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Localization in the Ground-State of the One Dimensional $X-Y$ Model with a Random Transverse Field

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Abstract. We consider the ground-state of the quantum spin model $H = -J\Sigma_{\langle i,j\rangle}[\sigma_x(i)\sigma_x(j) + \sigma_y(i)\sigma_y(j)] + \Sigma_i h_i\sigma_z(i)$ in one-dimension, where $\{h_i,\ i\in\mathbf{Z}\}$ are independent identically distributed random variables. By means of a Jordan–Wigner transformation the model is mapped into a free Fermi gas in the presence of a random external potential. We then use exponential localization of the one particle states to prove exponential decay for the spin–spin correlation functions.

1. Introduction

The Hamiltonian for the quantum $x-y$ model in the presence of a random transverse field is given by

$$H = -J\Sigma_{\langle x,y\rangle}[\sigma_1(x)\sigma_1(y) + \sigma_2(x)\sigma_2(y)] + \Sigma_x h(x)\sigma_3(x),$$

where $\sigma_1, \sigma_2, \sigma_3$, are the usual Pauli spin matrices, $x\in\mathbf{Z}^d$, $\langle x, y\rangle$ denotes a pair of nearest neighbors in $\mathbf{Z}^d$, and the $h(x)$, $x\in\mathbf{Z}^d$, are independent identically distributed random variables whose common probability distribution we will denote by μ.

The quantum $x-y$ model in the presence of a random transverse field was shown by Ma, Halperin and Lee [1] to be relevant when studying the effect of disorder upon superfluidity. It was argued there that at high disorder localization should take place destroying the longrange order of the $x-y$ components of the spin system.

In this paper we consider the ground state of the one-dimensional model and show that, for any non-zero disorder, the elementary excitations of the system are localized and the correlation functions decay exponentially. This is to be compared

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with the polynomial decay obtained by Lieb, Schultz and Mattis [2] for zero transverse field.

As in [2] we "solve" the model by means of a Jordan–Wigner transformation [3] which maps the system into a free Fermi gas in the presence of a random external potential. In one dimensional the one particle states are localized [4–7, 10] for any non-zero disorder and this entails exponential decay for the one-particle Green's function with probability one.

Since the spin operators are non-local functions of the Fermi creation and annihilation operators, the study of the spin–spin correlations is much subtler than determining the ground state energy and the excitation spectrum [2, 8, 9]. However, using Wick's Theorem and convenient resummations we are able to show that the exponential fall-off of the one-particle Green's function yields exponential decay of the spin–spin correlation functions.

Given a positive integer L, we denote by H_L the model Hamiltonian restricted to the box $\Lambda_L = \mathbf{Z} \cap [-L, L]$, with free boundary conditions. The corresponding ground state, which we will show to be unique with probability one, will by denoted by ψ_L. $\langle \cdot \rangle_L = (\psi_L, \cdot\, \psi_L)$ is the ground state expectation. We will also use $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$.

Our result is

Theorem. *Let $d = 1$. Suppose the support of μ is not concentrated on a single-point and $\int |h|^\eta \, d\mu(h) < \infty$ for some $\eta > 0$. Then for any J there exists $m_J > 0$ such that for almost every choice of the random transverse field h we have*

$$\sup_L |\langle \sigma_+(x)\sigma_-(y)\rangle_L| \leqq C_h e^{-m_J|x-y|}$$

for some $C_h < \infty$ and all $x, y \in \mathbf{Z}$.

Notice that we allow μ to have a delta function at zero, e.g., we can have $h(x)$ taking only the values 0 and 1 with nonzero probability.

This paper is organized as follows. In Sect. 2 we describe the model and review the Jordan–Wigner transformation. In Sect. 3 we discuss the properties of the one-particle Green's function and prove a folk theorem showing that exponential localization of states around the Fermi level implies exponential decay of correlation functions. In Sect. 4 we prove the theorem.

2. The Model and its Ground State

At each lattice site $x \in \mathbf{Z}$ we have a two dimensional space $\mathbf{C}^2 = \mathscr{H}_x$ and the Pauli spin matrices

$$\sigma_+(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_-(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $L \in \mathbf{Z}$, $L > 0$, we consider the finite system in $\Lambda_L = \mathbf{Z} \cap [-L, +L]$. In the Hilbert space $\mathscr{H}_L = \bigotimes_{x \in \Lambda_L} \mathscr{H}_x$, the Pauli space operators defined in the usual way satisfy the commutation rules

$$[\sigma_3(x), \sigma_\pm(y)] = \pm 2\sigma_\pm(x)\delta_{xy},$$

$$[\sigma_+(x), \sigma_-(y)] = \sigma_3(x)\delta_{xy}.$$

We shall also make use of the operator

$$n(x) = \frac{1+\sigma_3(x)}{2} = \sigma_+(x)\sigma_-(x).$$

The Hamiltonian in $\wedge_L$ with free boundary conditions is given (up to an energy shift) by:

$$H_L = \Sigma_{x=-L}^{L} h(x)n(x) - J\Sigma_{x=-L}^{L-1}[\sigma_+(x)\sigma_-(x+1) + \sigma_-(x)\sigma_+(x+1)].$$

The external transverse fields $\{h(x), x \in \mathbf{Z}\}$ are independent identically distributed random variables with common distribution $d\mu(h)$, which we will always assume to satisfy the hypotheses of the Theorem.

Following [2] we introduce fermion creation and annihilation operators by the Jordan–Wigner [3] transformation. For $-L < x \leqq L$, let

$$a^*(x) = \exp[i\pi\Sigma_{-L \leqq y < x} n(y)]\sigma_+(x),$$

$$a(x) = \exp[i\pi\Sigma_{-L \leqq y < x} n(y)]\sigma_-(x)$$

and,

$$a^*(-L) = \sigma_+(-L),$$

$$a(-L) = \sigma_-(-L).$$

The new operators satisfy canonical anticommutation relations (CAR):

$$\{a^*(x), a(y)\} = \delta_{xy},$$

$$\{a^*(x), a^*(y)\} = \{a(x), a(y)\} = 0$$

$\forall x, y \in \wedge_L$. Here $\{A, B\} = AB + BA$.

The Hamiltonian H_L can now be rewritten as:

$$H_L = \Sigma_{-L \leqq x \leqq L} h(x)a^*(x)a(x) - J\Sigma_{x=-L}^{L-1}[a^*(x)a(x+1) + a^*(x+1)a(x)]$$

which is the Hamiltonian for a gas of non-interacting spinless fermions in the presence of a random external potential $h(x)$.

We are thus led to consider the one-particle random Schrödinger operator

$$H_L^{(1)} = -J\Delta_L + h$$

in the Hilbert space $l^2(\wedge_L)$, where

$$(\Delta_L\varphi)(x) = \Sigma_{y:|y-x|=1, y \in \wedge_L}\varphi(y)$$

and h is the multiplication operator

$$(h\varphi)(x) = h(x)\varphi(x)$$

with $\varphi \in l^2(\wedge_L)$. The operator $-\Delta_L$, apart from a trivial additive constant, is the usual lattice Laplacian with Dirichlet boundary conditions.

Let now $\varphi_l(x)$ and $\varepsilon_l, l = 1, \ldots, 2L+1$ denote the normalized eigenfunctions and respective eigenvalues of the Schrödinger operator. Notice that without loss of generality $\varphi_l(x)$ can be assumed to be real. We then introduce

$$a_l^* = \Sigma_{x\in\wedge_L}\varphi_l(x)a^*(x), \quad a_l = \Sigma_{x\in\wedge_L}\varphi_l(x)a(x)$$

for $l = 1,\ldots,2L+1$, which also satisfy the CAR. The operator H_L can now be written as:

$$H_L = \Sigma_{l=1}^{2L+1}\varepsilon_l a_l^* a_l$$

so that its eigenvectors are

$$\psi_I = \prod_{l\in I} a_l^* \Omega$$

with eigenvalues

$$E_I = \Sigma_{l\in I}\varepsilon_l$$

where $I \subset \{1,2,\ldots,2L+1\}$ and Ω is the Fock (bare) vacuum:

$$a(x)\Omega = 0, \quad \forall x\in\wedge_L.$$

In particular the ground state is given by

$$\psi_0 \equiv \psi_{I_0} = \prod_{l\in I_0} a_l^*\Omega, \tag{2.1}$$

where $I_0 = \{l: \varepsilon_l < 0\}$, with energy

$$E_0 = \Sigma_{l\in I_0}\varepsilon_l.$$

More precisely, $\psi_{0,L}$ given by (2.1) is the ground state of H_L for all L large enough, with probability one. For uniqueness of the ground state (and (2.1)) is equivalent to $\varepsilon_l \neq 0$ $l = 1,\ldots,2L+1$. It follows from Theorem 2.1 in [10], by an application of the Borel Cantelli lemma, that, with probability one, zero is not in the spectrum of $H_L^{(1)}$ for all L large enough.

More symmetrical expressions are obtained by introducing the usual "particle-hole" operators;

$$b_l = a_l^* \quad \text{if} \quad l\in I_0, \quad b_l = a_l \quad \text{if} \quad l\notin I_0$$

which also satisfy CAR. For the new operators, the ground state is defined by the equations

$$b_l\psi_0 = 0, \quad l = 1,\ldots,2L+1.$$

3. Exponential Decay of The Fermi Two-Point Function

The two point function of the Fermi operators in the ground state can be computed to give

$$\begin{aligned}(\psi_0, a^*(x)a(y)\psi_0) &= \Sigma_{l,m}\varphi_l(x)\varphi_m(y)(\psi_0, a_l^* a_m\psi_0)\\ &= \Sigma_{l\in I_0}\varphi_l(x)\varphi_l(y) = P_{I_0}(x,y),\end{aligned}$$

where P_{I_0} is the operator, in $l^2(\wedge_L)$, projecting into the subspace generated by $\{\varphi_l, l\in I_0\}$, and $P_{I_0}(x,y)$ its kernel.

Let now S denote the support of the single site probability distribution $d\mu(h)$. The spectrum $\sigma(H_L^{(1)})$ of the operator $H_L^{(1)}$ is contained in the set

$$\{S+2J\}\cup\{S-2J\}=\{E\in\mathbf{R}:\operatorname{dist}(E,S)\leqq 2J\},$$

since $-\Delta_L$ is a bounded operator with $\|-\Delta_L\|=2$.

Let us now assume that $|h(x)|>2J+a$ for some $a>0$. In this situation the spectrum $\sigma(H_L^{(1)})$ is contained in the set

$$[-M_L,-a]\cup[a,M_L]$$

for some $M_L<\infty$, so that there is a gap of width $2a$ around zero.

Under these assumptions the operator P_{I_0} is given by the contour integral

$$P_{I_0}=\frac{1}{2\pi i}\oint_C R_L(z)dz, \tag{3.1}$$

where

$$R_L(z)=[(-J\Delta_L+h)-z]^{-1}$$

and C is a contour in the complex plane enclosing $[-M_L,-a]$ while leaving $[a,M_L]$ on the complement of its interior.

Let $0<\delta_0<a/2J$ be chosen, we will take the contour (we write $z=u+iv$):

$$\begin{aligned}C=&\{0+iv;\,-J(1+\delta_0)\leqq v\leqq 2J(1+\delta_0)\}\\&\cup\{u+i2J(1+\delta_u);u\leqq 0\}\cup\{u-i2J(1+\delta_u),u\leqq 0\},\end{aligned}$$

where $\delta_u=\delta_0(1+u^2)$. Notice that if $z+iv\in C$ we have $|z-h(x)|\geqq 2J(1+\delta_u)$.

For $z\in C$, the expansion

$$R_L(z)=(z-h)^{-1}\Sigma_{n=0}^{\infty}[(-J\Delta_L)(z-h)^{-1}]^n \tag{3.2}$$

is convergent in the operator norm. In particular (3.2) implies that for the Green's function

$$G_L(x,y;z)\equiv\langle x|R_L(z)|y\rangle$$

we have the absolute convergent expansion

$$G_L(x,y;z)=\Sigma_{w:x\to y}J^{|w|}\prod_{i=0}^{|w|}\frac{1}{z-h(w_i)}, \tag{3.3}$$

where the summation is taken over all walks w on Λ_L going from x to y, i.e. $w(n)\in\Lambda_L$ for $0\leqq n\leqq|w|$, $w(0)=x$, $w(|w|)=y$, where $|w|$ is any non-negative integer.

From (3.3), estimating the number of walks w with a fixed length $|w|$ by $2^{|w|}$ we have

$$|G_L(x,y;z)|\leqq\left(\frac{1}{2J\delta_u}\right)\left(\frac{1}{1+\delta_u}\right)^{|x-y|+1}. \tag{3.4}$$

From (3.1) and (3.4) it then follows that

$$|P_{I_0}(x,y)|\leqq\frac{1}{2\pi}\oint_C|G_L(x,y;z)|\,|dz|\leqq\frac{l(C)}{2J\delta_0}\left(\frac{1}{1+\delta_0}\right)^{|x-y|}$$

for some constant $l(C) < \infty$. Therefore

$$|P_{I_0}(x, y)| \leqq c_0 e^{-m_0|x-y|} \tag{3.5}$$

with constants $c_0 = l(C)/2J\delta_0$, $m_0 = \log(1 + \delta_0)$ independent of L.

We are now going to drop the assumption of a gap. In this situation we have to make use of the localization results for one dimensional random Schrödinger operators.

If the probability distribution μ is absolutely continuous with a bounded density, it follows [4] that, with probability one, $P_{I_0}(x, y)$ is exponentially decaying for all L large enough. For more general μ as in the Theorem, it follows from the results of [10] by using [5, 6, 7] that the exists $m > 0$, such that with probability one, given $\varepsilon_0 > 0$,

$$\sup_{|\varepsilon| \leqq \varepsilon_0} |G_L(x, y; E + i\varepsilon)| \leqq c(h, \varepsilon_0) e^{-m|x-y|}$$

for all L large enough and all $x, y \in \mathbf{Z}$, with some constant $c(h, \varepsilon_0) < \infty$. We can then use (3.1) with the same contour C, where we take δ_0 given by $m = \log(1 + \delta_0)$ and choose $\varepsilon_0 = 2J(1 + \delta_0)$. As before, we obtain (3.5) with probability one for all L large enough, with a different constant $c_0 = c_0(h)$.

Thus, under the hypotheses of the Theorem, there exists $m_J > 0$ such that, with probability one,

$$|\langle a^*(x) a(y) \rangle_L| \leqq C_h e^{-m_J|x-y|} \tag{3.6}$$

for all L large enough and all $x, y \in \mathbf{Z}$, with some constant $C_h < \infty$.

4. Correlation Functions

In this section we discuss the asymptotic behavior of the correlation function $(\psi_0, \sigma_+(x)\sigma_-(y)\psi_0)$. We first write the non-local expressions for products of spin operators in terms of fermion operators: for $x < y$,

$$\sigma_+(x)\sigma_-(y) = a^*(x) \prod_{x<z<y} \exp\{in(z)\} a(y),$$

$$\sigma_-(x)\sigma_+(y) = -a(x) \prod_{x<z<y} \exp\{in(z)\} a^*(y).$$

If $\psi \in \mathscr{H}_{\Lambda_L}$ is an eigenvector of the "total particle number" operator, i.e.

$$(\Sigma_{x \in \Lambda_L} n(x))\psi = N\psi$$

for some integer $N \geqq 0$, then

$$(\psi, \sigma_+(x)\sigma_-(y)\psi) = e^{i\pi(N-1)} \left(\psi, \prod_{-L \leqq z < x} \exp\{i\pi n(z)\} a^*(x) a(y) \prod_{y < z' \leqq L} \exp\{i\pi n(z')\} \psi \right)$$

so that, for $x < y$

$$(\psi, \sigma_+(x)\sigma_-(y)\psi) = -e^{i\pi(N-1)}(\psi_x^-, a^*(x) a(y) \psi_y^+) \tag{4.1}$$

and

$$(\psi, \sigma_-(x)\sigma_+(y)\psi) = e^{i\pi(N-1)}(\psi_x^-, a(x) a^*(y) \psi_y^+), \tag{4.2}$$

where

$$\psi_x^- = \prod_{-L \leqq z < x} \exp\{i\pi n(z)\}\psi$$

and

$$\psi_y^+ = \prod_{y < z \leqq L} \exp\{i\pi n(z)\}\psi.$$

Next, following [2] we introduce the operators:

$$A(z) = a^*(z) + a(z), \tag{4.3}$$
$$B(x) = a^*(z) - a(z), \tag{4.4}$$

which satisfy the anticommutation relations:

$$\{A(x), A(y)\} = 2\delta_{xy},$$
$$\{B(x), B(y)\} = -2\delta_{xy},$$
$$\{A(x), B(y)\} = 0.$$

Moreover

$$\exp[i\pi n(z)] = A(z)B(z).$$

Using (4.1), (4.2), (4.3) and (4.4) we get

$$\begin{aligned}(\psi, \sigma_+(x)\sigma_-(y)\psi) &= e^{i\pi(N-1)}(\psi_x^-, A(x)a(y)\psi_y^+) \\ &= e^{i\pi(N-1)}(\psi_x^-, A(x)B(y)\psi_y^+) + e^{i\pi(N-1)}(\psi_x^-, A(x)a^*(y)\psi_y^+) \\ &= -e^{i\pi(N-1)}(\psi_x^-, A(x)B(y)\psi_y^+) - (\psi, \sigma_-(x)\sigma_+(y)\psi),\end{aligned}$$

and so, for $x < y$

$$(\psi, [\sigma_+(x)\sigma_-(y) + \sigma_-(x)\sigma_+(y)]\psi)$$
$$= e^{i\pi N}\left(\psi, \prod_{-L \leqq z < x} (A(z)B(z))A(x)B(y) \prod_{y < z' \leqq L} (A(z')B(z'))\psi\right) \tag{4.5}$$

and

$$(\psi, [\sigma_+(x)\sigma_-(y) - \sigma_-(x)\sigma_+(y)]\psi)$$
$$= e^{i\pi N}\left(\psi, \prod_{-L \leqq z < x} (A(z)B(z))B(x)A(y) \prod_{y < z' \leqq L} (A(z')B(z'))\psi\right). \tag{4.6}$$

A crucial fact now is that in both expressions (4.5) and (4.6) all operators involved anti-commute.

If we now take ψ to be the ground state ψ_0 given by (2.1). We are in a position to apply Wick's Theorem. Indeed, if the operators $C_1, \ldots, C_{2n}$ satisfy $\{C_i, C_j\} = 0$, $i \neq j$, then

$$(\psi_0, C_1 C_2 \cdots C_{2n}\psi_0) = \Sigma_P \sigma(P)(\psi_0, C_{i_1} C_{j_1}\psi_0) \cdots (\psi_0, C_{i_n} C_{j_n}\psi_0),$$

where the summation is done over all permutations $P = (i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$ of $\{1, 2, \ldots, 2n\}, \sigma(P)$ being the corresponding signature.

We then notice that

$$(\psi_0, A(x)A(y)\psi_0) = \delta_{xy},$$
$$(\psi_0, B(x)B(y)\psi_0) = -\delta_{xy},$$
$$(\psi_0, B(x)A(y)\psi_0) = -(\psi_0, A(y)B(x)\psi_0) \equiv g_L(x, y),$$

where

$$g_L(x, y) = (\psi_0, a^*(x)a(y)\psi_0) + (\psi_0, a^*(y)a(x)\psi_0)$$
$$= P_{I_0}(x, y) - P_{I_0}^{\perp}(x, y).$$

The operator $P_{I_0} - P_{I_0}^{\perp}$ is unitary in $l^2(\wedge_L)$, since $(P_{I_0} - P_{I_0}^{\perp})^2 = I$, and therefore the kernel $g_L(x, y)$ satisfies

$$\Sigma_{z \in \wedge_L} g_L(x, z) g_L(z, y) = \delta_{xy}. \tag{4.7}$$

Now $P_{I_0}^{\perp}(x, y)$ clearly satisfy the same bound (3.6) with possibly different constants, so that there are constants $c(h)$ and $m > 0$ such that

$$|g_L(x, y)| \leqq c(h) e^{-m|x-y|}. \tag{4.8}$$

To simplify the notation let us rewrite (4.5) in the form

$$(\psi_0, [\sigma_+(x)\sigma_-(y) + \sigma_-(x)\sigma_+(y)]\psi_0)$$
$$= e^{i\pi N_0}(\psi_0, C_{2l}C_{2l-1} \cdots C_1 C_0 D_0 D_1 \cdots D_{2k}\psi_0) \tag{4.9}$$

with

$$C_0 = A(x),\ C_1 = B(x-1),\ C_2 = A(x-1), \ldots, C_{2l} = A(-L),$$

and

$$D_0 = B(y),\ D_1 = A(y+1), D_2 = B(y+1), \ldots, D_{2k} = B(-L).$$

A similar expression holds for (4.6).

Since both the number of C's and D's are odd in every term contributing to (4.9) through Wick's theorem, there is a number $m \geqq 1$ of pairings of C's with D's. We therefore write

$$(\psi_0, C_{2l} \cdots C_0 D_0 \cdots D_{2k}\psi_0)$$
$$= \sum_{m=1}^{\min\{2k+1, 2l+1\}} \sum_{\substack{(i_1, j_1)\\ \vdots \\ (i_m, j_m)}} \sigma_{i_1,\ldots,i_m} \sigma_{j_1,\ldots,j_m} \Big\{ (\psi_0, C_{i_1} D_{j_1}\psi_0) \cdots (\psi_0, C_{i_k} D_{j_k}\psi_0)$$
$$\cdot \Big(\psi_0, \prod_{i \neq i_1,\ldots,i_m} C_i \psi_0\Big)\Big(\psi_0, \prod_{j \neq j_1,\ldots,j_m} \psi_0\Big)\Big\}, \tag{4.10}$$

where $\sigma_{i_1,\ldots,i_m}$ and $\sigma_{j_1,\ldots,j_m}$ are such that

$$C_{2l}C_{2l-1} \cdots C_1 C_0 = \sigma_{i_1,\ldots,i_m} \Big(\prod_{i \neq i_1 \cdots i_m} C_i\Big) C_{im} \cdots C_{i_2} C_{i_1},$$

$$D_0 D_1 \cdots D_{2k} = \sigma_{j_1,\ldots,j_m} D_{j_1} D_{j_2} \cdots D_{j_m} \prod_{j \neq j_1,\ldots,j_m} D_j.$$

Formula (4.10) is proved using Wick's Theorem and resumming all contractions not involving $i_1,\dots,i_m$ and $j_1,\dots,j_m$, again with Wick's Theorem.

To estimate (4.10) we first notice that

$$\left|\left\langle \psi_0, \prod_{i\in I} C_i \psi_0 \right\rangle\right| \leqq 1 \tag{4.11}$$

and

$$\left|\left\langle \psi_0, \prod_{j\in J} D_j \psi_0 \right\rangle\right| \leqq 1 \tag{4.12}$$

for any collection of indeces I and J. This follows from the fact that both the right-hand side of (4.11) and (4.12) can be written as

$$\pm(\psi_0, A(x_1)B(y_1)\cdots A(x_n)B(y_n)\psi_0)$$

which by Wick's Theorem is given by the determinant

$$(\psi_0, A(x_1)B(y_1)\cdots A(x_n)B(y_n)\psi_0) = \Sigma_P \sigma(P) \prod_{i=1}^{n} g(x_i, y_{Pi}),$$

where the summation is taken over all permutations $(1,\dots,n)\to(P1,\dots,Pn)$.

Using Hadamard's Theorem

$$|\det C|^2 \leqq \prod_{i=1}^{n} (\Sigma_{j=1}^{n} C_{ij}^2)$$

and (4.7) we get (4.11) and (4.12).

Therefore (4.10) can be estimated, by

$$\begin{aligned}&|(\psi_0, C_{2l}\cdots C_0 D_0\cdots D_{2k}\psi_0)|\\ &= \sum_{m=1}^{\min(2k+1,2l+1)} \sum_{\substack{0\leqq i_1<i_2<\cdots<i_k\leqq 2l\\ 0\leqq j_1<j_2<\cdots<j_n\leqq 2k}} ((\psi_0, C_{i_1}D_{j_1}\psi_0)\cdots(\psi_0, C_{i_m}D_{j_m}\psi_0)).\end{aligned}$$

This implies

$$\begin{aligned}&|(\psi_0, [\sigma_+(x)\sigma_-(y) + \sigma_-(x)\sigma_+(y)]\psi_0)|\\ &\leqq \sum_{m=1}^{\min\{x+L,L-y\}} \sum_{\substack{-L\leqq z_1<z_2<\cdots<z_m\leqq x\\ y\leqq z_1'<z_2'<\cdots z_m'\leqq L}} \Sigma_P[|g_L(z_1, z_{P1}')|\cdots|g_L(z_m, z_{P_m}')|,\end{aligned}$$

the same estimate holding for $y<x$.

The number of pairs (z_i, z_j') such that $|z_i - z_j'| = |y-x| + R$ for some $R\geqq 0$ equals $R+1$. Using (4.8) we get the simple estimate

$$\begin{aligned}&|(\psi_0, [\sigma_+(x)\sigma_-(y) + \sigma_-(x)\sigma_+(y)]\psi_0)|\\ &\leqq \Sigma_{k=1}^{\infty}[\Sigma_{R=0}^{\infty} c(h) e^{-m|y-x|+R]}(R+1)]^k\\ &= \Sigma_{k=1}^{\infty} e^{-m|y-x|k} c(h)^k (\Sigma_{R=0}^{\infty} e^{-mR}(R+1))^k.\end{aligned}$$

For $|x-y|$ sufficiently large the right-hand side can be estimated yielding,

$$|(\psi_0, [\sigma_+(x)\sigma_-(y) + \sigma_-(x)\sigma_+(y)]\psi_0)| \leqq K(h) e^{-m|x-y|}$$

with

$$K(h) = dc(h)\frac{1}{1 - dc(h)e^{-m|x-y|}},$$

for a given constant $d < \infty$, thus concluding the proof.

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