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# The Universal $R$-Matrix for $U_q sl(3)$ and Beyond!

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**Abstract.** The $R$-matrices for the quantised Lie algebras $A_n$ are constructed through the quantum double procedure given by Drinfel'd [6]. The case of $U_q sl(3)$ is thoroughly analysed initially to demonstrate the more subtle points of the calculation. The ease of the calculation for $A_n$ is very dependent on a choice of generators for the Borel subalgebra $U_q b_+$ and its dual, and a certain ordering imposed on these generators which is related to the length of a certain word in the Weyl group.

## Introduction

To every Lia algebra and Kac Moody algebra $g$ there exists a unique Hopf algebra $A$; a one parameter deformation of the universal enveloping algebra of $g$. This is the quantisation of the algebra $g$, and was defined by Drinfel'd [6] and Jimbo [11]. In the terminology of [6], these Hopf algebras turn out to be (pseudo) quasi-triangular Hopf algebras, which means that there exists an element $R \in A \otimes A$, called the universal $R$-matrix, that satisfies certain properties. The recent interest in quantum groups and the associated quantised algebra appears to be based on two of these properties: the $R$-matrix is the quantisation of the classical $r$-matrix [2] associated with $g$, and $R$ satisfies the quantum Yang Baxter equation. The former property is important in attempts to quantise Toda field theories and related systems, since the classical $r$-matrix defines the Poisson structure of the monodromy matrix [8]:

$$\{T \overset{\otimes}{,} T\} = [r, T \otimes T], \tag{1}$$

where any variable dependence of the monodromy matrix $T$ and classical $r$-matrix $r$ (in some representation) has been suppressed. Quantisation is then achieved by interpreting $T$ as a matrix of operators that satisfies an appropriate quantum level

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version of the Poisson bracket [8]:

$$R T_1 T_2 = T_2 T_1 R. \tag{2}$$

This quantisation process quantises the $r$-matrix, interpreting it as a representation of the universal $R$-matrix associated with $A$. The representation corresponds to that initially used for $r$. Problems occur in this quantisation process when taking the continuum limit, and so has tended to be restricted to discrete models. However quantisations of the nonlinear Schrödinger equation and sinh-Gordon model have recently been proposed by Skylanin [17, 18]. Similar techniques and ideas occur in relation to the KdV hierarchy and $W$-algebras [1, 19]. If the representation of $R$ is fundamental, Eq. (2) defines a quantum group generated by the matrix elements $T_{ij}$ [9]. The ease with which this equation can be obtained from the quantised algebra $A$ given any representation of $A$, suggests that the Hopf algebra $A$ and the universal $R$-matrix are of deep significance in this quantisation process.

Secondly we have the quantum Yang Baxter equation, which is intimately linked with the Braid relation. This suggests that any system possessing a braiding may have an underlying quantum group interpretation. This is the case for conformal field theory, whose braiding matrix has a structure very reminiscent of the $R$-matrix provided that $q$ is a root of unity [16]. The connections between CFT and the KdV hierarchy [19] support this suggestion; the KdV hierarchy admitting hamiltonian structures defined by a classical $r$-matrix [13]. The $R$-matrix is, as before, in some representation. However it may be possible to remove the representation; lifting to some universal object encompassing the properties of the CFT.

From a more mathematical point of view, this relationship of the QYBE with the braid relation has been exploited to give universal invariants of links [15], the invariants being valued in $A \otimes A$. This construction uses the universal $R$-matrix and gives an universal invariant for each quantised algebra $A$. Given any representation of $A$, the more usual link invariants are obtained. This construction can be viewed as lifting the representation.

The connections between quantum groups and physics occur through a representation of the $R$-matrix, these normally being evaluated by indirect methods and not using the universal $R$-matrix. This is because although the universal $R$-matrix was defined by Drinfel'd [6], and a method of calculation given via the quantum double construction [6], the method is difficult, and some $R$-matrices in representation form were known [12], [14]. Hence motivation was lacking. However universal objects can display an underlying structure in a more succinct form and encompass representation independent properties. Hence they may ultimately prove useful.

This paper derives the universal $R$-matrix for the Lie algebra sequence $A_n$ by using the quantum double construction [6]. Initially the $U_q sl(3)$ case is analysed in detail, $U_q sl(3)$ having properties not present in the $U_q sl(2)$ case, which was given in [6] and is also analysed together with the quantum double construction in [4]. The main new feature is the $q$-analogue Serre relations [6, 11]. These arise because of the use of generators corresponding only to the simple roots. It is desired to avoid direct use of these Serre relations, and hence generators corresponding to each root of $A_n$ will be defined. It is necessary to order these generators, the chosen ordering being based on the length of the word of the element in the Weyl group that generates the root from $\alpha_1$; the end root of the Dynkin diagram. This ordering is necessary to make the calculation of dual bases of $U_q b_+, U_q b_-$ possible.

The following result is obtained:

$$R_{A_n} = q^{f(H_i)} \prod_{\alpha \in \Phi^+}^{<} E_{q^{-2}}(\lambda e_\alpha \otimes f_\alpha) \quad \text{with} \quad f(H_i) = \sum_{ij} a_{ij}^{-1} H_i \otimes H_j.$$

This involves a $q$-analogue of the exponential function $E_q$, non-coroot generators $\{e_\alpha\}, \{f_\alpha\}$ of $U_q b_+, U_q b_-$ respectively, a generator ordering implied by $<$ and a constant $\lambda \in C[[h]]$. $a_{ij}$ is the Cartan matrix corresponding to $A_n$.

This paper is constructed as follows. In Sect. 1 the Hopf structure of the quantised Lie algebra $U_q sl(3)$ is studied; a system of generators and an adjoint action being given. However the adjoint action does not appear to give a representation of the quantised algebra. Section 2 constructs the dual of the Borel subalgebra $U_q b_+$ of $U_q sl(3)$, the Hopf structure being explicitly constructed from the duality definition. The quantum double of $U_q b_+$ is constructed in Sect. 3 as defined in reference [6]. The $R$-matrix for the quantum double is constructed in Sect. 4 and then passed to the quotient to give that of $U_q sl(3)$. Emphasis is on the choice of generators and bases for the corresponding modules, an appropriate choice making this construction via the quantum double feasible. Section 5 demonstrates that the classical limit of the $R$-matrix reproduces the Lie bialgebra structure, i.e. it gives the classical $r$-matrix of $sl(3)$. Sections 6–8 extend the above constructions to the algebras $A_n$. Probably the essential feature of Sect. 6 is the ordering of all the positive roots and the corresponding generators defined by an adjoint action. The ordering is very important in the calculation of the structure of $U_q b'_+$ and in defining a system of dual bases. In Sect. 7 the dual algebra to $U_q b_+$ is analysed, again with an ordering of the generators. Here it is observed that $U_q b'_+ \cong U_q b_+$, hence only the coalgebra of the dual needs to be calculated; the algebra can be inferred. Finally the $R$-matrix for the quantised Lie algebras $A_n$ is calculated in Sect. 8. The fundamental representation is then used in Sect. 9 to project the universal $R$-matrix to the representation form, this agreeing with that given by Drinfel'd in [6].

The various definitions and quantities required to understand the form of the $R$-matrix are collected in an appendix for ease of reference.

## 0. Notation

It is necessary to assume a certain amount of prerequisites in order to limit the length of this work. The quantum group terminology used is defined in [6], while the structure of the root system may be found in [10]. The quantum double construction of the universal $R$-matrix is defined in [6], while an analysis can be found in [4] together with an explicit construction for the $U_q sl(2)$ case. $h$ will be the deformation parameter associated with the quantised algebra $A_n$. We shall find the following combination useful: $q = e^{h/2}$. This differs from some other works on quantum groups, there being no uniform notation.

In order to simplify notation we shall use the following convention; wherever indices appear we shall assume:

$$i,j\in\{1,2\} \quad \text{sum over simple roots.}$$
$$a,b\in\{1,2,3\} \text{ sum over positive roots.}$$

The explicit numerical values are for the $U_q sl(3)$ example. It must be stressed that in Sect. 1 through 5 the root $\alpha_3$ is not simple. This contrasts to the general case of $A_n$ where this root would be denoted $\alpha_{12}$.

## 1. The Hopf Structure of $U_q sl(3)$

The quantisation of the Lie algebra $sl(3)$ is achieved by defining generators for each coroot and simple root, the latter satisfying a $q$-analogue Serre relation [11]. In the construction of the quantum double, Sect. 3, it will be necessary to obtain the dual to $U_q b_+$, a Borel subalgebra of $U_q sl(3)$. Handling the Serre relation in this dualising process is immensely difficult; hence extra generators will be defined such that the Serre relation is reduced to commutation relations. These generators will be defined via an appropriate analogue of the commutator bracket.

We shall review the structure of the Hopf algebra $U_q sl(3)$ as given in [6]. $Sl(3)$ has 3 roots: $\alpha_1, \alpha_2$ and $\alpha_3 = \alpha_1 + \alpha_2$, with the following inner product structure:

$$(\alpha_i, \alpha_i) = 2, \quad (\alpha_1, \alpha_2) = -1,$$

where $\alpha_1, \alpha_2$ are a choice of simple roots. A suitable basis of generators for $U_q sl(3)$ is the $q$-analogue of the Chevalley basis of $sl(3)$. The generators are $H_1, H_2, X_1^\pm$, $X_2^\pm, X_3^\pm$, i.e. $H_i$ are coroots and the $X_a^\pm$ are the generators corresponding to the roots $\pm\alpha_a$.

The algebra structure is as follows:

$$[H_i, X_i^\pm] = \pm 2X_i^\pm, \quad [H_1, X_2^\pm] = \mp X_2^\pm, \quad [H_2, X_1^\pm] = \mp X_1^\pm,$$

$$[X_i^+, X_j^-] = \pm\delta_{ij} \frac{\sinh\left(\frac{hH_i}{2}\right)}{\sinh\left(\frac{h}{2}\right)}. \tag{3}$$

$X_i^\pm$ satisfy $q$-analogue Serre relations [6] and [11], in this case these are triple relations. However we shall define extra generators $X_3^\pm$ corresponding to the third root $\alpha_3$ such that a direct use of this triple relation is avoided. The extra generators are defined by a $q$-analogue of the adjoint action of $sl(3)$:

$$X_3^\pm = \mathrm{ad}_q X_1^\pm \cdot X_2^\pm := q^{1/2} X_1^\pm X_2^\pm - q^{-1/2} X_2^\pm X_1^\pm. \tag{4}$$

Then impose commutation relations between $X_3^\pm$ and the generators $X_i^\pm$ such that the $q$-analogue triple relations are reproduced. The total algebra structure of the generators $X_3^\pm$ is given by:

$$[H_i, X_3^\pm] = \pm X_3^\pm, \quad i = 1,2$$

$$
\begin{aligned}
&[H_1+H_2, X_3^\pm] = \pm 2X_3^\pm,\\
&\quad \mathrm{ad}_q X_1^\pm \cdot X_3^\pm := q^{-1/2} X_1^\pm X_3^\pm - q^{1/2} X_3^\pm X_1^\pm = 0,\\
&\quad \mathrm{ad}_q X_2^\pm \cdot X_3^\pm := q^{1/2} X_2^\pm X_3^\pm - q^{-1/2} X_3^\pm X_2^\pm = 0.
\end{aligned} \tag{5}
$$

This definition of the adjoint structure is the same as that given in [3].

The coalgebra structure is given by:

$$
\begin{aligned}
&\Delta: U_q sl(3) \to U_q sl(3) \otimes U_q sl(3),\\
&\Delta H_i = H_i \otimes 1 + 1 \otimes H_i,\\
&\Delta X_i^\pm = X_i^\pm \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i^\pm, \quad i=1,2,\\
&\Delta X_3^+ = X_3^+ \otimes q^{H_3/2} + q^{-H_3/2} \otimes X_3^+ + (q-q^{-1}) q^{-H_1/2} X_2^+ \otimes q^{H_2/2} X_1^+.
\end{aligned} \tag{6}
$$

The skew antipode $S_0$ [4], [6], is given by:

$$
\begin{aligned}
&S_0(H_i) = -H_i,\\
&S_0(X_i^\pm) = -q^{\mp 1} X_i^\pm, \quad i=1,2,\\
&S_0(X_3^\pm) = -q^{\mp 2} X_3^{\pm\prime},
\end{aligned} \tag{7}
$$

where $X_3^{\pm\prime} := q^{-1/2} X_1^\pm X_2^\pm - q^{1/2} X_2^\pm X_1^\pm$ is an alternative definition of the generator corresponding to the third root $\alpha_3$. This is further discussed in Sect. 8. These two generators are exchanged under the operation of the (skew) antipode.

There is an alternative choice of generators that is very useful. These are defined as follows:

$$
e_a = q^{H_a/2} X_a^+, \quad f_a = q^{-H_a/2} X_a^-, \quad a = 1,2,3. \tag{8}
$$

These satisfy the relations:

$$
\begin{aligned}
&e_3 = e_1 e_2 - q^{-1} e_2 e_1, \quad 0 = e_1 e_3 - q e_3 e_1,\\
&0 = e_2 e_3 - q^{-1} e_3 e_2 \quad \text{and} \quad e \to f,\\
&\Delta e_i = 1 \otimes e_i + e_i \otimes q^{H_i}, \quad \Delta f_i = q^{-H_i} \otimes f_i + f_i \otimes 1,\\
&\Delta e_3 = 1 \otimes e_3 + e_3 \otimes q^{H_1+H_2} + (q-q^{-1}) e_2 \otimes q^{H_2} e_1,\\
&[e_i, f_j] = \delta_{ij} \frac{2}{\lambda} \sinh\left(\frac{hH_i}{2}\right), \quad \text{where} \quad \lambda = (1-q^{-2}),\\
&S_0(e_i) = -q^{-H_i} e_i, \quad S_0(f_i) = -q^2 q^{H_i} f_i.
\end{aligned} \tag{9}
$$

There is a Borel structure of $U_q sl(3)$ denoted by $U_q b_\pm$. These are Hopf subalgebras of $U_q sl(3)$, $U_q b_+$ being generated by the coroots $H_i$, and the positive root generators $X_a^+$. Similarly $U_q b_-$ is generated by $H_i, X_a^-$. Since emphasis will now be on the Borel subalgebra $U_q b_+$ the $+$ superscript will be dropped for these generators.

## 2. The Dual to $U_q b_+$

The quantum double is isomorphic as a $C[[h]]$-module to $U_q b_+ \otimes U_q b_+^0$ [6], where $U_q b_+^0$ is the dual[1] $U_q b_+'$ with reversed comultiplication. Hence it is necessary

[1] The dual of a Hopf algebra will always refer to the maximal Hopf algebra contained in the dual

to evaluate the Hopf structure of $U_q b'_+$; more specifically obtain a basis of $U_q b'_+$ such that the evaluation map is known. In performing this process the generators are ordered to define a basis of the $C[[h]]$-module $U_q b_+$. The choice of ordering is initially arbitrary, however a more suitable choice for the general case emerges from the $U_q sl(3)$ example.

Consider the $C[[h]]$-module structure of $U_q b_+$. Since we may commute any of the generators $H_1, H_2, X_1^{\pm}, X_2^{\pm}, X_3^{\pm}$ a suitable basis for this module is:

$$H_1^a H_2^b X_1^c X_2^d X_3^e, \quad a,b,c,d,e \in Z_{\geqq 0}. \tag{10}$$

Any element of the dual is uniquely defined by its values on this system of basis elements[2]. Hence define the following dual elements $W_1, W_2, Y_1, Y_2, Y_3 \in U_q b'_+$ by:

$$W_i(H_j) = \delta_{ij}, \quad Y_a(X_b) = \delta_{ab} \tag{11}$$

with all other evaluations being zero on the chosen basis of the $C[[h]]$-module $U_q b_+$. Choose the following basis for the module generated by these elements:

$$W_1^a W_2^b Y_1^c Y_2^d Y_3^e, \quad a,b,c,d,e \in Z_{\geqq 0}.$$

There is a Hopf algebra structure induced on this $C[[h]]$-module; the algebra structure of $U_q b_+$ induces the coalgebra structure, and likewise the coalgebra structure of $U_q b_+$ induces the algebra structure. The Hopf structure will be evaluated, and it turns out to be isomorphic to that of $U_q b_+$. These elements span the Hopf dual to $U_q b_+$, this being obvious once we have established the existence of dual bases in Sect. 4a.

*The Commutation Rules for* $U_q b'_+$. The multiplication structure of the dual is defined as follows:

$$YY'(X) = Y \otimes Y'(\Delta X), \quad \forall X \in U_q b_+, \quad \forall Y, Y' \in U_q b'_+, \tag{12}$$

with the evaluation $Y \otimes Y'(X \otimes X') = (Y, X)(Y', X')$. To define $YY'$ the element $X$ may be taken to lie in the basis.

The duality relationship between the algebra structure of $U_q b'_+$ and the coalgebra of $U_q b_+$ means that the commutation rules in $U_q b'_+$ will depend on the relationship between the coalgebra maps $\Delta$ and $T \circ \Delta$ in $U_q b_+$. ($T$ is the transpose operator.) We note that the generators $\{H_i, X_i\}$ form Hopf subalgebras of $U_q b_+$, and that the generators of the dual have been defined such that the following can be deduced from the $U_q sl(2)$ case [4]:

$$[W_i, Y_i] = -\frac{h}{2} Y_i.$$

These are also easy to verify directly, using the techniques demonstrated below.

For the calculation of the commutation rule for $W_1$ and $Y_2$ we observe that the defining relation: $W_1 Y_2(X) = W_1 \otimes Y_2(\Delta X)$ is non-zero only if $X = H_1 X_2$. This

[2] For example if $\xi \in U_q b'_+$ then $\xi(X_1 H_1) = \xi(H_1 X_1) - 2\xi(X_1)$. All elements of the dual annihilate the commutation relations

is because the $Y_2$ requires an $X_2$ in $\Delta X$, and the $W_1$ term requires an $H_1$. By reversing the order of $H_1$ and $Y_2$ we have:

$$W_1 Y_2(H_1 X_2) = 1, \quad Y_2 W_1(H_1 X_2) = 1.$$

This implies that: $[W_1, Y_2] = 0$. A similar calculation holds for $W_2, Y_1$ giving the commutation relation: $[W_2, Y_1] = 0$. It can also be proved that $[W_1, W_2] = 0$. For the calculation of the commutation rule for $Y_1, Y_2$ we observe:

$$Y_1 Y_2(X) = Y_1 \otimes Y_2(\Delta X) \neq 0 \quad \text{only if} \quad X = X_1 X_2, X_3.$$

On inserting these two values we obtain:

$$Y_1 Y_2(X_3) = 0, \quad Y_2 Y_1(X_3) = (q - q^{-1}),$$
$$q^{1/2} Y_1 Y_2(X_1 X_2) = q^{-1/2} Y_2 Y_1(X_1 X_2) = 1,$$

the last two evaluations being deduced from:

$$\Delta(X_1 X_2) = q^{-1/2} q^{-H_2/2} X_1 \otimes q^{H_1/2} X_2 + q^{1/2} q^{-H_1/2} X_2 \otimes q^{H_2/2} X_1 + \cdots.$$

Hence this gives: $q^{1/2} Y_1 Y_2 - q^{-1/2} Y_2 Y_1 = -q^{-1/2}(q - q^{-1}) Y_3$. For the calculation of the commutation rule for $Y_i, Y_3$ we observe:

$$Y_3 Y_i(X) = Y_3 \otimes Y_i(\Delta X) \neq 0 \quad \text{only if} \quad X = X_i X_3.$$

This follows on noting that the generator $Y_a$ only has a non-zero evaluation on $X_a$. (For $Y_3$ this can occur in the combination $X_2 X_1$.) Note the occurrence of an $X_2 X_1$ combination due to the presence of $X_1, X_2$ in $\Delta X_3$ (6). The following evaluation maps are required:

$$Y_3 Y_1(X_1 X_3) = q^{-1/2}, \quad Y_1 Y_3(X_1 X_3) = q^{1/2},$$
$$Y_3 Y_2(X_2 X_3) = q^{-1/2}, \quad Y_2 Y_3(X_2 X_3) = q^{-3/2},$$

in order to deduce the following structure:

$$q^{1/2} Y_3 Y_1 - q^{-1/2} Y_1 Y_3 = 0, \quad q^{-1/2} Y_3 Y_2 - q^{1/2} Y_2 Y_3 = 0.$$

This is very similar to the adjoint structure defined before for $U_q sl(3)$; see (4), (5), and is in fact that of $U_q b_-$.

*The Coalgebra Structure of $U_q b'_+$.* The coalgebra structure of the dual to a Hopf algebra is defined as follows:

$$\Delta Y(X \otimes X') = Y(XX'), \quad \forall\, Y \in U_q b'_+, \quad X, X' \in U_q b_+. \tag{13}$$

Hence we may deduce (by using the evaluation structure given in the appendix) that:

$$\begin{aligned}
&\Delta W_i = 1 \otimes W_i + W_i \otimes 1, \\
&\Delta Y_1 = 1 \otimes Y_1 + Y_1 \otimes e^{W_2 - 2W_1}, \\
&\Delta Y_2 = 1 \otimes Y_2 + Y_2 \otimes e^{W_1 - 2W_2}, \\
&\Delta Y_3 = 1 \otimes Y_3 + Y_3 \otimes e^{-W_1 - W_2} - q Y_2 \otimes e^{W_1 - 2W_2} Y_1.
\end{aligned} \tag{14}$$

For example, to evaluate $\Delta Y_1$ we have:

$$\Delta Y_1(X \otimes X') = Y_1(XX') \neq 0 \quad \text{only if} \quad XX' = X_1 H_1^r H_2^s.$$

$X, X'$ are elements of the chosen basis, but $XX'$ is not necessarily so; hence reordering using the commutation relations is necessary. The product $XX'$ may be split in any fashion to give:

$$\Delta Y_1(1 \otimes X_1 H_1^r H_2^s) = Y_1(X_1 H_1^r H_2^s) = (-2)^r,$$
$$\Delta Y_1(X_1 H_1^r H_2^s \otimes H_1^{r'} H_2^{s'}) = (-2)^{r+r'}.$$

This implies: $\Delta Y_1 = 1 \otimes Y_1 + Y_1 \otimes e^{W_2 - 2W_1}$.

For $\Delta Y_3$ we may use the commutation relations. However the direct calculation of both the commutation relations and the comultiplication relations turns out to be unnecessary since $U_q b_+$ is self dual. This is shown in the general case of $A_n$.

## 3. The Quantum Double of $U_q b_+$

The algebra generated by $Y_1, Y_2$ is a QFSH algebra [6]. We require the QUE algebra equivalent with opposite comultiplication in order to build the quantum double [6]. The maximal ideal $m$ in $U_q b'_+$ is $\langle W_1, W_2, Y_1, Y_2, Y_3 \rangle$, (since $\langle W_1, W_2, Y_1, Y_2 \rangle$ only contains the combination $hY_3$). $m^r$ is the obvious $C[[h]]$-module, the $r^{\text{th}}$ power of $m$. So we construct the QUE equivalent as [4, 6]:

$$\left\{ \sum_{n=0}^{\infty} h^{-n} m^n \right\} \subset U_q b'_+ \otimes_{C[[h]]} C((h)),$$

which is generated by:

$$1, \frac{W_1}{h}, \frac{W_2}{h}, \frac{Y_1}{h}, \frac{Y_2}{h}, \frac{Y_3}{h}.$$

In more simplistic terms, the QUE algebra equivalent has a Hopf structure that is obtained from the commutation and comultiplication relations of the QFSH algebra by the transformation:

$$W_i \to hW_i, \quad Y_a \to hY_a.$$

The evaluations between the QUE algebra duals are also weighted by the above transformation. From now on the dual is interpreted either as the QFSH algebra or as the QUE algebra equivalent as appropriate.

It is useful to choose the following combinations:

$$J_1 = 2(2W_1 - W_2), \quad J_2 = 2(2W_2 - W_1), \quad J_3 = J_1 + J_2 = 2(W_1 + W_2). \qquad (15)$$

The generators $J_i, Y_a$ have an Hopf structure isomorphic to that of the $H_i, f_a$ generators of $U_q b_-$ (9).

The quantum double $D(U_q b_+)$, can be viewed as a lift of the original Hopf algebra $U_q sl(3)$ such that it separates the two Borel subalgebras, i.e. $U_q b_+ \cap U_q b_- = \emptyset$. This can be achieved in an infinite number of isomorphic ways (cf. separating $sl(3)$ Borel subalgebras $b_\pm$), however the requirement that the quantum double is quasi-triangular with the canonical element of $U_q b_+ \otimes U_q b_+^0$, [4], Sect. 4a fixes this uniquely. The quasi-triangular condition can be shown to specify the commutation relations between the sets of generators $H_i, e_a$ and $J_i, Y_a$

[4]. The prescription for obtaining these commutation relations can be manipulated into the following diagram [4]:

$$
\begin{tikzcd}
A^0 \otimes A \arrow[r, "\Delta^2 \otimes \Delta^2"] \arrow[drr, "\gamma"'] & A^{03} \otimes A^3 \arrow[r, "1^3 \otimes S_0 \otimes 1^2"] & A^{03} \otimes A^3 \arrow[r, "(1\,4)", "(3{,}6)"'] & A^0 \otimes A \arrow[dl, "T \circ \gamma"] \\
& & D(A) &
\end{tikzcd}
$$

$$A = U_q b_+ \qquad A^0 = U_q b_+^0 \cong U_q b_-$$

**Fig. 1**

The derivation and explanation of the usage of this diagram appear in [4]. Recall that in this diagram $A^0 = U_q b_+^0$ is the QUE algebra, $(n, m)$ is the evaluation between the entries in positions $n$ and $m$, and $\gamma$ is the multiplication map $\gamma: a \otimes b \to ab$. Knowledge of the $U_q sl(2)$ case [4] and the Hopf subalgebras $U_q b_+, U_q b_+^0$ means that the only relations that need to be calculated are those between the following pairs of elements:

$$H_i, J_j \quad J_1, e_2 \quad J_2, e_1 \quad H_1, Y_2 \quad H_2, Y_1 \quad e_1, Y_2 \quad e_2, Y_1 \quad e_a, Y_3. \tag{16}$$

The last pair may be evaluated by using the commutation relations from the previous pairs. The remaining pairs are evaluated by using the above diagram, Fig. 1. For the pair $J_1, e_2$ we require the following mappings:

$$S_0 \otimes 1^2 \circ \Delta^2 e_2 = 1 \otimes 1 \otimes e_2 + 1 \otimes e_2 \otimes q^{H_2} - q^{-H_2} e_2 \otimes q^{H_2} \otimes q^{H_2},$$
$$\Delta^2 J_1 = J_1 \otimes 1 \otimes 1 + 1 \otimes J_1 \otimes 1 + 1 \otimes 1 \otimes J_1.$$

On performing the evaluations we obtain the commutation rule: $[J_1, e_2] = -e_2$. For a more through exposition of this type of calculation see [4].

The only other mapping that is required to complete the algebra structure of the quantum double is:

$$\Delta^2 Y_1 = q^{-J_1} \otimes q^{-J_1} \otimes Y_1 + q^{-J_1} \otimes Y_1 \otimes 1 + Y_1 \otimes 1 \otimes 1.$$

This allows us to deduce that: $Y_1 e_2 = e_2 Y_1$.

All other pairs (16) have a commutation relation that can be obtained by dualising the two procedures above. Hence the final structure of the quantum double for $U_q sl(3)$ is:

$$
\begin{aligned}
&[H_i, J_j] = 0, \\
&[H_i, e_i] = 2e_i, \quad [J_i, e_i] = 2e_i, \\
&[H_1, e_2] = -e_2, \quad [J_1, e_2] = -e_2 \quad (1 \leftrightarrow 2), \\
&[H_i, Y_i] = -2Y_i, \quad [J_i, Y_i] = -2Y_i, \\
&[H_1, Y_2] = Y_2, \quad [J_1, Y_2] = Y_2 \quad (1 \leftrightarrow 2), \\
&[e_i, Y_j] = \frac{1}{h} \delta_{ij} (q^{H_i} - q^{-J_i}).
\end{aligned} \tag{17}
$$

The triple relations satisfied by $Y_i, e_i$ have been suppressed.

It is observed that $H_i - J_i$ commutes with all elements of the quantum double $D(U_q b_+)$. These generators produce an ideal and coideal, such that:

$$\frac{D(U_q b_+)}{\langle J_i - H_i \rangle} \cong U_q sl(3).$$

The Hopf structure for $U_q sl(3)$, as defined in Sect. (9), is reproduced under the following identification:

$$J_i, H_i \to H_i, \quad e_a \to e_a, \quad Y_i \to \frac{\lambda}{h} f_i, \quad Y_3 \to -\frac{\lambda}{h} f_3. \tag{18}$$

### 4a. The Universal *R*-Matrix for $D(U_q b_+)$

The quasi-triangular structure of the quantum double $D(U_q b_+)$ is given by the canonical element of $U_q b_+ \otimes U_q b_+^0$, i.e. the universal $R$-matrix is $R = \sum_s \zeta_s \otimes \zeta^s$, where $\{\zeta_s\} \in A$, $\{\zeta^s\} \in A^0$ are dual bases [6]. The practical procedure for finding the $R$-matrix is to choose bases for $U_q b_+$, $U_q b'_+$ and to calculate the evaluation matrix. ($U_q b'_+$ is the QFSH algebra.) This is[3]:

$$B^{\{k'\}}_{\{k\}} = W_1^k W_2^l Y_1^u Y_2^v Y_3^w (H_1^{k'} H_2^{l'} e_1^{u'} e_2^{v'} e_3^{w'}),$$
$$\{k'\} = \{\text{Indices for } U_q b_+\} \quad \{k\} = \{\text{Indices for } U_q b_+^0\}.$$

The $R$-matrix is then given by: $R = \sum_{st} B^{-1}{}^s_t \zeta_s \otimes \zeta^t$, where the bases are no longer necessarily dual. For the $U_q sl(3)$ case this is easily calculated to give:

$$B^{\{k'\}}_{\{k\}} = \delta^{\{k'\}}_{\{k\}} q^{-2vw} k!\, l!\, [u; q^{-2}]!\, [v; q^{-2}]!\, [w; q^{-2}]!,$$
$$[u; q]! = \prod_{i=1}^{u} [i; q], \quad [i; q] = \frac{(1-q^i)}{(1-q)}.$$

The $q^{-2vw}$ may be removed by a change of ordering of the $e_a \otimes Y_a$ generators from the 123 ordering to a 132 order. The bases with this ordering are then dual up to a normalization. Without the choice of $e_a$ and $Y_a$ as generators, the process of finding the $R$-matrix becomes much harder since the bases are no longer dual; the Cartan subalgebra generators cause mixing, and hence the semi-infinite matrix $B$ has to be inverted. The $q^{-2vw}$ factor demonstrates very nicely how important the ordering is, since now the summations in the $R$-matrix are independent. For the general case of $A_n$ a suitable ordering will be chosen from the outset. The $R$-matrix can now be written down as:

[3] The generators of $U_q h$ are chosen to be the generators $W$, as opposed to $J$. This is because $W_i$ is dual to $H_i$

$$R = e^{h(H_1 \otimes W_1 + H_2 \otimes W_2)} \sum_{uvw=0}^{\infty} Q'_{uvw} e_1^u e_3^v e_2^w \otimes Y_1^u Y_3^v Y_2^w,$$

$$Q'_{uvw} = \frac{(-1)^v h^{u+v+w}}{[u; q^{-2}]! [v; q^{-2}]! [w; q^{-2}]!}, \tag{19}$$

on summing the $k$, 1 indices. Note that the descendant of the simple root generators lies between those simple root generators, and in this form the summations are independent.

## 4b. The Universal $R$-Matrix for $U_q sl(3)$

So far we have a quasi-triangular Hopf algebra $D(U_q b_+)$ and a Hopf homomorphism $D(U_q b_+) \to U_q sl(3)$. Hence a quasi-triangular structure is induced on $U_q sl(3)$ by this homomorphism. The quotient map to $U_q sl(3)$ was obtained earlier, (18) and this gives the $R$-matrix for $U_q sl(3)$, from (19), as:

$$R_{U_q sl(3)} = q^{f(H_i)} \sum_{a,b,c=0}^{\infty} Q_{abc} e_1^a e_3^b e_2^c \otimes f_1^a f_3^b f_2^c,$$

$$Q_{abc} = (-1)^b ([a; q^{-2}]!, [b; q^{-2}]!, [c; q^{-2}]!)^{-1} (1 - q^{-2})^{a+b+c},$$

$$f(H_i) = \tfrac{2}{3} H_1 \otimes H_1 + \tfrac{2}{3} H_2 \otimes H_2 + \tfrac{1}{3} H_1 \otimes H_2 + \tfrac{1}{3} H_2 \otimes H_1.$$

This may be rewritten in a more succinct form using one of the $q$-analogues of the exponential function [7]:

$$R_{U_q sl(3)} = q^{f(H_i)} E_{q^{-2}}(\lambda e_1 \otimes f_1) E_{q^{-2}}(-\lambda e_3 \otimes f_3) E_{q^{-2}}(\lambda e_2 \otimes f_2),$$

where

$$\lambda = (1 - q^{-2}),$$

$$E_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{[r; q]!}. \tag{20}$$

The sign in the middle term can be removed by incorporating it in the generator $f_3$, i.e. by defining the composite root generators as: $e_{12} = \mathrm{ad}_q e_1 . e_2$, $f_{21} = \mathrm{ad}_q f_2 . f_1$, in the notation of Sect. 6.

The above $R$-matrix has been explicitly checked, and satisfies all the conditions required of it; thus making $U_q sl(3)$ a quasi-triangular Hopf algebra. We again note that the descendant of the simple root generators lies between them. This is a general feature.

## 5. Classical Limit $h \to 0$

Recall that as $h \to 0$ the Lie bialgebra structure of $sl(3)$ is obtained [6]. In particular we can obtain the classical $r$-matrix:

$$r = \left. \frac{R - 1}{h} \right|_{h \to 0},$$

which gives the Lie coalgebra structure:

$$\phi: sl(3) \to sl(3) \otimes sl(3) \quad \text{(a cocycle)},$$
$$\phi(a) = [\Delta_0 a, r] = \delta r(a), \quad \forall a \in sl(3),$$
$$\Delta_0 a = a \otimes 1 + 1 \otimes a.$$

By taking the $h \to 0$ limit:

$$R = 1 + \tfrac{1}{2} f(H_i) + h\{X_1^+ \otimes X_1^- + X_2^+ \otimes X_2^- - X_3^+ \otimes X_3^-\} + O(h^2).$$

Hence, the $r$-matrix is:

$$r = \tfrac{1}{3} H_1 \otimes H_1 + \tfrac{1}{3} H_2 \otimes H_2 + \tfrac{1}{6} H_1 \otimes H_2 + \tfrac{1}{6} H_2 \otimes H_1$$
$$+ X_1^+ \otimes X_1^- + X_2^+ \otimes X_2^- - X_3^+ \otimes X_3^-.$$

It is easily verified that this is the (quasi-triangular) $r$-matrix given in [6].

## 6. A System of Generators for the Quantisation of the Lie Algebras $A_n$

We have a system of generators for $U_q sl(3)$, Sect. 1; the $q$-analogue of a Chevalley basis. It is desired to achieve the same structure for $A_n = U_q sl(n+1)$ and build a dual basis for $U_q b_+, U_q b'_+$. Then the quantum double and finally the $R$-matrix can be constructed. In order to accomplish this, it is necessary to impose an ordering on the positive roots, and also on the generators of $U_q b_+, U_q b'_+$ corresponding to these roots. For instance, it is necessary to choose a basis for the $C[[h]]$-module $U_q b_+$ such that the generators of the dual can be defined; compare this to (10), (11). This will require an ordering of the generators of $U_q b_+$. An astute choice of ordering makes the following calculation of a system of dual bases for $U_q b_+, U_q b'_+$ much simpler than it would be on any arbitrarily imposed ordering.

First, we require some notation and conventions for the root system. Any classical properties of the root system can be found in [10]. Let $S$ be a choice of simple roots for $A_n$; these being numbered along the Dynkin diagram consecutively, and let $\Phi^+$ denote the positive roots. The roots for $A_n$ have the following form:

$$\alpha \in \Phi^+ \text{ iff } \alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \text{ for some } j \geqq i, \tag{21}$$

i.e. a consecutive sum of simple roots. Consider the roots being generated from the ultimate root of the Dynkin diagram $\alpha_1$ by an element of the Weyl group: $\alpha = \sigma(\alpha_1)$. The Weyl group $W_{A_n}$ is generated by the reflections $\sigma_i$ corresponding to the simple roots. Define $\mu(\sigma)$ as the length of the word $\sigma \in W_{A_n}$ with respect to these generators. Each root has a minimal length word associated with it which can be proved to be: $\mu(\alpha) = (j+i) - 2$ with $\alpha$ the root defined above. A partial order[4] is imposed by $\mu$: $\alpha < \beta$ if $\mu(\alpha) < \mu(\beta)$. If $\mu(\alpha) = \mu(\beta)$ then it can be proved that $(\alpha, \beta) = 0$ and vice versa. Hence this partial order is sufficient for the adjoint definition below (22). But for convenience we shall order this case by the number of simple roots in the expansion.

[4] This formulation of the ordering in terms of the length of a word in the Weyl group was suggested by Gérard Watts. The ordering may also be expressed diagrammatically as a pyramid of positive roots with the simple roots on the base, and projecting horizontally

Define the adjoint action of the generators in a Borel subalgebra by the following:

$$\mathrm{ad}_q P_\alpha \cdot O_\beta = P_\alpha O_\beta - q^{(\alpha,\beta)} O_\beta P_\alpha, \quad \alpha < \beta, \tag{22}$$

where $P_\alpha, O_\beta$ are generators corresponding to the roots $\alpha, \beta \in \Phi^+$. Supplement the definition with:

$$\mathrm{ad}_q P_\alpha \cdot O_\beta = -\mathrm{ad}_q O_\beta \cdot P_\alpha. \tag{23}$$

This is suitable for a $H_i, e_a$ system of generators, for a $H_i, X_a$ system the coefficients need to be altered. For instance compare the commutation relations in Sect. 1: (4), (5) and (9).

As in the classical theory the roots are the eigenvalues of the generators in the Cartan subalgebra. However as we are using the universal enveloping algebra the eigenvalues now lie in the $Z$-span of $\Phi^+$. We note that the eigenvalue of $\mathrm{ad}_q P_\alpha \cdot O_\beta$ is $\alpha + \beta$. This definition of adjoint generalises that of [3] and is suitable for all the quantum versions of the Lie algebras.

To ensure that the Serre triple relations do not appear directly in the algebra structure, we shall define a generator for each $\alpha \in \Phi^+$, and a corresponding generator for $-\alpha$. Within each Borel subalgebra we shall order the generators by the ordering induced from that of the corresponding roots. Hence we have the following:

$$P_\alpha < O_\beta \quad \text{iff} \quad \alpha < \beta, \tag{24}$$

for $P_\alpha, O_\beta$ any two generators of $U_q b_+$ (or $U_q b_-$) corresponding to the roots $\alpha, \beta \in \Phi^+$. Where necessary, the coroots will always be assumed to be placed before all other roots.

It is most convenient to extend the $H_i, e_a$ generator structure, since these led to dual bases in the $U_q sl(3)$ example, Sect. 4a. Hence define the following:

$$e_i, f_i \quad \text{for} \quad \alpha_i \in S,$$

$$e_\alpha = \prod_{s \in [i, j-1]}^{<} (\mathrm{ad}_q e_s) \cdot e_j,$$

$$f_\alpha = (-1)^{j-i} \prod_{s \in [i, j-1]}^{<} (\mathrm{ad}_q f_s) \cdot f_j, \quad \text{where} \quad \alpha = \sum_{\substack{s \in [i,j] \\ j > i}} \alpha_s \in \Phi^+. \tag{25}$$

The $<$ behind the product implies that the generators are ordered as above, in an ascending order when read from left to right. This is the successive adjoint action by descending simple root generators. For example in $A_3$ we have:

$$e_{12} = \mathrm{ad}_q e_1 \cdot e_2, \quad e_{23} = \mathrm{ad}_q e_2 \cdot e_3,$$
$$e_{123} = \mathrm{ad}_q e_1 \cdot \mathrm{ad}_q e_2 \cdot e_3 = \mathrm{ad}_q e_1 \cdot e_{23},$$

and the ordering $e_1 < e_{12} < e_2 < e_{123} < e_{23} < e_3$.

Also note that:

$$e_\alpha = \prod_{s \in [i, j-1]}^{<} (\mathrm{ad}_q e_s) \cdot e_j = (-1)^{j-i} \prod_{s \in [i+1, j]}^{>} (\mathrm{ad}_q e_s) \cdot e_i. \tag{26}$$

Thus the generators can be expressed either as the successive adjoint action by descending simple root generators (25), or ascending ones (26). Equation (26) implies that:

$$f_\alpha = \prod_{s\in[i+1,j]}^{>} (\mathrm{ad}_q f_s)\cdot f_i.$$

This differs in sign from the generator used in the $U_q sl(3)$ case, this change being advisable to avoid unpleasant signs in the $R$-matrix. To emphasise the difference between the definitions of the generators $e_\alpha, f_\alpha$, the notation $e_{12}, e_{23}, e_{123}$ and $f_{21}, f_{32}, f_{321}(A_3)$ is preferred for any specific quantised algebra. The proof of (26) is a simple consequence of the fact that: $\mathrm{ad}_q e_i\cdot \mathrm{ad}_q e_j = \mathrm{ad}_q e_j\cdot \mathrm{ad}_q e_i$ if $|i-j|>1$. More specifically we require: $\alpha_i+\alpha_j\notin\Phi^+, (\alpha_i,\alpha_j)=0$.

The generators corresponding to the simple roots have a simpler structure than the other roots because the adjoint structure was defined to reproduce the Serre triple relations. Thus we have, if $\alpha+\beta\notin\Phi^+$:

$$\begin{aligned}&\mathrm{ad}_q e_\alpha\cdot e_\beta = 0 \quad \text{if } \alpha \text{ or } \beta\in S,\\ &\mathrm{ad}_q e_\alpha\cdot e_\beta \neq 0 \text{ in general if neither } \alpha, \beta\notin S.\end{aligned} \tag{27}$$

This is illustrated in $A_3$ by: $\mathrm{ad}_q e_{12}\cdot e_{23} = (q-q^{-1})e_2 e_{123}$.

However we have the very important relation:

$$\mathrm{ad}_q e_\alpha\cdot e_\beta = \pm e_{\alpha+\beta} \quad \text{if} \quad \alpha+\beta\in\Phi^+, \quad \text{the sign for } \alpha \lessgtr \beta. \tag{28}$$

This implies that we may decompose a generator in any manner.

The generators have the following Hopf structure:

$$\begin{aligned}&[H,e_\alpha] = \alpha(H)e_\alpha, \quad [H,f_\alpha] = -\alpha(H)f_\alpha, \quad \forall H\in U_q h,\\ &[e_i,f_j] = \delta_{ij}\frac{2}{\lambda}\sinh\left(\frac{hH_i}{2}\right), \quad \text{where} \quad \lambda=(1-q^{-2}),\\ &\Delta e_i = 1\otimes e_i + e_i\otimes q^{H_i}, \quad \Delta f_i = q^{-H_i}\otimes f_i + f_i\otimes 1,\\ &\Delta e_\alpha = 1\otimes e_\alpha + e_\alpha\otimes q^{H_\alpha} + (q-q^{-1})\sum_{\substack{\beta>\beta'\\ \beta+\beta'=\alpha\\ \beta,\beta'\in\Phi^+}} e_\beta\otimes q^{H_\beta}e_{\beta'},\\ &\forall\alpha\in\Phi^+.\end{aligned} \tag{29}$$

The last equation can be proved by induction. The reason why the chosen ordering is useful is because of the structure of $\Delta e_\alpha$. Let $p_1$ be the projector on the first position of $U_q b_+^{\otimes 2}$, then the map $p_1\circ\Delta$ acting on the non-coroot generators increases the order of the element:

$$p_1\circ\Delta(x) \geqq x, \quad x\in U_q b_+. \tag{30}$$

This is also a characteristic of the dual comultiplication.

## 7. The Dual Structure of $U_q b_+$

In order to calculate the structure of the dual $U_q b'_+$ it is again necessary to choose a basis of the corresponding $C[[h]]$-module (10). This basis is chosen by using the root ordering above. This turns out to be a very useful choice as the adjoint structure is then self dual. The basis is:

$$\prod_{\alpha_i \in S} H_i^{r_i} \prod_{\alpha \in \Phi^+}^{<} e_\alpha^{s_\alpha}, \quad r_i, s_\alpha \in Z_{\geqq 0}. \tag{31}$$

For example, in the $U_q sl(3)$ case we obtain $H_1^{r_1} H_2^{r_2} e_1^{s_1} e_{12}^{s_{12}} e_2^{s_2}$, as suggested in Sect. 4a. Define the elements $W_i$, $Y_\alpha$ $\forall \alpha_i \in S, \alpha \in \Phi^+$ by the evaluation maps:

$$W_i(H_i) = 1, \quad Y_\alpha(e_\alpha) = 1, \tag{32}$$

will all other evaluations being zero on the chosen basis of the $C[[h]]$-module $U_q b_+$. Compare this to Eqs. (10) and (11). These generators will generate $U_q b'_+$ as a Hopf algebra, this being obvious once the existence of dual bases is demonstrated Sect. 8.

The comultiplication structure of $U_q b'_+$ will now be derived. For the generators $W$:

$$\Delta W_i(X \otimes X') = W_i(XX')$$
$$\text{implies that } X, X' = H_i, 1 \text{ or vice versa.}$$

And hence we obtain: $\Delta W_i = 1 \otimes W_i + W_i \otimes 1$.

Consider the generator $Y_\alpha$ for some root $\alpha \in \Phi^+$:

$$\Delta Y_\alpha(X \otimes X') = Y_\alpha(XX') \neq 0 \quad \text{only if} \quad XX' = e_\alpha f(H_i),$$

where $f$ an arbitrary function of the coroots. If a root is not simple we may decompose it into the sum of two other positive roots. This is reflected on the generator level by the adjoint action (28):

$$e_\alpha = \mathrm{ad}_q\, e_{\beta'} \cdot e_\beta, \quad \beta', \beta \in \Phi^+, \quad \beta' + \beta = \alpha, \quad \beta' < \beta.$$

This is valid for any decomposition of the root; hence in order to specify $\Delta Y_\alpha$ we must consider: $X = e_\beta, X' = e_{\beta'}$ for $\beta' < \beta$ as above (this order is reverse to that of the basis). This gives; on inserting two arbitrary functions of $H_i$:

$$Y_\alpha(e_\beta f(H_i) e_{\beta'} f'(H_i)) = -q^{(\beta', \beta)} f(-(\alpha_i, \beta)) f'(-(\alpha_i, \alpha)).$$

Hence it is deduced that:

$$\Delta Y_\alpha = 1 \otimes Y_\alpha + Y_\alpha \otimes e^{-J_\alpha/2} - q \sum_{\substack{\beta > \beta' \\ \beta + \beta' = \alpha \\ \beta, \beta' \in \Phi^+}} Y_\beta \otimes e^{-J_\beta/2} Y_{\beta'} \quad \forall \alpha \in \Phi^+, \tag{33}$$

where $J_\alpha = 2 \sum_k (\alpha, \alpha_k) W_k$. Compare this definition to (15).

The comultiplication (33), is identical to that of $H_i, e_\alpha$ (29), under the identification:

$$J_i \to -H_i, \quad Y_\alpha \to (-1)^m \frac{\lambda}{h} e_\alpha,$$

where $\alpha$ is a sum of $m+1$ simple roots. ($U_q b'_+$ is interpreted as the QUE algebra equivalent.) Hence the commutation structure of the dual is also identical to that of $U_q b_+$, being induced by the comultiplication of $U_q b_+$. This demonstrates that $U_q b'_+ \cong U_q b_+$. (The above method of finding the comultiplication can also be used to show the consistency of the Hopf structure of $U_q b_+$.)

Hence the Hopf structure of $U_q b'_+$ (interpreted as the QUE algebra) is given by:

$$[J_\alpha, Y_\beta] = -(\alpha, \beta) Y_\beta,$$

$$\mathrm{ad}_q Y_{\alpha_i} \cdot Y_\alpha = -\frac{\lambda}{h} Y_{\alpha_i+\alpha} \quad \text{for} \quad \alpha_i < \alpha, \alpha_i + \alpha \in \Phi^+,$$

$$\mathrm{ad}_q Y_{\alpha_i} \cdot Y_\alpha = 0 \quad \text{for} \quad \alpha_i + \alpha \notin \Phi^+,$$

$$\Delta Y_\alpha = 1 \otimes Y_\alpha + Y_\alpha \otimes q^{-J_\alpha} - hq \sum_{\substack{\beta > \beta' \\ \beta + \beta' = \alpha \\ \beta, \beta' \in \Phi^+}} Y_\beta \otimes q^{-J_\beta} Y_{\beta'} \quad \forall \alpha \in \Phi^+. \tag{34}$$

The adjoint structure of the commutation relations implies that the generators $Y_\alpha, \alpha \in S$ satisfy the triple relations [11]. Hence we could reverse the sequence of events in Sect. 1 and drop the non-simple root generators.

It now only remains to choose a basis of the dual, which will again be based on the root ordering defined in Sect. 6. The most convenient basis to choose is:

$$\prod_{\alpha_i \in S} W_i^{r_i} \prod_{\alpha \in \Phi^+}^{<} Y_\alpha^{s_\alpha}, \quad r_i, s_\alpha \in Z_{\geqq 0}, \tag{35}$$

Since it is dual (up to normalisation) to the basis chosen for $U_q b_+$ (31). This is shown in Sect. 8.

This completes the structure of the dual $U_q b'_+$ and so we may proceed with the calculation of the $R$-matrix and the quantum double.

## 8. The $R$-Matrix and the Quantum Double

The construction of the $R$-matrix is again practically trivial because the chosen bases for $U_q b_+, U_q b'_+$ are dual up to a normalisation. This is a consequence of the coalgebra structure (30): $p_1 \circ \Delta(x) \geqq x, x \in A = U_q b_+$ or $U_q b'_+$ with $p_1$ the projector onto the first position of $A^{\otimes 2}$. Observing this, we may calculate the evaluation matrix. Consider first the following evaluation (all generator products being assumed as ordered in the ascending fashion, (31), (35)):

$$\prod_{\alpha \geqq \delta} Y_\alpha^{r_\alpha} \left( \prod_{\alpha \geqq \eta} e_\alpha^{s_\alpha} \right) = Y_\delta^{r_\delta} \otimes \prod_{\alpha > \delta} Y_\alpha^{r_\alpha} \left( \Delta \left( \prod_{\alpha \geqq \eta} e_\alpha^{s_\alpha} \right) \right). \tag{36}$$

This implies that $\delta \geqq \eta$, by the above order increasing property of $\Delta$. The reverse equality is obtained by the dual procedure, and so $\delta = \eta$. Using the coalgebra structure (29), we continue this process to obtain:

$$\begin{aligned} Y_\delta^{r_\delta} \otimes \prod_{\alpha > \delta} Y_\alpha^{r_\alpha} \left( \Delta \left( \prod_{\alpha \geqq \delta} e_\alpha^{s_\alpha} \right) \right) &= Y_\delta^{r_\delta} \otimes \prod_{\alpha > \delta} Y_\alpha^{r_\alpha} \left( e_\delta^{s_\delta} \otimes q^{s_\delta H_\delta} \cdot 1 \otimes \prod_{\alpha > \delta} e_\alpha^{s_\alpha} \right) \\ &= Y_\delta^{r_\delta}(e_\delta^{s_\delta}) \prod_{\alpha > \delta} Y_\alpha^{r_\alpha} \left( \prod_{\alpha > \delta} e_\alpha^{s_\alpha} \right). \end{aligned} \tag{37}$$

Hence the evaluation matrix splits into several independent parts, one for each root $\alpha$.

The final step is to evaluate:

$$Y_\delta^{r_\delta}(e_\delta^{s_\delta}) = \delta^{r_\delta s_\delta}[r_\delta; q^{-2}]!,$$

which is identical to a calculation in [4]. Including the Cartan subalgebra generators is easily accomplished by using a contour integral technique. We evaluate:

$$\exp\left(\sum_i z_i W_i\right) \prod_{\alpha \in \Phi^+} Y_\alpha^{r_\alpha}\left(\exp\left(\sum_i w_i H_i\right) \prod_{\alpha \in \Phi^+} e_\alpha^{s_\alpha}\right)$$

with $w_i, z_i$ complex variables; the entries of the evaluation matrix being given by the coefficients of powers in $w_i, z_i$. These can be isolated by a contour integration about the origin.

The $R$-matrix for the quantum double can now be written down as:

$$R_{D(U_q b_+)} = \exp\left(h \sum_i H_i \otimes W_i\right) \prod_{\alpha \in \Phi^+}^{<} E_{q^{-2}}(h e_\alpha \otimes Y_\alpha). \tag{38}$$

On passing to the quotient Hopf algebra $A_n$ we use the following identification:

$$J_i, H_i \to H_i, \quad Y_\alpha \to \frac{\lambda}{h} f_\alpha,$$

which can be verified by a few calculations to obtain the quantum double structure of $D(U_q b_+)$. These are similar enough to those already evaluated in Sects. 3 and [4] that the proof is omitted.

Hence we obtain the Universal $R$-matrix for $A_n$, from (38) as:

$$R_{A_n} = q^{f(H_i)} \prod_{\alpha \in \Phi^+}^{<} E_{q^{-2}}(\lambda e_\alpha \otimes f_\alpha), \quad f(H_i) = \sum_{ij} a_{ij}^{-1} H_i \otimes H_j, \tag{39}$$

where $a_{ij}$ is the Cartan matrix and $\lambda = 1 - q^{-2}$. Recall that the generators have been defined differently from Sects. 1–5, see (9), (25), this accounting for the sign in (20). Without this absorption of the sign in the definition of $f_\alpha$ (25), we would obtain a sign in the arguments of the exponents: $(-1)^m$ for $\alpha$ the sum of $m+1$ simple roots.

From the universal $R$-matrix for the quantum double we can also obtain the $R$-matrix for any Hopf subalgebra of $A_n$ that is isomorphic to some quotient Hopf algebra of the double. This includes $A_m \subset A_n$ for $m < n$, the quotient being given by setting excess generators to zero. Does this suggest some infinitely generated quantum double?

The above construction probably follows through for the other quantised Lie algebras, the analysis being more complex because of quadratic/quartic Serre relations and branched Dynkin diagrams [5].

*Reversing the Order of the Roots.* The procedure for constructing the quantum double involved a number of choices to be made; in particular we have ordered the roots and defined an adjoint structure (22), (23) and (24). The order of the roots can be reversed and in doing so we transform the adjoint structure by a $q \to q^{-1}$ transformation:

$$\mathrm{ad}_q P_\alpha \cdot O_\beta = P_\alpha O_\beta - q^{-(\alpha,\beta)} O_\beta P_\alpha, \quad \alpha < \beta. \tag{40}$$

This is achieved by recalling that the antipode is an algebra anti-homomorphism [6], it exchanges the two definitions of the adjoint and that $S \otimes S(R) = R$ [4]. Hence the $R$-matrix is also equal to:

$$\begin{aligned} R_{A_n} &= \prod_{\alpha \in \Phi^+}^{>} E_{q^{-2}}(\lambda S(e_\alpha) \otimes S(f_\alpha))\, q^{f(H_i)} \\ &= \prod_{\alpha \in \Phi^+}^{>} E_{q^{-2}}(\lambda q^{-2m+2} q^{-H_\alpha} e'_\alpha \otimes q^{H_\alpha} f'_\alpha)\, q^{f(H_i)}, \end{aligned} \tag{41}$$

where the only changes are the descending order of the roots in the product of $q$-exponentials and the generators $e'_\alpha, f'_\alpha$ are now defined by the adjoint definition (40). The positive root $\alpha$ is the sum of $m+1$ simple roots.

## 9. The $R$-Matrix in the Fundamental Representation

The $R$-matrix in some representation $\rho: A_n \to \mathrm{End}(V, C[[h]])$ can be obtained by the projection of the universal $R$-matrix obtained above. For example using the fundamental representation [6]:

$$\rho(H_i) = E_{i,i} - E_{i+1,i+1}, \quad \rho(e_i) = q^{1/2} E_{i,i+1}, \quad \rho(f_i) = 1^{1/2} E_{i+1,i},$$

where $E_{ij}$ is the matrix with value one at position $i,j$, zero elsewhere, we obtain the $R$-matrix:

$$R_{\rho\otimes\rho} = q^{-1/(n+1)} \left( I \otimes I + (q-1) \sum_{i=1}^{n+1} E_{ii} \otimes E_{ii} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji} \right) \tag{42}$$

as given in [6]. Note that the normalization of the generators is different than those used in [6].

## Conclusion

The $R$-matrices for the quantised Lie algebras $A_n$ have been constructed through the quantum double. The final form involving $q$-analogues of the exponential function was dependent on the summations over generators being independent. A method for achieving this emerged naturally from the $U_q sl(3)$ example, namely choosing an ordering of the roots such that their descendants lie between them. This ordering also allowed the dual structure and the evaluation matrix to be evaluated. The final form is reminiscent of a symbolic structure $R \approx \exp(hr)$ with the "exp" suitably interpreted. The extension to the other Lie algebras is an obvious path to pursue [5], however complications arise: branched Dynkin diagrams do not allow a consecutive ordering of roots, and non-simply laced Lie algebras do not have such a concise form of the comultiplication structure (29). It is even possible that extended Dynkin diagrams can be treated; however the extra root requires special attention.

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## Appendix. The Evaluation Structure of $U_q b_+$ and $U_q b'_+$

The evaluation mappings below are necessary for the calculation of the coalgebra structure of the dual and algebra structure of the quantum double in the $U_q sl(3)$ example Sects. 2–4.

$$W_i^r(H_j^s) = \delta_{ij}\delta^{rs} r!, \quad Y_1(X_1 H_1^r H_2^s) = (-2)^r,$$
$$Y_2(X_2 H_1^r H_2^s) = (-2)^s, \quad Y_3(X_3 H_i^r) = (-1)^r,$$
$$Y_3(X_2 H_1^r H_2^s X_1 H_1^{r'} H_2^{s'}) = -q^{1/2}(-2)^s(-1)^{r'+s'}.$$

## Appendix. Summary of Definitions

The Universal $R$-matrix for the quantisation of the Lie algebra $A_n$ has been derived to be (39):

$$R_{A_n} = q^{f(H_i)} \prod_{\alpha\in\Phi^+}^{>} E_{q^{-2}}(\lambda e_\alpha \otimes f_\alpha),$$
$$f(H_i) = \sum_{ij} \alpha_{ij}^{-1} H_i \otimes H_j.$$

The meaning of the various quantities will be collected here for ease of reference.

$a_{ij}$ is the Cartan matrix and $\lambda = 1 - q^{-2}$.

The $q$-analogue of the exponential function is [7]: $E_q(x) = \sum_{r=0}^{u} \frac{x^r}{[r;q]!}$, where:

$$[u;q]! = \prod_{i=1}^{u} [i;q], \quad [i;q] = \frac{(1-q^i)}{(1-q)}.$$

The roots are ordered in Sect. 6, by the length of the minimal word in the Weyl group needed to generate it from the end root of the Dynkin diagram; $\alpha_1$. The length of the word for the root $\alpha = \sum_{s\in[i,j]} \alpha_s \in \Phi^+, j \geqq i$, is $\mu(\alpha) = (j+i) - 2$. So $\alpha < \beta$ if $\mu(\alpha) < \mu(\beta)$

The generators in each Borel subalgebra are ordered by (24): $P_\alpha < O_\beta$ iff $\alpha < \beta$.

Define the ordered products, Sect. $\prod^<, \prod^>$, where the $<, >$ denote an ascending order of generators and descending order of generators respectively, when read from left to right.

The adjoint map is defined, (22) by: $\mathrm{ad}_q P_\alpha \cdot O_\beta = P_\alpha O_\beta - q^{(\alpha,\beta)} O_\beta P_\alpha$ for $\alpha < \beta$, with the anti symmetry (23): $\mathrm{ad}_q P_\alpha \cdot O_\beta = -\mathrm{ad}_q O_\beta \cdot P_\alpha$. The generators are then defined by (25):

$$e_\alpha = \prod_{s\in[i,j-1]}^{<} (\mathrm{ad}_q e_s)\cdot e_j, \quad f_\alpha = \prod_{s\in[i+1,j]}^{>} (\mathrm{ad}_q f_s)\cdot f_i.$$

The generators $e_i, f_i$ have a structure given in (29), two generators for each simple root of $A_n$.

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