D}_{n+1}$ holds.*

Proof. $\mathscr{D}_0$ clearly holds since the critical points of the cosine are nondegenerate. Now suppose $\mathscr{D}_m$ holds and that $|E'_{m+1}(\theta)|\leq\delta_m^2$ for some θ such that $|\theta-\theta_*|\leq\delta_{m+1}/4$. Since $E'_{m+1}(\theta_s)=0$, where $\theta_s=-c_{m+1}\alpha$, Lemmas 5.3 and 5.4 imply that

$$|E'_{m+1}(\theta)|=|E''_{m+1}(\xi)|\,|\theta-\theta_s|\geq\tfrac{1}{2}\,|\theta-\theta_s|\,, \tag{5.11}$$

where ξ is some point between θ_s and θ. This establishes $\mathscr{D}_{m+1}$. In order to apply Lemma 5.3 and 5.4 we must check that $|\theta_s-\theta_*|\leq\delta_m/4$. When $s_m\leq 4l_m^2$ this follows easily from (5.8). If $s_m\geq 4l_m^2$ then (5.1) implies that $|E'_{m+1}(\theta)|\cong|E'_m(\theta)|\leq\delta_m^2$. By induction $\mathscr{D}_m$ implies that $|\theta-\theta_s|\leq\delta_m^2$ and since $|\theta-\theta_*|\leq\delta_{m+1}$, $|\theta_s-\theta_*|\leq\delta_m/4$ follows. Finally note that by the remarks following Lemma 5.3 ξ clearly belongs to the subinterval of $|\phi-\theta_*|\leq\delta_m/4$, where $|E'_{m+1}(\phi)|\leq\delta_m$ hence the lemmas can be applied as above. □

The next lemma gives a weak form of $\mathscr{C}_{n+1}$ and uses only the results of Sect. 4 and $\mathscr{C}_n$.

Lemma 5.6. *If c^i_{n+1} denotes the centers of S_{n+1}, then*

$$m(c^i_{n+1}, c^j_{n+1})^2 \le \text{const}\, \delta_n^3 . \tag{5.12}$$

Proof. Let I^k_{n+1}, $k = i, j$ belong to S_{n+1} and centered at c^k_{n+1}. Let E^k_{n+1} be the eigenvalue of $H(I^k_{n+1})$ closest to E_*. First suppose that $s_n \ge 4l_n^2$. By (5.1)

$$|E^i_n - E^j_n| \le |E^i_n - E^i_{n+1}| + |E^i_{n+1} - E^j_{n+1}| + |E^j_{n+1} - E^j_n| \le 2\delta_n^3 + 2\delta_{n+1} ,$$

hence $\mathscr{C}_n$ implies that $m^2(c^i_n, c^j_n) \le 3\delta_n^3$. Since $c^k_{n+1} = c^k_n$ the lemma is proved for this case. Now suppose $s_n \le 4l_n^2$. Then for each k,

$$|E^{2k}_n - E^k_{n+1}| \quad \text{or} \quad |E^{2k+1}_n - E^k_{n+1}| \le \mathcal{O}(\delta_n^3) ,$$

where I^{2k}_n, $I^{2k+1}_n \subset I^k_{n+1}$ are the two singular intervals in $\bar{S}_n$. This is easily established using $\psi_{n+1} \restriction I^{2k}_n$ or $\psi_{n+1} \restriction I^{2k+1}_n$ as a trivial wave function together with the estimates of Proposition 4.3. See the argument leading to (5.10). Let us label the centers $c^{2k+1}_n = c^{2k}_n + s_n$. Then it follows from $\mathscr{C}_n$ that either $m(c^{2i}_n, c^{2j}_n)$ or $m(c^{2i}_n, c^{2j+1}_{2n}) < \text{const}\, \delta_n^{3/2}$. By (2.2) we know that for $k = i, j$,

$$|(c^{2k}_n + c^{2k+1}_n)\alpha + 2\theta_*|_1 = |(2c^{2k}_n + s_n)\alpha + 2\theta_*|_1 \le \mathcal{O}(\delta_n^{1/2}) ,$$

which implies that $|(c^{2j}_n - c^{2i}_n)\alpha|_1 \le \mathcal{O}(\delta_n^{1/2})$. If we use this estimate together with those on m above we obtain

$$|(c^{2i}_n - c^{2j}_n)\alpha|_1 \le C\delta_n^{3/2} \quad \text{or} \quad |(c^{2i}_n + c^{2j+1}_n)\alpha + 2\theta_*|_1 \le C\delta_n^{3/2} . \tag{5.13}$$

Note that for example the case

$$|(c^{2i}_n - c^{2j+1}_n)\alpha| \le C\delta_n^{3/2}$$

is excluded because it and $|(c^{2j}_n - c^{2i}_n)\alpha|_1 \le \mathcal{O}(\delta_n^{1/2})$ would imply $|s_n\alpha|_1 \le \mathcal{O}(\delta_n^{1/2})$ which violates the Diophantine properties of α. Since $c^k_{n+1} = c^{2k}_n + \frac{1}{2} s_n$, (5.12) follows from (5.13). $\square$

The proof of the following elementary lemma will be left to the reader.

Lemma 5.7. *Let $E(\theta)$ be an even C^2 function for $\theta \in [a, b]$ such that whenever $|E'(\theta)| \le \delta$, then $|E''(\theta)| \ge 2/3$ and $E''(\theta)$ has a single sign for all such θ. Then*

$$|E(\theta) - E(\theta')| \ge \tfrac{1}{4} M^2(\theta, \theta') \equiv \tfrac{1}{4} \min(|\theta + \theta'|, |\theta - \theta'|)^2 ,$$

provided $M(\theta, \theta') \le \delta/4$.

We shall set $E_{n+1}(\theta - c_{n+1}\alpha) = E(\theta)$ and $\delta = \delta_n$. Note that $E(\theta) = E(-\theta)$ follows from (1.8). The hypothesis of Lemma 5.7 is met using Lemmas 5.3–5.6, where $[a, b] = [\theta_* - \delta_n/4, \theta_* + \delta_n/4]$.

Proof of Theorem 2.1. We claim that as in (1.7),

$$E^j_{n+1}(\theta_* + m) = E^i_{n+1}(\theta_*) , \qquad \tilde{E}^j_{n+1}(\theta_* + m) = \tilde{E}^i_{n+1}(\theta_*) , \tag{5.14}$$

where $|m| = m(c^i_{n+1}, c^j_{n+1})$. By Lemma 5.6 $|m| \leq c\delta_n^{3/2}$. Clearly $E^j_{n+1}(\theta_* + m)$ is an eigenvalue of $H(I^i_{n+1})$, by (1.7). To identify it with $E^i_{n+1}(\theta_*)$, note that using (5.12) and the bound $|dE(\theta)/d\theta| \leq 1$,

$$\begin{aligned}|E^j_{n+1}(\theta_* + m) - E_*| &\leq |E^j_{n+1}(\theta_* + m) - E^j_{n+1}(\theta_*)| + |E^j_{n+1}(\theta_*) - E_*| \\ &\leq C\delta_n^{3/2} + \delta_n^3 \leq \delta_n/3\,.\end{aligned}$$

By Propositions 4.2 and 4.3 we conclude that $E^j_{n+1}(\theta_* + m)$ equals either $E^i_{n+1}(\theta_*)$ or $\tilde{E}^i_{n+1}$. The geometric structure of E and $\tilde{E}$ (see Fig. 2) now allows us to conclude that (5.14) holds, so that in particular $E^j_{n+1}(\theta_* + m) \neq \tilde{E}^i_{n+1}(\theta_*)$. Since $E(\theta + c^j_n) \equiv E^j_{n+1}(\theta)$, where E satisfies the previous lemma,

$$\begin{aligned}2\delta_{n+1} &\geq |E^i_{n+1}(\theta_*) - E^j_{n+1}(\theta_*)| = |E^j_{n+1}(\theta_* + m) - E^j_{n+1}(\theta_*)| \\ &= |E(\theta_* + c^i_{n+1}\alpha) - E(\theta_* + c^j_{n+1}\alpha)| \geq \frac{|m|^2}{4}\,,\end{aligned}$$

where $m = (c^i_{n+1} - c^j_{n+1})\alpha$ or $m = -[2\theta_* + (c_i + c_j)\alpha]$. This completes our induction.

Appendix A. Proof of Theorem 2.2

The proof of Theorem 2.2 is by induction on n. For $n = 0$, see Sect. 1. Now let Λ be n-regular. We shall first suppose that Λ contains a single n-regular interval I_n and possibly many other m-regular intervals for $m < n$. The interval I_n contains either a single interval $I_{n-1} \in S_{n-1}$ or two intervals I^-_{n-1}, I^+_{n-1} in $\bar{S}_{n-1}$. There is an interval $\bar{I}_{n-1} \subset I_n$ which contains $I^+_{n-1} \cup I^-_{n-1}$ or I_{n-1} such that

a) $A \equiv \Lambda \backslash \bar{I}_{n-1}$ is $(n-1)$-regular,

b) $\mathrm{dist}(\partial A, \{x, y\}) \geq l_{n-1}$,

c) length $\bar{I}_{n-1} \leq 5 l_n^{1/2}$.

This set is easily constructed using Lemma 2.4. If we set $G = G_\Lambda$ and $\bar{G} = G_{\Lambda, \bar{I}_{n-1}}$ then by (2.8) with $\Gamma = \partial A$

$$G(x,y) = \{\bar{G} + \varepsilon\bar{G}\Gamma G\}(x,y) = \{\bar{G} + \varepsilon^2\bar{G}\Gamma\bar{G} + \varepsilon^4\bar{G}\Gamma G\Gamma\bar{G}\}(x,y)\,. \tag{A.1}$$

Both x and y cannot belong to $\bar{I}_{n-1}$ since that would violate $|x - y| \geq l_n^{5/6}$, by c). First consider the case where $x, y \in A$, then $\bar{G}$ can be replaced by G_A. By applying the induction hypothesis to G_A and expressing (A.1) in terms of its matrix elements we have

$$\begin{aligned}|G(x,y)| &\leq [\exp -\gamma_{n-1}|x-y|] + 3\exp[-\gamma_{n-1}(|x-y| - 5l_n^{1/2})] \cdot \exp\beta l_n^{2/3} \\ &\leq \exp -\gamma_n|x-y|\,,\end{aligned} \tag{A.2}$$

where $\gamma_n = \gamma_{n-1} - 2l_n^{2/3}/l_n^{5/6}$. Since $\Sigma l_n^{-1/6}$ is summable, we get a uniform lower bound on $\gamma_n \geq \frac{1}{2}\gamma_0 = \frac{1}{2}|l_n\varepsilon|$ for ε small. In (A.2) we have used a), b), c) above together with the bound $\|\Gamma G\Gamma\| \leq 2\delta_n^{-1} = 2\exp\beta l_n^{2/3}$. To obtain this bound we use the fact that I_n is regular, hence $\|G_n\| \equiv \|G_{I_n}(E,\theta)\| \leq 2\delta_n^{-1}$, and we express G using an alternating resolvent series

$$G = G_n + \varepsilon^2 G_n\Gamma_n G = G_n + \varepsilon^2 G_n\Gamma_n\bar{G} + \varepsilon^4 G_n\Gamma_n\bar{G}\Gamma G_n + \ldots\,. \tag{A.3}$$

Since $\|\Gamma_n \bar{G}\Gamma\| = \|\Gamma_n G_\Lambda \Gamma\| \le \exp(-\gamma_{n-1} l_n/2)$ holds by the induction hypothesis, our desired bound on $\|\Gamma G \Gamma\|$ follows from that on $\|G_n\|$. When x or $y \in \bar{I}_n$ we use the first equality of (A.2) and the rest of the proof goes as above.

Note that the bound $\|G_n(E_*, \theta_*)\| \le \delta_n^{-1}$ can be extended by using the resolvent series $\|G_n(E, \theta)\| \le 3\delta_n^{-1}$ provided that $\|v(\theta) - v(\theta_*)\| + |E - E_*| \le \frac{2\delta_n}{3}$. Hence the estimates of this appendix apply to those θ and E specified in Theorem 2.2.

The general case where Λ contains many n-regular interval I_n' is treated as in [10, 12]. We use the fact that the intervals I_n are separated by a distance $3l_n$ and then express G_Λ as a sum of products of Green's functions $G_{\Lambda'}$ where Λ' contains a single n regular interval. This is called the block resolvent expansion in [12]. Condition $\mathscr{C}_{n-1}$ is only needed to ensure the separation of the intervals I_m^i, $m \le n-1$.

For our analysis of H_c we apply (2.8) instead of (2.7) to obtain the identities corresponding to (A.1) and (A.3). Bounds on the derivatives of the Green's function follow by expressing $G(E) = G^0(E) - K^2 G^0(E) v G(E)$, where $G^0(E)$ is the Green's function with v set equal to zero. Since we consider $E \cong -2K^2 + \sigma(K)$, G^0 and its derivatives decay like $\exp(-|K|\,|x-y|)$.

Appendix B. Eigenvalue Splitting

Proof of Lemma 4.1. We first consider the finite difference case. Let $\tilde{\psi}_1(x) = -\psi(x)$ for $x \le a$ and $\tilde{\psi}_1(x) = \psi_1(x)$ for $x > a$. Then

$$\begin{aligned}\langle \tilde{\psi}_1, H\psi_2\rangle &= E_2\langle \tilde{\psi}_1, \psi_2\rangle \\ &= E_1\langle \tilde{\psi}_1, \psi_2\rangle + 2\varepsilon^2[\psi_2(a)\psi_1(a+1) - \psi_1(a)\psi_2(a+1)]\,. \end{aligned} \tag{B.1}$$

Let us normalize ψ_1, and ψ_2 so that for $i = 1, 2$, $\psi_i(a) = \cos(\theta_i)$, $\psi_i(a+1) = \sin\theta_i$. Since the transfer matrix has norm less than $[\varepsilon^{-2} + 4]$ we have

$$\|\psi_i\| \le 2\,\|\chi\psi_1\| \le 3\,|\varepsilon^{-2} + 4|\,.$$

Therefore for small ε, (B.1) and the above inequality imply that

$$|\sin(\theta_1 - \theta_2)| \le |E_1 - E_2|\,\|\langle\tilde{\psi}_1, \psi_2\rangle\| \le \frac{1}{4}\,|E_2 - E_1|\,\|\psi_1\|^{1/2}\,\|\psi_2\|^{1/2}\left[\frac{\varepsilon^2}{2}\right]^{-(l+1)}. \tag{B.2}$$

If $|E_1 - E_2|$ is very small, then ψ_1 and ψ_2 satisfy nearly the same initial data and nearly the same equation. Thus ψ_1 and ψ_2 are nearly equal on the support of χ, which contradicts the orthogonality of ψ_1 and ψ_2. In fact it is straightforward to see that

$$\begin{aligned}\|\chi(\psi_1 - \psi_2)\| &\le [|E_1 - E_2|l + |\sin(\theta_1 - \theta_2)|]\,[\varepsilon^{-2} + 4]^l \\ &\le |E_1 - E_2|\,\|\psi_1\|^{1/2}\|\psi_2\|^{1/2}\left[\frac{\varepsilon^2}{2}\right]^{-2(l+1)} \end{aligned} \tag{B.3}$$

On the other hand $\|\psi_1\|^2 + \|\psi_2\|^2 \le 2\,\|\chi(\psi_1 - \psi_2)\|^2$ because

$$\begin{aligned}\|\psi_1\|^2 + \|\psi_2\|^2 = \|\psi_1 - \psi_2\|^2 &= \|\chi(\psi_1 - \psi_2)\|^2 + \|(1-\chi)(\psi_1 - \psi_2)\|^2 \\ &\le \|\chi(\psi_1 - \psi_2)\|^2 + \tfrac{1}{2}\left(\|\psi_1\|^2 + \|\psi_2\|^2\right).\end{aligned}$$

In the last inequality we used (4.1). The lemma now follows by combining the above inequality and (B.3).

In the continuum we use the identity

$$\frac{d}{dx}\left[\psi_1'(x)\psi_2(x) - \psi_1(x)\psi_2'(x)\right] = (E_2 - E_1)\psi_1(x)\psi_2(x) \tag{B.4}$$

to estimate $|E_2 - E_1|$. Integrating both sides of (B.4) from a to the endpoint of Λ we obtain the analogue of (B.1),

$$|\psi_1'(a)\psi_2(a) - \psi_1(a)\psi_2'(a)| \le |E_2 - E_1|\,\|\psi_1\|\,\|\psi_2\|\,. \tag{B.5}$$

We normalize $\psi_i(a) = \cos\theta_i$, $\psi_i'(a) = \sin\theta_i$. Since $|E_* - V|^{1/2} \le 2K$, solutions grow at most like $\exp 2K|x|$ and we obtain the analogue of (B.3), as above.

Remarks. This theorem uses the Dirichlet boundary condition at the boundary of Λ in the derivation of B.1 and B.5. The theorem gives eigenvalue splittings for any bounded potential on the line in terms of the growth of the respective eigenfunctions.

Appendix C. Estimates on E_0^i (Continuum)

In this appendix we shall establish (1.6) for the case $n = 0$, and obtain bounds on $E_0''(\theta)$, where E_0 is the lowest eigenstate of $H(I_0)$ and $I_0 \in S_0$. Let $v(x)$ be given by (1.2) and let $v(x(\theta),\theta)$ be the minimum of $v(x,\theta)$ restricted to I_0. By definition of S_0,

$$(1 - \cos 2\pi x(\theta)) + (1 - \cos 2\pi(x(\theta)\alpha + \theta)) \le \frac{12}{K}\,. \tag{C.1}$$

This implies $x(\theta)$ must be nearly integer valued. In fact $|x(\theta)|_1 \le K^{-1/2}$ and $|x(\theta)\alpha + \theta_*|_1 \le K^{-1/2}$, hence $2|c_0\alpha + \theta_*|_1 \le 2K^{-1/2(1+\alpha)}$. This yields (1.6) for $n = 0$ with $\delta_0 = K^{-1}$ provided $|\alpha| \le 1$. Note that this also implies that the centers of S_0 are separated by at least $C_1 K^{1/4}/4$.

It is well-known that $E_0(\theta)$ has a standard asymptotic expansion

$$\begin{aligned}E_0 &= K^2 v(x(\theta),\theta) + K(v''/2)^{1/2} + \mathcal{O}(1) \\ &= K^2 v(x(\theta),\theta) + K\pi(2 + 2\alpha^2)^{1/2} + \mathcal{O}(1)\,,\end{aligned} \tag{C.2}$$

where

$$v''(x(\theta),\theta) = \frac{d^2}{dx^2}\,v(x(\theta),\theta) \cong 4\pi^2(1+\alpha^2)\,.$$

Note that the next eigenvalue of $H(I_0)$ is larger than

$$-2K^2 + 3\pi(2 + 2\alpha^2)^{1/2} \ge -2K^2 + 10K\,.$$

This explains our constraint on E_* in Theorem 2.3 and allows us to establish (5.13) for E_0^i.

Next we consider derivatives of $E(\theta)$. It is easy to check that $dx(\theta)/d\theta \cong -\alpha/(1+\alpha^2)$. If we formally differentiate both sides of (C.2) we see that

$$\frac{d^2}{d\theta^2} E_0(\theta) = (2\pi K)^2/(1+\alpha^2) + \mathcal{O}(K), \tag{C.3}$$

which is the desired lower bound. This identity may be justified using (2.11) and the asymptotics for ψ. This bound is used to establish (5.4) for the $n = 0$ case.

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