Commun. Math. Phys. 128, 77–97 (1990)

Communications in
Mathematical
Physics

L^2-Index Formulae for Perturbed Dirac Operators

N. Anghel*
Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

Abstract. The Callias index theorem is generalized from the Euclidean case to certain spin manifolds with warped ends, making use of certain index-preserving deformations.

0. Introduction

Physical considerations led C. Callias to the following open space index theorem (cf. [C]):

Theorem 0.1. *Let Σ be the spinor space over $\mathbb{R}^n$, n odd, and D the Dirac operator on $C^\infty(\mathbb{R}^n, \Sigma \otimes \mathbb{C}^m)$. Let L be the perturbation of D by $\sqrt{-1}\,\mathrm{Id} \otimes \Phi$, where $\Phi \in C^\infty(\mathbb{R}^n, \mathrm{End}(\mathbb{C}^m))$ is Hermitian, asymptotically homogeneous of degree 0, and Φ^2 is positive outside some compact set. Then L is a Fredholm elliptic differential operator, and if U is the unitarization of Φ at infinity,* i.e., *$U = |\Phi|^{-1}\Phi$ outside a compact set, one has*

$$L^2\text{-index}(L) = \frac{1}{2\left(\frac{n-1}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{(n-1)/2} \lim_{R\to\infty} \int_{\mathbb{S}_R^{n-1}} \operatorname{tr} U(dU)^{n-1}. \tag{0.2}$$

In (0.2), $\mathbb{S}_R^{n-1}$ stands for the sphere centered at the origin and of radius R in $\mathbb{R}^n$. As remarked by H. Moscovici, the formula (0.2) can be rewritten as follows:

$$L^2\text{-index}(L) = \mathrm{ch}(V_+)[\mathbb{S}_\infty^{n-1}]. \tag{0.3}$$

The right-hand side of (0.3) represents the evaluation on $\mathbb{S}_\infty^{n-1}$, the sphere at infinity in $\mathbb{R}^n$, of the Chern character of the subbundle V_+ of $\mathbb{C}^m$ over $\mathbb{S}_\infty^{n-1}$ given by $V_+ = \{U = \mathrm{Id}\}$.

While Callias' result and method of proof attracted a lot of interest (see [Bo–S], [S]), there is no direct generalization of (0.1) that we know of. In this paper we attempt a generalization of Theorem 0.1 based on the observation (0.3) to a

* Present address: Department of Mathematics, University of North Texas, Denton, TX 76203, USA

class of perturbed Dirac operators on spinor bundles over odd dimensional spin-manifolds with warped ends.

We work with manifolds M which outside some compact set are geometrically isometric to warped products $(\varepsilon, \infty) \times_f N$, $\varepsilon \in \mathbb{R}$. Here N is some compact manifold and $f \in C^\infty((\varepsilon, \infty))$, $f > 0$. In Callias' case, $\mathbb{R}^n \setminus \{0\} \equiv (0, \infty) \times_f \mathbb{S}_1^{n-1}$, $f(r) = r$, $r \in (0, \infty)$. Our perturbed Dirac operators are operators L of type $L = D + A$, where D is a (generalized) Dirac operator on some Dirac bundle S, as defined by M. Gromov and H. B. Lawson in [G–L], and A is a bundle morphism. Our main result can then be stated as follows:

Theorem 0.4. *Let M be an odd dimensional Riemannian spin manifold with a warped end $W = (\varepsilon, \infty) \times_f N$, $f \in C^\infty((\varepsilon, \infty))$, $f > 0$ and $f(r) \to \infty$ if $r \to \infty$. Let $\mathbf{S} = \Sigma \otimes V$ be the spinor-type bundle over M obtained by twisting the spinor bundle Σ on M with a trivial Hermitian bundle V. If $A \in C^\infty(M, \mathrm{End}(V))$ is a skew-Hermitian endomorphism such that $A|_W$ is independent of the radial direction r, for $r \geqq R$, $R \in (\varepsilon, \infty)$, and $-A^2$ is positive at infinity, then the perturbed Dirac operator $D + A$, where D is the Dirac operator on $\mathbf{S}$, is a Fredholm operator and*

$$L^2\text{-index}\,(D + A) = \int_N \hat{A}(N) \wedge \mathrm{ch}\,(V_R)_+ .$$

Here $\hat{A}(N)$ stands for the total $\hat{A}$-class of N, $(V_R)_+$ is the bundle over $N \equiv \{R\} \times N$, given by $(V_R)_+ = \{v \in V_R \equiv V|_{\{R\} \times N} : 1/\sqrt{-1}(-A^2)^{-1/2} A\, v = v\}$, *and $\mathrm{ch}\,(V_R)_+$ is the Chern character of $(V_R)_+$.*

The proof is based on a series of index-preserving deformations of L. One key deformation rests on an odd-dimensional variant of Gromov–Lawson's relative index theorem [G–L].

1. Perturbed Dirac Operators

The generalized Dirac operators and their perturbations form a broad class of first order elliptic differential operators. Their importance in global geometry and mathematical physics is fundamental. Since they are basic in our approach and in order to establish necessary notations we recall briefly here their main properties. For details and proofs we refer to the beautiful references [G–L] and [L–M].

Let (M, g) be a complete Riemannian manifold of dimension n. Let $Cl(M)$ be the Clifford bundle of algebras induced by the tangent bundle TM and the Riemannian metric g. There is a canonical embedding $TM \to Cl(M)$, and then the Riemannian metric and Levi–Civita connection extended from TM to $Cl(M)$. The connection ∇^{LC} on $Cl(M)$ preserves the metric and acts as a derivation.

A bundle of left modules over the bundle of algebras $Cl(M)$, say $S \to M$, will be called a (generalized) Dirac bundle if S is furnished with a Hermitian metric $\langle , \rangle$ and a metric connection ∇^S such that

The action on S by unit vectors in $TM \subset Cl(M)$ is a pointwise isometry. (1.1)

$$\nabla_e^S(\phi \circ s) = \nabla_e^{\mathrm{LC}}(\phi) \circ s + \phi \circ \nabla_e^S(s) \quad \text{for all} \quad e \in C^\infty(TM),\ \phi \in C^\infty(Cl(M)),\ s \in C^\infty(S). \tag{1.2}$$

The "$\circ$" indicates the Clifford multiplication on S. Equation (1.2) is simply saying that ∇^S acts as a derivation with respect to the Clifford action on S.

There are two categories of Dirac bundles:

a) The fundamental ones, like $Cl(M)$ itself, or the spinor bundle Σ, if M happens to be a spin manifold. To be more specific in this second case, in order that M be a spin manifold, the principal $SO(n)$-bundle $P_{SO}(M)$ of oriented frames of TM must lift to a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}}(M)$ equivariantly with respect to $\mathrm{Spin}(n) \to SO(n)$. The spinor bundle is then the fibre product $\Sigma = S_{\mathrm{Spin}}(M) \times_{\mu} \Delta$, of $P_{\mathrm{Spin}}(M)$ with a n-dimensional spinor space (Δ, μ). Recall that the pair (Δ, μ) is a spinor space if the complex vector space Δ is an irreducible module over the algebra $Cl(\mathbb{R}^n) \otimes \mathbb{C}$ and μ is the unitary representation $\mu: \mathrm{Spin}(n) \to U(\Delta)$ induced by the left multiplication with elements of $\mathrm{Spin}(n) \subset Cl(\mathbb{R}^n) \otimes \mathbb{C}$. When n is odd there are two inequivalent irreducible $Cl(\mathbb{R}^n) \otimes \mathbb{C}$-modules, but they induce the same group representation μ, which is also irreducible [L–M].

Lifting the Riemannian connection on $P_{SO}(M)$ to $P_{\mathrm{Spin}}(M)$ via the Lie algebra isomorphism $so(n) \cong \mathrm{Spin}(n)$, we get the canonical connection ∇^{Σ} of Σ. In fact, any local section $\mathbf{e} = \{e_1, \ldots, e_n\}$ of $P_{SO}(M)$ can be lifted up to $P_{\mathrm{Spin}}(M)$ and then embedded into the P_{Σ}—the principal $SO(N)$-bundle, $N = 2^{(n-1)/2}$, of orthonormal bases in Σ—. Doing so we get a local section $s = \{s_1, \ldots, s_N\}$ in P_{Σ}, called a spinor basis. Then

$$\nabla_e^{\Sigma} s_\alpha = \tfrac{1}{2} \sum_{i<j} g(\nabla_e^{LC} e_i, e_j) e_i e_j \circ s_\alpha \quad e \in C^\infty(TM) \quad \alpha = 1, 2, \ldots, N. \tag{1.3}$$

b) The generated ones, by algebraic operations, out of old ones. For example, if S is a Dirac bundle and $E \to M$ is any complex vector bundle with Hermitian connection ∇^E, then the tensor product $S \otimes E$ is again a Dirac bundle with respect to the tensor product metric and connection

$$\nabla^{S \otimes E} = \nabla^S \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^E. \tag{1.4}$$

Another example, which will be used later, is $\mathrm{End}(S)$, the bundle of endomorphisms of a given Dirac bundle S. Here

$$(\nabla_e^{\mathrm{End}(S)} A)s = \nabla_e^S(As) - A(\nabla_e^S s), \quad e \in C^\infty(TM),\ s \in C^\infty(S), A \in C^\infty(\mathrm{End}(S)). \tag{1.5}$$

We could have considered the Dirac bundle S^* dual to S and then view $\mathrm{End}(S)$ as $S^* \otimes S$. Finally the spinor bundle in a) can be generalized, giving up the irreducibility of Δ. The connection formula (1.3) is preserved. Such a bundle will be referred to as a spinor-type bundle and denoted by $\mathbf{S}$.

Any Dirac bundle S generates a distinguished first order differential operator $D^S = D: C^\infty(S) \to C^\infty(S)$, called the (generalized) Dirac operator. Locally it can be expressed by

$$D \equiv \sum_{i=1}^{n} e_i \circ \nabla_{e_i}^S. \tag{1.6}$$

Equation (1.6) is clearly independent of the local frame $\{e_1, \ldots, e_n\}$. $D^{\mathrm{End}(S)}$ will be denoted shortly by $\mathscr{D}$. D is elliptic. In fact the principal symbol $\sigma_\xi(D) \in \mathrm{End}(S)$, $\xi \in T^*M$, is the Clifford multiplication by the tangent vector metric equivalent to ξ. This can be seen from the following obvious formula:

$$D(fs) = \mathrm{grad}\, f \circ s + f Ds, \quad f \in C^\infty(M),\ s \in C^\infty(S). \tag{1.7}$$

Let $\Omega \subset M$ be any open subset of M. The usual inner product in $C^\infty(\Omega, S)$ will be denoted by $(,)_\Omega$, i.e.,

$$(s_1, s_2)_\Omega = \int_\Omega \langle s_1, s_2 \rangle d\,\mathrm{vol}, \quad s_1, s_2 \in C^\infty(\Omega, S). \tag{1.8}$$

If $\Omega = M$, we will write $(,)$ instead of $(,)_M$. The Dirac operator is then seen to be formally selfadjoint, i.e.,

$$(Ds_1, s_2) = (s_1, Ds_2), \quad \text{for any} \quad s_1, s_2 \in C^\infty(S), \text{ one of them compactly supported} \tag{1.9}$$

Equation (1.9) is a consequence of the following integration by parts formula for Dirac operators

$$(Ds_1, s_2)_\Omega = (s_1, Ds_2)_\Omega + (\mathbf{n} \circ s_1, s_2)_{\partial\Omega}, \quad s_1, s_2 \in C^\infty(S). \tag{1.10}$$

In (1.10) Ω is assumed to be any relatively compact open subset of M with piecewise smooth boundary $\partial\Omega$ and $\mathbf{n}$ denotes the outward unit normal vector field to $\partial\Omega$. The integration on $\partial\Omega$ is carried out with respect to the measure induced from Ω.

We will be interested in the class of perturbed Dirac operators.

Definition 1.11. An operator $L: C^\infty(S) \to C^\infty(S)$, S any Dirac bundle, will be called a *perturbed Dirac operator* if $L = D + A$, where D is the Dirac operator associated to S and A is a 0^{th} order differential operator on S, i.e., $A \in C^\infty(\mathrm{End}(S))$.

The properties (1.7), (1.9), and (1.10) can be adjusted to perturbed Dirac operators. To this end we introduce for any $s_1, s_2 \in C^\infty(S)$, the vector field V_{s_1,s_2} on M defined by

$$\langle V_{s_1,s_2}, X \rangle = \langle X \circ s_1, s_2 \rangle, \quad \text{for any tangent vector field } X, \tag{1.12}$$

and div V_{s_1,s_2}, its divergence.

Proposition 1.13. *The following statements are equivalent*:

(i) *L is a perturbed Dirac operator.*
(ii) *L satisfies (1.7).*
(iii) *The formal adjoint $L^\dagger$ of L is a perturbed Dirac operator.*
(iv) *Pointwise, for $s_1, s_2 \in C^\infty(S)$, $\langle Ls_1, s_2 \rangle = \langle s_1, Ls_2 \rangle + \mathrm{div}\, V_{s_1,s_2}$.*
(v) *$(Ls_1, s_2)_\Omega = (s_1, Ls_2)_\Omega + (\mathbf{n} \circ s_1, s_2)_{\partial\Omega}$, under the assumptions of (1.10).*

Proof. The proof is identical to the one for Dirac operators (see [G–L] and [Ch]). □

We now consider extensions of perturbed Dirac operators to L^2-sections. Let $C_0^\infty(S) \subset C^\infty(S)$ denote the space of C^∞-sections of S with compact support, and let $L^2(S)$ denote the Hilbert space completion of $C_0^\infty(S)$ in the nom (,). The operator $L_0 = L|_{C_0^\infty(S)}$ has two natural extensions to an unbounded operator to $L^2(S)$: a minimal one, $\bar{L}_0$, obtained by taking the L^2-closure of the graph of L_0, and a maximal one obtained by taking the domain to be all $s \in L^2(S)$ such that the distributional image Ls is also in $L^2(S)$. In order words, the maximal extension equals $(L_0^\dagger)^*$, the Hilbert space adjoint of $(L^\dagger)_0$. By definition these two extensions are closed. Clearly the maximal extension contains the minimal one, and any other closed extension lies in between. The remarkable fact is that they coincide for complete manifolds M [G–L].

Our manifolds will be complete and so there will be no ambiguity when talking about closed extensions of various operators.

Let $L = D + A$ be a perturbed Dirac operator on a complete manifold M, as introduced in Definition 1.11. Let $L^{r,2}(S)$ be the r^{th}-Sobolev space, defined as the completion of $C_0^\infty(S)$ in the norm

$$\|s\|_r^2 = \int_M (\langle s, s\rangle + \langle Ds, Ds\rangle + \cdots + \langle D^r s, D^r s\rangle).$$

We will be interested in the C^∞- and L^2-solution spaces of L and $L^\dagger$. Let us denote these kernel spaces by

$$\ker(L) = \{s \in C^\infty(S) \mid Ls = 0\}, \quad L^2\text{-}\ker(L) = \{s \in L^{1,2}(S) \mid Ls = 0\}).$$

They relate nicely if we make the following assumption on A:

Assumption 1.14. a) A is pointwise skew-Hermitian i.e., $A^* = -A$, and A and $\mathscr{D}(A)$ are uniformly bounded on M in the pointwise norm.

b) A commutes with the Clifford multiplication on S, i.e., $A(m)\phi\circ = \phi\circ A(m)$ for any $m \in M$ and any $\phi \in Cl_m(M)$.

Assumption 1.14 is motivated by the following proposition:

Proposition 1.15. *Let L be a perturbed Dirac operator satisfying Assumption 1.14. Then*

a) *The domain of the unique closed extension of L to $L^2(S)$ is $L^{1,2}(S)$ and $L: L^{1,2}(S) \to L^2(S)$ is a bounded operator.*

b) *The commutator $[D, A]$ is a 0^{th} order differential operator,* i.e., $[D, A] \in C^\infty(\mathrm{End}(S))$. *Moreover, $[D, A]$ is seen to be equal to $\mathscr{D}(A)$; recall that $\mathscr{D}$ was the generalized Dirac operator on $C^\infty(\mathrm{End}(S))$.*

c) $L^\dagger L = D^2 + \mathscr{D}(A) - A^2$.

d) $\ker(L) \cap L^2(S) = L^2\text{-}\ker(L) = \ker(L^\dagger L) \cap L^2(S)$.

Proof. a) The domain of the closed extension of L to the L^2-space consists of sections $s \in L^2(S)$ such that $Ls \in L^2(S)$. But if A is uniformly bounded on M, $Ls \in L^2(S)$ if and only if $Ds \in L^2(S)$. M being complete, $s \in L^{1,2}(S)$. The continuity of this extension is obvious.

b) The claim amounts to the linearity of $[D, A]$ with respect to functions $f \in C^\infty(M)$. If $s \in C^\infty(S)$, $[D, A](fs) = DA(fs) - AD(fs) = D(fAs) - A(\mathrm{grad}\, f\circ s + fDs) = \mathrm{grad}\, f\circ As + fDAs - \mathrm{grad}\, f\circ As - fADs = f[D, A]$. Moreover, $[D, A] = \sum_i (e_i\circ\nabla_{e_i}^S A - Ae_i\circ\nabla_{e_i}^S) = \sum_i e_i\circ(\nabla_{e_i}^S A - A\nabla_{e_i}^S) = \sum_i e_i\circ\nabla_{e_i}^{\mathrm{End}(S)}(A) = \mathscr{D}(A)$.

c) $L^\dagger L = (D + A^*)(D + A) = (D - A)(D + A) = D^2 + [D, A] - A^2 = D^2 + \mathscr{D}(A) - A^2$.

d) If $s \in \ker(L) \cap L^2(S)$, then $Ds = -As \in L^2(S)$, i.e., $s \in L^2\text{-}\ker(L)$. The opposite inclusion follows from the regularity property of any elliptic system. Obviously $\ker(L) \subset \ker(L^\dagger L)$. Let $s \in \ker(L^\dagger L) \cap L^2(S)$. M being complete, we can choose compactly supported bump functions $f \in C^\infty(M)$, $0 \leqq f \leqq 1$, $f = 1$ on any prescribed compact subset of M, such that $|\mathrm{grad}\, f|_\infty \overset{\text{def}}{=} \sup_{m \in M} \langle \mathrm{graf}_m f, \mathrm{grad}_m f\rangle$ is as small as we wish [G–L]. Then $\|Ls\|^2 = \lim_f \|fLs\|^2 = \lim_f (L^\dagger(fLs), s) = \lim_f 2(\operatorname{grad} f \circ fLs, s) \leqq \lim_f |\operatorname{grad} f|_\infty (\|fLs\|^2 + \|Ls\|^2) = 0$. $\square$

Remark 1.16. In Proposition 1.15 we can replace L by $L^\dagger$. In view of the skew-symmetry of A, any statement about $L^\dagger$ can be obtained from the corresponding statement for L, simply by replacing A with $-A$.

We now proceed to describe sufficient conditions for $L: L^{1,2}(S) \to L^2(S)$ to be a Fredholm operator. Let us denote by $\mathbf{R}_A$ the Hermitian, uniformly bounded (on M), bundle morphism $\mathscr{D}(A) - A^2$. 1.15c) becomes then

$$L^\dagger L = D^2 + \mathbf{R}_A \tag{1.17}$$

on $C^\infty(S)$. Notice that

$$\|Ls\|^2 = \|Ds\|^2 + (\mathbf{R}_A s, s), \quad s \in L^{1,2}(S). \tag{1.18}$$

Certainly (1.18) holds for $s \in C_0^\infty(S)$, from (1.17). However, any element $s \in L^{1,2}(S)$ is a L^2-limit of some sequence $s_n \in C_0^\infty(S)$ such that $Ds_n \xrightarrow{L^2} Ds$. The claim follows.

Remark 1.19. The following positivity assumption on $\mathbf{R}_A$ ensures that as a bounded operator L has finite dimensional kernel and closed range. Thus L is a semi-Fredholm operator. The proof of this fact can be fashioned out after the similar one in [G–L]. We will not repeat the details here.

Assumption 1.20. There exists a compact subset $K \subset\subset M$ and a constant $c > 0$ such that $\mathbf{R}_A \geqq c\,\mathrm{Id}$ on $M - K$, i.e.,

$$\langle \mathbf{R}_A v, v \rangle_m \geqq c \langle v, v \rangle_m, \quad m \in M - K, \quad v \in S_m.$$

Assumption (1.20) will be referred to as the positivity at infinity of $\mathbf{R}_A$.

Proposition 1.21. a) *If $\mathbf{R}_{\pm A}$ is positive at infinity, then L is a Fredholm operator.* b) *If $-A^2$ is positive at infinity (assumption 1.20 for $-A^2$ instead of $\mathbf{R}_A$) and $\mathscr{D}(A)(m) \to 0$ as $m \to \infty$, then L is a Fredholm operator.*

Proof. a) L is a semi-Fredholm operator, by Remark 1.19. If $\mathbf{R}_{-A}$ is also positive at infinity, L^2-ker$(L^\dagger)$ is finite dimensional as well, thus L is a Fredholm operator. b) The hypotheses in b) obviously suffice for $\mathbf{R}_{\pm A}$ to be positive at infinity, since $\mathbf{R}_{\pm A} = \pm \mathscr{D}(A) - A^2$. $\square$

The size of the L^2-solution spaces for an operator is usually difficult to compute. The correct object to look to is their analytic or Fredholm index, when interested in solution spaces.

Definition 1.22. Let L be a Fredholm operator of the type described in Proposition 1.21. We define the *analytic* or *Fredholm* L^2-index of L by the formula

$$\begin{aligned} L^2\text{-index}(L) &= \dim L^2\text{-ker}(L) - \dim L^2\text{-coker}(L) \\ &= \dim L^2\text{-ker}(L) - \dim L^2\text{-ker}(L^\dagger). \end{aligned}$$

Our goal in the next sections well be to evaluate this index for particular classes of perturbed Dirac operators.

Remark 1.24. So far, we viewed $L^{\dagger}L$ and $LL^{\dagger}$ as acting on $C^{\infty}(S)$ only. Just as with L, we can consider their closed extensions to the L^2-space. For operators satisfying the assumption 1.14, it is an exercise to see that they admit a unique closed extension to $L^2(S)$, with domain $L^{2,2}(S)$. Alternatively, if L^* is the Hilbert space adjoint of $L|_{C_0^{\infty}(S)}$, these unique extensions equal L^*L, respectively LL^*. It is a fact [G] that L is a Fredholm operator if and only if L^*L is so.

2. Index-Preserving Deformations

Using deformation theory, in this section we move toward evaluating the L^2-index of a perturbed Dirac operator $L = D + A$. We "unitarize" A outside a compact set and then "diagonalize" L with respect to the bundle splitting induced by the unitarization U of A, without changing the index.

Definition 2.1. The problem of evaluating the L^2-index of a perturbed Dirac operator $L = D + A$, A being subject to Assumption 1.14 and to the hypotheses of Proposition 1.21 will be more simply referred to as a *Callias-type index problem.* The operator L itself will be called a *Callias-type operator.*

The basic result in index preserving deformation theory is the following [G]:

(2.2) *The Homotopy Invariance.* If $T_t, 0 \leqq t \leqq 1$, is a continuous homotopy of Fredholm operators between two Hilbert spaces, then

$$\operatorname{index}(T_0) = \operatorname{index}(T_1).$$

(2.3) *The Invariance Under Compact Deformations.* If T is a Fredholm operator and C is a T-compact operator, i.e., C is compact in the graph norm $\|\cdot\| + \|T\cdot\|$, then $T + C$ is a Fredholm operator and

$$\operatorname{index}(T) = \operatorname{index}(T + C).$$

Notice that for our operator L, the L-compactness is equivalent to the D-compactness.

Remark 2.4. The Callias type index problem is uninteresting if the manifold M is a) compact, or b) even dimensional.

a) For M compact, D itself is a Fredholm operator and by Rellich's Lemma, A is D-compact. Thus $\operatorname{index}(L) = \operatorname{index}(D) = 0$, since D is selfadjoint.
b) For M even dimensional, the "volume form" on M given by $e = (\sqrt{-1})^{n/2} e_1 \cdots e_n \in C^{\infty}(Cl(M))$ anticommutes with D and commutes with A. As a result $Le = -eL^{\dagger}$. Thus e is an isometry from L^2-$\ker(L)$ to L^2-$\ker(L^{\dagger})$, which implies L^2-$\operatorname{index}(L) = 0$.

The next proposition shows that only the behavior of A at infinity matters for a Callias-type index problem.

Proposition 2.5. *Let L be a Callias-type operator and $\Psi \in C_0^{\infty}(\operatorname{End}(S))$ such that $\Psi^* = -\Psi$ and Ψ commutes with the Clifford multiplication. Then $L + \Psi$ is a Callias-type operator and*

$$L^2\text{-}\operatorname{index}(L) = L^2\text{-}\operatorname{index}(L + \Psi).$$

Proof. It is clear that if A satisfies the requirements of Definition 2.1, so does $A + \Psi$. In order to prove the index invariance we could use either (2.2) or (2.3). For instance, $L_t = L + t\Psi, 0 \leqq t \leqq 1$, is obviously seen to be a continuous homotopy of Fredholm operators linking L and $L + \Psi$. $\square$

We can simplify A at infinity by unitarizing it. To this end let us consider the polar decomposition of the skew-Hermitian endomorphism A outside the compat set K, where $-A^2$ is strictly positive. Thus on $M - K, A = PU$, with $P, U \in C^\infty(M - K, S), P$ uniformly positive, and U unitary. Clearly, $P = (-A^2)^{1/2}$ and $U = (-A^2)^{-1/2}A$.

Proposition 2.6. *Let $L = D + A$ be a Callias-type operator and $\chi \in C^\infty(M)$ a bump function which vanishes on K and is identically 1 outside some relatively compact set Ω. Then $D + \chi U$ is a Callias-type operator and*

$$L^2\text{-index}\,(L) = L^2\text{-index}\,(D + \chi U)$$

Proof. χU must satisfy the requirements of Definition 2.1. All of them, except one, are immediate. In particular χU commutes with the Clifford multiplication because any power of A does so, and $-(\chi U)^2 \equiv \mathrm{Id}$, outside some compact set. Not so obvious is the fact that $\mathcal{D}(\chi U)(m) \to 0$ as $m \to \infty$, or equivalently $\mathcal{D}(U)(m) \to 0$ as $m \to \infty$. It is clear that $\mathcal{D}$ acts as a derivation on the subspace of bundle morphisms of S commuting with the Clifford multiplication, i.e., for $V, W \in C^\infty(\mathrm{End}(S))$, commuting with the Clifford multiplication, $\mathcal{D}(VW) = \mathcal{D}(V)W + V\mathcal{D}(W)$. Therefore, at infinity, $\mathcal{D}(U) = \mathcal{D}((-A^2)^{-1/2})A + A\mathcal{D}((-A^2)^{-1/2})$, and thus $\mathcal{D}(U)$ has the required decay property if $\mathcal{D}((-A^2)^{-1/2})$ does so. Now a variant of Cauchy's integral formula [K] gives $(-A^2)^{-1/2} = (1/\pi) \int_0^\infty \lambda^{-1/2}(-A^2 + \lambda)^{-1} d\lambda$ and so $\mathcal{D}((-A^2)^{-1/2}) = (1/\pi) \int_0^\infty \lambda^{-1/2} \mathcal{D}((-A^2 + \lambda)^{-1}) d\lambda$. Using again the derivation property we get $\mathcal{D}((-A^2 + \lambda)^{-1}) = (-A^2 + \lambda)^{-1} \mathcal{D}(-A^2 + \lambda)(-A^2 + \lambda)^{-1}$. Thus pointwise, outside some compact set where $\| -A^2 \| \geqq c$, we have $\| \mathcal{D}((-A^2 + \lambda)^{-1}) \| \leqq$ const. $\|(-A^2 + \lambda)^{-1}\|^2 \times \|\mathcal{D}(A)\| \leqq$ const. $(c + \lambda)^{-2} \|\mathcal{D}(A)\|$. Finally $\|\mathcal{D}((-A^2)^{-1/2})\| \leqq$ const. $\int_0^\infty \lambda^{-1/2}(c + \lambda)^{-2} d\lambda \times \|\mathcal{D}(A)\|$ shows that $\mathcal{D}((-A^2)^{-1/2})$ vanishes at infinity. Now we consider the continuous homotopy $A_t = tA + (1 - t)\chi U$, $0 \leqq t \leqq 1$, in $C^\infty(\mathrm{End}(S))$. $\mathcal{D}(A_t)$ clearly vanishes at infinity. In order to prove positivity for $-A_t^2$, notice that

$$-A^2(m) \geqq c\,\mathrm{Id} \Leftrightarrow \left\{ \begin{array}{c} \lambda^2 \geqq c \quad \text{for any} \\ \text{(pure imaginary) eigenvalue} \\ \sqrt{-1}\lambda \text{ of } A(m) \end{array} \right\}.$$

Then $A_t(m)$ has eigenvalues $\sqrt{-1}t\lambda + \sqrt{-1}(1 - t)(\lambda/|\lambda|)$, so the elementary inequality

$$\left(t\lambda + (1 - t)\frac{\lambda}{|\lambda|} \right)^2 \geqq \min(1, c), \quad 0 \leqq t \leqq 1$$

proves that $-A_t^2 \geqq \min(1, c)$, outside some compact set. Now (2.2) applied to the homotopy $L_t = D + A_t$ yields the desired result. $\square$

The Dirac bundle $S|_{M-K}$ splits into a direct sum $S_+ \oplus S_-$ of Dirac subbundles over $M - K$. $L|_{M-K}$ can then be written in a 2×2 matrix form whose off-diagonal terms are bundle morphisms. Another deformation will wipe out the off diagonal terms and so facilitate, in the next section, a separation of variables on manifolds with warped ends.

Lemma 2.7. *Let $U \equiv (-A^2)^{-1/2} A$ on $M - K$, with A as in Definition 2.1. Then the bundles S_+ over $M - K$,*

$$(S_\pm)_m \overset{\text{def}}{=} \{v \in S_m \mid Uv = \pm\sqrt{-1}v\}, \quad m \in M - K$$

are in a canonical way Dirac bundles.

Proof. Clearly $S_\pm = (\mathrm{Id} \mp \sqrt{-1}U)S|_{M-K}$ and the splitting $S|_{M-K} = S_+ \oplus S_-$ is orthogonal. The functions $m \to \operatorname{rank}((\mathrm{Id} \mp \sqrt{-1}U)(m)$, $m \in M - K$ are upper semi-continuous and $\operatorname{rank}((\mathrm{Id} + \sqrt{-1}U)(m) + \operatorname{rank}((\mathrm{Id} - \sqrt{-1}U)(m) = \dim S_m$ (constant). Thus these functions must be locally constant on $M - K$ and this ensures (cf. [A]) that $S_\pm$ are bundles. Since $S_\pm = (\mathrm{Id} \mp \sqrt{-1}U)S$, and U commutes with the Clifford multiplication, $S_\pm$ are invariant $Cl(M)$-modules. They inherit a Hermitian scalar product from S and, as already noticed, $S_+ \perp S_-$. The condition 1.1 in the definition of a Dirac bundles is trivially satisfied. We endow $S_\pm$ with the connection

$$\nabla_e^\pm s_\pm = \mathrm{proj}_{S_\pm}(\nabla_e^S s_\pm) = \tfrac{1}{2}(\mathrm{Id} \mp \sqrt{-1}U)(\nabla_e^S s_\pm),$$
$$e \in C^\infty(T(M-K)),\ s_\pm \in C^\infty(M - K, S_\pm).$$

The check that $\nabla^\pm$ satisfies Eq. 1.2 is immediate. It involves the basic properties of ∇^S and U. $\square$

Let us denote now by $D_\pm$ the corresponding Dirac operators on $(M - K, S_\pm)$.

Proposition 2.8. *Relative to the orthogonal decomposition $S|_{M-K} = S_+ \oplus S_-$ we have the following matrix representations associated with $L = D + \chi U$:*

$$L|_{M-\Omega} = \begin{pmatrix} D_+ + \sqrt{-1} & \dfrac{\sqrt{-1}}{2}\mathscr{D}(U)|_{S_-} \\ -\dfrac{\sqrt{-1}}{2}\mathscr{D}(U)|_{S_+} & D_- - \sqrt{-1} \end{pmatrix},$$

$$L^\dagger|_{M-\Omega} = \begin{pmatrix} D_+ - \sqrt{-1} & \dfrac{\sqrt{-1}}{2}\mathscr{D}(U)|_{S_-} \\ -\dfrac{\sqrt{-1}}{2}\mathscr{D}(U)|_{S_+} & D_- + \sqrt{-1} \end{pmatrix}.$$

Proof. Let $s_\pm \in C^\infty(M-\Omega, S_\pm)$. Locally,

$$\begin{aligned} Ls_\pm &= \sum_i e_i \nabla^S_{e_i} s_\pm + Us_\pm = \sum_i \tfrac{1}{2}(\mathrm{Id} \mp \sqrt{-1}U) e_i \nabla^S_{e_i} s_\pm \\ &\quad + \sum_i \tfrac{1}{2}(\mathrm{Id} \pm \sqrt{-1}U) e_i \nabla^S_{e_i} s_\pm \pm \sqrt{-1} s_\pm \\ &= \left(\sum_i e_i \tfrac{1}{2}(\mathrm{Id} \mp \sqrt{-1}U) \nabla^S_{e_i} s_\pm \pm \sqrt{-1} s_\pm \right) \\ &\quad + \sum_i e_i \tfrac{1}{2}(\mathrm{Id} \pm \sqrt{-1}U) \nabla^S_{e_i} s_\pm + (D_\pm \pm \sqrt{-1}) s_\pm \mp (\sqrt{-1}/2)[D, U] s_\pm . \quad \square \end{aligned}$$

Now we are ready to prove our main deformation result:

Theorem 2.9. *Let $L = D + \chi U, K$, and Ω be as in Proposition 2.6. There exists a Fredholm perturbed Dirac operator $T: L^{1,2}(S) \to L^2(S)$ such that*

$$T|_{M-\Omega} = \begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-1} \end{pmatrix} \quad T^\dagger|_{M-\Omega} = \begin{pmatrix} D_+ - \sqrt{-1} & 0 \\ 0 & D_- + \sqrt{-1} \end{pmatrix}$$

and L^2-index $(L) = L^2$-index (T).

Proof. Clearly $\Psi \equiv \chi \mathscr{D}(U) = \begin{pmatrix} 0 & (\sqrt{-1}/2)\mathscr{D}(U)|_{S_-} \\ -(\sqrt{-1}/2)\mathscr{D}(U)|_{S_+} & 0 \end{pmatrix}$ belongs to C^∞ (M, SymmEnd(S)). Set $T \overset{\text{def}}{=} L - \chi\mathscr{D}(U)$. T is obviously a perturbed Dirac operator and by Proposition 2.8, T and $T^\dagger$ have the stated matrix representations on $M - \Omega$. We can use a slight variant of Proposition 1.21 to prove that T is a Fredholm operator. In fact the arguments in [G–L], Theorem 3.2 ff., carry through if we can write $T^\dagger T$ as a sum of a positive operator and a bundle morphism which is positive at infinity. In our case $T^\dagger T = (D - \chi\mathscr{D}(U))^2 + ([D - \chi\mathscr{D}(U), \chi U] + \chi^2)$. Now $(D - \chi\mathscr{D}(U))^2$ is a positive operator and $[D - \chi\mathscr{D}(U), \chi U] + \chi^2$ is positive at infinity, since on $M - \Omega$, $[D - \chi\mathscr{D}(U), \chi U] + \chi^2 = \mathscr{D}(U) - [\mathscr{D}(U), U] + \mathrm{Id}$, and $\mathscr{D}(U)$ vanishes at infinity. The two indices are seen to be equal, by applying (2.2) to the homotopy $T_t = L - t\chi\mathscr{D}(U)$, $0 \leqq t \leqq 1$. $\square$

We conclude this section with the following corollary:

Corollary 2.10. *Let us assume that globally on M we have $U^2 = -\mathrm{Id}$ and $\mathscr{D}(U)(m) \to 0$ as $m \to \infty$. Then L^2-index $(L) = L^2$-index $(D + U) = 0$.*

Proof. By Theorem 2.9, L^2-index $(L) = L^2$-index $\begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{1} \end{pmatrix}$. As self-adjoint operators $D_\pm$ admit only real eigenvalues. Thus 0 cannot be an eigenvalue for $\begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-1} \end{pmatrix}$, i.e., L^2-ker $\begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-} \end{pmatrix} = 0$. $\square$

3. Separation of Variables on Warped Ends

In this section we will study Dirac bundles and Dirac operators defined on a special class of manifolds: the warped products of type $(\varepsilon, \infty) \times_f N$. Using parallel transport along the radial geodesics, any Dirac bundle can be viewed as a one-parameter family of Dirac bundles over N. Accordingly, for any Dirac operator the variables can be separated; this is particularly insightful for spinor-type bundles and Clifford bundles.

Definition 3.1. Let (N, ds^2) be a compact Riemannian manifold and $f \in C^\infty((\varepsilon, \infty))$, $\varepsilon \in \mathbb{R}$, a positive function. The product $(\varepsilon, \infty) \times N$, equipped with the Riemannian metric $dr^2 + f^2(r) ds^2$, r being the coordinate in (ε, ∞), will be called a *warped product* and denoted by $W = (\varepsilon, \infty) \times_f N$.

A basic example of a warped product of this type is ($\mathbb{R}^n - \{0\}$, Euclidean metric) $\equiv (0, \infty) \times_f \mathbb{S}_1^{n-1}$; here $f(r) = r$, and $\mathbb{S}_1^{n-1}$ is the standard $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

Let S be any Dirac bundle over a warped product W. Fix an $R \in (\varepsilon, \infty)$ and denote by S_R the restriction of S to $\{R\} \times N$. Changing the metric on N from ds^2 to $f^2(R) ds^2$, we can assume that $f(R) = 1$ and then identify metrically N and $\{R\} \times N$. The bundle $S_R \to N$ inherits a canonical structure of Dirac bundle, under a mild restriction on the curvature tensor on S, which we describe next.

For any section $s \in C^\infty(N, S_R)$, define $s^\sim$ in $C^\infty(W, S)$, as being the parallel transport of s along the radial geodesics i.e., $s^\sim$ is the unique section in $C^\infty(W, S)$, subject to

$$s^\sim|_{\{R\} \times N} = s, \quad \nabla^S_{\partial/\partial r} s^\sim = 0.$$

Note that for the bundle TW, we have $TN \subset (TW)_R$ and if $e \in C^\infty(N, TN)$, then $e^\sim = E/f$, where E is the standard lift of e from TN to TW. Also $(\partial/\partial r)^\sim|_N = \partial/\partial r$. These are immediate consequences of the properties of the Levi–Cività connection on warped products [O]. Another important fact is: $(e \circ s)^\sim = e^\sim \circ s^\sim$.

We assume now that there exists a real valued function $g \in C^\infty((\varepsilon, \infty))$, $g(R) = 1$ such that the curvature tensor $\mathcal{R}$ of S satisfies the equation:

$$\mathcal{R}_{\partial/\partial r, e^\sim} s^\sim = g(r)(\mathcal{R}_{\partial/\partial r, e} s)^\sim, \quad e \in C^\infty(TN),\ s \in C^\infty(N, S_R). \tag{3.2}$$

Proposition 3.3. *If the above assumption (3.2) holds, then $g(r)$ must satisfy the equation*

$$g(r) = \frac{f''(r)}{f(r) f''(R)}. \tag{3.4}$$

Remark 3.5. Equation (3.4) makes sense only if $f''(R) \neq 0$; this is unnecessarily restrictive and artificial. In fact, for the *metric cone* ($f(r) = r$), it never holds. As we shall see, however, in many examples of interest (in particular the geometric Dirac bundles) we have $\mathcal{R}_{\partial/\partial r, e} s = f''(R) \mathcal{A}_e$, where $\mathcal{A}_e$ is another tensor having the same properties as $\mathcal{R}_{\partial/\partial r, e}$ (i.e., it is antisymmetric, a derivation with respect to the Clifford multiplication, etc.). Thus the assumption (3.2) shall be broadened to accommodate a formal cancellation of $f''(R)$, but we prefer it that way for esthetic reasons.

Proof of Proposition 3.3. Assume (3.2), and $f''(R) \neq 0$. We will evaluate $\mathcal{R}_{\partial/\partial r, e^\sim}(e^\sim \circ s^\sim)$ in two different ways using the derivation property of the curvature tensor $\mathcal{R}$. On one hand

$$\mathcal{R}_{\partial/\partial r, e^\sim}(e^\sim \circ s^\sim) = \mathcal{R}^W_{\partial/\partial r, e^\sim}(e^\sim) \circ s^\sim + e^\sim \circ \mathcal{R}_{\partial/\partial r, e^\sim} s^\sim = -\frac{f''(r)}{f(r)} \mathbf{n} \circ s^\sim + e^\sim \circ \mathcal{R}_{\partial/\partial r, e^\sim} s^\sim,$$

where $\mathbf{n} = \partial/\partial r$. On the other hand

$$\mathscr{R}_{\partial/\partial r, e^{\sim}}(e^{\sim} \circ s^{\sim}) = \mathscr{R}_{\partial/\partial r, e^{\sim}}(e \circ s)^{\sim} = g(r)(\mathscr{R}_{\partial/\partial r, e}(e \circ s))^{\sim}$$

$$= g(r)\{\mathscr{R}^{W}_{\partial/\partial r, e}(e) \circ s + e \circ \mathscr{R}_{\partial/\partial r, e}(s)\}^{\sim} = -g(r)\frac{f''(R)}{f(R)}\mathbf{n} \circ s^{\sim} + e^{\sim} \circ \mathscr{R}_{\partial/\partial r, e^{\sim}} s^{\sim}.$$

Comparing the two results we get the desired identity (3.4). $\square$

Example 3.6. $S = Cl(W)$, the Clifford bundle of exterior algebras of W.

For $Cl(W)$ we have [L–M], $\mathscr{R}_{\partial/\partial r, e^{\sim}} s^{\sim} = \frac{1}{2}(f''(r))/f(r)\,\mathrm{ad}_{\mathbf{n} \circ e^{\sim}} s^{\sim}$, where $\mathrm{ad}_{\mathbf{n} \circ e^{\sim}} s^{\sim} = \mathbf{n} \circ e^{\sim} \circ s^{\sim} - s^{\sim} \circ \mathbf{n} \circ e^{\sim}$. Thus $Cl(W)$ satisfies the assumption (3.2) since

$$R_{\partial/\partial r, e^{\sim}} s^{\sim} = \frac{1}{2}\frac{f''(r)}{f(r)}(\mathrm{ad}_{\mathbf{n} \circ e} s)^{\sim}.$$

Example 3.7. $S = Spin(W)$, the spinor bundle of W. Again, for $\mathrm{Spin}(W)$ the assumption (3.2) holds since

$$\mathscr{R}_{\partial/\partial r, e^{\sim}} s^{\sim} = \frac{1}{2}\frac{f''(r)}{f(r)}\mathbf{n} \circ e^{\sim} \circ s^{\sim}. \tag{3.8}$$

Formula (3.8) remains true for any spinor-type bundle S.

Theorem 3.9. *Let W, S, and S_R be as above and assume that (3.2) holds. Then $S_R \to N$ admits a canonical structure of Dirac bundle, and the associated Dirac operator on S admits the following separation of variables*

$$Ds^{\sim} = \frac{(\not{\partial}_R s)^{\sim}}{f} + \frac{f'}{f}(\Xi_R s)^{\sim}, \quad s \in C^{\infty}(N, S_R), \tag{3.10}$$

where $\not{\partial}_R$ is the Dirac operator on S_R and Ξ_R is an element of $C^{\infty}(N, \mathrm{End}(S_R))$. If $f''(R) \neq 0$, then

$$\Xi = \frac{1}{f''(R)} \sum_{i=1}^{\dim N} e_i \circ \mathscr{R}_{\partial/\partial r, e_i}. \tag{3.11}$$

Here $\{e_i\}_i$ is a local orthonormal basis in TN.

Proof. $S_R \to N$ will be given the induced structure of $Cl(N)$-module, as $Cl(N) \subset Cl(W)$ and $S_R \subset S$. Thus (1.1) holds trivially. The key point in the proof is the definition of the connection on S_R. A careful analysis of $\nabla^S_{\partial/\partial r} \nabla^S_{e} s^{\sim}$ suggests how such a connection comes around and why assumption (3.2) is necessary.

$$\nabla^S_{\partial/\partial r} \nabla^S_{e^{\sim}} s^{\sim} = \nabla^S_{\partial/\partial r} \nabla^S_{e^{\sim}} s^{\sim} - \nabla^S_{e^{\sim}} \nabla^S_{\partial/\partial r} s^{\sim} - \nabla^S_{[\partial/\partial r, e]} s^{\sim} + \nabla^S_{[\partial/\partial r, e^{\sim}]} s^{\sim} = \mathscr{R}_{\partial/\partial r, e^{\sim}} s^{\sim}$$

$$- \frac{f'}{f}\nabla^S_{e^{\sim}} s^{\sim} \Rightarrow f\nabla^S_{\partial/\partial r} \nabla^S_{e^{\sim}} s^{\sim} + f'\nabla^S_{e^{\sim}} s^{\sim} = f\mathscr{R}_{\partial/\partial r, e^{\sim}} s^{\sim}$$

$$\Rightarrow \nabla^S_{\partial/\partial r}(f\nabla^S_{e^{\sim}} s^{\sim}) = f\mathscr{R}_{\partial/\partial r, e^{\sim}} s^{\sim}.$$

Assume for convenience that $f''(R) \neq 0$. Then from (3.2) and Proposition 3.3 we get $\nabla^S_{\partial/\partial r}(f\nabla^S_{e^{\sim}} s^{\sim}) = (f''/f''(R))(\mathscr{R}_{\partial/\partial r, e} s)^{\sim}$ or equivalently

$$\nabla^S_{\partial/\partial r}\left(f\nabla^S_{e^{\sim}} s^{\sim} - \frac{f'}{f''(R)}(\mathscr{R}_{\partial/\partial r, e} s)^{\sim}\right) = 0. \tag{3.12}$$

Define the connection ∇^R on S_R by

$$\nabla^R_e s \overset{\text{def}}{=} \left\{ f\nabla^S_{e^\sim} s^\sim - \frac{f'}{f''(R)}(\mathscr{R}_{\partial/\partial r,e})^\sim \right\}|_N = \nabla^S_{e^\sim} s^\sim|_N - \frac{f'(R)}{f''(R)}\mathscr{R}_{\partial/\partial r,e}s. \tag{3.13}$$

Equation (3.12) merely says that

$$(\nabla^R_e s)^\sim = f\nabla^S_{e^\sim} s^\sim - \frac{f'}{f''(R)}(\mathscr{R}_{\partial/\partial r,e}s)^\sim.$$

The verification that ∇^R is a metric connection compatible with the Clifford action is a simple exercise. We will check only property (1.2) i.e.,

$$\nabla^R_e(\phi\circ s) = (\nabla^N_e\phi)\circ s + \phi\circ\nabla^R_e s, \quad \phi\in C^\infty(Cl(N)), \quad s\in C^\infty(N,S_R).$$

Indeed $(\nabla^R_e(\phi\circ s))^\sim = f\nabla^S_{e^\sim}(\phi\circ s)^\sim - (f'/f''(R))(\mathscr{R}_{\partial/\partial r,e}(\phi\circ s))^\sim = f(\nabla^W_{e^\sim}\phi^\sim)\circ s^\sim + f\phi^\sim\circ\nabla^S_{e^\sim}s^\sim - (f'/f''(R))(\mathscr{R}^N_{\partial/\partial r,e}(\phi))^\sim\circ s^\sim - (f'/f''(R))\phi^\sim\circ(\mathscr{R}_{\partial/\partial r,e}(s))^\sim = (f\nabla^W_{e^\sim}\phi^\sim - (f'/f''(R))(\mathscr{R}^N_{\partial/\partial r,e}(\phi))^\sim)\circ s^\sim + \phi^\sim\circ(\nabla^R_e s)^\sim$.

It remains to be checked that

$$f\nabla^W_{e^\sim}\phi^\sim - \frac{f'}{f''(R)}(\mathscr{R}^N_{\partial/\partial r,e}(\phi))^\sim = (\nabla^N_e\phi)^\sim. \tag{3.14}$$

Since $\nabla^W, \mathscr{R}^N$, and ∇^N are derivations with respect to the Clifford multiplication, it suffices to check (3.14) on elements $\phi\in TN$ only. There, it follows because [O],

$$\nabla^W_{e^\sim}\phi^\sim = -\frac{\langle e^\sim,\phi^\sim\rangle}{f}f'\frac{\partial}{\partial r} + \frac{(\nabla^N_e\phi)^\sim}{f}, \quad \mathscr{R}^N_{\partial/\partial r,e}(\phi) = -\frac{f'(R)}{f''(R)}\langle e,\phi\rangle\frac{\partial}{\partial r}.$$

Now we prove the separation of variables formula (3.10). For s in $C^\infty(N,S_R)$

$$Ds^\sim = \frac{\partial}{\partial r}\circ\nabla^S_{\partial/\partial r}s^\sim + \sum_{i=1}^{\dim N} e_i^\sim\circ\nabla^S_{e_i}s^\sim = \frac{1}{f}\sum_{i=1}^{\dim N} e_i^\sim\circ(\nabla^R_{e_i}s)^\sim + \frac{f'}{ff''(R)}$$
$$\cdot\sum_{i=1}^{\dim N} e_i^\sim\circ(\mathscr{R}_{\partial/\partial r,e_i}s)^\sim = \frac{(\not{\partial}_R s)^\sim}{f} + \frac{f'}{f}(\Xi_R s)^\sim, \quad \text{where} \quad \Xi_R s = \frac{1}{f''(R)}$$
$$\cdot\sum_{i=1}^{\dim N} e_i\circ\mathscr{R}_{\partial/\partial r,e_i}s \quad \text{on} \quad C^\infty(N,S_R). \quad \square$$

It is interesting to trace down the Dirac bundle on (N,S_R) and the corresponding endomorphism Ξ_R, when $S=Cl(W)$ or $S=\mathrm{Spin}(W)$.

(3.15) a) $S=\mathrm{Spin}(W)$. This situation is worked out by Chou [Ch], when W is even dimensional. If $\dim W$ is odd, then $S_R\equiv\mathrm{Spin}(N)$, $\not{\partial}_R$ is the classical Dirac operator on N, and from (3.8) and (3.11) we get $\Xi_R = (\dim N/N)\mathbf{n}\circ$.

(3.15) b) $S=Cl(W)$. In this case $S_R\equiv Cl(N)\oplus Cl(N)$, under the identification $S_R\ni\omega=\omega_0+\omega_1\wedge(\partial/\partial r)\to(\omega_0,\omega_1)\in Cl(N)\oplus Cl(N)$. The connection ∇^R is simply the (direct sum of) Levi–Civitá connection(s) and $\not{\partial}_R$ is a direct sum of two copies of the Gauss–Bonnet operator on N. Ξ_R then becomes $\frac{1}{2}\sum_{i=1}^{\dim N} e_i\circ\mathrm{ad}_{\mathbf{n}\circ e_i}$. If ω_p is a

Clifford section of degree p on N, then

$$\mathrm{ad}_{\mathbf{n}\circ e_i}(\omega_p)=\begin{cases}2\mathbf{n}\circ e_i\circ\omega_p & \text{if} \quad e_i\in\omega_p\\ 0 & \text{if} \quad e_i\notin\omega_p\end{cases}.$$

Thus $\Xi_R\omega_p = p\mathbf{n}\circ\omega_p$. Similarly $\Xi_R(\mathbf{n}\circ\omega_p)=(\dim N - p)\mathbf{n}\circ(\mathbf{n}\circ\omega_p)$. Closely related formulas appear also in [B].

The parallel transport introduced before allows us to trivialize S in the radial direction. Precisely, if $\pi:(\varepsilon,\infty)\times N\to N$ is the projection, then $\pi^*(S_R)$ is canonically isomorphic to S. When S is viewed as $\pi^*(S_R)$, any section in $C^\infty(W,S)$ can be viewed as an element in $C^\infty((\varepsilon,\infty),C^\infty(N,S_R))$ and the separation of variables formula (3.10) extends to

$$Ds=\mathbf{n}\circ s'(r)+\frac{\not{\partial}_R s}{f}+\frac{f'}{f}\Xi_R s,\quad s\in C^\infty((\varepsilon,\infty),C^\infty(N,S_R)). \tag{3.16}$$

An easy consequence of (3.13) is that $\mathbf{n}$ and $\not{\partial}_R$ anticommute.

4. Index Formulae

In this paragraph we will solve the Callias-type index problem for a triple (M,S,L) consisting in an odd dimensional Riemannian spin manifold M with a warped end, a spinor-type bundle S, and a perturbed Dirac operator $L=D+A$, for which the potential A is independent of the radial direction on the warped end.

Definition 4.1. The Riemannian manifold M is said to have a *warped end* if there is a compact set $K\subset\subset M$ and a warped product W, such that $M-K$ and W are isometric as Riemannian manifolds. Thereafter we will identify $M-K$ and W. Any such manifold is complete [O].

For the rest of the section we assume that M is a $(n+1)$-dimensional spin manifold, n even, with a warped end W and S is a spinor-type bundle over M. As shown in the previous section, $S|_{M-K}\equiv\pi^*(S_R)$, with $S_R=S|_{\{R\}\times N}$ also a spinor-type bundle—afortiori N is spin too—. The separation of variables for the Dirac operator restricted to sections in $C^\infty(M-K,S)$ yields $\Xi_R=(n/2)\mathbf{n}\circ$ (see 3.15*a*). We also assume that A is skew-Hermitian, commutes with the Clifford action and is independent of the radial direction r, i.e., $A(r,x)=A(R,x)\equiv A(x)$, $r\geqq R$, $x\in N$. Then $-A^2$ is positive at infinity if and only if $-A^2(R,\cdot)$ is positive on N, which will also be assumed.

Proposition 4.2. *Under the above hypotheses the operator $L=D+A$ is a Fredholm operator if the warping function f on $M-K\equiv W$ has the property that $f(r)\to\infty$, if $r\to\infty$.*

Proof. According to Proposition 1.21b), it is enough to show that $\mathscr{D}(A)\equiv[D,A]$ goes to zero pointwise, as we approach the end of the manifold. Since $D=\mathbf{n}\circ(\partial/\partial r)+(\not{\partial}_R/f)+(f'/f)(n/2)\mathbf{n}\circ$ on $M-K$, we see that $[D,A](r,x)=(1/f(r))[\not{\partial}_R,A](x)$, $r\geqq R$, $x\in N$. Thus $[D,A](r,x)\to 0$ as $r\to\infty$, if $f(r)\to\infty$ as $r\to\infty$. □

We will need a variant of the relative index theorem of Gromov and Lawson ([G–L]). This theorem is proved in [G–L] for generalized Dirac operators on even dimensional manifolds only. However the proof goes through mutatis mutandis for more general differential operators, for example those whose principle symbol is given by the Clifford multiplication. This allows us to consider odd dimensional manifolds as well, in which case the theorem really amounts to an index-preserving deformation theorem. We also notice that the manfolds involved need not be connected.

(4.3) *The Relative Index Theorem.* Let (M_i, S_i, L_i), $i=0,1$, be two Callias-type operators which coincide outside compact sets, where manifolds and bundles are assumed to be compatibly isometric ([G–L], Assumption III, 4.1). Suppose that $L_i^* L_i$ and $L_i L_i^*$, $i=0,1$, are strictly positive at infinity (i.e., Assumption 1.20 holds) Then L_i is a Fredholm operator and

$$L^2\text{-index}(L_0) - L^2\text{-index}(L_1) = \mathrm{ind}_t(L_0, L_1). \tag{4.4}$$

□

In Eq. (4.4) $\mathrm{ind}_t(L_0, L_1)$ is the relative topological index. One way to define it is the following: let $\omega_i, i=0,1$, be the Atiyah–Singer index form on M_i associated to L_i [A–B–P], i.e., ω_i is the coefficient of t^0 in the local asymptotic expansion of the heat kernel $\mathrm{tr}\,[e^{-tL_i^* L_i}(m_i) - e^{-tL_i L_i^*}(m_i)]$, $m_i \in M_i$, as $t \to 0$. $\omega_0 = \omega_1$ on the common portion of M_i, and so $\int_{M_0} \omega_0 - \int_{M_1} \omega_1 \overset{\text{def}}{=} \mathrm{ind}_t(L_0, L_1)$ makes sense.

Proposition 4.5. *In the above relative index theorem assume that* $\dim M_0 = \dim M_1 =$ *odd number. Then*

$$L^2\text{-index}(L_0) = L^2\text{-index}(L_1)$$

Proof. It is well-known that $\omega_i, i=0,1$, vanishes on odd dimensional manifolds. (see [A–B–P]) □

We now return to the L^2-index for the Callias-type operator L considered in this chapter. The basic idea is to link, via the relative index theorem, two copies of M with their respective Callias-type operators to an operator (M_1, S_1, L_1) on a manifold with two ends and whose index is easily computable. Next we describe in detail this (M_1, S_1, L_1).

Let us take $M_1 \overset{\text{def}}{=} \mathbb{R} \times_{f_1} N$, where $f_1 \in C^\infty(\mathbb{R})$, $f_1 > 0$ and $\lim_{|t| \to \infty} f_1(t) = \infty$. Assume that the portion $(\varepsilon, \infty) \times_{f_1} N$ of M_1 is the end of a manifold M equipped with a spinor-type bundle $\mathbf{S}$, and $f_1(t) = f(|t|)$ for $|t| \geqq R$. If $\pi: \mathbb{R} \times N \to \{R\} \times N \equiv N$, $\pi(t,x) = (R,x)$ is the projection on N, we take $S_1 \overset{\text{def}}{=} \pi^*(\mathbf{S}_R)$.

Proposition 4.6. *$S_1 \to M$ can be made a Dirac bundle in a canonical way.*

Proof. The proof is of course similar to the corresponding part in Theorem 3.9. We only sketch it. Fix $(t,x) \in M_1$ and let $\{e_1, \dots, e_n\}$ be a local orthonormal basis in $T_x N$. Then $\{\partial/\partial t, (1/f_1)e_1, \dots, (1/f_1)e_n\}$ is a local orthonormal basis in $T_{(t,x)}(M_1)$ and we define the Clifford multiplication in S_1 by

$$\mathrm{cliff}\left(\frac{\partial}{\partial t}\right) = \mathbf{n}\circ, \quad \mathrm{cliff}\left(\frac{1}{f_1} e_i\right) = e_i \circ, \quad i = 1, \dots, n.$$

Also define the connection $\nabla_e^{S_1}$ by the formulas:

$$\nabla_{\partial/\partial t}^{S_1} s^{\sim} = 0, \quad \nabla_{(1/f_1)/e_i}^{S_1} s^{\sim} = \frac{1}{f_1}(\nabla_{e_i}^{S_R} s)^{\sim} + \frac{1}{2}\frac{f_1'}{f_1}(\mathbf{n}\circ e_i \circ s)^{\sim}.$$

Here s is a local section in $\mathbf{S}_R$ and $s^{\sim}$ is its parallel lift to $\pi^*(\mathbf{S}_R) = S_1$. Now the necessary verifications are identical to those in Theorem 3.9. $\square$

Let D_1 be the Dirac operator associated to $S_1 \to M_1$ and $L_1 = D_1 + \Psi$, $\Psi(t,x) = \chi(t)A(x)$, $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = \begin{cases} 1 & \text{if } t \geq R \\ -1 & \text{if } t \leq -R \end{cases}$. We can write $L_1 = \mathbf{n}\circ \partial/\partial r + \partial\!\!\!/_R/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ + \chi A$ on $C^\infty(M_1, S_1)$. Since $\lim_{|t|\to\infty} f_1(t) = \infty$, L_1 is a Fredholm operator (Proposition 4.2). The notations being those of Lemma 2.7, Theorem 2.9, and (3.16), L_1 is homotopic to the operator

$$T_1 = \begin{pmatrix} \mathbf{n}\circ\dfrac{\partial}{\partial t} + \dfrac{(\partial\!\!\!/_R)_+}{f_1} + \dfrac{f_1'}{f_1}\dfrac{n}{2}\mathbf{n}_0 + \sqrt{-1}\chi & 0 \\ 0 & \mathbf{n}\circ\dfrac{\partial}{\partial t} + \dfrac{(\partial\!\!\!/_R)_-}{f_1} + \dfrac{f_1'}{f_1}\dfrac{n}{2}\mathbf{n}\circ - \sqrt{-1}\chi \end{pmatrix} \tag{4.7}$$

globally defined on $C^\infty(M_1, S_1) \equiv C^\infty(M_1, (S_1)_+ \oplus (S_1)_-)$. In writing out (4.7) we used the fact $(S_1)_\pm = \pi^*((\mathbf{S}_R)_\pm)$, where $(\mathbf{S}_R)_\pm \overset{\text{def}}{=} \{U(R,\cdot) = \pm\sqrt{-1}\,\mathrm{Id}\} \subset \mathbf{S}_R$. $(\partial\!\!\!/_R)_\pm$ is the Dirac operator associated to $(\mathbf{S}_R)_\pm \to N$ and therefore $(\partial\!\!\!/_R)_\pm = \frac{1}{2}(\mathrm{Id} \mp \sqrt{-1}\,U) \times \partial\!\!\!/_R|_{(\mathbf{S}_R)_\pm}$. Set now $(\mathbf{S}_R)_+^\pm = \{s \in (\mathbf{S}_R)_+ \mid \mathbf{n}\circ s = \pm\sqrt{-1}s\}$ and let $(\partial\!\!\!/_R)_+^\pm$ be the restriction of $(\partial\!\!\!/_R)_+$ to $(\mathbf{S}_R)_+^\pm$. Similarly define $(\partial\!\!\!/_R)_-^\pm$ and $(S_R)_-^\pm$.

Theorem 4.8. *If (M_1, S_1, L_1) is as above, then*

$$L^2\text{-index}(L_1) = \text{index}(\partial\!\!\!/_R)_+^+ + \text{index}(\partial\!\!\!/_R)_-^-.$$

Proof. Clearly L^2-index$(L_1) = L^2$-index$(T_1) = L^2$-index$(\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_+/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ + \sqrt{-1}\chi) + L^2$-index$(\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_-/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ - \sqrt{-1}\chi)$. We concentrate next on the L^2-index of the operator $\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_+/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ + \sqrt{-1}\chi$ defined on $C^\infty(M_1, \pi^*(S_R)_+)$. The obvious Hilbert space isometry

$$\begin{cases} L^2(M_1, \pi^*(S_R)_+) \to L^2(\mathbb{R}\times N, \pi^*(S_R)_+) \\ \qquad\qquad \omega \to f_1^{n/2}\omega \end{cases}$$

takes the operator $\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_+/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ + \sqrt{-1}\chi$ into the operator $\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_+/f_1 + \sqrt{-1}\chi$ defined on $C^\infty(\mathbb{R}\times N, \pi^*(\mathbf{S}_R)_+)$. Let us denote by Q the closure of the operator $\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_+/f_1 + \sqrt{-1}\chi$ in $L^2(\mathbb{R}\times N, \pi^*(\mathbf{S}_R)_+)$. Then $Q^* = \mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_+/f_1 - \sqrt{-1}\chi$, and the task reduced to that of finding L^2-index(Q).

As a selfadjoint differential operator on a compact manifold, $(\partial\!\!\!/_R)_+$ has a discrete spectrum located on the real line. Since $(\partial\!\!\!/_R)_+\mathbf{n}\circ = -\mathbf{n}\circ(\partial\!\!\!/_R)_+$, if $\lambda \neq 0$ is an eigenvalue corresponding to the eigenvector $s_\lambda \in C^\infty(N, (\mathbf{S}_R)_+)$, then $(-\lambda)$ is also an eigenvalue corresponding to the eigenvector $\mathbf{n}\circ s_\lambda$. Let then $\{(\lambda, s_\lambda) \mid \lambda > 0\}_\lambda \cup$

$\{(-\lambda, \mathbf{n}\circ s_\lambda)\}_\lambda \cup \{(0, s_\alpha)\}_\alpha$ be the spectral decomposition of $(\partial\!\!\!/_R)_+$. $\{s_\lambda, \mathbf{n}\circ s_\lambda, s_\alpha\}$ is then a Hilbert basis for $L^2(N, (S_R)_+)$, $\{s_\alpha\}_\alpha$ generates the finite dimensional space $\ker(\partial\!\!\!/_R)_+$ and $\{s_\lambda, \mathbf{n}\circ s_\lambda\}_\lambda$ is a Hilbert basis for $[\ker(\partial\!\!\!/_R)_+]^\perp$, the orthogonal complement of $\ker(\partial\!\!\!/_R)_+$ in $L^2(N, (\mathbf{S}_R)_+)$. Moreover, in the decomposition $L^2(\mathbb{R}\times N,\ \pi^*(\mathbf{S}_R)_+) \equiv L^2(\mathbb{R}, L^2(N, (\mathbf{S}_R)_+)) = L^2(\mathbb{R}, \ker(\partial\!\!\!/_R)_+) \oplus L^2(\mathbb{R}, [\ker(\partial\!\!\!/_R)_+]^\perp)$, $L^2(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)$ and $L^2(\mathbb{R}, [\ker(\partial\!\!\!/_R)_+]^\perp)$ are left invariant by Q and Q^*. Thus

$$L^2\text{-index}(Q) = L^2\text{-index}(Q|_{C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)}) + L^2\text{-index}(Q|_{C^\infty(\mathbb{R}, [\ker(\partial\!\!\!/_R)_+]^\perp)}). \tag{4.9}$$

It is easily seen that the L^2-isometry of $[\ker(\partial\!\!\!/_R)_+]^\perp$ into itself given by $\begin{cases} s_\lambda \to \mathbf{n}\circ s_\lambda \\ \mathbf{n}\circ s_\lambda \to s_\lambda \end{cases}$ induces an isometry on $L^2(\mathbb{R}, [\ker(\partial\!\!\!/_R)_+]^\perp)$ which identifies $L^2\text{-ker}(Q|_{C^\infty(R, [\ker(\partial\!\!\!/_R)_+]^\perp)})$ and $L^2\text{-ker}(Q^*|_{C^\infty(\mathbb{R}, [\ker(\partial\!\!\!/_R)_+]^\perp)})$. Equation (4.9) becomes then $L^2\text{-index}(Q) = L^2\text{-index}(Q|_{C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)})$. We can choose $s_\alpha \in C^\infty(N, (S_R)_+)$, $(\partial\!\!\!/_R)_+ s_\alpha = 0$, such that $s_\alpha \in C^\infty(N, (S_R)_+^+)$ or $s_\alpha \in C^\infty(N, (S_R)_+^-)$. This is always possible since $\mathbf{n}$ preserves $\ker(\partial\!\!\!/_R)_+$. Then

$$Q|_{C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+^\pm)} = \pm\sqrt{-1}\frac{\partial}{\partial t} + \sqrt{-1}\chi.$$

Any solution $\sigma \in C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+^+)$ of the equation $\sqrt{-1}\sigma' + \sqrt{-1}\chi\sigma = 0$ is of the form $\sigma(t,x) = \exp\left(-\int_0^t \chi(\tau)d\tau\right)s(x), s \in \ker(\partial\!\!\!/_R)_+^+$, thus always in $L^2(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)$, and any solution $\sigma \in C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+^-)$ of the equation $-\sqrt{-1}\sigma' + \sqrt{-1}\chi\sigma = 0$ is of the form $\sigma(t,x) = \exp\left(\int_0^t \chi(\tau)d\tau\right)s(x), s \in \ker(\partial\!\!\!/_R)_+^-$, thus never in $L^2(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)$. As a result

$$\dim L^2\text{-ker}(Q|_{C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)}) = \dim\ker(\partial\!\!\!/_R)_+^+.$$

Similarly

$$\dim L^2\text{-ker}(Q^*|_{C^\infty(\mathbb{R}, \ker(\partial\!\!\!/_R)_+)}) = \dim\ker(\partial\!\!\!/_R)_+^-$$

i.e.,

$$L^2\text{-index}\left(\mathbf{n}\circ\frac{\partial}{\partial t} + \frac{(\partial\!\!\!/_R)_+}{f_1} + \frac{f_1'}{f_1}\frac{n}{2}\mathbf{n}\circ + \sqrt{-1}\chi\right) = \text{index}(\partial\!\!\!/_R)_+^+.$$

The whole argument can be repeated for the operator $\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_-/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ - \sqrt{-1}\chi$ and gives $L^2\text{-index}(\mathbf{n}\circ\partial/\partial t + (\partial\!\!\!/_R)_-/f_1 + (f_1'/f_1)(n/2)\mathbf{n}\circ - \sqrt{-1}\chi) = \text{index}(\partial\!\!\!/_R)_-^-$. Theorem 4.8 follows. □

Corollary 4.10. *Let L_1 be the Fredholm operator considered in Theorem 4.8. Then*

$$L^2\text{-index}(L_1) = 2\,\text{index}(\partial\!\!\!/_R)_+^+.$$

Proof. According to Theorem 4.8, it is enough to show that $\text{index}(\partial\!\!\!/_R)_+^+ = \text{index}(\partial\!\!\!/_R)_-^-$. The operator $(\partial\!\!\!/_R)^+ : C^\infty(N, (S_R)^+) \to C^\infty(N, (S_R)^-)$ is cobordant to 0 (cf. [A–S]). Thus (loc. cit.) $\text{index}(\partial\!\!\!/_R)^+ = 0$. Now $(\partial\!\!\!/_R)^+$ and $\begin{pmatrix} (\partial\!\!\!/_R)_+^+ & 0 \\ 0 & (\partial\!\!\!/_R)_-^+ \end{pmatrix}$ are homotopic (a compact version of Theorem 2.9). As a result, $0 = \text{index}(\partial\!\!\!/_R)^+ = \text{index}(\partial\!\!\!/_R)_+^+ + \text{ind\^{e}x}(\partial\!\!\!/_R)_-^+ = \text{index}(\partial\!\!\!/_R)_+^+ - \text{index}(\partial\!\!\!/_R)_-^-$. □

Let us now introduce the manifold $M_0 \overset{\text{def}}{=} M \coprod M$, the disjoint union of two copies of M and the Dirac bundle $S_0 \overset{\text{def}}{=} \mathbf{S} \coprod \mathbf{S}$. The operator $L_0 \overset{\text{def}}{=} L \coprod (-L)$ is clearly a Fredholm operator and L^2-index$(L_0) = L^2$-index$(L) + L^2$-index$(-L) = 2L^2$-index(L). If we define $T_0 = T \coprod (-T)$ on S_0, where T is as in Theorem 2.9, we also have L^2-index$(T_0) = 2L^2$-index(L).

Proposition 4.11. *The operators (M_0, S_0, T_0) and (M_1, S_1, T_1) introduced above are isometric outside compact sets* i.e., *there are compact sets $K_0 \subset M_0$ and $K_1 \subset M_1$ and a manifold isometry $F: M_0|_{M_0 - K_0} \to M_1|_{M_1 - K_1}$ covered by a bundle isometry $\tilde{F}: S_0|_{M_0 - K_0} \to S_1|_{M_1 - K_1}$ such that*

$$T_1 = \tilde{F} T_0 (\tilde{F})^{-1}. \tag{4.12}$$

Proof. The obvious isometry F between $\mathsf{C}K_0 \overset{\text{def}}{=} \{(r,x) \in M | r > R\} \coprod \{(r,x) \in M | r > R\} \subset M_0$ and $\mathsf{C}K_1 \overset{\text{def}}{=} \{(t,x) \in M_1 | \, |t| > R\} \subset M_1$,

$$F(m) = \begin{cases} (r,x) & \text{if} \quad m \in 1^{\text{st}}\ M \text{ in } M_0 \quad \text{and} \quad m = (r,x) \\ (-r,x) & \text{if} \quad m \in 2^{\text{nd}}\ M \text{ in } M_0 \quad \text{and} \quad m = (r,x) \end{cases}$$

is covered by the bundle isometry $\tilde{F}$ defined by the formula

$$\tilde{F}(v_m) = \begin{cases} v_m & \text{if} \quad m \in 1^{\text{st}}\ M \text{ in } M_0 \quad \text{and} \quad v_m \in \mathbf{S}_R \equiv (S_0)_m \equiv (S_1)_m \\ \mathbf{n} \circ v_m & \text{if} \quad m \in 2^{\text{nd}}\ M \text{ in } M_0 \quad \text{and} \quad v_m \in \mathbf{S}_R \equiv (S_0)_m \equiv (S_1)_{F(m)} \end{cases}.$$

Equation (4.12) is obvious for sections in $C^\infty(\{(t,x) | t > R\}, S_1)$, since $\tilde{F} = \text{Id}$ there. Consider now a section s in $C^\infty(\{(t,x) | t < -R\}, S_1)$. For simplicity we assume that $s \in C^\infty(\{(t,x) | t < -R\}, (S_1)_+)$. Then $(\tilde{F}^{-1}s)(r,x) = -\mathbf{n} \circ s(-r,x)$ and

$$(T_0 \tilde{F}^{-1} x)(r,x) = (-T\tilde{F}^{-1}s)(r,x) = -\mathbf{n} \circ \frac{\partial \sigma}{\partial r}(r,x) - \frac{((\not{\partial}_R)_+ \sigma)(r,x)}{f(r)}$$
$$-\frac{f'(r)}{f(r)} \frac{n}{2} \mathbf{n} \circ \sigma(r,x) - \sqrt{-1}\sigma(r,x),$$

where $\sigma = \tilde{F}^{-1}s$. Thus

$$(T_0 \tilde{F}^{-1} s)(r,x) = \frac{\partial s}{\partial r}(-r,x) - \frac{\mathbf{n} \circ ((\not{\partial}_R)_+ s)(-r,x)}{f_1(-r)}$$
$$+\frac{f_1'(-r)}{f_1(-r)} \frac{n}{2} s(-r,x) + \sqrt{-1}\mathbf{n} \circ s(-r,x).$$

Finally

$$(\tilde{F} T_0 \tilde{F}^{-1} s)(t,x) = \mathbf{n} \circ \frac{\partial s}{\partial t}(t,x) - \frac{((\not{\partial}_R)_+ s)(t,x)}{f_1(t)} + \frac{f_1'(t)}{f_1(t)} \frac{n}{2} \mathbf{n} \circ s(t,x) - \sqrt{-1} s(t,x)$$
$$= (T_1 s)(t,x). \quad \square$$

We summarize the results obtained in Proposition 4.5, Theorem 4.8, Corollary 4.10, and Proposition 4.11 in the following main theorem:

Theorem 4.13. *Let M be an odd dimensional Riemannian spin manifold with a warped end $W=(\varepsilon,\infty)\times_f N$, $f\in C^\infty((\varepsilon,\infty))$, $f>0$ and $f(r)\to\infty$ if $r\to\infty$. Let $\mathbf{S}$ be a spinor-type bundle over M and let $A\in C^\infty(M,\mathrm{End}(\mathbf{S}))$ be a skew-Hermitian endomorphism such that A commutes with the Clifford action on $\mathbf{S}$, $A|_W$ is independent of the radial direction r, and $-A^2$ is positive at infinity. Then the perturbed Dirac operator $D+A$, where D is the Dirac operator on $\mathbf{S}$, is a Fredholm operator and*

$$L^2\text{-index}(D+A)=\mathrm{index}(\partial\!\!\!/_R)^+_+.$$

Here $(\partial\!\!\!/_R)^+_+ : C^\infty(N,(\mathbf{S}_R)^+_+)\to C^\infty(N,(\mathbf{S}_R)^-_+)$ is the Dirac operator on the bundle $(\mathbf{S}_R)^+_+=\{s\in\mathbf{S}|_{\{R\}\times N\equiv N}|(-A^2)^{-1/2}As=\sqrt{-1}s,\ \mathrm{cliff}(\partial/\partial r)s=\sqrt{-1}s\}$ over the compact even dimensional spin manifold $\{R\}\times N\equiv N$. $\square$

Theorem 4.13 indicates that the L^2-index depends only on the spin geometry of the cross section of the manifold M, and on the spectral properties of the potential A at infinity. This fact is even better outlined by the following particular case which also leads to an elegant derivation of Callias' index formula (0.2).

Corollary 4.14. a) *In Theorem 4.13 assume in addition that $\mathbf{S}=\Sigma\otimes V$, where Σ is the spinor bundle on M, V is a trivial bundle over M, and $A\in C^\infty(M,\mathrm{End}(V))$. Then*

$$L^2\text{-index}(D+A)=\int_N \hat{A}(N)\wedge\mathrm{ch}(V_R)_+=\int_N \hat{A}(N)\wedge\mathrm{tr}\exp\frac{\sqrt{-1}}{2\pi}\left(\frac{\mathrm{Id}+\Phi}{8}\right)(d\Phi)^2, \tag{4.15}$$

where $\hat{A}(N)$ is the total $\hat{A}$-class of N, $\Phi=1/\sqrt{-1}(-A^2)^{-1/2}A$, $(V_R)_+$ is the bundle over N given by $(V_R)_+=\{v\in V_R\equiv V|_{\{R\}\times N}|\Phi v=v\}$, and $\mathrm{ch}(V_R)_+$ is the Chern character of $(V_R)_+$.

b) *If $\hat{A}(N)=1$, then (4.15) can be written:*

$$L^2\text{-index}(D+A)=\frac{1}{2\left(\frac{n}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{n/2}\int_N \mathrm{tr}\,\Phi(d\Phi)^n.$$

Proof. Since $U=\sqrt{-1}\Phi$, we have $(\mathbf{S}_R)_+=\Sigma\otimes(V_R)_+$. We prefer to work with Φ instead of U here for esthetical reasons. The first half of Eq. (4.15) follows now from Theorem 4.13 and the Atiyah–Singer index theorem for twisted Dirac operators on compact manifolds [A–S]. The second half is a consequence of the following lemma:

Lemma 4.16. *Let $\xi\to N$ be a trivial bundle equipped with the flat connection with respect to a fixed trivialization. If $P\in C^\infty(N,\mathrm{End}(\xi))$ is a projection then the curvature associated to the induced connection on the subbundle $\xi_+=\{P=\mathrm{Id}\}$ is given by $P\cdot(dP)^2$.*

Proof of Lemma 4.16. Let ∇ denote the flat connection on ξ. Then $P\nabla$ is the induced connection on ξ_+ whose curvature $\mathbf{r}_+$ is given for any $X,Y\in TN$, by

$$\begin{aligned}\mathbf{r}_+(X,Y)&=[P\nabla_X,P\nabla_Y]-p\nabla_{[X,Y]}=P\nabla_XP\nabla_Y-P\nabla_YP\nabla_X-P\nabla_{[X,Y]}=PX(P)\nabla_Y\\&\quad-P^2\nabla_X\nabla_Y-PY(P)\nabla_X+P^2\nabla_Y\nabla_X-P\nabla_{[X,Y]}=PX(P)\nabla_Y-PY(P)\nabla_X,\end{aligned}$$

since ∇ is flat and P is a projection. On the other hand

$$\begin{aligned} PX(P)\nabla_Y - PY(P)\nabla_X = PX(P)\nabla_Y P - PY(P)\nabla_X P &= PX(P)Y(P) + PX(P)P\nabla_Y \\ &- PY(P)X(P) - PY(P)P\nabla_X + P(dP)^2(X,Y) \\ &+ PX(P)P\nabla_Y - PY(P)P\nabla_X = P(dP)^2(X,Y), \end{aligned}$$

since $PX(P)P$ and $PY(P)P$ are $0, P$ being a projection. $\square$

Proof of Corollary 4.14 continued. Put now $\xi = V_R$ and $P = (\mathrm{Id} + \Phi)/2$ in Lemma 4.16. Then $\xi_+ = (V_R)_+$, and a representative for the Chern character $\mathrm{ch}\,(V_R)_+$ is $\mathrm{tr}\exp((\sqrt{-1}/2\pi)\mathbf{r}_+) = \exp(\sqrt{-1}/2\pi)((\mathrm{Id} + \Phi)/8)(d\Phi)^2$.
b) If $\hat{A}(N) = 1$ then $L^2\text{-index}(D+A) = \int_N \mathrm{ch}\,(V_R)_+ = \int_N \mathrm{tr}\exp\sqrt{-1}/2\pi((\mathrm{Id}+\Phi)/8)(d\Phi)^2$.
Since only the component of top degree of $\mathrm{ch}\,(V_R)_+$ matters in the above integration and $\mathrm{tr}\exp\sqrt{-1}/2\pi((\mathrm{Id} + \Phi)/8)(d\Phi)^2$ is a form of degree 2, we get

$$\begin{aligned} L^2\text{-index}\,(D + A) &= \frac{1}{\left(\frac{n}{2}\right)!} \int_N \mathrm{tr}\left(\frac{\sqrt{-1}}{2\pi}\left(\frac{\mathrm{Id} + \Phi}{8}\right)(d\Phi)^2\right)^{n/2} \\ &= \frac{1}{\left(\frac{n}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{n/2} \int_N \mathrm{tr}\left(\frac{\mathrm{Id} + \Phi}{2}(d\Phi)^2\right)^{n/2} \\ &= \frac{1}{\left(\frac{n}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{n/2} \int_N \mathrm{tr}\frac{\mathrm{Id} + \Phi}{2}(d\Phi)^n \\ &= \frac{1}{2\left(\frac{n}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{n/2} \int_N \mathrm{tr}\,(d\Phi)^n + \frac{1}{2\left(\frac{n}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{n/2} \int_N \mathrm{tr}\,\Phi(d\Phi)^n \\ &= \frac{1}{2\left(\frac{n}{2}\right)!}\left(\frac{\sqrt{-1}}{8\pi}\right)^{n/2} \int_N \mathrm{tr}\,\Phi(d\Phi)^n, \end{aligned}$$

since $\mathrm{tr}\,(d\Phi)^n$ is an exact form. Taking $M = \mathbb{R}^n, n$ odd, and $A = \sqrt{-1}\,\Phi$ outside some compact set, we recover Callias' index formula, since $N \equiv \mathbb{S}_1^{n-1}$ and $\hat{A}(\mathbb{S}_1^{n-1}) = 1$. $\square$

Acknowledgement. This paper is based upon the author's doctoral dissertation who wishes to thank his advisor Professor H. Moscovici for his continuing help and encouragement.

References

[A] Atiyah, M. F.: *K*-theory. New York: W. A. Benjamin 1967
[A–B–P] Atiyah, M. F., Bott, R., Singer, I. M.: On the heat equation and the index theorem. Invent. Math. **19**, 279–330 (1973)

[A–S] Atiyah, M. F., Singer, I. M.: The index of elliptic operators on compact manifolds. Bull. Am. Math. Soc. **69**, 422–433 (1963)
[B] Bruning, J.: Index theory for regular singular operators and applications. Universitat Augsburg, Preprint no. 110, 1986
[Bo–S] Bott, R., Seeley, R.: Some remarks on the paper of Callias. Commun. Math. Phys. **62**, 235–245 (1978)
[C] Callias, C.: Axial anomalies and index theorems on open spaces. Commun. Math. Phys. **62**, 213–234 (1978)
[Ch] Chou, A. W.: The Dirac operator on spaces with conical singularities and positive scalar curvatures. Trans. Am. Math. Soc. **289**, 1–40 (1985)
[G] Goldberg, S.: Unbounded linear operators. New York: Mc Graw Hill 1966
[G–L] Gromov, N., Lawson, H. B. Jr.: Positive scalar curvature and the Dirac operators on complete Riemannian manifolds. Publ. Math. IHES **58**, 295–408 (1983)
[K] Kato, T.: Perturbation theory for linear operators. 2nd ed. Berlin, Heidelberg, New York: Springer 1980
[L–M] Lawson, H. B. Jr., Michaelson, L.: Spin geometry. Universidade Federal do Ceara (1983)
[O] O'Neil, B.: Semi-Riemannian geometry with applications to relativity. New York: Academic Press 1983
[S] Stern, M. A.: L^2-index theorems on warped products. Dissertation, Princeton Univ. 1984

Communicated by K. Gawedzki

Received April 21, 1989