Commun. Math. Phys. 125, 227–237 (1989)

Communications in Mathematical Physics

The Topological Sigma Model

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Abstract. We obtain the invariants of Witten's topological σ-model by gauge fixing a topological action and using BRST symmetry. The fields and the BRST formalism are interpreted geometrically.

1. Introduction

In [1], Witten introduced the idea of topological quantum field theory and showed how to obtain the Donaldson polynomials from his action on four dimensional vector potentials. In [2], we showed how to obtain his result by starting with a purely topological action $I_{\text{top}} = (8\pi)^{-2} \int_{M_4} \operatorname{Tr} F \wedge F$, i.e. the second chern class. Since the action is a constant function on vector potentials, the path integral $\int \mathscr{D}A \exp -I_{\text{top}}$ needs interpretation. We did so by choosing appropriate gauge functions and applying the BRST formalism. The topological invariance followed from the BRST symmetry. This symmetry and the ghost fields introduced by gauge fixing has a geometric interpretation on $M_4 x \mathscr{A}/\mathscr{G}$, where M_4 is the 4-manifold and $\mathscr{A}/\mathscr{G}$ is the orbit space of vector potentials equivalent under gauge transformations.

In this paper we present the case of Witten's topological σ-model [3] in the same spirit. Again we start with a topological action, which we gauge fix. The resultant ghost fields have a geometric interpretation and the cocycles of the BRST symmetry lead to topological invariants.

As we might expect, the moduli space of (anti) selfdual Yang Mills fields is replaced by the space of (antiholomorphic) pseudo-holomorphic maps (in the sense of Gromov [4]) and the topological invariants are multilinear maps on the cohomology of M.

As in the case of the topological Yang-Mills theory, the topological properties of "physical observables" computed by functional integration are guaranteed from the Ward identities of the BRST symmetry corresponding to the enlarged gauge

* Supported by DOE grant 709821

symmetry which characterizes a classical topological action. What truly remain from functional integration are the contributions from zero modes of Dirac operators. All other modes compensate one against another due to the BRST invariance. The existence of these zero modes implies a breaking of ghost number conservation and leads eventually to non-vanishing expectation value for the cocycles of the BRST symmetry. This phenomenon is at the heart of the correspondence between the BRST symmetry cocycles and the cohomology of moduli space.

When we take the Legendre transform and look at the Hamiltonian formalism, we find that the classical Hamiltonian H_{cl} corresponding to the topological action vanishes. There is no classical time evolution and the functional integral is not well defined. Our gauge fixing procedure giving an action I_{GF} leads to a non-vanishing gauge fixed Hamiltonian H_{GF}. Since I_{GF} is s- and d-exact, we expect $H_{\mathrm{GF}}=\frac{1}{2}[Q,\ ]_+$. In fact $H_{\mathrm{GF}}=\frac{1}{2}[Q,\bar{Q}]_+$, where $\bar{Q}$ is an adjoint to the topological BRST charge operator Q. The Hamiltonian form is useful for the geometrical interpretation of our BRST formalism. We see that the gauge function was (and should be chosen) so that the anti-BRST charge operator $\bar{Q}$ is the adjoint of the BRST charge Q. (Of course, as usual in index theory, the nonzero eigenstates can be paired into BRST symmetry doublets, and only the zero modes matter.) The same discussion applies to Yang-Mills topological theory, although we do not display it here.

Since the σ-model can be interpreted as moving strings, it is instructive to firstly apply our procedure to the case of particles, i.e. quantum mechanics. Doing so, we recover supersymmetric quantum mechanics [5]. The σ-model can be seen as a straightforward generalization, at least when the Riemann surface is a torus.

The paper is organized as follows. In Sect. 2, we set up our formalism by considering the simplest case of supersymmetric quantum mechanics. This permits us in Sect. 3 to construct a gauge fixed action from the topological Lagrangian of a σ-model suitable for functional integration. We determine cocycles in a way similar to our earlier treatment of the Yang-Mills case. In Sect. 4 we interpret our results geometrically. (See also Sect. 2.)

Other authors have been studying topological field theories from several viewpoints. The ones we know about are listed in [6].

2. The Supersymmetric Quantum Action as a Topological Action

Let M be a given manifold with a metric $g_{\mu\nu}[X]$ in local coordinates X^μ. Consider the space $\mathscr{L}(M)$ of closed parametrized curves Γ in M. It would be natural in general to consider an action which is constant on components of $\mathscr{L}(M)$, i.e. independent of deformations of $\Gamma\in\mathscr{L}(M)$. More specifically, the natural topological actions I come from closed 1-forms ω on M with $I[\mathrm{X}]=\int_\Gamma dt\omega_\mu(X(t))dX^\mu/\mathrm{d}t$. Since I is topological, any infinitesimal variation

$$\delta X^\mu=\varepsilon^\mu \tag{1}$$

leaves I invariant. Although $H_1(M,R)$ is interesting, for our purpose we only consider the case $I=0$.

The BRST symmetry differential operator s associated to the gauge symmetry (1) is defined as follows:

$$sX^\mu = \psi^\mu, \quad s\psi^\mu = 0, \quad s\bar{\psi}^\mu = b^\mu, \quad sb^\mu = 0. \tag{2}$$

$\psi^\mu(t)$ is the ghost field associated with ε^μ in (1) and the s-variation of X^μ reproduces (1) when one substitutes ψ to ε. The ghost numbers are respectively $0, 1, -1, 0$ for $X^\mu, \psi^\mu, \bar{\psi}^\mu, b^\mu$. As usual, grading properties are determined by the ghost number plus the form degree modulo two.

In order to gauge fix the symmetry (1) and eventually obtain an action quadratic in the velocities while maintaining covariance in the target space M we choose as a gauge function:

$$\mathscr{F}^\mu = \dot{X}^\mu + \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho. \tag{3}$$

The symbol $\dot{}$ means d/dt, so that $\dot{X}^\mu = d/dtX^\mu$. As usual $\Gamma_{\mu\sigma\rho} \equiv \frac{1}{2}(\partial_\sigma g_{\mu\rho} + \partial_\rho g_{\mu\sigma} - \partial_\mu g_{\sigma\rho})$ is the Christoffel symbol and $\Gamma^\nu{}_{\sigma\rho} \equiv g^{\mu\nu}\Gamma_{\mu\sigma\rho}$. In our gauge function (3) the presence of terms quadratic in the ghosts is necessary for general covariance, as will be seen geometrically at the end of this section.

To gauge fix the vanishing actions corresponding to the symmetry (1) by means of the gauge function (3) we postulate BRST invariance. We thus consider as a gauge fixed action:

$$I_{\mathrm{GF}} = \int_\Gamma dts(\bar{\psi}^\mu(g_{\mu\nu}\dot{X}^\nu + \tfrac{1}{2}\Gamma_{\mu\sigma\rho}\bar{\psi}^\sigma\psi^\rho - \tfrac{1}{2}g_{\mu\nu}b^\nu)). \tag{4-a}$$

By using the definition (3) of the BRST operation s one can easily expand (4). In doing so, one must vary the metric $g_{\mu\nu}$ as well as the Christoffels $\Gamma_{\mu\sigma\rho}$ since these objects depend on X^μ and thus transform under the action of s. One gets the following action:

$$\begin{aligned} I_{\mathrm{GF}} = \int_\Gamma dt(&\tfrac{1}{2}g_{\mu\nu}b^\nu b^\mu + b^\mu(g_{\mu\nu}\dot{X}^\nu + \Gamma_{\mu\sigma\rho}\bar{\psi}^\sigma\psi^\sigma) \\ &- \bar{\psi}^\mu(g_{\mu\nu}\dot{\psi}^\nu + \partial_\rho g_{\mu\nu}\dot{X}^\nu\psi^\rho) + \tfrac{1}{2}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\tau\partial_\tau\Gamma_{\mu\sigma\rho}). \end{aligned} \tag{4-b}$$

The b dependence of I_{GF} is purely algebraic. One can thus eliminate the field b from the action by using its equation of motion, $b^\mu = -\dot{X}^\mu - \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho$. One finally gets:

$$I_{\mathrm{GF}} = \int_\Gamma dt(\tfrac{1}{2}g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - \bar{\psi}^\mu(g_{\mu\nu}\dot{\psi}^\nu + \Gamma_{\mu\sigma\rho}\dot{X}^\sigma\psi^\rho) + \tfrac{1}{2}R_{\mu\rho\sigma\tau}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\tau). \tag{5}$$

In (4) $R_{\mu\rho\sigma\tau}$ is the Riemann curvature tensor. I_{GF} can be recognized as the supersymmetric action for quantum mechanics [5]. I_{GF} is invariant under general coordinate transformations in M, owing to our choice of gauge function $\mathscr{F}^\mu$ in (3).

Momenta are defined as:

$$\begin{aligned} p_\mu &= \delta I_{\mathrm{GF}}/\delta\dot{X}^\mu = g_{\mu\nu}\dot{X}^\nu - \Gamma_{\sigma\mu\rho}\bar{\psi}^\sigma\psi^\rho, \\ p_{\psi^\mu} &= \delta I_{\mathrm{GF}}/\delta\dot{\psi}^\mu = \bar{\psi}_\mu. \end{aligned} \tag{6}$$

The canonical commutation relations are $[p_\mu, X^\nu] = [\bar{\psi}_\mu, \psi^\nu]_+ = \delta_\mu{}^\nu$. The Hamiltonian ghosts can of course be represented as Pauli matrices. The Hamiltonian H_{GF} corresponding to I_{GF} is obtained by the Legendre transform and can be written as follows:

$$H_{\mathrm{GF}} = \tfrac{1}{2}(p_\mu + \Gamma_{\sigma\mu\rho}\bar{\psi}^\sigma\psi^\rho)^2 - \tfrac{1}{2}R_{\mu\rho\sigma\tau}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\tau. \tag{7-a}$$

As a matter of fact, one has:

$$H_{\mathrm{GF}} = \tfrac{1}{2}[Q, \bar{Q}]_+, \tag{7-b}$$

where

$$Q = p_\mu\psi^\mu, \tag{8-a}$$

and

$$\bar{Q} = \psi_\nu g^{\mu\nu}(p_\mu + \Gamma^\sigma{}_{\mu\rho}\bar{\psi}_\sigma\psi^\rho).$$

The BRST symmetry of I_{GF} implies $[Q, H_{\mathrm{GF}}] = 0$, also obtainable from the nilpotency property $Q^2 = \frac{1}{2}[Q, Q]_+ = 0$.

One can furthermore verify the nilpotency of the anti-BRST operator $\bar{Q}$: $\bar{Q}^2 = \frac{1}{2}[\bar{Q}, \bar{Q}]_+ = 0$. This result also follows from the fact that the operator $\bar{Q}$ equates $-Q^*$, i.e. minus the adjoint of Q, as can be proven directly. In fact, the BRST charge Q is the differential operator d on forms. The dictionary is as follows: ψ^μ is multiplication by dx^μ and $\bar{\psi}_\nu g^{\mu\nu}$ is interior product by dx^μ. The ψ's being annihilation operators, a differential form μ in local coordinates is $\lambda_{\mu_1\cdots\mu_r}dx^{\mu_1}\cdots dx^{\mu_r} = \lambda_{\mu_1\cdots\mu_r}\psi^{\mu_1}\cdots\bar{\psi}^{\mu_r}|0\rangle$. The inner product of two forms λ and τ is $\langle\lambda,\tau\rangle = \langle 0|\lambda_{\mu_1\cdots\mu_r}\psi^{\mu_1}\cdots\psi^{\mu_r}|\tau_{v_1\cdots v_r}\psi^{v_1}\cdots\psi^{v_r}|0\rangle = \int_M \lambda_{\mu_1\cdots\mu_r}\tau^{\mu_1\cdots\mu_r}\sum(\pm\prod g^{\mu_i\mu_j})$. Now, write the exterior differential as $d = p_\mu\psi^\mu = \psi^\mu\partial/\partial x^\mu$. Then $d^* = \psi^{\mu *}(\partial/\partial x^\mu)^* = g_{\mu\nu}\psi^\nu(\partial/\partial x^\mu)^*$. To check $Q^* = -\bar{Q}$, we need only verify that $(\partial/\partial x^\mu)^* = -(\partial/\partial x^\mu + \Gamma^\sigma{}_{\mu\rho}\bar{\psi}_\sigma\psi^\rho)$, which of course is integration by parts. Note that the Hamiltonian H_{GF} quantizes $\frac{1}{2}(dd^* + d^*d)$. Our choice of gauge function in (3) is the only choice for which $\bar{Q} = -Q^*$. Other choices of gauge functions would also lead to a Q-invariant Hamiltonian of the form $H = \frac{1}{2}[Q, \bar{Q}']_+$, but the operator $\bar{Q}'$ would not be nilpotent and our geometrical interpretation of the gauge fixing would be lost.

Notice that the action of Q reproduces the action of the operation s, up to the change of the auxiliary field b into the momentum p. Moreover the expression of the Hamiltonian in the form $H_{\mathrm{GF}} = \frac{1}{2}[Q, \bar{Q}]_+$ means that H_{GF} is Q-exact.

Finally, it is of interest to write the equations of motion stemming from our gauge fixed action:

$$\begin{aligned} \ddot{X}^\mu + \Gamma^\mu{}_{\sigma\rho}\dot{X}^\sigma\dot{X}^\rho &= g^{\mu\lambda}D_\lambda R_{\mu\rho\sigma\tau}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\tau, \\ \dot{\psi}^\mu + \Gamma^\mu{}_{\sigma\rho}\dot{X}^\sigma\psi^\rho &= R^\mu{}_{\rho\sigma\tau}\psi^\sigma\bar{\psi}^\sigma\psi^\tau. \end{aligned} \tag{9}$$

In the linearized ghost approximation, geodesic motion is a stationary point of our action.

To conclude this section, let us observe that we can introduce a real potential V by the standard substitution of Q by $(\exp - V)Q(\exp V)$ and $\bar{Q}$ by $(\exp + V)\bar{Q}(\exp - V)$.

3. The Topological σ-Model

Let M be a compact n-dimensional manifold with a closed symplectic two form ω equal to $\omega_{\mu\nu} dX^\mu \wedge dX^\nu$ in local coordinates X^μ, $1 \leqq \mu \leqq n$. One has

$$d\omega = d(\omega_{\mu\nu} dX^\mu \wedge dX^\nu) = 0, \tag{10}$$

and $\omega_{\mu\nu}$ is non-singular. The symbol $\wedge$ stands for the wedge product.

Consider the space σ of all smooth maps of N into M, where N is a compact two dimensional oriented manifold. The "winding number" of the map X, a constant function on components of σ, is defined as follows:

$$I_t[X] = \int_N X^*\omega = \int_N \omega_{\mu\nu} dX^\mu \wedge dX^\nu. \tag{11}$$

We wish to define $\int_\sigma \mathscr{D}X \exp -I_t[X]$. Although $I_t[X]$ is independent of metrics, the "measure" $\mathscr{D}X$ requires a choice of metric on N and on M. We thus choose a metric on N making N into a complex manifold with local complex coordinates z and $\bar{z}$ and partial derivatives ∂ and $\bar{\partial}$. From a choice of the metric g or $\langle , \rangle$ and the given non-singular two form ω, we get the non-singular skewsymmetric linear transformation W on $T(M,N)$ characterized by $\omega(u,v) = \langle Wu, v\rangle$. Let $W = J|W| \equiv J(-W^2)^{1/2}$ be the polar decomposition, so that J is skew and orthogonal and $J^2 = -\mathbb{1}$. Hence M has an almost complex structure given by the field J. It will be convenient in what follows to change the metric g to g_1 (i.e. $\langle , \rangle$ to $(\langle , \rangle_1)$, where $\langle u, v\rangle_1 = \langle |W|u, v\rangle$. Then $\omega(u,v) = \langle Wu, v\rangle = \langle |W|Ju, v\rangle = \langle Ju, v\rangle_1$. That is, relative to g_1 the skewsymmetric W becomes J, with $J^2 = -\mathbb{1}$, and we have a Hermitian inner product $g_1 + i\omega$. In the terminology of Gromov [4], J is tamed by ω; ω calibrates J. Henceforth we drop the subscript on g_1.

Generalizing our previous analysis of the Yang-Mills case and the Quantum Mechanics case we thus consider the gauge fixing problem for the following σ-model topological action:

$$I_t = \int_N \omega_{\mu\nu} dX^\mu \wedge dX^\nu = \int_N dz d\bar{z} \omega_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu. \tag{11}$$

To determine an action which is suitable for functional integration from (10) we shall proceed in close analogy with what we have done in Sect. 2. The gauge symmetry which characterizes the action (11) is:

$$\delta X^\mu = \varepsilon^\mu, \tag{12}$$

where ε^μ is infinitesimal and satisfies the required boundary conditions. The corresponding BRST symmetry is thus defined as:

$$sX^\mu = \psi^\mu, \quad s\psi^\mu = 0, \quad s\bar{\psi}^\mu = b^\mu, \quad sb^\mu = 0. \tag{13}$$

We want to probe the moduli space of pseudoholomorphic maps in the sense of Gromov [4]. That is X is pseudoholomorphic if $dX \circ J = J \circ dX$; In other words one has $-\partial_x X = J\partial_y X$ with $z = x + iy$. Equivalently $-i\bar{\partial} X = J\bar{\partial} X$ or $(\mathbb{1} - iJ)\bar{\partial} X = 0$, where $\bar{\partial}X$ means dX restricted to $T^{0,1}(N)$. Let then:[1]

$$\dot{X} = (J - i\mathbb{1})\bar{\partial}X + (J + i\mathbb{1})\partial X = \partial_y X + J\partial_x X \tag{14}$$

so that $\dot{X}$ vanishing means that X is holomorphic. ($\frac{1}{2}(\mathbb{1} - iJ)$ and $\frac{1}{2}(\mathbb{1} + iJ)$ are projectors).

It is easy to check that $\frac{1}{2}\dot{X}^2 \equiv \frac{1}{2}\langle \dot{X}, \dot{X}\rangle_1 = -2iJ_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu {}_+ 2g_{1\mu\nu}\partial X^\mu \bar{\partial} X^\nu$. The first term is the integrand of the topological action I_t and the second one the bosonic string Lagrangian in the background g_1. In formal analogy with Yang-Mills, we define $F_+ \equiv \frac{1}{2}(J - i\mathbb{1})\bar{\partial}X + \frac{1}{2}(J + i\mathbb{1})\partial X$. If we let $F_- \equiv \frac{1}{2}(J + i\mathbb{1})\bar{\partial}X + \frac{1}{2}(J - i\mathbb{1})\partial X$. Then $\frac{1}{2}(F_+{}^2 + F_-{}^2) = 2g_{1\mu\nu}\partial X^\mu \bar{\partial} X^\nu$ and $\frac{1}{2}(F_+{}^2 - F_-{}^2) = -2i\omega_{\mu\nu}\,\partial X^\mu \bar{\partial} X^\nu$. Moreover, the energy is minimized when $\dot{X} = 0$, i.e. when X is holomorphic. We have chosen our orientation of N so that $I_t \geqq 0$.

As in supersymmetric quantum mechanics of Sect. 2, we now add a fermionic term to $\dot{X}$ and obtain the gauge function:

$$\mathscr{F}^\mu = \dot{X}^\mu + \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma \psi^\rho. \tag{15}$$

Here $\Gamma^\mu{}_{\sigma\rho}$ is an orthogonal connection compatible with J so that $DJ = 0$. It is the unitary connection obtained from reducing the frame bundle with Riemannian connection to the unitary group. So $J\Gamma = \Gamma J$ and the curvature 2-form has values in skew adjoint complex matrices. In the Kähler case it is the Riemannian connection, i.e. the Christoffel symbol computed with the metric g_1.

The BRST invariant gauge fixed action corresponding to the gauge function (15) is obtained as in (4), by adding to the classical action an s-gauge fixing action:

$$I_{\mathrm{GF}} = \int_N dz d\bar{z}\omega_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu - \frac{i}{2}\int_N dz d\bar{z} s(\bar{\psi}^\mu(g_{\mu\nu}\dot{X}^\nu + \tfrac{1}{2}\Gamma_{\mu\sigma\rho}\bar{\psi}^\sigma\psi^\rho - \tfrac{1}{2}g_{\mu\nu}b^\nu)). \tag{16}$$

Using the definition (13) of the BRST operation s, and eliminating as in Sect. 2 the field b by its algebraic equation of motion $b^\mu = -\dot{X}^\mu - \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho$, we can rewrite the action in (16) as:

$$I_{\mathrm{GF}} = \int_N -\frac{i}{2} dz d\bar{z}(2g_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu - \bar{\psi}^\mu(g_{\mu\nu}\dot{\psi}^\nu + \Gamma_{\mu\sigma\rho}\dot{X}^\sigma\psi^\rho) + \tfrac{1}{2}R_{\mu\rho\sigma\tau}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi). \tag{17}$$

$\dot{X}^\mu$ has been defined in (14) and $\dot{\psi}^\mu$ means:

$$\dot{\psi}^\mu = s\dot{X}^\mu. \tag{18}$$

Analogous to $D = d + \Gamma$, we have $S = s + \Gamma$ with $\Gamma = \Gamma\psi$. Since $DJ = 0$, $S(J) = 0$ and J and Γ commute, $J\Gamma = \Gamma J$. Hence $\dot{\psi} = (J - i\mathbb{1})\bar{\partial}\psi + (J + i\mathbb{1})\partial\psi + J\Gamma\psi(\partial + \bar{\partial})X$. Equation (17) shows that the gauge fixed version of the topological σ-model action (10) is a string action plus a ghost dependent action. This action can be used for computing the partition function. The procedure followed here is much similar to that used in [2] where we showed that the topological Yang-Mills action

[1] We have chosen to develop the real version, a chiral version $\dot{X} = (J - i\mathbb{1})\bar{\partial}X$ also works. We have also chosen to probe the space of holomorphic solutions and its normal boundle. To probe the space of antiholomorphic maps, change J to $-J$ in (14) with a corresponding change of gauge function

can be gauge fixed into an ordinary Yang-Mills action plus action plus some ghost dependent terms. In particular when eliminating the field b, we found that the classical action I_t cancels against part of the gauge fixing action, in such a way that one eventually recovers the ordinary classical string action. As in the Yang-Mills case, the BRST invariance of the action (17) guarantees that one can compute from this gauged fixed action a set of expectation values which only depends on the topology of the target space M. These Green functions will be shortly determined by closed forms on M.

The action (17) corresponds to a Feynman type gauge with the gauge function (14). In a Landau type gauge the term quadratic in b^μ would have been omitted and the b field equation of motion would have enforced $\mathscr{F}^\mu = \dot{X}^\mu + \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho = 0$ as a gauge condition. The Feynman type gauge that we have obtained in (17) smears this gauge condition.

Cocycles such as those discussed in the Yang-Mills case can now be determined. We first observe that the topological BRST symmetry equations (13) can be rewritten as:

$$(d+s)X^\mu = dX^\mu + \psi^\mu, \tag{19-a}$$

$$(d+s)(dX^\mu + \psi^\mu) = 0. \tag{19-b}$$

Equations (19) are similar in nature with those introduced in [2] for the Yang-Mills theory. We will see in the next section that these equations have a geometrical interpretation.

Since $(d+s)^2 = 0$, the condition (10) satisfied by the 2-form $\omega_{\mu\nu}$ can be rewritten as:

$$(d+s)(\omega_{\mu\nu}(d+s)X^\mu(d+s)X^\nu) = 0. \tag{20}$$

We can replace $(d+s)X^\mu$ by $dX^\mu + \psi^\mu$ in (20). This yields:

$$(d+s)\{\omega_{\mu\nu}(dX^\mu + \psi^\mu)(dX^\mu + \psi^\mu)\} = 0. \tag{21}$$

This equation is the analog for the σ-model of Eq. (11) in [2]. By expansion in ghost number, we have:

$$\begin{aligned} s(\omega_{\mu\nu}dX^\mu \wedge dX^\nu) &= -2d(\omega_{\mu\nu}dX^\mu\psi^\nu), \\ s(\omega_{\mu\nu}dX^\mu\psi^\nu) &= -d(\omega_{\mu\nu}\psi^\mu\psi^\nu), \\ s(\omega_{\mu\nu}\psi^\mu\psi^\nu) &= 0. \end{aligned} \tag{22}$$

Consider now the cocycles $\Delta_2{}^0 = \omega_{\mu\nu}dX^\mu \wedge dX^\nu$, $\Delta_1{}^1 = \omega_{\mu\nu}dX^\mu\psi^\nu$ and $\Delta_0{}^2 = \omega_{\mu\nu}\psi^\mu\psi^\nu$ defined in (22) and respectively integrated over N, on a 1-cycle and at a point in N. (In our notation the upper index stands for ghost number and the lower one for form degree.) The expectation values computed by functionally integrating these objects with the weight $[dX]\ d\psi d\bar{\psi} \exp -I_{\mathrm{GF}}$ will depend only on the topology of the target space and not on the choice of our adapted metric, since the metric has been introduced through a s-exact term. To prove this result, repeat the same argument as in [2] for the Yang-Mills case using the BRST Ward identities.

Generalizations of the cocycles defined in (12, 13) exist if the target space M

has $H^q(M) \neq 0$, i.e. forms $\omega_{\mu_1 \cdots \mu_q}[X]$ which satisfy:

$$d(\omega_{\mu_1 \cdots \mu_q} dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_q}) = 0 \tag{23}$$

but are not globally exact. One gets then the cocyles:

$$\begin{aligned} \Delta_2^{q-2} &= \omega_{\mu_1 \cdots \mu_q} dX^{\mu_1} \wedge dX^{\mu_1} \psi^{\mu_2} \cdots \psi^{\mu_q}, \\ \Delta_1^{q-1} &= \omega_{\mu_1 \cdots \mu_q} dX^{\mu_1} \psi^{\mu_2} \cdots \psi^{\mu_2}, \\ \Delta_0^q &= \omega_{\mu_1 \cdots \mu_q} \psi^{\mu_1} \psi^{\mu_2} \cdots \psi^{\mu_q}. \end{aligned} \tag{24}$$

The Green functions we compute are the expectation values of products of the $\int_{C_i} \Delta_i^{q-i}$, where the σ_i are i-cycles on N, $i = 0, 1, 2$ with respect to the measure $\exp -I_{\mathrm{GF}} \mathscr{D}\psi \mathscr{D}\bar{\psi} \mathscr{D}X$.

We turn to the Hamiltonian formalism. The experience gained in Sect. 2 can now be used to convert our action (17) into a Hamiltonian. We use y as a time variable for performing the Legendre transform. Momenta are defined as:

$$\begin{aligned} p_\mu &= \delta I_{\mathrm{GF}} / \delta \partial_g X^\mu = g_{\mu\nu} \partial_y X^\nu - \Gamma_{\sigma\mu\rho} \bar{\psi}^\sigma \psi^\rho, \\ p_{\psi^\mu} &= \delta I_{\mathrm{GF}} / \delta \partial_y \psi^\mu = g_{\mu\nu} \bar{\psi}^\nu. \end{aligned} \tag{25}$$

The Hamiltonian H_{GF} corresponding to I_{GF} is thus:

$$\begin{aligned} H_{\mathrm{GF}} = \int dx [&\tfrac{1}{2}(p_\mu + \Gamma_{\sigma\mu\rho} \bar{\psi}^\sigma \psi^\rho)^2 + \tfrac{1}{2}(g_{\mu\nu} \partial_x X^\mu \partial_x X^\nu)^2 \\ &+ \bar{\psi}_\sigma J_\mu^\sigma (\partial_x \psi^\mu + 2\Gamma^\mu_{\lambda\rho} (\partial_x X^\lambda) \psi^\rho) - \tfrac{1}{2} R_{\mu\rho\sigma\tau} \bar{\psi}^\mu \psi^\rho \bar{\psi}^\sigma \psi^\tau]. \end{aligned} \tag{26}$$

One has also, using the commutation relations $[p_\mu, X^\nu] = [\psi^\nu, \bar{\psi}_\mu]_+ \sim \delta_\mu^\nu$:

$$H_{\mathrm{GF}} = \tfrac{1}{2}[Q, \bar{Q}]_+ \tag{27}$$

with

$$Q = \int dx \psi^\mu (p_\mu + J_{\mu\nu} \partial_x X^\nu) \tag{28}$$

and

$$\bar{Q} = \int dx \bar{\psi}_\nu g^{\mu\nu} (p_\mu - J_{\mu\nu} \partial_x X^\nu + \Gamma^\sigma_{\mu\rho} \bar{\psi}_\sigma \psi \rho). \tag{29}$$

The BRST operators Q and $\bar{Q}$ can be reduced to the form given in (8) by a substitution $Q \to e^\nu Q e^{-\nu}$, $\bar{Q} \to e^{-\nu} \bar{Q} e^{+\nu}$. The BRST invariance of the action has been transposed into the property that H_{GF} commutes with the nilpotent BRST charge operators Q and $\bar{Q}$. These results are formally similar to those encountered in Sect. 2 for the case of supersymmetric quantum mechanics. Of course, the moduli space can be much richer in the case of the σ-model.

4. Geometrical Interpretation

We now interpret Sect. 3 geometrically.

4a. We first describe the topological invariants that will be obtained from the functional integral based on the action (17). Let us recall that σ is the space of all smooth maps X of N into M, N and M being defined in Sect. 3. We have the evaluation map $e: N \times \sigma \to M$, where $e(n, X) = X(n)$. One pulls back cohomology on M to cohomology on N. Integration over cycles of N gives cohomology on σ. In terms of differential forms, take a closed q-form μ (Eq. 23) on M. $e^*\mu$ is a sum of three forms $\mu_2^{q-2}, \mu_1^{q-1}, \mu_0^q$ graded by $N \times \sigma$ (Eq. 24). Let C_i be i-cycle on N. Then $\int_{C_i} \mu_i^{q-i}$ gives a closed $q-i$ form on σ. The interesting case is $i = 1$.

Let $\mathscr{M} \subset \sigma$ denote the moduli space of pseudo-holomorphic maps. Suppose $\sum_{k=1}^{n} q_k - i_k = \dim \mathscr{M}$. We then get a map of $\bigotimes_1^N H^{j_k}(M, \mathbb{C}) \to \mathbb{C}$ given by $\int_{\mathscr{M}} \left(\int_{C_{i_1}} \mu^{i_1}{}_{q_1} \wedge \cdots \wedge \int_{C_{i_1}} \mu^{i_N}{}_{q_N} \right)$. Integration over $\mathscr{M}$ may not be well defined, but formally this is what the functional integral gives. We do not discuss the infrared problems in the final integration, and focus only on formal properties. One could compute the formal dimension of $\mathscr{M}$ by Riemann Roch [4]. Even when this dimension is positive, $\mathscr{M}$ might be empty. Again see [4] for existence theorems.

4b. We have already motivated our choice of gauge function to probe the space $\mathscr{M}$. Let $T(\sigma, X)$ denote the tangent space of σ at X. An element $\psi \in T(\sigma, X)$ is a vector field along $\sigma(N)$, i.e. a cross section of the vector bundle $X(T(M))$. Since $dX: T(N, n) \to T(M, X(n))$, we can consider dX as a 1-form on $Nx\sigma$ with values in $T(\sigma)$ but with a component only on the N direction.

Normally $\mathscr{F}^\mu$ is a gauge function; here it is a 1-form on $N \times \sigma$ with values in $T(\sigma)$. It has two terms, the first $\dot{X}^\mu$ is discussed above. The second term $\Gamma^\mu{}_{\sigma\rho} \bar{\psi}^\sigma \psi^\rho$ expresses in terms of fermionic variables the connection one form on $e^*T(M)$ inherited from $T(M)$. Here $\bar{\psi}$ is a one form on $N \times \sigma$ with values in $T(\sigma)$, but with a component only in the N direction. Note that $\mathscr{F}^{-1}(0)$ contains $(X, \psi, \bar{\psi})$ with X pseudo-holomorphic and $\psi = 0, \bar{\psi} = 0$.

4c. Once one has a gauge function $\mathscr{F}$, one proceeds as in the Yang Mills case. One introduces a Lagrange multiplier b for $\mathscr{F}$ in order to concentrate the integral on the zeros of $\mathscr{F}$. The new action is the old action plus a gauge fixing action $-\langle b, b \rangle + \langle b, \mathscr{F} \rangle - \mathrm{Log}\,\mathrm{det} d_v \mathscr{F} = s(\bar{\psi}(\mathscr{F} + b))$. In the Yang–Mills case $d_v \mathscr{F}$ means the differential $d_v \mathscr{F}$ of $\mathscr{F}$ in the gauge orbit direction and $sA = D_{A\rho}$, $s\rho = -\frac{1}{2}[\rho, \rho]$, $s\bar{\rho} = b$, $sb = 0$. The $\mathrm{det} d_v \mathscr{F}$ is replaced by the fermionic integral of $\exp \bar{\rho} d_v \mathscr{F}_\rho$.

The cohomology interpretation of the above is as follows:[2] s is differentiation along the gauge orbit $\mathscr{J}A$. Let $\Lambda_v(\mathscr{A})$ be forms along the orbits with coboundary operator s. Let $\Lambda(g)$ be the Grassmann algebra over g, the Lie algebra of infinitesimal gauge transformations and $S(g)$ be polynomials over g. We have the Koszul complex $\Lambda(g) \otimes S(g)$ with coboundary operator induced by $\delta(\bar{\rho} \otimes 1) = 1 \otimes b$ and $\delta(1 \otimes b) = 0$, for $\bar{\rho} \in \Lambda^1(g) = g$ and $b \in S^1(g) = g$ with $b = \bar{\rho}$.

The total complex is $\Lambda_v(\mathscr{A}) \otimes \Lambda(g) \otimes S(g)$ with coboundary operator $d_v \otimes I + I \otimes \delta$. Hence the gauge fixing action above is exact in this cohomology and the total action is closed. The grading is given by degree or ghost number.

[2] We have chosen not to develop the equivarient view point

In our present case we have formally Eq. (13) which we interpret cohomologically as follows:

(i) s means exterior differentiation on σ and sX^μ means the derivative in the ψ direction, $\psi \in T(\sigma, X)$. Thus $0 = s^2 X = s(sX) = s(\psi)$.

(ii) $\bar{\psi}$ is a 1-form on $N \times \sigma$ in the N direction with values in $T(\sigma)$. Let V denote the vector space of such forms. $s\bar{\psi} = b$ and $sb = 0$ is to be interpreted as the coboundary operator $s = \delta$ on the Koszul complex $\Lambda(V) \otimes S(V)$.

The total complex is $\Lambda(N \times \sigma) \otimes \Lambda(V) \otimes S(V)$ with $d + s$ the exterior differential on $N \times \sigma$ and δ on $\Lambda(V) \otimes S(V)$. Since $H^*(\Lambda(V) \otimes S(V), \sigma)$ is trivial, the cohomology of this complex is $H^*(N) \otimes H^*(\sigma)$. As mentioned above, integrating over cycles of N maps $H^*(N) \otimes H^*(\sigma) \to H^*(\sigma)$.

As in Yang Mills, once the gauge function is chosen we set the gauge fixing action as $s(\bar{\psi}(\mathscr{F} + b))$ (Eq. 16). The quadratic part of this action (Eq. 17) is $\langle dX, dX \rangle + \bar{\psi} D \psi$. Here D means $(\mathbb{1} - J)\bar{\partial} + (I + iJ)\partial$, an elliptic operator. D maps section of $T(\sigma, X) \otimes \mathbb{C} = E$ into $E \otimes \Lambda^1(\Sigma)$. E has connection inherited from the unitary connection on $T(M)$ and decomposes into $E^{1,0} \oplus E^{0,1} \xrightarrow{D} E^{1,0} \otimes \Lambda^{0,1}(\Sigma) \oplus E^{0,1} \otimes \Lambda^{1,0}(\Sigma)$. Note that the 0-modes of $\bar{\partial}'$, the kernel of $\bar{\partial}'$, is exactly $T(\mathscr{M}, X) \subset T(\sigma, X)$.

4d. We finally explain what the path integral does. Consider the weak coupling limit. That is, replace σ by the normal bundle to the moduli space $\mathscr{M}$ of holomorphic maps and replace the action by the quadratic part of the action normal to the moduli space at each $X \in \mathscr{M}$. When $X \in \mathscr{M}$ is fixed, (17) gives the measure $\exp \bar{\psi} \bar{\partial} \psi \mathscr{D} \bar{\psi} \mathscr{D} \psi$.

The fermionic integral $\int \mathscr{D} \bar{\psi} \mathscr{D} \psi \exp \bar{\psi} \bar{\partial} \psi \psi_1 \wedge \cdots \wedge \psi_k$ will be 0 unless $\psi_1, \ldots, \psi_k \in \operatorname{Ker} D$ and $\psi_1 \wedge \cdots \wedge \psi_k \neq 0$ in $\Lambda^{\dim(\operatorname{Ker} D)}(\operatorname{Ker} D)$, i.e., unless $\psi_1, \ldots, \psi_k$ is a basis of $\operatorname{Ker} D$. In that case the integral is $\det' D(\psi_1, \ldots, \psi_k/\text{vol. elem. } \operatorname{Ker} D)$. (We are assuming the kernel of D^* is zero for simplicity.)

Consider now a closed form μ on M (Eq. (23) in local coordinates). Pull back μ to $e^*(\mu)$ on $N \times \sigma$ via the evaluation map c. The form $e^*(\mu)$ can be split into ${\mu^2}_0 + {\mu^1}_1 + {\mu^0}_2$ of type $(2,0) + (1,1) + (0,2)$ on $N \times \sigma$. In local coordinates, $e^*(dx^\mu) = (d_N + d_\sigma)(X^\mu) = dX^\mu + \psi^\mu$. Thus Eq. (24) represents the pull back of a closed form on M to $Nx\sigma$. The fermionic integral gives projection on the zero modes, i.e. restriction to $\mathscr{M}$. Thus the path integral in the weak coupling limit gives the integrand in (24). This is Witten's formula (3.49) in [1]. The determinant contribution is cancelled (up to a phase) by $\det(D^* D)^{1/2}$ coming from the kinetic term in the action $\langle dX, dX \rangle$.

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Communicated by A. Jaffe

Received January 10, 1989