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The Holonomy of the Determinant of Cohomology of an Algebraic Bundle

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Abstract. The purpose of this note is to remark that Theorem 3.7 in [1], when combined with the work of Bismut and Freed [2], leads, in the algebraic case, to an improvement of both results concerning the holonomy of determinant line bundles.

So let $f: X \to Y$ be a smooth proper map between projective complex manifolds. Choose a metric h on the relative tangent space $T_{X/Y}$ and a smooth complement $T^H X$ to $T_{X/Y}$ in TX. We assume that $(f, T^H X, h)$ is a Kähler fibration in the sense of [3], i.e. there exists a closed $(1,1)$ form ω on X for which $T_{X/Y}$ and $T^H X$ are orthogonal, and ω restricts to the $(1,1)$ form associated to h on $T_{X/Y}$.

Let E be an algebraic vector bundle on X, endowed with a smooth Hermitian metric h_E. The (algebraic) determinant line bundle

$$\lambda(E) = \det Rf_*(E)$$

may then be equipped with its Quillen metric [3], whose associated connection we denote by ∇_Q.

Given a smooth loop

$$\gamma: S^1 \to Y$$

we want to compute the holonomy of ∇_Q along γ. By pulling back f along γ we get a commutative diagram of real manifolds,

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\gamma}} & X \\ \downarrow{\scriptstyle f_\gamma} & & \downarrow{\scriptstyle f} \\ S^1 & \xrightarrow[\gamma]{} & Y \end{array}$$

with $TM \cong \tilde{\gamma}^*(T_{X/Y}) \oplus f_\gamma^*(TS^1)$ (because of the choice of $T^H X$). Endow TM with the orthonormal direct sum of $\tilde{\gamma}^*(h)$ with the metric on TS^1 giving norm one to $\frac{d}{dt}$ and invariant by rotation. Let D be the Dirac operator acting on the sections of

$E \otimes F \otimes \xi^{-1}$ on M, where F is the (locally defined) bundle of spinors and ξ a square root of $\det(T^{*(0,1)}_{X/Y})$. Denote by $\eta(0)$ the η-invariant of D, by $h(D)$ the dimension of the kernel of D, and by χ the Euler characteristic of E on any fiber of f on $\gamma(S^1)$.

Theorem. *The holonomy* $\mu(\gamma)$ *of* V_Q *around* γ *is*

$$\mu(\gamma) = (-1)^\chi \exp(2\pi i(\eta(0) + h(D))/2) \in S^1 .$$

To prove this theorem, for every real number $\varepsilon > 0$, we define D^ε as D above, except that $\frac{d}{dt}$ has norm ε^{-1} (so $D = D^1$). If $\eta^\varepsilon(0)$ is the êta invariant of D^ε, and $\bar{\eta}^\varepsilon(0) = (\eta^\varepsilon(0) + \dim(D^\varepsilon)/2)$, we know from [2] that

$$\lim_{\varepsilon \to 0} (-1)^\chi \exp(2\pi i \bar{\eta}^\varepsilon(0)) = \mu(\gamma) .$$

On the other hand, the proof of Theorem 3.7 in [1] shows that, for any $\varepsilon > 0$,

$$\mu(\gamma) \equiv \exp(2\pi i \bar{\eta}^\varepsilon(0))$$

is $S^1 \otimes \mathbb{Q}$. Since we know from [2] that $\bar{\eta}^\varepsilon(0)$ is a continuous function of ε, the number $\exp(2\pi i \bar{\eta}^\varepsilon(0)) \in S^1$ depends continuously on ε. It is constant modulo roots of unity, therefore it is constant. We conclude that

$$\mu(\gamma) = \lim_{\varepsilon \to 0} (-1)^\chi \exp(2\pi i \bar{\eta}^\varepsilon(0)) = (-1)^\chi \exp(2\pi i \bar{\eta}^\varepsilon(0))$$

for all ε, in particular for $\varepsilon = 1$. In other words, the adiabatic limit of [2] is stationary under our hypotheses.

References

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