Commun. Math. Phys. 127, 225–238 (1990)

Communications in
Mathematical
Physics

Hierarchy Structure in Integrable Systems of Gauge Fields and Underlying Lie Algebras

K. Takasaki
Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto-shi 606, Japan

Abstract. An improved version of Nakamura's self-dual Yang–Mills hierarchy is presented and its symmetry contents are studied. The new hierarchy as well as the previous one represents a set of commuting dynamical flows in an infinite dimensional manifold of "loop type," but includes a larger set of dependent variables. Because of new degrees of freedom the theory acquires a more symmetric form with richer structures. For example it allows a larger symmetry algebra of Riemann–Hilbert type, which is actually a direct sum of two subalgebras ("left" and "right"). This phenomenon is basically the same as observed recently by Avan and Bellon in the case of principal chiral models. In addition to these rather familiar symmetries, a new type of symmetries referred to as "coordinate transformation type" are also introduced. Generators of the above dynamical flows are all included therein. These two types of symmetries altogether form a big Lie algebra, which leads to more satisfactory understanding of symmetry properties of integrable systems of gauge fields.

I. Introduction

In recent work Nakamura [N] introduced an infinite system of gauge field equations as a candidate for the "hierarchy" of the self-duality equation in four dimensions. This would be presumably the first paper published on the construction of a hierarchy for nonlinear integrable systems in gauge theory. For "soliton equations" such as the KdV equation the notion of hierarchy has been playing, though under different names, a fundamental role since early days. The terminology "hierarchy" itself is rather new and perhaps became more familiar in the beginning of the eighties in the course of the study of the KP hierarchy [SS, DJKM]. Its usage is however occasionally somewhat vague. We now use it to mean a system of differential equations that describe, directly or indirectly, a commuting set of dynamical flows in some (possibly infinite dimensional) manifold, in which the original equation (say, the KdV equation) is included as a special sector. In the classical theory of nonlinear integrable systems the existence of such flows are recognized as a *consequence* of sufficiently many constants of motion (or conservation laws). We now adopt a reversed point of view, namely we consider a hierarchy as a *primary* object and seek to derive its "integrability" from some other principle. As Nakamura argued in the above paper on the basis of his previous work with Ueno [UN] and of the present author [T1–3], richness of "symmetries" or "transformation groups" appears to be the most promising standpoint as such an alternative principle. This observation is also advocated by the work on the KP hierarchy cited above. The present paper is intended to give a more detailed analysis on this issue.

We first introduce an improved version of Nakamura's hierarchy (Sect. II). His hierarchy contains an infinite number of independent variables as well as dependent ones (unknown functions), but from a twister-theoretical point of view (which can also be restated in the language of the Riemann–Hilbert problem of Ueno and Nakamura) they are still just part of a larger set of variables that carry full information of gauge fields. These variables altogether satisfy a closed system of differential equations, which gives our improved hierarchy.

The remaining part of this paper is focussed on symmetry properties of this new hierarchy. An introductory overview is given in Sect. III. As indicated therein, the hierarchy allows two distinct types of symmetries. The first one is referred to as "Riemann–Hilbert (RT) type;" it basically consists of Riemann–Hilbert transformations as introduced by Ueno and Nakamura. We will however see a new circumstance (Sect. IV) that these RT type symmetries further decompose into two pieces, "left" and "right," which mutually commute. More precisely, the symmetry algebra of RT type becomes a direct sum of two subalgebras made up of these one-sided members. The occurrence of similar two distinct RH transformations had been discovered by several people, but "left" and "right" symmetries in their formulation do not commute in general. From our point of view they are still "mixed," i.e. linear combinations of both "left" and "right" pieces, so that their commutations relations take largely different forms. The same direct sum structure was also recently pointed out by Avan and Bellon [AB] for principal chiral models. The second class of symmetries introduced in Sect. III are referred to as "coordinate transformation (CT) type." A detailed analysis is done in Sect. V. It appears that no one has ever investigated this type of symmetries. The dynamical flows represented by the hierarchy are nothing other than the action of an abelian subalgebra therein; this type of symmetries is therefore rather of fundamental relevance. A preferable formulation of a symmetry algebra for the present case should naturally incorporate these CT type symmetries as well.

In Sect. VI we shall summarize these analyses to identify a full symmetry algebra of the hierarchy. This Lie algebra consists of both the above two types of symmetries, and accordingly is much larger than what has ever been considered in the literature. This observation was already announced in [T3], though the setting therein is rather parallel to Nakamura's hierarchy. Accordingly the symmetry algebra contained only one-sided RH transformations and only part of the symmetries of CT type allowed in the present function. We shall also finally see several interesting possibilities to connect these results with some other Lie algebras of CT type that arise in seemingly distinct contexts.

II. Nakamura's Hierarchy in Improved Form

To find an improved version of Nakamura's gauge field hierarchy, we now start from the "zero-curvature system,"

$$[\partial_{am} - \lambda\partial_{a,m-1} + A_{am}, \partial_{bn} - \lambda\partial_{b,n-1} + A_{bn}] = 0, \tag{2.1}$$

and the associated linear system,

$$(\partial_{am} - \lambda\partial_{a,m-1} + A_{am})\Psi = 0, \tag{2.2}$$

where t^{am} ($m \in \mathbf{Z}$, $1 \leqq a \leqq s$) are independent variables, A_{am} "gauge potentials" with values in, say, $gl(r)$ and m, n, a and b run over such ranges that $m, n \in \mathbf{Z}$, $1 \leqq a, b \leqq s$. The r and s are positive integers which are henceforth fixed throughout; we are mostly interested in the nonabelian case (i.e. $r \geqq 2$). The λ is a spectral parameter taking values in a Riemann sphere, and ∂_{am} denotes the derivatives

$$\partial_{am} \stackrel{\text{def}}{=} \partial/\partial t^{am}. \tag{2.3}$$

All independent variables in this formulation are treated on an equal footing. To extract its dynamical contents, however, one has to break this symmetry to identify what are "space variables" and "time variables" among the whole set of the independent variables. Our task below is to find such a "space-time" framework and an evolutionary system that underlie the above equations.

To this end, we now choose t^{a0} ($1 \leqq a \leqq s$) as "space variables" and rewrite:

$$t^{a0} = x^a. \tag{2.4}$$

The zero-curvature equations and the linear system have the following equivalent expression:

$$[\partial_{am} - A_{am}(\lambda), \partial_{bn} - A_{bn}(\lambda)] = 0 \quad (m, n \neq 0), \tag{2.1$'$}$$

$$(\partial_{am} - A_{am}(\lambda))\Psi = 0 \quad (m \neq 0), \tag{2.2$'$}$$

where

$$A_{am}(\lambda) \stackrel{\text{def}}{=} \begin{cases} \lambda^m \partial_a - \sum\limits_{n=0}^{m-1} \lambda^n A_{a,m-n} & \text{for} \quad m \geqq 1 \\ \lambda^m \partial_a + \sum\limits_{n=m}^{-1} \lambda^n A_{a,m-n} & \text{for} \quad m \leqq -1 \end{cases} \tag{2.5}$$

and

$$\partial_a \stackrel{\text{def}}{=} \partial/\partial x^a. \tag{2.6}$$

We further consider the following two types of solutions of the linear system, which are Laurent-expanded with respect to λ in a neighborhood of, respectively, $\lambda = \infty$ and $\lambda = 0$ in the Riemann sphere,

$$W(\lambda) = \sum_{n=0}^{\infty} w_n \lambda^{-n}, \quad w_0 = \mathbf{1}, \tag{2.7a}$$

$$\hat{W}(\lambda)=\sum_{n=0}^{\infty}\hat{w}_n\lambda^n. \tag{2.7b}$$

The zero-curvature system ensure that the linear system does have such solutions. (Actually one may take W and $\hat{W}$ to be merely formal Laurent series; see [T1–3] for example of such a point of view.) Conversely if the linear system has such solutions, the zero-curvature equations are accordingly satisfied. As widely recognized, this kind of relation is the most basic postulate for a number of solution techniques of nonlinear integrable systems to apply to a nonlinear system. Our aim here is somewhat different; we consider what differential equations the Laurent coefficients of W and $\hat{W}$ satisfy in themselves. As it turns out below, the indeed obey a nonlinear evolutionary system, which is to be a hierarchy for the present case.

A key is the following relations, which are immediate consequences of the linear system for W and $\hat{W}$:

$$\sum_{n=0}^{m-1}\lambda^n A_{a,m-n}=(\lambda^m\partial_a W(\lambda)\cdot W(\lambda)^{-1})_+ \quad (m\geqq 1), \tag{2.8a}$$

$$\sum_{n=m}^{-1}\lambda^n A_{a,m-n}=-(\lambda^m\partial_a\hat{W}(\lambda)\cdot\hat{W}(\lambda)^{-1})_- \quad (m\leqq -1), \tag{2.8b}$$

where $(\ )_+$ and $(\ )_-$ denote the projection operators

$$(\sum u_n\lambda^n)_+\stackrel{\text{def}}{=}\sum_{n\geqq 0}u_n\lambda^n,$$

$$(\sum u_n\lambda^n)_-\stackrel{\text{def}}{=}\sum_{n<0}u_n\lambda^n$$

on the space of Laurent series of λ. Substituting the A_{am}'s with the above expression, one finds that the linear system for W and $\hat{W}$ actually becomes a *nonlinear* evolutionary system of the form.

$$\partial_{am}w_n=\partial_a w_{n+m}-\sum_{l=0}^{m-1}A_{a,m-l}w_{n+l} \quad (m\geqq 1), \tag{2.9a}$$

$$\partial_{am}w_n=\partial_a w_{n+m}+\sum_{l=m}^{-1}A_{a,m-l}w_{n+l} \quad (m\leqq -1), \tag{2.9b}$$

$$\partial_{am}\hat{w}_n=\partial_a\hat{w}_{n-m}-\sum_{l=0}^{m-1}A_{a,m-l}\hat{w}_{n-l} \quad (m\geqq 1), \tag{2.9c}$$

$$\partial_{am}\hat{w}_n=\partial_a\hat{w}_{n-m}+\sum_{l=m}^{-1}A_{a,m-l}\hat{w}_{n-l} \quad (m\leqq -1). \tag{2.9d}$$

The Laurent coefficients $w_n=w_n(t,x)$ and $\hat{w}_n=\hat{w}_n(t,x)$ are now unknown functions, $t=(t^{am})$ $(1\leqq a\leqq s, m\neq 0)$ being "time variables" and $x=(x^a)$ $(1\leqq a\leqq s)$ "space variables."

We adopt this nonlinear system as a hierarchy for the present setting. Geometrically this certainly represents a commuting set of dynamical flows in an infinite dimensional manifold, whose points are parametrized by the Laurent coefficients of the W and $\hat{W}$. From this standpoint the zero-curvature system is rather a secondary object.

It turns out that Nakamura's hierarchy may be identified with the $s = \infty$ case in our setting, but with only a smaller set of flows (time evolutions). More precisely, the space variables in his formulation are made up of two groups as $x^a = (y^a, z^a)$ $(a = 1, 2, \ldots)$. Accordingly let us write the full time variables as $t^{am} = (u^{am}, v^{am})$. Then the flows in Nakamura's hierarchy are limited to those that correspond to (u^{aa}, v^{aa}). Thus those with $a \neq m$ are missing. This is however a rather superficial difference; it seems that Nakamura dared to take such a formulation so as to respect a direct connection with the original self-dual Yang–Mills equations. A more relevant difference is that we allow time variables with *negative indices*, i.e. t^{am} for $m \leqq -1$. As one can readily check, the system of evolutionary equations above contains a subsystem made up of only $w_n (n \geqq 1)$ and their t^{am}-$(m \geqq 1)$ and x^a-derivatives. (Nakamura's hierarchy is included therein). These variables are insufficient to deal with a larger set of flows indexed by all integers. For this reason we introduce both the W and $\hat{W}$ as dynamical variables.

Algebraic structures of this evolutionary system become more understandable if one introduces the following operators:

$$Q_a(\lambda) \overset{\text{def}}{=} \partial_a - \partial_a W(\lambda) \cdot W(\lambda)^{-1}, \tag{2.10a}$$

$$\hat{Q}_a(\lambda) \overset{\text{def}}{=} \partial_a - \partial_a \hat{W}(\lambda) \cdot \hat{W}(\lambda)^{-1}. \tag{2.10b}$$

Actually the operators Q_a and $\hat{Q}_a$ turn out to play almost the same role as the "Lax operator" in the theory of the KP hierarchy [SS, DJKM]. Firstly, Eqs. (2.8) take, in terms of them, a more concise form:

$$A_{am}(\lambda) = (\lambda^m Q_a(\lambda))_+ \quad (m \geqq 1), \tag{2.8a$'$}$$

$$A_{am}(\lambda) = -(\lambda^m \hat{Q}_a(\lambda))_- \quad (m \leqq -1). \tag{2.8b$'$}$$

Acordingly the nonlinear system for the Laurent coefficients of the W and $\hat{W}$ can be now neatly written:

$$\partial_{am} W(\lambda) = (\lambda^m Q_a(\lambda))_+ W(\lambda) \quad (m \geqq 1), \tag{2.11a}$$

$$\partial_{am} W(\lambda) = (\lambda^m \hat{Q}_a(\lambda))_- W(\lambda) \quad (m \leqq -1), \tag{2.11b}$$

$$\partial_{am} \hat{W}(\lambda) = (\lambda^m Q_a(\lambda))_+ \hat{W}(\lambda) \quad (m \geqq 1), \tag{2.11c}$$

$$\partial_{am} \hat{W}(\lambda) = (\lambda^m \hat{Q}_a(\lambda))_- \hat{W}(\lambda) \quad (m \leqq -1). \tag{2.11d}$$

Secondly, the zero-curvature system has the following equivalent expression, which corresponds to the "Lax system" of the KP hierarchy:

$$\partial_{am} Q_b(\lambda) = [A_{am}(\lambda), Q_b(\lambda)] \quad (m \neq 0). \tag{2.12}$$

The occurrence of such "multi-component" (carried by the index a) and "multi-centered" (Laurent expanded around $\lambda = \infty$ and $\lambda = 0$) Lax operators is due to the structure of an underlying group or Lie algebra, which is to be discussed later.

To make the above analogy more accurate, a comparison with a generalized

AKNS system (cf. [FNR]) will be rather helpful. For such a generalized AKNS system a zero-curvature system and a linear system take the following form:

$$[\partial_{am} - A_{am}(\lambda), \partial_{bn} - A_{bn}(\lambda)] = 0 \quad (m, n \neq 0), \tag{2.13}$$

$$(\partial_{am} - A_{am}(\lambda))\Psi = 0 \quad (m \neq 0), \tag{2.14}$$

where

$$A_{am}(\lambda) \overset{\text{def}}{=} \begin{cases} \lambda^m C_a + \sum_{n=0}^{m-1} \lambda^n A_{a,m-n} & \text{for} \quad m \geqq 1 \\ \sum_{n=m}^{-1} \lambda^n A_{a,m-n} & \text{for} \quad m \leqq -1 \end{cases}, \tag{2.15}$$

and $C_a (1 \leqq a \leqq s)$ are a commuting set of constant $r \times r$ matrices. Now in terms of

$$Q_a(\lambda) \overset{\text{def}}{=} W(\lambda) C_a W(\lambda)^{-1}, \tag{2.16a}$$

$$\hat{Q}_a(\lambda) \overset{\text{def}}{=} \hat{W}(\lambda) C_a \hat{W}(\lambda)^{-1}, \tag{2.16b}$$

one can write $A_{am}(\lambda)$ as

$$A_{am}(\lambda) = (Q_a(\lambda)\lambda^m)_+ \quad (m \geqq 1), \tag{2.17a}$$

$$A_{am}(\lambda) = (\hat{Q}_a(\lambda)\lambda^m)_- \quad (m \leqq -1). \tag{2.17b}$$

A counterpart of Eq. (I.11) now reads:

$$\partial_{am} Q_b(\lambda) = [A_{am}(\lambda), Q_b(\lambda)] \quad (m \neq 0). \tag{2.18}$$

A Hamiltonian structure of Kostant–Kirillov type of this hierarchy is presented in [FNR].

III. Preliminary Consideration on Symmetries

Suppose that the $W(\lambda)$ and $\hat{W}(\lambda)$ have a common domain of convergence (which becomes an annulus in the Riemann sphere). Their "matrix ratio"

$$g(\lambda) = W(\lambda)^{-1} \hat{W}(\lambda) \tag{3.1}$$

obeys the linear system

$$(\partial_{am} - \lambda \partial_{a,m-1}) g(\lambda) = 0. \tag{3.2}$$

[*Notational remark.* As in the previous section, only the λ is explicitly indicated though the g, W and $\hat{W}$ also depend on the (x, t) variables; this is merely for the sake of simplifying notations.] In other words $g(\lambda)$ is a function with special dependence on t^{am} and λ as:

$$g(\lambda) = g(t(\lambda), \lambda), \quad t^a(\lambda) = \sum_{m \in \mathbf{Z}} t^{am} \lambda^m. \tag{3.3}$$

The $GL(r)$-valued function $g(x, \lambda)$ of $s+1$ independent variables is nothing other than the patching function of a vector bundle over an associated twistor space.

Conversely if one can find for such a patching function a pair of $GL(r)$-valued functions $W(\lambda)$ and $\hat{W}(\lambda)$ that satisfy Eq. (3.1), the $W(\lambda)$ and $\hat{W}(\lambda)$ then give rise to a solution of our hierarchy. This is a hierarchy version of the twistor-theoretical method of Ward [Wa].

As Ueno and Nakamura [UN] argued in the case of self-dual Yang–Mills fields, one can now derive "hidden symmetries" of the hierarchy as a consequence of this basic correspondence between solutions of the hierarchy and patching functions of twistor spaces. Following the terminology of Ueno and Nakamura, we call them "Riemann–Hilbert transformations." Their infinitesimal generators form a Lie algebra of Kac–Moody type (though the occurrence of the x-dependence in $g = g(x, \lambda)$ is a new circumstance).

Actually one can find two types of Riemann–Hilbert transformations. In the language of the patching function $g(\lambda)$ these transformations are the right and left multiplication of matrix functions $g_{L,R}(\lambda)$ with the same property, i.e. Eq. (3.2) or Eq. (3.3), as:

$$g(\lambda) \mapsto g_L(\lambda) g(\lambda) g_R(\lambda)^{-1}. \tag{3.4}$$

Let us call these transformations, respectively, *left* and *right* Riemann–Hilbert transformations. The symmetries first considered by Ueno and Nakamura [UN] and later by [N] as well as by the present author in [T1–3] are in fact only one of them. The set of all patching functions evidently forms a group (of loop type) under matrix multiplication. At the level of patching functions the above transformations are thus simply the right and left translations on this group. An obvious consequence of this observation is that these two sets of transformations *commute* with each other, thus the whole transformation group of RH type decomposes into a direct product of two copies of the above loop-type group. An explicit form of their infinitesimal action is to be given later on. As mentioned in the Introduction, such a direct product structure of a transformation group was already discovered by Avan, Bellon and the present author.

In view of the theory of the KP hierarchy, however, an important class of symmetries is still missing here, which are translations in the space of independent variables (i.e. "time evolutions" and "spatial translations"). These are manifest (rather than "hidden") symmetries of the evolutionary system of w_n and $\hat{w}_n$. In terms of the patching function the effect of these symmetries is simply a translation in the first s variables:

$$g(t(\lambda), \lambda) \mapsto g(t(\lambda) + \varepsilon(\lambda), \lambda), \quad \varepsilon^a(\lambda) = \sum \varepsilon^{am} \lambda^m, \tag{3.5}$$

where ε^{am} are translation parameters. At the level of the W and $\hat{W}$ this should result in the original evolutionary system itself. This type of transformations can be further extended so as to include any coordinate transformation (with parameter λ) in the x-space, $x \mapsto \varphi(x, \lambda)$. A detailed analysis will also be done later on.

To summarize, our hierarchy of integrable gauge field equations possesses the following two distinct types of symmetries:

1. RH (*Riemann–Hilbert*) *type*. They are further classified into "left" and "right" ones, and a general transformation is a mixture of them.

2. CT (*Coordinate Transformation*) *type.* They contain translations as a special subset, which reproduce the dynamical flows of the original hierarchy.

A full Lie algebra (or group) of symmetries should include both these two types of transformations. This point of view appears to be absolutely lacking in preceding studies on integrable systems of gauge fields. In the case of the KP hierarchy ([SS, DJKM]) an infinite dimensional matrix algebra, $gl(\infty)$, occurs as such a Lie algebra. We shall identify a candidate of its counterpart for the present case.

From both technical and conceptual standpoints, it is more natural to represent the effect of the above symmetries as an action on the "initial values" $W(x,\lambda) = W|_{t=0}$ and $\hat{W}(x,\lambda) = \hat{W}|_{t=0}$ at $t = (t^{am}, m \neq 0) = 0$. With this formulation one can understand the symmetries as "kinematical" ones; the previous setting is, so to speak, a "dynamical" realization which represents symmetries on the set of "trajectories" rather than "phase space points." In the "kinematical" picture the relation of the W and $\hat{W}$ to the patching function again becomes a Riemann–Hilbert problem of the form

$$g(x,\lambda) = W(x,\lambda)^{-1}\hat{W}(x,\lambda). \tag{3.1$'$}$$

The three types of symmetries above now take, at the level of $g(x,\lambda)$, the following form:

$$g(x,\lambda) \mapsto g_L(x,\lambda) g(x,\lambda) g_R(x,\lambda)^{-1}, \tag{3.4$'$}$$

$$g(x,\lambda) \mapsto g(x+\varepsilon(\lambda),\lambda), \quad \varepsilon^a(\lambda) = \sum \varepsilon^{am}\lambda^m. \tag{3.5$'$}$$

In the following sections we mostly adopt this "kinematical" formulation of symmetries rather than the previous one.

Remark. As observed in [AB] and [T2], the construction of RT type transformations can be further extended to the case where the domains of convergences of $W(\lambda)$ and $\hat{W}(\lambda)$ do not overlap or, actually, even if they are merely formal Laurent series of λ. Of course the argument based on the use of the patching function $g(\lambda)$ then breaks down, but an alternative method in the above papers leads to such a generalized formulation. Even without such an analysis, in fact, one can infer this fact from a number of formulae of infinitesimal action to be derived in the following sections, because these formulae themselves are built up from quantities which are also meaningful in such a purely formal setting.

IV. Transformations of RH Type

To derive an infinitesimal form of these symmetries, we take

$$g_L(x,\lambda) = \exp \varepsilon P_L(x,\lambda), \tag{4.1a}$$

$$g_R(x,\lambda) = \exp \varepsilon P_R(x,\lambda), \tag{4.1b}$$

where $P_{R,L}(x,\lambda)$ are $gl(r)$-valued functions with the same analytical properties as $g_{R,L}(x,\lambda)$, and compute the associated transformation of $W(x,\lambda)$ and $\hat{W}(x,\lambda)$ up to the first order of the infinitesimal parameter ε. Remember that we are now dealing with a "kinematical" representation of symmetries, as mentioned at the end of the last section, in terms of the "initial values" $W(x,\lambda)$ and $\hat{W}(x,\lambda)$ at $t=0$. Now a transformation of RH type, $(W,\hat{W})\mapsto(W_\varepsilon,\hat{W}_\varepsilon)$, with the above (g_L,g_R) is determined by the relation

$$W_\varepsilon^{-1}\hat{W}_\varepsilon=\exp(\varepsilon P_L)W^{-1}\hat{W}\exp(-\varepsilon P_R). \tag{4.2}$$

Rewriting it as

$$W_\varepsilon\exp(\varepsilon P_L)W^{-1}=\hat{W}_\varepsilon\exp(\varepsilon P_R)\hat{W}^{-1} \tag{4.3}$$

and Laurent-expanding both sides to pick out the coefficients of ε, one obtains the basic relation

$$\delta W\cdot W^{-1}+WP_LW^{-1}=\delta\hat{W}\cdot\hat{W}^{-1}+\hat{W}P_R\hat{W}^{-1} \tag{4.4}$$

for the associated infinitesimal transformation,

$$\delta W\overset{\text{def}}{=}\frac{d}{d\varepsilon}W_\varepsilon\bigg|_{\varepsilon=0},\qquad \delta\hat{W}\overset{\text{def}}{=}\frac{d}{d\varepsilon}\hat{W}_\varepsilon\bigg|_{\varepsilon=0}.$$

From the $(\quad)_-$-part of Eq. (4.4) one readily finds that

$$\delta W\cdot W^{-1}=(\hat{W}P_R\hat{W}^{-1}-WP_LW^{-1})_-, \tag{4.5}$$

because $(\delta W\cdot W^{-1})_-=\delta W\cdot W^{-1}$ and $(\delta\hat{W}\cdot\hat{W}^{-1})_+=\delta\hat{W}\cdot\hat{W}^{-1}$. Likewise

$$\delta\hat{W}\cdot\hat{W}^{-1}=(WP_LW^{-1}-\hat{W}P_R\hat{W}^{-1})_+. \tag{4.6}$$

One can thus obtain an explicit expression of the action of the infinitesimal transformation $\delta=\delta(P_L,P_R)$ with generators (P_L,P_R).

We now list several basic properties of these RH type transformations:

i) $\delta(P_L,P_R)$ is bilinear in their arguments. In particular,

$$\delta(P_L,P_R)=\delta(P_L,0)+\delta(0,P_R). \tag{4.7}$$

Thus any transformation of RH type splits into two "purely one-sided" pieces.
ii) Two infinitesimal transformations $\delta(P_L,P_R)$ and $\delta(Q_L,Q_R)$ obey the commutation relation

$$[\delta(P_L,P_R),\delta(Q_L,Q_R)]=\delta([Q_L,P_L],[Q_R,P_R]). \tag{4.8}$$

In particular the "left-sided" and "right-sided" transformations commute with each other. (This commutation relation can be checked with direct calculations.)

iii) The leading coefficient $J\overset{\text{def}}{=}\hat{W}_0$ of $\hat{W}$ transforms as

$$\delta(P_L,P_R)J\cdot J^{-1}=\oint(WP_RW^{-1}-\hat{W}P_L\hat{W}^{-1})\frac{d\lambda}{2\pi i\lambda}. \tag{4.9}$$

For the next-to-leading coefficient $K\overset{\text{def}}{=}W_1$ of W, likewise,

$$\delta(P_L,P_R)K=\oint(\hat{W}P_L\hat{W}^{-1}-WP_RW^{-1})\frac{d\lambda}{2\pi i}. \tag{4.10}$$

Here $\oint d\lambda/2\pi i$ denotes the operation of taking, formally, the coefficient of λ^{-1}.

iv) The action of $\delta(P_L, P_R)$ on $W(t,x,\lambda)$ and $\hat{W}(t,x,\lambda)$ (i.e. the "dynamical representation") obeys exactly the same formulae as above except that the $P_{L,R}(x,\lambda)$ should be replaced by $P_{L,R}(t(\lambda),\lambda)$.

From i) and ii) above, in particular, one finds that the δ gives a Lie algebra anti-homomorphism of the *direct sum*

$$\mathbf{g}^{\mathrm{RH}} \stackrel{\mathrm{def}}{=} \mathbf{g}_L^{\mathrm{RH}} \oplus \mathbf{g}_R^{\mathrm{RH}} \tag{4.11}$$

of two loop algebras (from which P_L and P_R are taken) into the set of vector fields on the manifold of $(W, \hat{W})$. (At the group-level this fact is already pointed out in the last section.) As an algebraic model one may take, for example,

$$\mathbf{g}_L^{\mathrm{RH}} = \left\{ P = \sum_{-\infty < n \ll \infty} p_n(x)\lambda^n; p_n(x) \in gl(r, \mathbf{C}[[x]]) \right\}, \tag{4.12a}$$

$$\mathbf{g}_R^{\mathrm{RH}} = \left\{ P = \sum_{-\infty \ll n < \infty} p_n(x)\lambda^n; p_n(x) \in gl(r, \mathbf{C}[[x]]) \right\}, \tag{4.12b}$$

where "$n \ll (\gg) \infty$" means that the range of n is limited by an upper (respectively lower) bound which may depend on P; $\mathbf{C}[[x]]$ denotes the set of all formal power series of x. This setting is stuited for the treatment of formal power series solutions of the hierarchy, and one can indeed check that all the formulae above are still meaningful for such formal Laurent series. This fact further tells us that the separation of RH type transformations into the left and right pieces is effectively a kind of "localization" principle on the Riemann sphere. Namely, left (right) transformations are sitting at the infinity $\lambda = \infty$ (respectively the origin $\lambda = 0$), as one infers from the form of the Laurent expansion in Eq. (4.12) as well as the Laurent-expanded form of the $W(\lambda)$ and $\hat{W}(\lambda)$. Of course if they are merely formal series, the method due to the patching function $g(\lambda)$ obviously breaks down, but the formulae for infinitesimal action of RH type transformations are still meaningful in themselves, and one can indeed rejustify them (cf. the final remark in the last section). From a more general standpoint what we have seen is quite special; the present situation can be further extended to allow an arbitrary number of centers of Laurent expansion on the Riemann sphere (or even on a general Riemann surface) with the same number of W's and g's.

As mentioned in the previous section, the existence of these two types of Riemann–Hilbert transformations has remained to be noticed in almost all literature. Actually some people pointed out a similar situation (cf. [C, Wu, SJ]), but a careful analysis (we omit its detail) shows that their infinitesimal transformations belong to the diagonal subalgebra

$$\mathbf{g}_{\mathrm{diag}}^{\mathrm{RH}} \stackrel{\mathrm{def}}{=} gl(r, \mathbf{C}[[x]] \otimes \mathbf{C}[\lambda, \lambda^{-1}]) \hookrightarrow \mathbf{g}_L^{\mathrm{RH}} \oplus \mathbf{g}_R^{\mathrm{RH}}, \tag{4.13}$$

where the inclusion map sends $P \in \mathbf{g}_{\mathrm{diag}}^{\mathrm{RH}}$ to (P, P). At the group level this corresponds to the case where $g_L = g_R$. It seems somewhat strange that no one in the above literature noticed this simple fact at all. This would be presumably because the main concern of these authors was confined to the effect of transformations on the J-potential (cf. (4.9)), which historically played a central role, rather than the

W and $\hat{W}$. From our point of view the transformations constructed therein correspond to those with generators of the form $(p(x)\lambda^n, p(x)\lambda^n)$ $(p(x)\in gl(r, \mathbf{C}[[x]])$, $n\in\mathbf{Z})$ of $\mathbf{g}^{\mathrm{RH}}_{\mathrm{diag}}$ (though the x dependence is mostly ignored). As remarked in item i) above, their action splits into left and right pieces, but if $n>0$ (<0) the right (respectively left) piece has no effective contribution to the right-hand side of Eq. (4.9). Therefore the situation looked as if there were three distinct classes of transformations, two infinite series indexed with positive or negative integers and an exceptional set corresponding to $n=0$; see, in particular, [SJ]. Our observation above concludes that they in fact belong to the same Lie algebra, $\mathbf{g}^{\mathrm{RH}}_{\mathrm{diag}}$.

Finally we should also note that the above "kinematical representation" of hidden symmetries of RH type are of purely algebraic nature and independent of a system of differential equations under consideration. In particular all the formulae above can be applied to, for example, both the self-dual Yang–Mills case and the principal chiral models. The only difference is that there are no x-variables in the latter case; the presence of x-dependence is an essential feature of the former case. This will be one of the most clear ways to recognize the difference of symmetry properties of these two cases. In fact the difference is two-fold; symmetries of CT type, which are to be discussed in the next section, are also based upon the presence of these x-variables. Principal chiral models have in principle no such symmetries.

V. Transformations of CT Type

Infinitesimal generators for these transformations are vector fields $p(x,\lambda,\partial_x)=\sum p^a(x,\lambda)\partial_a$ in the x-space with the additional loop parameter λ. As discussed in Sect. III, the 1-parameter group of coordinate transformation $\exp \varepsilon p(x,\lambda,\partial_x)$ yields a transformation $(W(x,\lambda), \hat{W}(x,\lambda))\mapsto(W_\varepsilon(x,\lambda), \hat{W}_\varepsilon(x,\lambda))$ with the following relation:

$$W_\varepsilon^{-1}\hat{W}_\varepsilon = \exp\varepsilon p(x,\lambda,\partial_x)(W^{-1}\hat{W}) = \exp\varepsilon p(x,\lambda,\partial_x)W^{-1}\cdot\exp\varepsilon p(x,\lambda,\partial_x)\hat{W}. \quad (5.1)$$

One can derive an infinitesimal form of this transformation in much the same way as in the previous section: First rewrite the above relation as

$$W_\varepsilon\cdot\exp\varepsilon p(x,\lambda,\partial_x)W^{-1} = \hat{W}_\varepsilon\cdot\exp\varepsilon p(x,\lambda,\partial_x)\hat{W}^{-1}, \quad (5.2)$$

then compute both sides up to the first order of ε to pick out their coefficients. This result in the relation

$$\delta W\cdot W^{-1} + Wp(x,\lambda,\partial_x)W^{-1} = \delta\hat{W}\cdot\hat{W}^{-1} + \hat{W}p(x,\lambda,\partial_x)\hat{W}^{-1} \quad (5.3)$$

for the infinitesimal variations

$$\delta W \overset{\mathrm{def}}{=} \frac{d}{d\varepsilon}W_\varepsilon|_{\varepsilon=0}, \quad \delta\hat{W} \overset{\mathrm{def}}{=} \frac{d}{d\varepsilon}\hat{W}_\varepsilon|_{\varepsilon=0}.$$

One can readily solve this equation, too, and finds the following expression of the infinitesimal transformation $\delta=\delta(p)$:

$$\delta W\cdot W^{-1} = (\hat{W}p(x,\lambda,\partial_x)\hat{W}^{-1} - Wp(x,\lambda,\partial_x)W^{-1})_-, \quad (5.4a)$$

$$\delta\hat{W}\cdot\hat{W}^{-1} = (Wp(x,\lambda,\partial_x)W^{-1} - \hat{W}p(x,\lambda,\partial_x)\hat{W}^{-1})_+. \quad (5.4b)$$

As one will readily notice, there is no separation into "left" and "right" pieces unlike the case of RH type; both the W and $\hat{W}$ come to obey seemingly almost the same rule of transformations. The vector field $p(x, \lambda, \partial_x)$ occurs as a differential operator of first order in the above formulae, in contrast to the $\delta(P_L, P_R)$, for which $P_{L,R}(x, \lambda)$ are merely matrix-valued functions. Apart from this difference the $\delta(p)$ satisfies a similar communication relation as:

$$[\delta(p), \delta(q)] = \delta([q, p]), \tag{5.5}$$

where the last commutator is understood as that of differential operators, which again becomes a vector field. Furthermore, these formulae are also valid for the case where the W and $\hat{W}$ are merely formal Laurent series of λ.

An algebraic interpretation of these results is obvious: The infinitesimal transformations of CT type, in the "kinematical picture," gives a Lie algebra anti-homomorphism of

$$\mathbf{g}^{\mathrm{CT}} \stackrel{\mathrm{def}}{=} \sum_{a=1}^{s} \mathbf{C}[[x]][\lambda, \lambda^{-1}]\partial_a \tag{5.6}$$

into vector fields on the manifold of $W(\lambda)$ and $\hat{W}(\lambda)$.

In particular the generators $\lambda^n \partial_a$ ($n \in \mathbf{Z}$, $1 \leqq a \leqq s$) with x-independent coefficients cause a commuting set of flows, which is exactly the dynamics of our hierarchy. For these generators, indeed, Eqs. (5.4) give exactly the right-hand side of evolution equations (2.11). Being "exponentiated" as $\exp(\sum t^{am}\lambda^m \partial_a)$, they give rise to a hierarchy version of the exponential operator $\exp(\bar{z}\lambda\partial_y - \bar{y}\lambda\partial_z)$ (where $(y, z, \bar{y}, \bar{z})$ are complex space-time coordinates) of [T1–2]. Using this exponential operator one can rewrite all results presented therein (for example, construction of solutions as Sato [SS] did in the case of the KP hierarchy) in a fully analogous way.

To go to a "dynamical picture" one just has to replace the x^a in $p(x, \lambda)$ by $t^a(\lambda) = \sum_{n=-\infty}^{\infty} t^{an}\lambda^n$ in the above results. Note that the generators of the time evolutions above are free of this adjustment.

VI. Conclusion

Summarizing the previous consideration, we find that the big Lie algebra

$$\mathbf{g}^{\mathrm{tot}} \stackrel{\mathrm{def}}{=} \mathbf{g}^{\mathrm{RH}} + \mathbf{g}^{\mathrm{CT}}, \tag{6.1}$$

which is now understood as a Lie algebra of matrix differential operators of first order, becomes a symmetry algebra of our gauge field hierarchy.

The structure of this Lie algebra is reminiscent of several Lie algebras that arise in two-dimensional quantum field theories. For example, this Lie algebra is by no means "simple" in the sense that the first piece $\mathbf{g}^{\mathrm{RH}}$ on the right-hand side is an ideal of the whole Lie algebra. A similar situation occurs when one extends a current algebra into a larger Lie algebra by the Sugawara construction. This comparison will become more suggestive if one notices that the second piece $\mathbf{g}^{\mathrm{CT}}$ represents infinitesimal reparametrization of coordinates, which is similar to the

Virasoro algebra. Of course the analogy is not very accurate, because whereas the $\mathbf{g}^{\mathrm{CT}}$ gives rise to reparametrization of the x-variables, Virasoro generators are responsibile for reparametrization of the loop variable λ. In fact, one can further incorporate into the above $\mathbf{g}^{\mathrm{tot}}$ such reparametrization transformations of λ to obtain a still larger Lie algebra, though we shall not go into its detail here. Presumably that would be a maximally enlarged symmetry algebra of the gauge field hierarchy.

Even apart from such a possibility, the result above appears to tell us several significant features that have not been noticed or mentioned in such a clear form. First of all, a comparison with the KP hierarchy can now take a more complete form under the above construction. The second piece $\mathbf{g}^{\mathrm{CT}}$ carries all generators of time evolutions of our gauge field hierarchy; thus one obtains a unified framework to treat them on an equal footing with hidden symmetries of RH type. Actually it appears that even the interpretation as "dynamical flows" is itself a rather new standpoint in the study of integrable systems in gauge theory; efforts in the past were mostly devoted to the study of symmetries of RH type. Because of this the meaning of "integrability" of these equations remained somewhat obscure. This circumstance was in sharp contrast with a number of soliton equations including the KP hierarchy, for which such a dynamical picture has been a basic point of view since the seventies. In this respect Nakamura's formulation of a self-dual Yang–Mills hierarchy, as well as its improved version presented in Sect. II of this paper, is of fundamental importance in itself. Nakamura's work however, as well as others, still lacked a Lie algebraic formulation of the dynamics of flows, which has been one of our main concerns. Our answer to this equation is now summarized in the structure of the Lie algebra $\mathbf{g}^{\mathrm{tot}}$.

Secondly, the occurrence of a Lie algebra of CT type suggests a link with gravitation theory. Typical examples of "integrable systems" therein are the field equations of self-dual Einstein and Hyper-Kähler metrics [HKLR]. A hierarchy structure [T4] and a symmetry algebra [T5] have been discovered. Their structures are very similar to the present case, but the Lie algebra is of pure CT type, i.e., it contains no part like $\mathbf{g}^{\mathrm{RH}}$. This does not mean that the symmetry algebra for such a pure gravitational system is smaller than the case of gauge fields. Rather one may recognize the latter case as a result of "symmetry breaking" of the former. To be precise, one starts from a pure gravitational case with D "space variables" $(\xi^1,\dots,\xi^D)$ from which a symmetry algebra of CT type is to be constructed (note that just as in the preceding sections, these "space variables" do not agree with those in an ordinary physical interpretation); one then divides the totality of these space variables into two pieces as $D=r+s$, $(\xi^1,\dots,\xi^D)=(\psi^1,\dots,\psi^r,\ x^1,\dots,x^s)$, requiring the whole manifold to form a vector bundle in such a way that the first r variables be "fiber coordinates" and the others "base manifold coordinates;" this accordingly reduces the original transformations of CT type to those that retain the vector bundle structure as $\psi\mapsto g(x,\lambda)\psi$, $x\mapsto\varphi(x,\lambda)$; the Lie algebra thus obtained is nothing other than the above $\mathbf{g}^{\mathrm{tot}}$. In the course of this reduction process an associated system of differential equations also comes to take a special form, and finally part of it that corresponds to a special sector of "time variables" decouples to reproduce our gauge field hierarchy. This is simply a conceptual way of understanding and of no practical use, but one can thus anyway recognize a link with integrable systems in gravitation theory. Further this shows an origin of elements of $\mathbf{g}^{CT}$ with x-dependent coefficients $p^a(x, \lambda)$ which seemingly play no substantial role in the description of the flows. They are, so to speak, residual symmetries that survived the above reduction process. One might be able to give them a more active role in some other context.

Acknowledgements. The author is deeply grateful to J. Avan and M. Jacques for discussions.

References

[AB] Avan, J., Bellon, M.: Group of dressing transformations for integrable models in dimension two. Phys. Lett. **213B**, 459–465 (1988)

[C] Chau, L.-L.: Chiral fields, self-dual Yang–Mills fields as integrable systems, and the role of the Kac–Moody algebra. In: Nonlinear Phenomena. Wolf, K. B. (ed.). Lecture Notes in Physics, Vol. **189** Berlin, Heidelberg, New York: Springer 1983

[DJKM] Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. In: Nonlinear integrable systems—Classical theory and quantum theory. Jimbo, M., Miwa, T. (eds.). Singapore: World Scientific 1983

[FNR] Flaschka, H., Newell, A. C., Ratiu, T.: Kac–Moody Lie Algebras and Soliton Equations, Physica **9D**, 300–323 (1983)

[HKLR] Hitchin, N.J., Kahlhede, A., Lindström, U., Roček, M.: Hyperkähler Metrics and Supersymmetry. Commun. Math. Phys. **108**, 535-589 (1987)

[N] Nakamura, Y.: Transformation groups acting on a self-dual Yang–Mills hierarchy. J. Math. Phys. **29**, 244–248 (1988)

[SJ] Saint-Aubin, Y., Jacques, M.: Infinite-dimensional Life algebras acting on the solution space of various σ models. J. Math. Phys. **28**, 2463-2479 (1987)

[SS] Sato, M., Sato, Y.: Soliton equations as dynamical systems on infinite dimensional Grassmann manifold. In: Nonlinear partial differential equations in applied mathematics. Tokyo 1982, Lax, P. D., Fujita, H., Strang, G. (eds.), North-Holland and Kinokuniya 1982

[T1] Takasaki, K.: A new approach to the self-dual Yang–Mills equations. Commun. Math. Phys. **94**, 35–59 (1984)

[T2] Takasaki, K.: ——. II, Saitama Math. J. **3**, 11–40 (1985)

[T3] Takasaki, K.: Issues of multi-dimensional integrable systems. In: Prospect of algebraic analysis. Kashiwara, M., Kawai, T. (eds.). New York: Academic Press

[T4] Takasaki, K.: An infinite number of hidden variables in Hyper–Kähler metrics. J. Math. Phys. **30**, 1515–1521 (1989)

[T5] Takasaki, K.: Hidden symmetries of Hyper–Kähler metrics. RIMS-625, May 1988

[UN] Ueno, K., Nakamura, Y.: Transformation theory for anti-self-dual equations and the Riemann–Hilbert problem. Phys. Lett. **109B**, 273–278 (1982)

[Wa] Ward, R.S.: On the self-dual gauge fields. Phys. Lett. **61A**, 81-82 (1977); Completely solvable gauge-field equations in dimension greater than four. Nucl. Phys. **B236**, 381–396 (1984)

[Wu] Wu,: Y.-S.: The group theoretical aspects of Infinitesimal Riemann–Hilbert transforms and hidden symmetries. Commun. Math. Phys. **90**, 461–472 (1983)

Communicated by H. Araki

Received October 11, 1988; in revised form March 17, 1989