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A Note on the Global Structure of Supermoduli Spaces*

Gregorio Falqui and Cesare Reina
S.I.S.S.A, International School for Advanced Studies, Strada Costiera 11, I-34014 Trieste, Italy

Abstract. We recall some deformation theory of Susy-curves and study obstructions to projectedness of supermoduli spaces, both from a general standpoint and by means of the local "coordinate charts" most commonly used in the physical literature. We prove that these give rise to a projected atlas for genus $g=2$ only.

Introduction

Although the burst of interest for string theory in the last years seems to be fading away, there are still interesting open problems to be solved. Among others, the question of the global structure of supermoduli space arises, which is believed to play in superstring theory the rôle of moduli space of algebraic curves in the bosonic model. For instance, when computing amplitudes in superstring theory via a path integral approach, one faces the problem of dealing with odd variables. While the bosonic piece of the Polyakov path integral is well understood as an integral over moduli spaces of algebraic curves, the fermionic part is more embarrassing as the discovery of ambiguities in performing the integration over odd variables pointed out (for a review see e.g. [AMS], [DP] and references quoted therein). To cut a long story short, the basic trouble comes from the fact that in a given supersymmetric gauge the measure for superstrings reduces to a Berezin form, which unluckily is gauge dependent. This is because a supersymmetry transformation induces a small variation of one's gauge choice in a "non-split" way. In other words, the modular parameters change by a nilpotent contribution which in turn induces a change of the string measure by a "total derivative."

Besides these local problems, on which most of the physical literature was focused, there is an even more serious global obstacle to give a mathematically sound definition of the path integral. This may arise from a Gribov-like ambiguity

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in supersymmetry gauge fixing, which would imply non-split "coordinate transformations" in the overlap of any pair of local charts on "supermoduli" spaces.

Although the local problem may be handled [MT] within the correct treatment of Berezin forms under non-split coordinate transformations [R], yet it is not clear how to handle the global issue. Anyway, this approach is not satisfactory for the following reasons. First, due to the non-naturality of the splitting of the Berezinian sequence even in the smooth case, possible divergences in the amplitudes (before GSO-projection) may give rise to boundary terms. Second, one would like to set up a formalism which manifestly keeps into account super-holomorphic structures step by step.

In this paper we consider such problems from a more mathematical point of view, by studying the global structure of supermoduli "spaces". Our strategy is to describe the supersymmetric analogue of the moduli stack, but we do not attempt to formalize such a structure here. We choose instead a direct coordinate approach. Although a stack theoretical approach is the natural arena for the present problem, we feel that the more informal way we are going to follow here is closer to the physical literature. We limit ourselves to recall some ideas about moduli stack in the Appendix, where we also comment on the $\mathbb{Z}_2$ "global" ambiguity as described in [LR].

With this *proviso* in mind, our basic result is that the usual choices done in the physical literature yield non-projected structures for genus $g \geqq 3$.

This paper is organized as follows. In Sect. 1 we collect some basic facts about susy-curves and their deformations and we discuss the actual meaning of putting coordinates on supermoduli space, studying in particular the first obstruction to projectedness. In Sect. 2 we construct a particular class of universal deformations, which generalize to the $\mathbb{Z}_2$-commutative case *Schiffer* deformations of algebraic curves; these give the mathematical counterpart of the choice of δ-function gravitino zero-modes done in the physical literature. We then show, by means of explicit computations, that such choices cannot give projected atlases for supermoduli. The only exception is the case of even θ-characteristics at genus 2 where a careful choice of the "gravitino" supports proves, in a constructive way, the splitness of even supermoduli for smooth curves of genus $g = 2$.

1. Susy-Curves and their Deformations

To make the paper self-consistent, we will recall some basic facts on susy-curves [D, F, BMFS] which will be needed in the following. For a more complete treatment see [LR, FR], which we briefly summarize here. The structure sheaf $\mathscr{A}_X$ of a supermanifold is a sheaf of $\mathbb{Z}_2$-graded commutative algebras over a manifold X, which is locally isomorphic to the sheaf of Grassmann algebras $\Lambda^*\mathscr{E}$ generated by a locally free sheaf $\mathscr{E}$ on X. Taking even and odd parts one has $\mathscr{A}_X \simeq \mathscr{A}_X^0 \oplus \mathscr{A}_X^1$; $\Lambda^*\mathscr{E} \simeq (\Lambda^*\mathscr{E})^0 \oplus (\Lambda^*\mathscr{E})^1$. A supermanifold is called *split* iff $\mathscr{A}_X \simeq \Lambda^*\mathscr{E}$ and *projected* iff $\mathscr{A}_X^0 \simeq (\Lambda^*\mathscr{E})^0$.

[1] Recall that this is actually a morphism of ringed spaces; together with the map $X \xrightarrow{\pi} B$ of topological spaces, one has a homomorphism of sheaves of $\mathbb{Z}_2$-graded algebras $\pi^{\#}: \pi^{-1}\mathscr{A}_B \to \mathscr{A}_X$

1.1. A *family of susy-curves* X parametrized by a complex superspace B is a proper surjective flat map $\pi: X \to B$ of complex superspaces[1] having $1|1$ dimensional fibres, together with a $0|1$ dimensional distribution $\mathscr{D}_\pi$ in the relative tangent sheaf $\mathscr{T}_\pi X$ such that the supercommutator mod $\mathscr{D}_\pi$, $[,]_{\mathscr{T}_\pi}: \mathscr{D}_\pi^{\otimes 2} \to \mathscr{T}_\pi X/\mathscr{D}_\pi$ is an isomorphism (see [LR] for more details).

A susy-curve over a point $\{*\}$ will be called a *single* susy-curve. A *deformation* of a single susy-curve is a family $X \xrightarrow{\pi} (B, b_0)$ of susy-curves over a pointed superspace together with a fixed identification of the central fibre, i.e. a diagram

$$\begin{array}{ccc} C & \overset{\iota}{\hookrightarrow} & X \\ \downarrow & & \downarrow \\ \{*\} & \hookrightarrow & B \end{array}$$

which commutes as a diagram of maps of complex superspaces. Notice that, whenever $(B, \mathscr{A}_B)$ is a superdomain, (e.g. a superpolydisk with coordinates (t, η)), we can cover X with open sets of the form $\mathscr{U}_\alpha = U_\alpha \times B$, where $\{U_\alpha, z_a, \vartheta_\alpha\}$ is an atlas for C and use relative *canonical* coordinates[2] in such a way that $(x_\alpha, \phi_\alpha)\upharpoonright_{b_0} = i(z_\alpha, \vartheta_\alpha)$. Then the clutching function on $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$ takes the form[3]

$$\begin{cases} x_\alpha = f_{\alpha\beta}(x_\beta; t, \eta) \mp \sqrt{f'_{\alpha\beta}(x_\beta; t, \eta)}\mu_{\alpha\beta}(x_\beta; t, \eta)\phi_\beta \\ \phi_\alpha = \pm\sqrt{f'_{\alpha\beta}(x_\beta; t, \eta) + \mu_{\alpha\beta}\mu'_{\alpha\beta}(x_\beta; t, \eta)} \cdot \phi_\beta + \mu_{\alpha\beta}(x_\beta; t, \eta) \end{cases}. \tag{1}$$

Having a single susy-curve no extra parameters, its clutching functions read

$$x_\alpha = f_{\alpha\beta}(x_\beta); \quad \phi_\alpha = \pm\sqrt{f'_{\alpha\beta}(x_\beta)}\phi_\beta$$

and therefore it is trivially split.

1.2. On a family of susy-curves there are two natural subsheaves of the tangent sheaf $\mathscr{T} X$, namely the subsheaves $\mathscr{T}^{\mathscr{D}}$ $(\mathscr{T}_\pi^{\mathscr{D}})$ of the (relative) germs of derivations which commute with $\mathscr{D}_\pi$. Associated to the exact sequence

$$0 \to \mathscr{T}_\pi^{\mathscr{D}} \to \mathscr{T}^{\mathscr{D}} \to \pi^* \mathscr{T} B \to 0$$

we have a coboundary map (the *Kodaira–Spencer* map) $KS: H^0(\pi^*(\mathscr{T}\mathscr{B}) \to H^1(\mathscr{T}_\pi^{\mathscr{D}})$. The family is called versal whenever KS is an isomorphism (see [W] for a study deformation theory of super-holomorphic structures). When $C \hookrightarrow X \xrightarrow{\pi} B$ is considered as a deformation of C, we can restrict KS to the central fibre C_0 getting $KS_0: T_{b_0}B \to H^1(C_0, \mathscr{T}_\pi^{\mathscr{D}})$; one can show that if KS_0 is an isomorphism, then X is a universal deformation of C. Some relevant examples of such deformations will be constructed in Sect. 2. Notice that there is an isomorphism [LR] between $\mathscr{T}_\pi^{\mathscr{D}}$ and $\mathscr{D}_\pi^{\otimes 2}$. This fact enables us to speak of an *even* (respectively *odd* Kodaira–Spencer map KS_0^{ev} (respectively KS_0^{odd}) by projection on the even (odd) part of $\mathscr{D}_\pi^{\otimes 2}$.

[2] A relative coordinate system is called canonical if the generator for $\mathscr{D}\upharpoonright_{\mathscr{U}_\alpha}$ takes the form $\partial/\partial\phi_\alpha + \phi_\alpha \cdot \partial/\partial x_\alpha$. Such coordinates always exist [LR]

[3] The overall sign $\pm$ will be left implicit. $\mathbb{Z}_2$-ambiguity is not going to be tackled in this paper

1.3. The global structure of supermoduli spaces is quite subtle because of the presence of automorphisms of susy-curves, and in particular of the canonical $\mathbb{Z}_2$ automorphism. Here we will limit ourselves to a somewhat simpler problem in the framework of the theory of moduli *stacks* [Mu], a very brief sketch of which will be given in the appendix. To make things intuitive, a "covering" on the moduli stack is a collection of versal families $\{X^j \xrightarrow{\pi^j} B^j\}_{j\in I}$ of genus g susy-curves such that

1. the reduced families $\{X^j_{\text{red}} \xrightarrow{\pi^j_{\text{red}}} B^j_{\text{red}}\}_{j\in I}$ give a covering of the stack Σ_g of genus g spin curves [C];
2. whenever X^j_{red} and X^i_{red} contain isomorphic spin curves, the isomorphism of reduced families

$$\begin{array}{ccc} X^j_{\text{red}} & \xrightarrow{L_{ji}} & X^i_{\text{red}} \\ \downarrow{\scriptstyle \pi^j_{\text{red}}} & & \downarrow{\scriptstyle \pi^i_{\text{red}}} \\ B^j_{\text{red}} & \xrightarrow{h_{ji}} & B^i_{\text{red}} \end{array}$$

over suitably restricted base spaces, comes together with morphisms $L^{\#}_{ji}: L^{-1}_{ji}\mathscr{A}_{X_i} \to \mathscr{A}_{X_j}$ and $h^{\#}_{ji}: h^{-1}_{ji}\mathscr{A}_{B_i} \to \mathscr{A}_{B_j}$ which make the two families superconformally isomorphic. We shall say for short that, in this latter case, X_j and X_i *partially overlap*. Notice that with two modular families, h_{ji} and $h^{\#}_{ji}$ are essentially unique.

An obvious way to construct an "atlas" on the stack is to choose the B_j's to be superpolydisks and the X_j's to be universal deformations of susy-curves C_j. So each X_j comes with the open covering $\{\mathscr{U}_{\alpha_j}\}$ with coordinates $\{x_{\alpha_j}, \varphi_{\alpha_j}, t_j, \eta_j\}$. We can as well cast the maps $L^{\#}_{\alpha_j\alpha_i}$ in the form of (1). Setting $\alpha \rightsquigarrow \alpha_j$ $\beta \rightsquigarrow \beta_j$ we get

$$\begin{cases} x_{\alpha_j} = f_{\alpha_j\beta_j}(x_{\beta_j}; t_j, \eta_j) - \sqrt{f'_{\alpha_j\beta_j}(x_{\beta_j}; t_j, \eta_j)}\,\mu_{\alpha_j\beta_j}(x_{\beta_j}; t_j, \eta_j)\phi_{\beta_j} \\ \phi_{\alpha_j} = \sqrt{f'_{\alpha_j\beta_j}(x_{\beta_j}; t_j, \eta_j) + \mu_{\alpha_j\beta_j}\mu'_{\alpha_j\beta_j}(x_{\beta_j}; t_j, \eta_j)} + \mu_{\alpha_j\beta_j}(x_{\beta_j}; t_j, \eta_j) \\ t_j = t_j \\ \eta_j = \eta_j. \end{cases} \tag{2a}$$

As a morphism of complex supermanifolds is completely specified by expressing its effect on the "coordinate functions," when X_j and X_k partially overlap one can locally describe $h^{\#}_{kj}$ and $L^{\#}_{kj}$ in terms of the following maps:

$$\begin{cases} x_{\beta_k} = g_{\beta_k\beta_j}(x_{\beta_j}; t_j, \eta_j) - \sqrt{g'_{\beta_k\beta_j}(x_{\beta_j}; t_j, \eta_j)}\,\sigma_{\beta_k\beta_j}(x_{\beta_j}; t_j, \eta_j)\phi_{\beta_j} \\ \phi_{\beta_k} = \sqrt{g'_{\beta_k\beta_j}(x_{\beta_j}; t_j, \eta_j) + \sigma_{\beta_k\beta_j}\sigma'_{\beta_k\beta_j}(x_{\beta_j}; t_j, \eta_j)} + \sigma_{\beta_k\beta_j}(x_{\beta_j}; t_j, \eta_j) \\ t_k = h_{kj}(t_j, \eta_j) \\ \eta_k = \Xi_{kj}(t_j \eta_j). \end{cases} \tag{2b}$$

Notice that the last two lines give the coordinate description of $h^{\#}_{kj}$. These give rise to a superconformal isomorphism provided that

$$L^{\#}_{\alpha_j\beta_j} \circ L^{\#}_{\beta_j\beta_k} = L^{\#}_{\alpha_j\alpha_k} \circ L^{\#}_{\alpha_k\beta_k}.$$

More explicitly, considering a common open covering $\mathscr{U}_\alpha$ for the reduced families X_i and $h^*(X_j)$, redefining $L^{\#}_{\alpha_j\beta_j} \equiv L^{j\#}{}_{\alpha\beta}, L^{\#}_{\beta_j\beta_k} \equiv L^{\#}_{\beta}, t_j \equiv t, \eta_j \equiv \eta$, $t_k \equiv s$, $\eta_k \equiv \xi$, these equations take the form

$$L^j_{\alpha\beta}(L_\beta(x_\beta, \phi_\beta; t, \eta); t, \eta) = L_\alpha(L^k_{\alpha\beta}(x_\beta, \phi_\beta; h(t,\eta), \Xi(t,\eta)); t, \eta). \tag{3}$$

An easy computation shows that this is equivalent to imposing the following conditions on the building blocks of the superconformal maps (2a) and (2b):

$$\begin{aligned} &f_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k} - \sqrt{f'_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k}} \cdot (\mu_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k}) \cdot \sigma_{\beta_j\beta_k} \\ &= g_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k} - \sqrt{g'_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k}} \cdot (\sigma_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k}) \cdot \mu_{\alpha_k\beta_k}, \end{aligned} \tag{4a}$$

where the meaning of $\circ$ is as given in Eq. (3) and

$$\begin{aligned} &\sqrt{f'_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k} + \mu_{\alpha_j\beta_j}\mu'_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k}} \cdot \sigma_{\beta_j\beta_k} + \mu_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k} \\ &= \sigma_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k} + \sqrt{g'_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k} + \sigma_{\alpha_j\alpha_k}\sigma'_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k}}\, \mu_{\alpha_k\beta_k}. \end{aligned} \tag{4b}$$

Notice that, as in the ordinary case (see e.g. [K]), when dealing with universal deformations these relations actually determine the maps $t_k = h_{kj}(t_j, \eta_j), \eta_k = \Xi_{kj}(t_j \eta_j)$. It is convenient to expand these equations in powers of odd generators. One possibility is to quotient Eqs. 4a and 4b by $\pi^{-1}\mathscr{N}^n_{B_j}$ $(n = 1, 2, \dots)$, $\mathscr{N}_{B_j}$ being the ideal of nilpotents in the base $(B_j, \mathscr{A}_{B_j})$, which leads to the notion of susy-curves over a thickened basis as in [LR]. For our purposes, however, it is sufficient to quotient the equations above by the full 3rd power $\mathscr{N}^3_{X_j}$ of the nilpotent ideal in $\mathscr{A}_{X_j}$. In fact, as one easily checks, up to $\mathscr{N}^3_{B_j}$ the maps $h^{\#}_{jk}$ are the same in both procedures. On the other hand, we gain a nice intuitive description of what is going on in terms of the (ètale) sheaf cohomology on the moduli stack, entering the details of which is outside the aims of this paper.

Equation (3) modulo $\mathscr{N}_{X_j}$ give us the reduced structure of the moduli stack of spin curves. Up to $\mathscr{N}^2_{X_j}$ we get

$$\sigma^1_{\alpha_j\alpha_k} \circ f^0_{\alpha_k\beta_k} + \sqrt{g'^0_{\alpha_j\alpha_k} \circ f^0_{\alpha_k\beta_k}} \cdot \mu^1_{\alpha_k\beta_k} = \sqrt{f'^0_{\alpha_j\beta_j} \circ g^0_{\beta_j\beta_k}} \cdot \sigma^1_{\beta_j\beta_k} + \mu^1_{\alpha_j\beta_j} \circ g^0_{\beta_j\beta_k}$$

(here superscripts denote the order of the η-expansion) which, upon tensorization with $\partial/\partial\phi_{\alpha_k}$ tells us that $\mu_{\alpha_i\beta_i}\partial/\partial\phi_{\alpha_i}$ and $L^*_{ij}\mu_{\alpha_j\beta_j}\partial/\partial\phi_{\alpha_j}$ are cohomologous in $H^1(C_t, \mathscr{L}^{-1})$ via the coboundary $\sigma_{\alpha_i\alpha_j}\partial/\partial\phi_{\alpha_i}$. Then we get the well known fact that, up to order 1, we can safely take cohomology classes getting the 1rst infinitesimal neighbourhood of Σ_g as $R^1\pi_*\mathscr{L}^{-1}$.

The first obstruction to projectedness of the supermoduli stack can be seen at the next order. One gets that h^2_{ij} must satisfy the condition

$$\frac{\partial g_{\alpha_i\alpha_j}}{\partial t_j} h^2_{ij} = f^2_{\alpha_i\beta_i} - f^2_{\alpha_j\beta_j} + \sigma_{\alpha_i\alpha_j}\mu_{\alpha_i\beta_i} - \mu_{\alpha_j\beta_j}\sigma_{\beta_i\beta_j}.$$

A clearer coordinate free description of this condition can be given by noticing that modulo $\mathscr{N}^3_{X_j}$, the glueing maps give rise to the data of vector fields in each "overlap" as follows:

i) on $\mathscr{U}_{\alpha_i} \cap \mathscr{U}_{\beta_i}$ $\quad V_{\alpha_i\beta_i} = f^2_{\alpha_i\beta_i}\dfrac{\partial}{\partial x_{\alpha_i}} + \phi_{\alpha_i}\mu^1_{\alpha_i\beta_i}\dfrac{\partial}{\partial x_{\alpha_i}} + \mu^1_{\alpha_i\beta_i}\dfrac{\partial}{\partial \phi_{\alpha_i}}$,

ii) on[4] $\mathscr{U}_{\alpha_i}\cap\mathscr{U}_{\alpha_j}$ $\quad V_{\alpha_i\alpha_j} = g^2_{\alpha_i\alpha_j}\dfrac{\partial}{\partial x_{\alpha_i}} + \phi_{\alpha_i}\sigma^1_{\alpha_i\alpha_j}\dfrac{\partial}{\partial x_{\alpha_i}} + \sigma^1_{\alpha_i\alpha_j}\dfrac{\partial}{\partial\phi_{\alpha_i}} + h^2_{ij}\dfrac{\partial}{\partial t_i}.$

The cocycle rule for $L^{\#}_{\alpha.\beta_*}$ shows that these may be interpreted as a one-cycle on the supermoduli stack with values in $\mathscr{T}^{\mathscr{D}}$. In the same spirit base changes on the families of the "atlas" mod $\mathscr{N}^3_{X_J}$ induce the action of coboundaries. Indeed, these correspond to vector fields of the form

$$V_{\alpha_i} = g^2_{\alpha_i}\frac{\partial}{\partial x_{\alpha_i}} + \phi_{\alpha_i}\sigma^1_{\alpha_i}\frac{\partial}{\partial x_{\alpha_i}} + \sigma^1_{\alpha_i}\frac{\partial}{\partial\phi_{\alpha_i}} + h^2_i\frac{\partial}{\partial t_i}$$

acting on the cocycle $V_{\alpha_i\beta_j}$ by

$$\tilde{V}_{\alpha.\beta_*} = V_{\alpha.\beta_*} + V_{\alpha.} - V_{\beta_*}.$$

In particular we see that this induces the action of coboundaries on the cocycle $\mu_{\alpha_i\beta_i}$ so that these representatives can be chosen at will. In other words, changing representatives of $\mu_{\alpha_i\beta_i}$ "fibrewise" in $R^1\pi_*\mathscr{L}^{-1}$ yields non-equivalent families [LR] and must be compensated by suitable base changes.

In spite of the fact that this description deserves a detailed formalization, we will use it in its present rough form as an intuitive clue to construct an explicit representative of the first obstruction to projectedness of supermoduli stack.

2. Schiffer Deformations and Obstructions to Projectedness

This section is devoted to study what happens when we try to "glue" two universal deformations of susy-curves of a special kind which extend to the supersymmetric case the so-called Schiffer deformations of ordinary curves. The interest in this construction is that it is the mathematical counterpart of the choice of δ-function gravitino zero-modes usually done in the physical literature (see, e.g. [B]). We recall the following [FR].

Lemma 2.1. *Let L be a θ-characteristic on a curve C_{red} of genus g. For a generic point $p\in C_{\mathrm{red}}$ and $k\geqq 1$ the connecting homomorphism $\delta^k_p:\mathbb{C}\to H^1(C_{\mathrm{red}},\mathscr{L}^{-k})$ associated to the exact sequence*

$$0\to\mathscr{L}^{-k}\to\mathscr{L}^{-k}(p)\to\mathscr{L}^{-k}(p)/\mathscr{L}\to 0$$

is injective. The map $\delta^k: C_{\mathrm{red}}\to H^1(C_{\mathrm{red}},\mathscr{L}^{-k})$ given by $\delta^k(p)=\delta^k_p(1)$ is full, i.e. *there are $(k+1)(g-1)$ points $p_i\in C_{\mathrm{red}}$ such that $\delta^k_{p_i}(1)$ gives a basis of $H^1(C_{\mathrm{red}},\mathscr{L}^{-k})$.*

Thanks to this lemma one can easily construct deformations of a susy-curve C by choosing $n+m$ generic points $p_i\in C_{\mathrm{red}}$ with local supercoordinates (z_i,θ_i), $(z_i(p_i)=0)$, and glueing $(1|1)$-dimensional superdisks $\Delta^{(1|1)}_i$ with local coordinates (x_i,φ_i) with $C_{\mathrm{red}}\setminus\{p_i\}$ by means of the maps

[4] These intersections do not really exist. Ours is a heuristic notation denoting patches on families which partially overlap in the sense specified above

$$\begin{cases} x_i = z_i + \dfrac{t_i}{z_i} \\ \varphi_i = \sqrt{1 + \dfrac{t_i}{z_i^2}} \cdot \theta_i \end{cases}$$

for $i = 1, \ldots, n$ and

$$\begin{cases} x_i = z_i + \theta_i \dfrac{\eta_{i-n}}{z_i} \\ \varphi_i = \theta_i \dfrac{\eta_{i-n}}{z_i} \end{cases}$$

for $i = n+1, \ldots, m$. Since these maps are manifestly superconformal (wherever defined), we get a family of susy-curves over a superpolydisk $\Delta^{n|m}$. Moreover, the Kodaira–Spencer map restricted to the central fiber reads

$$KS_0^{\mathrm{ev}}\left(\frac{\partial}{\partial t_i}\right) = \left[\frac{1}{z_i}\frac{\partial}{\partial z_i}\right] = \delta_{p_i}^2(1) \qquad i = 1, \ldots, n,$$

$$KS_0^{\mathrm{odd}}\left(\frac{\partial}{\partial \eta_{i-n}}\right) = \left[\frac{1}{z_i}\theta_i \otimes \frac{\partial}{\partial z_i}\right] = \delta_{p_i}^1(1) \quad i = n+1, \ldots, m.$$

So the deformation is universal for $n|m = 3g-3|2g-2$ and will be called a *Schiffer* deformation of the susy-curve C. Notice that if $n|m \geqq 3g-3|2g-2$ the deformation is complete but overparametrized by an amount of $n-(3g-3)|m-(2g-2)$ parameters. An atlas for $X \to \Delta^{n|m}$ can be constructed along the lines discussed in Subsect. 1.3 as follows. We can cover C_0 with an atlas $(U_\alpha; z_\alpha, \theta_\alpha)$ such that for $1 \leqq \alpha, \beta \leqq n+m$, $U_\alpha \cap U_\beta = \emptyset$, and with transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$; $\theta_\alpha = \sqrt{f'_{\alpha\beta}(z_\beta)} \cdot \theta_\beta$. Then an atlas for X is given by $\mathcal{U}_\alpha = U_\alpha \times \Delta_{\mathrm{red}}^{m|n}$ with coordinates $(x_\alpha, \varphi_\alpha; t_i, \eta_i)$ and with transition functions

$$\begin{cases} x_\alpha = F_{\alpha\beta}(x_\beta) + \varphi_\alpha \sqrt{F'_{\alpha\beta}(x_\beta)}\mu_{\alpha\beta} \\ \varphi_\alpha = \sqrt{F'_{\alpha\beta}(x_\beta)}\varphi_\beta + \mu_{\alpha\beta}(x_\beta) \end{cases}$$

where:
for $1 \leqq \alpha \leqq n$ and any $\beta \geqq m+n$

$$\begin{cases} F_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta) + \dfrac{t_\alpha}{f_{\alpha\beta}(x_\beta)} \\ \mu_{\alpha\beta}(x_\beta) = 0, \end{cases}$$

for $n+1 \leqq \alpha \leqq n+m$ and any $\beta \geqq n+m$

$$\begin{cases} F_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta) \\ \mu_{\alpha\beta}(x_\beta) = \dfrac{\eta_{\alpha-n}}{f_{\alpha\beta}(x_\beta)}, \end{cases}$$

for $\alpha, \beta \geqq n+m$

$$\begin{cases} F_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta) \\ \mu_{\alpha\beta}(x_\beta) = 0 \end{cases}.$$

The crucial property of such a deformation we want to capture can be summarized in the following.

Proposition 2.2. *There exist universal deformations $X \to \Delta^{n|m}$, $(n|m = 3g-3|2g-2)$ of a susy-curve C whose transition functions*

i) *depend linearly on the odd deformations parameters;*
ii) *are split but for a finite number of intersections*
iii) *the relative one-cocycle with values in $\mathscr{L}_\pi^{-1}\mu_{\alpha\beta}$ has $m = 2g-2$ simple poles on each fibre of X.*

More generally, we have complete deformations with the same properties when $n|m \geqq 3g-3|2g-2$. Needless to say, given any other universal deformation $X' \to \Delta'^{n|m}$ of C (e.g. one given by a generic choice of non-trivial $\mu_{\alpha\beta}$'s) there is a unique base change $h: \Delta^{n|m} \to \Delta'^{n|m}$ such that h^*X' is isomorphic to X (possibly after a suitable shrinking of the bases). This makes us free to choose Schiffer "atlases" on the supermoduli stack of the form $\{C_j; X_j \to \Delta_j^{n|m}\}$, the $X_j \to \Delta_j^{n|m}$ being universal Schiffer deformations of C_j. To handle such deformations, and in particular the choices of the $\mu_{\alpha\beta}$, we need some more technicalities. Given two different choices of $m = 2g-2$ points $\{p_i\}$, $\{p_{m+i}\}$ $i = 1, 2, \ldots, 2g-2$ on each fibre[5] C_t of a deformation X, we have a *Stein* covering of C_{red} made of the disjoint union of $U_0 \equiv C_{\text{red}} \setminus \{p_k\}_{k=1,\ldots,4g-4}$, and small disks $\{U_k\}_{k=1,\ldots,4g-4}$ around each p_k such that the only non-empty intersections are punctured disks given by $U_{0k} = U_0 \cap U_k$. Now two choices of one-cocycles $\mu_{\alpha\beta}$ can be represented on this covering by a collection of *meromorphic* sections of $\mathscr{L}_\pi^{-1} \restriction_{U_k}$ with simple poles at p_i, $1 \leqq i \leqq 2g-2$ and at p_j, $2g-1 \leqq j \leqq 4g-4$, i.e.

$$\begin{cases} \mu_{i0} = \sum_{j=1}^{2g-2} \delta_i^j \frac{\eta^i}{x_i} \varphi_i \frac{\partial}{\partial x_i} \\ \nu_{i0} = \sum_{j=2g-1}^{4g-4} \delta_i^j \frac{\varepsilon^{i-m}}{x_i} \varphi_i \frac{\partial}{\partial x_i} \end{cases}.$$

We shall need the following

Lemma 2.3. *There exists a unique (up to a sign) linear map $\varepsilon^k = A_i^k \eta^i$ such that μ_{0i} and ν_{0i} are cohomologous.*

Proof. The difference $\lambda_{i0} \equiv \mu_{0i} - \nu_{0i}$ has simple poles at $4g-4$ points. From the exact sequence

$$0 \to \mathscr{L}^{-1} \to \mathscr{L}^{-1}\left(\sum_{i=1}^{4g-4} p_i\right) \to \mathscr{L}^{-1}\left(\sum_{i=1}^{4g-4} p_i\right) \Big/ \mathscr{L}^{-1} \to 0$$

[5] These actually can be made to define a divisor on X, but we prefer to work fibrewise

we have

$$0 \to H^0\left(\mathscr{L}^{-1}\left(\sum_{i=1}^{4g-4} p_i\right)\right) \to \mathbb{C}^{4g-4} \xrightarrow{\delta} H^1(\mathscr{L}^{-1}) \to 0$$
$$\|$$
$$\mathbb{C}^{2g-2}$$

so that $\lambda \in \ker \delta$ iff there are $2g-2$ sections $s_0^i \in H^0\left(\mathscr{L}^{-1}\left(\sum_{i=1}^{4g-4} p_i\right)\right)$ such that[6]

$$\lambda_{i0} = \sum_k \eta_k (s_0^k - s_i^k),$$

where s_i^k denotes the holomorphic tail of s_0^k on $U_{i.}$, i.e. $s_0^k \restriction_{U_i} = \mathbb{B}_i^k / x_i + s_i^k$, with $\mathbb{B} = (\mathbb{I}, \mathbb{A})$. ■

We can now prove the following

Proposition 2.4. *For $g \geqq 3$ Schiffer atlases are not projected.*

Proof. Let X_j and X_k be two universal Schiffer deformations which partially overlap, and let $\mu_{\alpha_j \beta_j}, \mu_{\alpha_k \beta_k}$ the corresponding $\mathscr{L}^{-1}$-valued one-cocycles. From Sect. 2 we know that there exists an $\mathscr{L}^{-1}$-valued coboundary which make them cohomologous. Then the obstruction to projectedness mod $\mathscr{N}^3$ now reads

$$\tau^2 = \frac{\partial g_{\alpha_i \alpha_j}}{\partial t_j} h_{ij}^{(2)} = \sigma_{\alpha_i \alpha_j} \mu_{\alpha_i \beta_i} - \mu_{\alpha_j \beta_j} \sigma_{\beta_i \beta_j} \stackrel{\text{def}}{=} \sigma_\alpha \mu_{\alpha\beta} - \nu_{\alpha\beta} \sigma_\beta .$$

As in Proposition 2.3 we can take $\mu_{\alpha\beta}$ ($\nu_{\alpha\beta}$) with support on the punctured disks $U_{0\alpha}$ for $\alpha = 1, \ldots, 2g-2$ and for $\alpha = 2g-1, \ldots, 4g-4$ respectively. Then, after the suitable identification of the odd parameters given by Lemma 2.3, we have $\mu_{\alpha\beta} - \nu_{\alpha\beta} = \sum_k \eta^k (s_\beta^k - s_\alpha^k)$ and $\sigma_\alpha = \sum_k \eta^k s_\alpha^k$, i.e.

$$\mu_{\alpha\beta} - \nu_{\alpha\beta} = \mu_{\alpha 0} - \nu_{\alpha 0} \restriction_{U_\beta} = \sum_k \eta^k (s_0^k - s_\alpha^k).$$

Now τ^2 reads

$$\tau^2 = \begin{cases} \sum_l \eta^l s_\alpha^l \left(\sum_k \eta^k (s_0^k - s_\alpha^k) \right) \restriction_{U_\beta} & 1 \leqq \alpha \leqq 2g-2 \\ \sum_k \eta^k s_0^k \left(\sum_l \eta^l (s_0^l - s_\alpha^l) \right) \restriction_{U_\beta} & 2g-1 \leqq \alpha \leqq 4g-4 \end{cases},$$

[6] We benefit here from the fact that for generic points p_i, $H^0\left(\mathscr{L}^{-1}\left(p_j + \sum_{i=1}^{2g-2} p_{m+i}\right)\right)$ is one-dimensional for $j = 1, \ldots, 2g-2$

i.e.

$$\tau^2 = \begin{cases} \sum_{kl} \eta^l \eta^l (s^l_\alpha s^k_0 - s^l_\alpha s^k_\alpha) & 1 \leqq \alpha \leqq 2g-2 \\ \sum_{kl} \eta^k \eta^l (s^k_0 s^l_0 - s^k_0 s^l_\alpha) & 2g-1 \leqq \alpha \leqq 4g-4 \end{cases}$$

As the last row can be written as $\sum_{kl} \eta^l \eta^k (s^k_0 s^l_\alpha - s^k_0 s^l_0)$, then $\tau^2 = \xi_{0\alpha} - \xi_0 + \xi_\alpha$, where

$$\xi_{0\alpha} = \sum_{kl} \eta^l \eta^k s^k_0 s^l_\alpha \quad 1 \leqq \alpha \leqq 4g-4,$$

$$\xi_0 = \sum_{kl} \eta^l \eta^k s^k_0 s^l_0 \quad 1 \leqq \alpha \leqq 4g-4,$$

$$\xi_\alpha = \begin{cases} -\sum_{kl} \eta^l \eta^k s^l_\alpha s^k_\alpha & 1 \leqq \alpha \leqq 2g-2 \\ 0 & 2g-1 \leqq \alpha \leqq 4g-4 \end{cases}$$

This shows that τ^2 is a one-cocycle with values in $\mathscr{L}^{-2}$ cohomologous to $\xi_{0\alpha}$, which, in turn can be identified with a family of local meromorphic sections of $\mathscr{L}^{-2}$ with simple poles at the points p_α. This is cohomologous to zero if and only if the $\frac{1}{2}(2g-2)(2g-3) = (g-1)(2g-3)$ sections $s^k_0 s^l_\alpha - s^l_0 s^k_\alpha$ are cohomologous to zero. The exact sequence

$$0 \to \mathscr{L}^{-2} \to \mathscr{L}^{-2}\Big(\sum_\alpha p_\alpha\Big) \to \mathscr{L}^{-2}\Big(\sum_\alpha p_\alpha\Big)\Big/\mathscr{L}^{-2} \to 0$$

gives us on each fibre

$$0 \to H^0\Big(\mathscr{L}^{-2}\Big(\sum_\alpha p_\alpha\Big)\Big) \to \mathbb{C}^{4g-4} \to H^1(\mathscr{L}^{-2}). \tag{4}$$

Now for any choice of p_α's, $\dim H^0\Big(\mathscr{L}^{-2}\Big(\sum_\alpha p_\alpha\Big)\Big) \leqq g$ and so for $g \geqq 3$, τ^2 cannot be cohomologous to 0. Since the representative for the splitting cocycle is $h^{(2)}_{ij} \partial/\partial t_j = (KS^{ev}_t)^{-1}(\tau^2)$, we see that $h^{(2)}_{ij}$ cannot vanish. ■

For $g = 2$ this dimensional argument clearly breaks down. In fact, special choices can be made (see e.g. [GIS]) which allow us a constructive proof of Deligne's result[7] on the splitness of even-supermoduli space at $g = 2$. This runs as follows.

Proposition 2.5. *There exist split Schiffer "atlases" on the even supermoduli stack at $g = 2$.*

Proof. Let $p_\pm$ be two conjugate points under the hyperelliptic involution h. The

[7] Deligne's proof is contained in an unpublished letter to D. Kazhdan [AG], whose content is actually unknown to us. Anyway the splitness of genus 2 supermoduli spaces can be understood also in terms of the holomorphic geometry of moduli of spin curves [FMRT]

exact sequence

$$0 \to \mathscr{L}^{-1} \to \mathscr{L}^{-1}(p_- + p_+)) \to \mathscr{L}^{-1}(p_- + p_+))/\mathscr{L}^{-1} \to 0$$

yields

$$\cdots \to 0 \to H^0(\mathscr{L}^{-1}(p_- + p_+)) \to \mathbb{C}^2 \overset{\delta}{\to} H^1(\mathscr{L}^{-1}) \to H^1(\mathscr{L}^{-1}(p_- + p_+)) \to \cdots .$$

Now $\deg \mathscr{L}^{-1}(p_- + p_+) = 1 = g - 1$, and therefore $H^n(\mathscr{L}^{-1}(p_- + p_+))(n = 0, 1)$ have the same dimension. So to prove that δ is an isomorphism we notice that $p_- + p_+$ is a canonical divisor, so that $H^1(\mathscr{L}^{-1}(p_- + p_+)) \equiv H^1(\mathscr{L}^{-1} \otimes \mathscr{K}) \simeq H^0(\mathscr{L}) = \{0\}$ because genus two even θ-characteristics are non-singular.

Accordingly we can choose the $\mu_{\alpha_i \beta_i}$ defining the Schiffer deformations X_i with simple poles at $p^i_\pm$.[8] Now glueing two such families X_j and X_k as in Subsect. 1.3 we see that the obstruction cocycle $\tau^2_{0\alpha} = s^1_o s^2_\alpha - s^2_0 s^1_\alpha$ to splitness (which is the same thing as projectedness at $g = 2$) is an $\mathscr{L}^{-2}$-valued one-cocycle with simple poles at $p^j_\pm$ and $p^k_\pm$ depending linearly on two parameters.

Now τ^2 is cohomologous to a global section in $H^0(\mathscr{L}^{-2}(p^j_- + p^j_+ + p^k_- + p^k_+))$ if and only if the family $\omega_\alpha \equiv z_\alpha \cdot \tau^2_{0\alpha}$ is cohomologous to a global section of $\mathscr{K}$ (indeed $\mathscr{L}^{-2}(p^j_- + p^j_+ + p^k_- + p^k_+) \equiv \mathscr{K}$). Since abelian differentials are anti-invariant under the hyperelliptic involution h we need only to show that the leading terms of ω_α satisfy $\omega_1 = -h^*\omega_2$ and $\omega_3 = -h^*\omega_4$. This turns out to be true for the following reason. The section s^1_0 (respectively s^2_0) is the *unique* (up to multiplication by a constant) meromorphic section in $\mathscr{L}^{-1}$ having simple poles at p^j_-, p^k_-, p^k_+ (respectively at p^j_+, p^k_-, p^k_+.) Since $h^*(s^1_0)$ has the same behaviour as s^2_0 and conversely, there must hold $h^*(s^1_0) = \lambda s^2_0$ and $h^*(s^2_0) = \mu s^1_0$ with $\lambda\mu = 1$. Now a glance at the expression of $\tau^2_{0\alpha}$ shows that $h^*(\tau^2_{0\alpha}) = -\lambda\mu\tau^2_{0\alpha}$ and this concludes the proof. ■

3. Final Remarks

But for $g = 2$, this paper contains results which are essentially of a negative type and namely that the simplest choices one can make in defining universal deformations of susy-curves give non-projected "atlases" on supermoduli spaces, a fact that has been suspected for a long time in the physical literature.

This does not mean, of course, that there are no subtler choices of gravitino zero-modes which may eventually lead to projected atlases. For instance, our proof of Proposition 3.3 heavily relies on a dimensional argument that can be easily bypassed by suitably enlarging the functional dependence of the gravitino choices. Namely, one could allow gravitinos to have simple poles at $N \geqq (g-1)(2g-3)$ points. The nontrivial part is however to study whether the obstruction cocycle can be made trivial, and, in any case, this may possibly remove only the first obstruction to projectedness leaving open the question of higher order obstructions.

[8] We stress that genus $g = 2$ is the only case for which this choice is significant. In fact, for $g \geqq 3$ the variety of curves admitting singular θ-characteristics is a *divisor* in Σ^g_{ev}

Needless to say, a full study of the problem requires a more careful handling of sheaves on the moduli stack of spin curves and their cohomology, including families with singular curves. Anyhow, the fact that the most natural choices one can make do not work has severe consequences for physical applications, where one needs a detailed computational control of the matter. Accordingly, we feel rather than looking for abstruse fine tunings of choices, the strategy of setting up a formalism which is insensitive to splitness obstructions is more promising. In other words one may try to work on the split model of supermoduli, and extract from it the relevant information. The results on super-Mumford forms of Manin and others [BMFS, M, V] seem to support this possible way out.

Appendix. A Glimpse to Moduli Stacks

If the moduli spaces M_g of genus g curves existed as manifolds, one would have been able to use the bases B of universal deformations $C \stackrel{i}{\hookrightarrow} X \to B$ to give local coordinates on M_g by setting [isomorphism class of $\pi^{-1}(b)$]$\rightsquigarrow b \in B$. Unfortunately this is not the case because of the presence of automorphism. Indeed, for any automorphisms $\alpha \in \mathrm{Aut}(C)$, we get another deformation $C \stackrel{i \circ \alpha}{\hookrightarrow} X \to B$ of C and by the universal property there is a base change $\phi(\alpha): B \to B$ making the two deformations isomorphic. So, Aut(C) acts on B; in other words, B overparametrizes the curves "near" C, while the correct local model turns out to be $B/\mathrm{Aut}(C)$. This way of thinking leads to the construction of the "coarse moduli spaces." Incidentally, these turn out to be complex spaces but generically not smooth manifolds.

Another possibility of dealing with the moduli problem is to enlarge the very concept of "manifolds" by first enlarging that of the underlying topological space. This generalization is actually a *stack* and we want to describe here its basic features, referring to the literature for the complete set up [Mu, DM, P].

Let us first work at the topological level. The basic idea of Grothendieck is to forget about points (i.e. isomorphism classes of curves in our case) and to construct a generalized topology by allowing more open sets and "inclusions" than usual. Recall that a topology on a space can be considered as a category, whose objects are open sets and morphisms are inclusions. Intersections and unions correspond to products and sums in the category and of course finite products and any sum exist in the category itself. The basic property one is going to generalize is that, in ordinary topologies, the morphisms between two objects U, V are either empty or consist of a single morphism; namely the inclusion of U in V. One gets in this way a category $\mathcal{M}_g$, which in our case (for $g \geqq 2$) can be described as follows;

1. objects ("open sets") are versal families of smooth curves of genus g $\pi_j: X_j \to B_j$ over smooth bases B_j, with final object X,
2. morphisms ("inclusions") are morphisms of families of curves and projections on the final object X,
3. the category is closed under finite product ("intersections") and generic sums ("unions")
4. a collection of morphisms

$$\begin{array}{ccc} X_\alpha & \xrightarrow{G_\alpha} & X \\ \downarrow & & \downarrow \\ S_\alpha & \xrightarrow{g_\alpha} & S \end{array}$$

is a covering of S if $S = \bigcup_\alpha g_\alpha(S_\alpha)$. A collection of projections of families onto the final object X is a covering if every curve occurs in at least one of the families.

Loosely speaking, one forgets about automorphisms by using versal families (instead of their bases up to automorphisms) as open sets. Notice that this is by no means an ordinary topology, that is π_g is not an ordinary topological space, because morphisms of two objects are not required to unique be when they are defined.

This topology for moduli problems comes together other properties, which are the distinctive features of stacks. Among these, we want to recall that;

5. for any morphism $\phi: B' \to B''$ and any family $X'' \to B''$, there is a unique pull-back family $\phi^* X'' \to B'$ over B'.
6. for any covering $\phi_j: B_j \to B'$ of B', denote by

$$B_{ij} = B_i \times_{B'} B_j =: \{(b_i, b_j) \in B_i \times B_j | \phi_i(b_i) = \phi_j(b_j)\},$$
$$B_{ijk} = B_i \times_{B'} B_j \times_{B'} B_k.$$

Then, there exist some family $X' \to B'$ and isomorphisms $\Phi_{ij}: \phi_i^* X' \to \phi_j^* X'$ over B_{ij}, satisfying an obvious cocycle condition over B_{ijk}.

Notice that for susy-curves the "reduced" moduli stack is the stack Σ_g of smooth spin curves, whose objects are families as in 1) above together with a choice $\mathscr{L}_j$ of a θ-characteristics on each of them [C]. Although this is not quite the datum of an invertible sheaf over the stack, because of the presence of a $\mathbb{Z}_2$-ambiguity in the notion of isomorphism of spin-curves, it induces as well as a class in the Picard group $\mathrm{Pic}(\Sigma_g)$ (see [C] Sect. 7 for more details).

Finally we remark that a naive generalization of the supermanifold structure would lead to defining the moduli "super-stack" at genus g as a "super-conformal" sheaf $\mathscr{A}_{\Sigma_g}$ over Σ_g, i.e.

a) for any family $X_i \to B_i$ of spin-curves, a susy-structure $\mathscr{A}_i$ as in Definition 1.1,
b) for any commutative diagram

$$\begin{array}{ccc} X_i & \longrightarrow & X_j \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{\phi_{ij}} & B_j \end{array}$$

an isomorphism between $\phi_{ij}^{-1} \mathscr{A}_j$ and $\mathscr{A}_k$.

This isomorphism should be "natural" in a precise technical sense [Mu, DM] to call $\mathscr{A}_{\Sigma_g}$ a sheaf on the stack; and here again the $\mathbb{Z}_2$-ambiguity of [LR] plays a subtle rôle. Nonetheless, this $\mathbb{Z}_2$-ambiguity is mild enough not to affect most of the physical computations which involve even powers of the odd generators. Accordingly, a study of the global aspects of this intersecting problem was not even attempted here.

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