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Spectral Asymptotics for the Schrödinger Operator with Potential which Steadies at Infinity*

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Abstract. We consider the discrete spectrum of the selfadjoint Schrödinger operator $A_h = -h^2\Delta + V$ defined in $L^2(\mathbb{R}^m)$ with potential V which steadies at infinity, i.e. $V(x) = g + |x|^{-\alpha} f(1+o(1))$ as $|x| \to \infty$ for $\alpha > 0$ and some homogeneous functions g and f of order zero. Let $\mathfrak{N}_h(\lambda)$, $\lambda \geqq 0$, be the total multiplicity of the eigenvalues of A_h smaller than $M - \lambda$, M being the minimum value of g over the unit sphere S^{m-1} (hence, M coincides with the lower bound of the essential spectrum of A_h). We study the asymptotic behaviour of $\mathfrak{N}_1(\lambda)$ as $\lambda \downarrow 0$, or of $\mathfrak{N}_h(\lambda)$ as $h \downarrow 0$, the number $\lambda \geqq 0$ being fixed. We find that these asymptotics depend essentially on the structure of the submanifold of S^{m-1}, where the function g takes the value M, and generically are nonclassical, i.e. even as a first approximation $(2\pi)^m \mathfrak{N}_h(\lambda)$ differs from the volume of the set $\{(x, \xi) \in \mathbb{R}^{2m} : h^2|\xi|^2 + V(x) < M - \lambda\}$.

1. Introduction

Let $\mathfrak{A}_h \equiv -h^2\Delta + V$ be the Schrödinger operator with domain $C_0^\infty(\mathbb{R}^m)$, $m \geqq 3$. Here $h > 0$ is a constant parameter, Δ is the Laplacian, and V is a real-valued potential which is supposed to possess the following properties:

i) $V \in L^{m/2}_{\text{loc}}(\mathbb{R}^m)$;
ii) V steadies at infinity, i.e. there exist two continuous real-valued functions f and g over the unit sphere S^{m-1} and a positive number α such the asymptotic relation

$$\lim_{|x| \to \infty} |x|^\alpha (V(x) - g(\hat{x})) = f(\hat{x}),\ \hat{x} \equiv x/|x| ,$$

holds uniformly with respect to $\hat{x} \in S^{m-1}$;

Then $\mathfrak{A}_h$ is symmetric and semibounded from below in $L^2(\mathbb{R}^m)$. Denote by A_h the selfadjoint Friedrichs extension of $\mathfrak{A}_h$.

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Set $M = \min_{\omega \in S^{m-1}} g(\omega)$. It is easy to verify that the essential spectrum of the operator A_h coincides with the semiaxis $[M, \infty)$ for each $h > 0$.

Denote by $\mathfrak{N}_h(\lambda)$, $\lambda \geqq 0$, the total multiplicity of the eigenvalues of A_h which lie to the left of the point $M - \lambda$. We shall deal with the asymptotic behaviour of $\mathfrak{N}_1(\lambda)$ as $\lambda \downarrow 0$, or of $\mathfrak{N}_h(\lambda)$ as $h \downarrow 0$, the number $\lambda \geqq 0$ being fixed. There exists extensive literature concerning analogous asymptotics for the case $g \equiv 0$, i.e. for a potential which vanishes at infinity (cf. e.g. [10, Theorems XIII.80 and XIII.82]; see also the more recent works [6, 15, 16] which contain more precise asymptotics formulas). The essentially new point in the present paper is that the function g is not supposed to be equal identically to its minimum value M.

Various aspects of the spectral theory of some quantum-mechanics operators which have much in common with the operator A_h considered here, are discussed in [1, 3, 5, 12]. The interest in such operators has arisen, in particular, in connection with the famous Klein paradox (see [7]) related to the Dirac-type ordinary differential operator with scalar potential $V(x)$ such that $V(x) \to a_\pm$ as $x \to \pm\infty$ with $a_+ \neq a_-$. However, most of the works on the spectral theory of the Schrödinger operator with asymptotically steady potential deal with the continuous spectrum, usually aiming at some scattering-theory results, while the discrete spectrum has not been investigated in detail yet.

2. Statement of Main Results

2.1. The results of the paper are valid under some complementary natural assumptions about the structure of the set $\Phi = \{\omega \in S^{m-1} : g(\omega) = M\}$ and the behaviour of the function g near this set. In this subsection we formulate these assumptions.

In the sequel we suppose that the following condition is fulfilled:

iii) Φ is either a closed connected C^2-submanifold of S^{m-1} of dimension d such that $1 \leqq d \leqq m-2$, or a one-point set $\{z_0\}$ (in the latter case we set $d=0$);

Denote by $\mathfrak{G}$ the standard covariant metric tensor over $S^{m-1} \times S^{m-1}$. Let $d \geqq 1$. For a given $z \in \Phi$ set $\mathcal{N}_z = \mathcal{T}_z S^{m-1} \ominus \mathcal{T}_z \Phi$, where $\mathcal{T}_z S^{m-1}$ and $\mathcal{T}_z \Phi$ are the tangent spaces at z respectively to S^{m-1} and Φ, and the orthogonal completion is taken with respect to the inner product generated by $\mathfrak{G}(z)$. If $\Phi = \{z_0\}$, then $\mathcal{N}_{z_0} \equiv \mathcal{T}_{z_0} S^{m-1}$.

For brevity's sake put $t = m-1-d \equiv \operatorname{codim} \Phi$. Fix $z \in \Phi$. Denote by $\mathfrak{M}(z)$ the restriction of $\mathfrak{G}(z)$ onto $\mathcal{N}_z \times \mathcal{N}_z$. We consider $\mathcal{N}_z$ as a t-dimensional linear space with a norm $|\cdot|_z$ generated by $\mathfrak{M}(z)$. Put $\mathcal{S}_z = \{y \in \mathcal{N}_z : |y|_z = 1\}$ (note that if $t=0$, then $\mathcal{S}_z \equiv \{v_-, v_+\}$ is just a two-point set). We introduce a trivial Riemannian structure on $\mathcal{N}_z$, identifying for each $y \in \mathcal{N}_z$ the tangent space $\mathcal{T}_y \mathcal{N}_z$ with $\mathcal{N}_z$ itself. The canonical measures on Φ (for $d \geqq 1$) and $\mathcal{S}_z$ (for $t > 1$) are denoted respectively by $d\phi$ and $d\sigma_z$.

Next, we describe our assumptions about the asymptotic behaviour of g near Φ. For a sufficiently small number $\varrho > 0$ and given $z \in \Phi$ and $v \in \mathcal{S}_z$ define the unique point $\omega(\varrho, v; z)$ which is connected with z by a geodesic curve of length ϱ such that the tangent vector to this curve at z coincides with v.

Henceforth we suppose that the following assumption holds:

iv) For some $\beta>0$ there exists a finite, positive and uniform with respect to $z\in\Phi$ and $v\in\mathscr{S}_z$ limit

$$\mathscr{K}(v;z)=\lim_{\varrho\downarrow 0}\varrho^{-\beta}(g(\omega(\varrho,v;z))-M) ;$$

For a fixed $z\in\Phi$ and any $y\in\mathscr{N}_z$ set $\mathscr{V}_z(y)=|y|_z^\beta\mathscr{K}(y/|y|_z;z)$ and introduce the selfadjoint in $L^2(\mathscr{N}_z)$ operator $\mathscr{A}(z)=-\Delta_z+\mathscr{V}_z$, where Δ_z is the Laplace-Beltrami operator. Obviously, for each $z\in\Phi$ the operator $\mathscr{A}(z)$ is positively definite and its resolvent is compact. Denote by $\{\Lambda_k(z)\}_{k\geqq 1}$ the nondecreasing sequence of the eigenvalues of $\mathscr{A}(z)$ counted with the multiplicities.

Example. Assume $g\in C^2(S^{m-1})$. Let ζ be some local coordinates on Φ. On a sufficiently small vicinity of Φ on S^{m-1} introduce the geodesic coordinates (ζ,y) such that $y\in\mathscr{N}_{z(\zeta)}$ and Φ is defined by $y=0$. For any given $z\equiv z(\zeta)\in\Phi$ define the "partial" Hesse matrix $\mathfrak{H}(z)=\{\mathfrak{h}_{ij}\}_{i,j=1}^t$, where $\mathfrak{h}_{ij}(\zeta)=\partial^2 g(\zeta,y)/\partial y_i\partial y_j|_{y=0}$. Suppose that the Hermitian in $\mathscr{N}_z$ matrix $\tilde{\mathfrak{H}}(z)=\mathfrak{M}^{-1}(z)\mathfrak{H}(z)$ is positively definite for each $z\in\Phi$. Then condition iv) is fulfilled for $\beta=2$ and $\mathscr{K}(v;z)=\frac{1}{2}\sum_{i,j=1}^{t}\mathfrak{h}_{ij}(z)v^iv^j$.

Denote by $\{\Omega_j^2(z)\}_{j=1}^t$ the eigenvalues of the matrix $\tilde{\mathfrak{H}}(z)$; it is clear that they are independent of the choice of the local coordinates ζ. The operator $\mathscr{A}(z)$ is unitarily equivalent to the harmonic oscillator $-\Delta+\frac{1}{2}\sum_{i=1}^{t}\Omega_i^2(z)x_i^2$, Δ being here the Laplacian in $L^2(\mathbb{R}^t)$. Hence, in this case the eigenvalues $\Lambda_k(z)$ of $\mathscr{A}(z)$ can be computed explicitly and have the form $2^{-\frac{1}{2}}\sum_{i=1}^{t}|\Omega_i(z)|(2k_i+1)$ with nonnegative integer k_i.

2.2. Introduce the numbers $\gamma=\dfrac{1}{2}+\dfrac{1}{\beta}-\dfrac{1}{\alpha}$ and $\theta=-m\gamma+(d+1)/\beta$ which will be met often in the formulation of our main results.

Our first theorem concerns the behaviour of $\mathfrak{N}_1(\lambda)$ as $\lambda\downarrow 0$.

Theorem 2.1. *Let* i)–iv) *hold. Assume* $1\leqq d<m-2$.

a) *Let* $\gamma<0$ (*hence* $\theta>0$). *Then we have*

$$\lim_{\lambda\downarrow 0}\lambda^\theta\mathfrak{N}_1(\lambda)=\mathscr{C}_1\int_\Phi d\phi(z)\int_{\mathscr{S}_z}d\sigma_z(v)(f(z))_-^{m/\alpha}\mathscr{K}^{-t/\beta}(v;z) , \tag{2.1}$$

where $\mathscr{C}_1=\Gamma(\theta)\Gamma(t/\beta)/\alpha\beta(4\pi)^{m/2}\Gamma(1+m/\alpha)$.

b) *Let* $\gamma=0$ (*hence* θ *is positive again*). *Then we have*

$$\lim_{\lambda\downarrow 0}\lambda^\theta\mathfrak{N}_1(\lambda)=\mathscr{C}_2\sum_{k\geqq 1}\int_\Phi d\phi(z)(f(z)+\Lambda_k(z))_-^{(d+1)/\alpha} , \tag{2.2}$$

where $\mathscr{C}_2=\Gamma(\theta)/\alpha(4\pi)^{(d+1)/2}\Gamma(1+(d+1)/\alpha)$.

c) *If* $\gamma>0$, *we have*

$$\mathfrak{N}_1(\lambda)=O(1),\ \lambda\downarrow 0 , \tag{2.3}$$

i.e. the isolated eigenvalues of the operator A_1 *do not accumulate to its essential-spectrum lower bound.*

2.3. In the remaining two theorems we deal with the asymptotics of the quantity $\mathfrak{N}_h(\lambda)$ as $h\downarrow 0$ for a fixed $\lambda \geqq 0$.

Theorem 2.2. *Let* i)–iv) *hold. Assume that either* $\theta<0$ *(hence* $\gamma>0$*) and* $\lambda=0$, *or* $\lambda>0$. *Then we have*

$$\lim_{h\downarrow 0} h^m \mathfrak{N}_h(\lambda)=\mathscr{C}_3 \int_{\mathbb{R}^m} (M-V-\lambda)_+^{m/2}\,dx\ , \tag{2.4}$$

where $\mathscr{C}_3=1/(4\pi)^{m/2}\Gamma(1+m/2)$.

Theorem 2.3. *Let* i)–iv) *hold. Assume* $1\leqq d<m-2$.

a) *Let* $\theta=0$ *(hence* $\gamma>0$*). Then we have*

$$\lim_{h\downarrow 0} h^m |\log h|^{-1}\mathfrak{N}_h(0)=\mathscr{C}_4 \int_{\Phi} d\phi(z) \int_{\mathscr{S}_z} d\sigma_z(v)(f(z))_-^{m/\alpha}\mathscr{K}^{-t/\beta}(v;z)\ , \tag{2.5}$$

where $\mathscr{C}_4=\Gamma(t/\beta)/\alpha\beta\gamma(4\pi)^{m/2}\Gamma(1+m/\alpha)$.

b) *Let* $\theta>0$ *but still* $\gamma>0$. *Set* $\mu=m+\theta/\gamma$ *and* $\kappa=2\beta/(\beta+2)$. *Then we have*

$$\lim_{h\downarrow 0} h^\mu \mathfrak{N}_h(0)=\mathscr{C}_5 \sum_{k\geqq 1} \int_{\Phi} d\phi(z) \Lambda_k^{-(\mu-d-1)/\kappa}(z)(f(z))_-^{\mu/\alpha}\ , \tag{2.6}$$

where $\mathscr{C}_5=\Gamma((\mu-d-1)/\kappa)/\alpha\gamma\kappa(4\pi)^{(d+1)/2}\Gamma(1+\mu/\alpha)$.

Remark 2.1. The hypotheses of Theorem 2.3b) imply that the series at the right-hand side of (2.6) is absolutely convergent uniformly with respect to $z\in\Phi$.

2.4. Now we pass to some brief comments of our results.

Remark 2.2. Theorems 2.1 and 2.3 are valid also in the case $\Phi=\{z_0\}$ (i.e. $d=0$) but the integration over Φ in (2.1)–(2.2) and (2.5)–(2.6) must be omitted and the integrands have to be evaluated for $z=z_0$. Similarly, if $t=1$ so that $\mathscr{S}_z$ coincides with the two-point set $\{v_-, v_+\}$, Theorems 2.1 and 2.3 remain valid if we replace in (2.1) and (2.5) the integration over $\mathscr{S}_z$ by summation of the values of the integrands for v_- and v_+.

Remarks 2.3. Analogues of Theorems 2.1–2.3 are valid also if the manifold Φ consists of several disjoint connected components $\Phi^{(i)}$ of dimension d_i $(0\leqq d_i\leqq m-2)$, provided that on each $\Phi^{(i)}$ condition iv) is fulfilled for $\beta=\beta_i>0$ and a strictly positive function $\mathscr{K}=\mathscr{K}_i$. We do not describe in detail these quite obvious generalizations in order to avoid bulky formulations.

Remark 2.4. For $h>0$ and $\lambda\geqq 0$ denote by $I_h(\lambda)$ the volume of that part of the cotangent bundle $\mathscr{T}^*\mathbb{R}^m\equiv\mathbb{R}^{2m}$, where the value of the complete symbol of the operator A_h is smaller than $M-\lambda$, i.e.

$$I_h(\lambda)=\mathrm{vol}\,\{(x,\xi)\in\mathbb{R}^{2m}: h^2|\xi|^2+V(x)<M-\lambda\}\ .$$

We say that a given asymptotic formula describing the behaviour of $\mathfrak{N}_h(\lambda)$ is classical, if it can be written as $\lim_{\lambda\downarrow 0}(2\pi)^m\mathfrak{N}_1(\lambda)/I_1(\lambda)=1$, or as $\lim_{\lambda\downarrow 0}(2\pi)^m\mathfrak{N}_h(\lambda)/I_h(\lambda)=1$, the number $\lambda\geqq 0$ being fixed.

From this point of view, the formulas (2.1) and (2.4) are classical and the formulas (2.2), (2.5) and (2.6) are nonclassical. Analogously, the asymptotic estimate (2.3) for the case $\theta<0$ (when the phase volume $I_1(0)$ is finite) can be regarded as classical, while the same estimate for $\theta \geqq 0$ but $\gamma>0$ is nonclassical since now $I_1(0)$ is infinite, provided that the set $\{\omega \in S^{m-1} : f(\omega)<0\}$ is not empty.

2.5. The proof of Theorem 2.1 can be found in Sect. 4 and the proofs of Theorems 2.2–2.3 – in Sect. 5, while Sect. 3 contains some necessary auxiliary concepts and results.

Remark 2.5. In the proofs of Theorems 2.1–2.3 we assume $M=0$ without any loss of generality.

The validity of Theorems 2.1–2.3 is established by means of variational methods of Weyl-Courant type (see [10], ch. XIII, and [2]). Besides, we employ the scheme of the Schrödinger operator with operator-valued potential. Such a scheme has been applied recently by many authors (see [4, 8, 11, 13, 14]) in their studies of the spectral asymptotics for Schrödinger operators with "degenerate" potentials, i.e. potentials which grow unboundedly at infinity everywhere except some conic set.

The results of the paper were announced at the Conference on Partial Differential Equations held in Holzhau, GDR, in 1988 and will appear in its proceedings as the short communication [9].

3. Some Auxiliary Concepts and Results

3.1. Let $\mathscr{E} \subseteqq \mathbb{R}^k$, $k \geqq 1$, be an open set; if $k>1$ and the boundary $\partial\mathscr{E}$ is not empty, we suppose that $\partial\mathscr{E}$ is of Lipschitz class and the number of its connected components is finite. Then we say that $\mathscr{E}$ is a regular domain.

Let $\mathscr{E}$ be a regular domain. The standard Lebesgue spaces over $\mathscr{E}$ are denoted by $L^p(\mathscr{E})$, $p \in [1,\infty]$. If $\mathscr{J} \geqq 0$ is a measurable function over $\mathscr{E}$, then $L^p(\mathscr{E};\mathscr{J})$ is the corresponding $\mathscr{J}$-weighted Lebesgue space. The usual Sobolev spaces of functions which are in $L^2(\mathscr{E})$ together with all their derivatives of order l are denoted by $H^l(\mathscr{E})$, $l \in \mathbb{Z}$, $l \geqq 1$; besides, $H_0^l(\mathscr{E})$ is the closure of $C_0^\infty(\mathscr{E})$ in the H^l-norm. Let $B_R \subset \mathbb{R}^k$, $k>1$, be the ball of radius R, centred at the origin. If $\mathscr{E}$ is unbounded, then $\mathring{C}^\infty(\mathscr{E})$ is the class of functions $C^\infty(\overline{\mathscr{E}})$ which vanish outside $\mathscr{E} \cap B_R$ for some $R>0$; if $\mathscr{E}$ is bounded, then $\mathring{C}^\infty(\mathscr{E}) \equiv C^\infty(\overline{\mathscr{E}})$.

3.2. Let $\mathbb{A}=\mathbb{A}^*$ be a linear operator in some Hilbert space $\mathscr{H}$. Denote by $P_{\mathscr{I}}(\mathbb{A})$ its spectral projection corresponding to the set $\mathscr{I} \subseteqq \mathbb{R}$. For $\eta \in \mathbb{R}$ put

$$N_\eta(\mathbb{A}) = \dim P_{(-\infty,\eta)}(\mathbb{A})\mathscr{H} . \tag{3.1}$$

Let q be a semibounded closed quadratic form (QF) in $\mathscr{H}$. Then q generates by the Lax-Milgram theorem a unique selfadjoint operator $\mathbb{A}$. We shall discuss the spectral properties of the QF q meaning the corresponding properties of $\mathbb{A}$ and shall write q instead of $\mathbb{A}$ in the notations of the type of (3.1).

The domain of the semibounded closed QF q is denoted by $D[q]$. The value of the QF q for a given $u \in D[q]$ is denoted by $q[u]$; if q depends on some additional parameters p, we write $q[u;p]$. When we need to indicate only the dependence of q on the parameters p, we use the notation $q(p)$.

3.3. Let $\mathscr{E}\subseteq\mathbb{R}^k$, $k\geqq 1$, be a regular domain. Suppose that for each $x\in\mathscr{E}$ the Hermitian $k\times k$ matrix $\mathscr{W}(x)$ is positively definite and W is a real-valued function over $\mathscr{E}$. Introduce the notation

$$\omega[u;\mathscr{E},\mathscr{W},W]=\int_{\mathscr{E}}\{\langle\mathscr{W}\nabla u,\nabla u\rangle+W|u|^2\}dx\ , \tag{3.2}$$

where $\langle,.,\rangle$ is the usual inner product in $\mathbb{C}^k$.

Let the potential V possess the properties i)–ii). Define the closed lower-bounded in $L^2(\mathbb{R}^m)$ QF

$$a[u;h,V]=\omega[u;\mathbb{R}^m,h^2E,V],\ D[a]=H^1(\mathbb{R}^m)\ , \tag{3.3}$$

where E is the unit $m\times m$ matrix. Evidently, the operator generated by the QF $a(h,V)$ coincides with A_h. Then according to our notations we have

$$\mathfrak{N}_h(\lambda)=N_{-\lambda}(a(h,V)),\ \lambda\geqq 0\ . \tag{3.4}$$

3.4. This subsection contains estimates and asymptotic results concerning the eigenvalues of selfadjoint second-order differential operators.

Let $k\in\mathbb{Z}$, $k\geqq 1$. Set $\imath_k=k/2$ if $k\geqq 3$, $\imath_2>1$ and $\imath_1=1$.

Lemma 3.1. *Let $\mathscr{E}\subseteq\mathbb{R}^k$, $k\geqq 1$, be a regular domain (if $k<3$, we suppose that $\mathscr{E}$ is bounded). Let the matrix-valued function $\mathscr{Q}_1=\mathscr{Q}_1^*>0$ satisfy $\mathscr{Q}_1\in L^1_{\rm loc}(\mathscr{E})$, $\mathscr{Q}_1^{-1}\in L^\infty(\mathscr{E})$, and the scalar function $Q_1=\bar{Q}_1$ satisfy $(Q_1)_+\in L^1_{\rm loc}(\mathscr{E})$, $(Q_1)_-\in L^{\imath}(\mathscr{E})$, $\imath=\imath_k$. Define the* QF $q_1(h,Q_1,Q_1)=\omega(\mathscr{E},h^2Q_1,Q_1)$, $h>0$, *on* $C_0^\infty(\mathscr{E})$ *and close it in* $L^2(\mathscr{E})$.

a) *The estimate*

$$N_0(q_1(h,\mathscr{Q}_1,Q_1))\leqq h^{-k}c_1\|\mathscr{Q}_1^{-1}\|^{k/2}_{L^\infty(\mathscr{E})}\|(Q_1)_-\|^{k/2}_{L^{\imath}(\mathscr{E})}\ ,\qquad \imath=\imath_k\ ,$$

holds with constant c_1 which is independent of h, $\mathscr{Q}_1$, Q_1.

b) *Moreover, we have*

$$\lim_{h\downarrow 0} h^kN_0(q_1(h,\mathscr{Q}_1,Q_1)) \\ =\int_{\mathscr{E}}(\det\mathscr{Q}_1)^{-1/2}(Q_1)_-^{k/2}dx/(4\pi)^{k/2}\Gamma(1+k/2)\ . \tag{3.5}$$

Lemma 3.2. *Let $\mathscr{E}\subseteq\mathbb{R}^k$, $k\geqq 1$, be a regular domain. Assume that the scalar function $Q_2=\bar{Q}_2$ satisfies $(Q_2)_+\in L^1(\mathscr{E}\cap B_R)$, $\forall R>0$, $(Q_2)_-\in L^{\imath}(\mathscr{E})$, $\imath=\imath_k$, and $\tilde{\mathscr{E}}\equiv\operatorname{supp}(Q_2)_-$ is bounded. Let the matrix-valued function $\mathscr{Q}_2=\mathscr{Q}_2^*\geqq 0$ satisfy $\mathscr{Q}_2\in L^\infty(\mathscr{E})$, $\mathscr{Q}_2^{-1}\in L^\infty(\tilde{\mathscr{E}})$. Define the* QF $q_2(h,\mathscr{Q}_2,Q_2)=\omega(\mathscr{E},h^2\mathscr{Q}_2,Q_2)$, $h>0$, *on* $\mathring{C}^\infty(\mathscr{E})$ *and close it in* $L^2(\mathscr{E})$.

a) *The estimate*

$$N_0(q_2(h,\mathscr{Q}_2,Q_2))\leqq h^{-k}\cdot c_2\|\mathscr{Q}_2^{-1}\|^{k/2}_{L^\infty(\tilde{\mathscr{E}})}\|(Q_2)_-\|^{k/2}_{L^{\imath}(\mathscr{E})}+1,\quad \imath=\imath_k\ ,$$

is valid for a constant c_2 which is independent on h and $\mathscr{Q}_2$ and may depend on Q_2 only through $\tilde{\mathscr{E}}$.

b) *Moreover, the asymptotic formula* (3.5) *remains valid if we replace in it* q_1, $\mathscr{Q}_1$ *and* Q_1 *respectively by* q_2, $\mathscr{Q}_2$ *and* Q_2.

Lemmas 3.1–3.2 follow quite easily from Theorems 4.1, 4.6, 4.14, and 4.15 in [2].

Let $\mathscr{I} \subseteqq \mathbb{R}_+$ be an arbitrary interval and $\mathscr{E} = \mathscr{E}(r)$, $r \in \mathscr{I}$, be a regular domain. Assume that for each $r \in \mathscr{I}$ $\mathscr{W}(r) > 0$ is a Hermitian matrix-valued function and $p(r) > 0$ and $W(r)$ are real-valued functions over $\mathscr{E}(r)$. Let $s \in \mathbb{R}$. Put

$$\mathfrak{w}[u; \mathscr{I}, s, p, \mathscr{E}, \mathscr{W}, W] = \int_{\mathscr{I}} \{\|\partial_r u\|^2 + \omega[u; \mathscr{E}(r), \mathscr{W}(r), W(r)]\} r^s dr \ , \qquad (3.6)$$

where $\|.\|$ is the norm in $L^2(\mathscr{E}(r); p(r))$.

Introduce the QFs

$$q_3^{\pm}(h, R, s, \not{p}, \mathscr{E}, \mathscr{Q}_3, Q_3) = \mathfrak{w}(\mathscr{I}, s, h^2 \not{p}, \mathscr{E}, h^2 r^{-2} \mathscr{Q}_3, Q_3)$$

with $\mathscr{I} = (0, R)$ where $R \leqq R_0 < \infty$, $\mathscr{E}$ coincides either with $\mathbb{R}^t \times \mathscr{O}$ ($t \geqq 1$) or just with $\mathscr{O}$, where $\mathscr{O} \subset \mathbb{R}^d$ ($d \geqq 1$) is an independent of r regular bounded domain. Set $k = 1 + \dim \mathscr{E}$. Further, the number $\not{p}$ and the matrix $\mathscr{Q}_3$ are supposed to be constant and, moreover, $s \neq 1$. Finally, we assume $(Q_3)_+ \in L^1((\mathscr{I} \times \mathscr{E}) \cap B_R; r)$, $\forall R > 0$, and $(Q_3)_- \in L^{k/2}(\mathscr{I} \times \mathscr{E}; r^{k-1})$ if $k > 2$, and $(Q_3(r, \zeta))_- \leqq \tilde{Q}_3(r) \leqq c r^{-\alpha}$, $\forall \zeta \in \mathscr{E}$, for some $\alpha \in [0, 2)$ and $c \geqq 0$, if $k = 2$. Define the QF q_3^+ on functions $u \in \mathring{C}^\infty(\mathscr{I} \times \mathscr{E})$ which vanish near $\partial \mathscr{I} \times \mathscr{E}$ and then close it in $L^2(\mathscr{I} \times \mathscr{E}; r^s)$. Respectively, $D[q_3^-] = \{u \in D[q_3^+] : u_{|\mathscr{I} \times \partial \mathscr{E}} = 0\}$.

Lemma 3.3. *Let* $\|.\|$ *denote the norm in* $L^{k/2}(\mathscr{E})$.

a) *The estimates*

$$N_0(q_3^{\pm}(h)) \leqq h^{-k} c_3 \int_0^R \|(Q_3)_-\|^{k/2} r^{k-1} dr, \quad k \geqq 3 \ ,$$

$$N_0(q_3^{\pm}(h)) \leqq h^{-2} c_4 \int_0^\infty X^{-1/2} \sum_{l=0}^\infty \left(\int_{\mathscr{I}_l} \tau^{l/2} (X r^{-2} - \tilde{Q}_3(r))_- r dr \right)^{1/2} dX, \ k = 2 \ ,$$

where $\tau > 1$ *and* $\mathscr{I}_0 = (R\exp(-1), R)$, $\mathscr{I}_l = (R\exp(-\tau^l), R\exp(-\tau^{l-1}))$, $l \geqq 1$, *hold with* $c_3 = \max\{\not{p}^{-k/2}, \|\mathscr{Q}_3\|^{-k/2}\} c_3'$, $c_4 = \max\{\not{p}^{-1}, \mathscr{Q}_3^{-1}\} c_4'$, *and some numbers* c_i', $i = 3, 4$, *independent of* h, R, $\not{p}$, $\mathscr{Q}_3$ *and* Q_3.

b) *Moreover, we have*

$$\lim_{h \downarrow 0} h^k N_0(q_3^{\pm}(h))$$
$$= (\not{p} \det \mathscr{Q}_3)^{-1/2} \int_0^R \|(Q_3)_-\|^{k/2} r^{k-1} dr / (4\pi)^{k/2} \Gamma(1 + k/2), \quad k \geqq 2 \ .$$

In order to prove Lemma 3.3, shift the trial function $u \to r^{(1-s)/2} u$ in the QFs $q_3^{\pm}$, change the variable $r = R\exp(-\varrho)$, $\varrho \in \mathbb{R}_+$, and apply a variational technique which is quite similar to the methods developed in Chap.4 of [2].

Next, we discuss the spectral asymptotics for operators which generalize the operator $\mathscr{A}(z)$ defined in Subsect. 2.1.

Let $t \equiv \operatorname{codim} \Phi > 1$. Fix some $z \in \Phi$. Let $\mathfrak{R} = \mathfrak{R}(z)$ be a positive homogeneous function of order zero which is continuous over $\mathscr{N}_z \setminus \{0\}$. Define the selfadjoint in

$L^2(\mathcal{N}_z)$ operator

$$\tilde{\mathscr{A}}(z) = -\Delta_z + \tilde{\mathscr{V}}_z, \tilde{\mathscr{V}}_z = |y|_z^\beta \mathfrak{K}(y;z) \ , \quad \beta > 0 \ . \tag{3.7}$$

Let $\{\tilde{\Lambda}_k\}_{k \geqq 1}$ be the nondecreasing eigenvalue sequence for the operator $\tilde{\mathscr{A}}(z)$.

Lemma 3.4. a) *Let $\lambda \geqq \lambda_0 > 0$. Then there exists a number c_5 independent of λ, such that we have*

$$N_\lambda(\tilde{\mathscr{A}}(z)) \leqq \lambda^{t/\kappa} c_5 \|(\mathscr{V}_z - 1)_-\|^{t/2} \ , \tag{3.8}$$

where κ is introduced in the hypotheses of Theorem 2.3b) and $\|.\|$ denotes the norm in $L^{\imath}(\mathcal{N}_z)$, $\imath = \imath_t$.

b) *Moreover, the asymptotics*

$$\lim_{\lambda \to \infty} \lambda^{-t/\kappa} N_\lambda(\tilde{\mathscr{A}}) = \mathfrak{C}(\mathfrak{K}) \tag{3.9}$$

hold with

$$\mathfrak{C}(\mathfrak{K}) = \Gamma(t/\beta) \int_{\mathscr{S}_z} d\sigma_z \mathfrak{K}^{-t/\beta} / \beta (4\pi)^{t/2} \Gamma(1 + t/\kappa) \ . \tag{3.10}$$

Hence the eigenvalues $\tilde{\Lambda}_k$ satisfy the relation

$$\lim_{k \to \infty} k^{-\kappa/t} \tilde{\Lambda}_k = \mathfrak{C}(\mathfrak{K})^{-\kappa/t} \ . \tag{3.11}$$

Lemma 3.4 follows quite easily from Lemma 3.2.

Remark 3.1. Lemmas 3.3–3.4 do not concern the ordinary differential operators which are analogous respectively to the operators generated by the QFs $q_3^\pm$ or to the operator $\tilde{\mathscr{A}}$ since we shall use these lemmas for the proof of Theorems 2.1–2.3 where such versions of the lemmas will not be needed. However, if we want to handle the cases $d = 0$ or $d = m - 2$ (see Remark 2.2), we have to use the obvious analogues of Lemmas 3.3 and 3.4 for ordinary differential operators.

4. Proof of Theorem 2.1

4.1. In the present and the following three subsections we assume that the hypotheses of Theorem 2.1 a)–b) hold and verify the corresponding assertions.

For $R > 0$ put $\mathscr{B}_R = \mathbb{R}^m \backslash B_R$. Choose some R_0 and denote by χ_0 the characteristic function of the set $\mathscr{B}_{R_0}$. Define the potential

$$V_1(x;\varepsilon) = g(\hat{x}) + \chi_0 |x|^{-\alpha}(f(\hat{x}) - \varepsilon), \ \hat{x} \equiv x/|x|, \ x \in \mathbb{R}^m, \ \varepsilon \in \mathbb{R} \ .$$

Let $\mathcal{O} \subseteqq \mathbb{R}^m$ be a regular domain. For $\varepsilon_1 \in \mathbb{R}$ and $\varepsilon_2 < 1$ define the lower bounded in $L^2(\mathcal{O})$ QFs $a_1^\pm(\mathcal{O}, \varepsilon_1, \varepsilon_2) = \omega(\mathcal{O}, (1 - \varepsilon_2)E, V_1(\varepsilon_1))$ (see (3.2)) on the domains $D[a_1^-] = H_0^1(\mathcal{O})$, $D[a_1^+] = H^1(\mathcal{O})$. If $\mathcal{O} = \mathbb{R}^m$, then $D[a_1^+] = D[a_1^-]$, so that we put $a_1 = a_1^+(\mathbb{R}^m) \equiv a_1^-(\mathbb{R}^m)$.

Lemma 4.1. *For each $\varepsilon_1 > 0$ and $\varepsilon_2 \in (0, 1)$ we have*

$$\pm N_{-\lambda}(a) \leqq \pm N_{-\lambda}(a_1(\pm\varepsilon_1, \pm\varepsilon_2)) + O(1) \ , \quad \lambda \downarrow 0 \ , \tag{4.1$_\pm$}$$

where the QF $a \equiv a(1, V)$ *is defined in* (3.3).

Proof. Let $\mathcal{O}\subseteqq\mathbb{R}^m$ be a regular domain. Set $a^{\pm}=\omega(\mathcal{O},E,V)$, $D[a^{\pm}]=D[a_1^{\pm}]$; $a^{+}(\mathbb{R}^m)\equiv a^{-}(\mathbb{R}^m)=a$. Clearly, the QFs $a^{\pm}$ are closed and lower-bounded in $L^2(\mathcal{O})$.

Applying the Dirichlet-Neumann bracketing procedure, we find that the estimates

$$N_{-\lambda}(a)\leqq N_{-\lambda}(a^{+}(\mathscr{B}_R))+N_0(a^{+}(B_R))\ , \tag{4.2$_+$}$$

$$N_{-\lambda}(a)\geqq N_{-\lambda}(a^{-}(\mathscr{B}_R))\ , \tag{4.2$_-$}$$

hold for each $\lambda>0$ and $R>0$.

Fix $\varepsilon_1>0$ and choose R great enough so that the inequality $\||x|^{-\alpha}(V(x)-g(\hat{x}))-f(\hat{x})|<\varepsilon_1$ is valid for each $x\in\mathscr{B}_R$. Thus, $\pm a^{\pm}[u;\mathscr{B}_R]\geqq\pm a_1^{\pm}[u;\mathscr{B}_R,\pm\varepsilon_1,0]$, $\forall u\in D[a^{\pm}(\mathscr{B}_R)]$, and hence we have

$$\pm N_{-\lambda}(a^{\pm}(\mathscr{B}_R))\leqq\pm N_{-\lambda}(a_1^{\pm}(\mathscr{B}_R,\pm\varepsilon_1,0))\ ,\quad\forall\varepsilon_1>0\ ,\quad\forall\lambda>0\ . \tag{4.3$_\pm$}$$

Next, we show that the asymptotic estimates

$$\pm N_{-\lambda}(a_1^{\pm}(\mathscr{B}_R,\pm\varepsilon_1,0))\leqq\pm N_{-\lambda}(a_1^{\mp}(\mathscr{B}_R,\pm\varepsilon_1,\pm\varepsilon_2))+O(1),\ \lambda\downarrow 0\ , \tag{4.4$_\pm$}$$

hold for each $\varepsilon_1>0$ and $\varepsilon_2\in(0,1)$. For this purpose we shall use a simple method ("the truncation trick") whose various versions will be applied in the sequel. Fix any $R_1>R$. Denote by χ_1 the characteristic function of $\mathscr{B}_{R_1}$. For $\varepsilon\in\mathbb{R}$ and $c\geqq 0$ set

$$V_{1,1}(x;\varepsilon,c)=g(\hat{x})+\chi_1\chi_0|x|^{-\alpha}(f(\hat{x})-\varepsilon)+c(1-\chi_1)\ ,$$

$$V_{1,2}(x;\varepsilon,c)=V_1(x;\varepsilon)-V_{1,1}(x;\varepsilon,c)\ .$$

For $\varepsilon_1\in\mathbb{R}$, $\varepsilon_2\in(0,1)$ and $c\geqq 0$ introduce the QFs

$$a_{1,1}^{\pm}[u;R]\equiv a_{1,1}^{\pm}[u;R,\varepsilon_1,\varepsilon_2,c]=\omega[u;\mathscr{B}_R,(1-\varepsilon_2)E,V_{1,1}(\varepsilon_1,c)]\ ,$$

$$a_{1,2}^{\pm}[u;R]\equiv a_{1,2}^{\pm}[u;R,\varepsilon_1,\varepsilon_2,c]=\omega[u;\mathscr{B}_R,\varepsilon_2 E,V_{1,2}(\varepsilon_1,c)]\ ,$$

on the domains $D[a_{1,k}^{\pm}(R)]=D[a_1^{\pm}(\mathscr{B}_R)]$, $k=1,2$. Obviously, we have

$$\pm N_{-\lambda}(a_1^{\pm}(\mathscr{B}_R,\pm\varepsilon_1,0))\leqq\pm N_{-\lambda}(a_{1,1}^{\pm}(R,\pm\varepsilon_1,\pm\varepsilon_2,c))$$
$$+N_0(a_{1,2}^{\pm}(R,\pm\varepsilon_1,\varepsilon_2,c)),\ \forall\lambda>0,\ \forall\varepsilon_1>0,\ \forall\varepsilon_2\in(0,1),\ \forall c\geqq 0\ . \tag{4.5$_\pm$}$$

Since the support of $V_{1,2}$ is compact, Lemmas 3.1–3.2 imply $N_0(a_{1,2}^{\pm}(R))<\infty$, so that $(4.5)_\pm$ entails the asymptotic estimates

$$\pm N_{-\lambda}(a_1^{\pm}(\mathscr{B}_R,\pm\varepsilon_1,0))\leqq\pm N_{-\lambda}(a_{1,1}^{\pm}(R,\pm\varepsilon_1,\pm\varepsilon_2,c))+O(1),\ \lambda\downarrow 0\ , \tag{4.6$_\pm$}$$

which hold for each $\varepsilon_1>0$, $\varepsilon_2\in(0,1)$ and $c\geqq 0$.

Now, choose a "cutting" function $\Theta\in C_0^\infty(\mathbb{R}^m)$ such that $\Theta(x)=0$ if $x\in B_R$, $\Theta(x)=1$ if $x\in\mathscr{B}_{R_1}$, and $\Theta(x)\in[0,1]$ otherwise. Then we have $a_{1,1}^{-}[\Theta u;R,\varepsilon_1,\varepsilon_2,0]+\lambda\|\Theta u\|^2\leqq a_{1,1}^{+}[u;R,\varepsilon_1,\varepsilon_2,\tilde{c}]+\lambda\|u\|^2$, $u\in D[a_{1,1}^{+}(R)]$, where $\|.\|$ is the norm in $L^2(\mathscr{B}_R)$, $\tilde{c}=\max_{x\in\mathbb{R}^m}|\Theta\Delta\Theta|$, the numbers $\lambda>0$, $\varepsilon_1\in\mathbb{R}$, $\varepsilon_2<1$ are arbitrary. Hence, we have

$$N_{-\lambda}(a_{1,1}^{+}(R,\varepsilon_1,\varepsilon_2,\tilde{c}))\leqq N_{-\lambda}(a_{1,1}^{-}(R,\varepsilon_1,\varepsilon_2,0))\ . \tag{4.7}$$

By analogy with $(4.6)_\pm$, one easily finds that the estimates

$$N_{-\lambda}(a_{1,1}^-(R,\varepsilon_1,\varepsilon_2,0)) \leqq N_{-\lambda}(a_1^-(\mathscr{B}_R,\varepsilon_1,\varepsilon_2')) + O(1) \ , \qquad (4.8)_+$$

$$N_{-\lambda}(a_{1,1}^+(R,-\varepsilon_1,-\varepsilon_2,\tilde{c})) \geqq N_{-\lambda}(a_1^+(\mathscr{B}_R,-\varepsilon_1,-\varepsilon_2')) + O(1) \ , \qquad (4.8)_-$$

hold as $\lambda\downarrow 0$ for each $\varepsilon_1>0$, $\varepsilon_2\in(0,1)$ and $\varepsilon_2'\in(\varepsilon_2,1)$. The combination of $(4.5)_\pm - (4.8)_\pm$ yields $(4.4)_\pm$.

Further, it is clear that the inequalities

$$N_{-\lambda}(a_1^-(\mathscr{B}_R,\varepsilon_1,\varepsilon_2)) \leqq N_{-\lambda}(a_1(\varepsilon_1,\varepsilon_2)) \ , \qquad (4.9)_+$$

$$N_{-\lambda}(a_1^+(\mathscr{B}_R,-\varepsilon_1,-\varepsilon_2)) \geqq N_{-\lambda}(a_1(-\varepsilon_1,-\varepsilon_2)) - N_0(a_1^+(B_R,-\varepsilon_1,-\varepsilon_2)) \ , \qquad (4.9)_-$$

are valid for each $\varepsilon_1>0$ and $\varepsilon_2\in(0,1)$. To complete the demonstration of $(4.1)_\pm$, we have just to put together $(4.2)_\pm$–$(4.4)_\pm$ and $(4.9)_\pm$ and to notice that Lemma 3.2 entails $N_0(a_1^+(B_R,-\varepsilon_1,-\varepsilon_2))<\infty$ for each $\varepsilon_1\in\mathbb{R}$ and $\varepsilon_2>-1$. □

4.2. For $T>0$ put $\Phi_T=\{\omega\in S^{m-1}: \mathrm{dist}(\omega,\Phi)<T\}$ and $S_T=S^{m-1}\backslash\bar{\Phi}_T$. Obviously, if $T>0$ is small enough, then Φ_T and S_T are $(m-1)$-dimensional submanifolds of S^{m-1} with C^2-boundaries.

Let $\mathcal{O}$ be a $(m-1)$-dimensional submanifold of S^{m-1}. Denote by $\mathcal{O}^*$ the conic set $\{x\in\mathbb{R}^m: \hat{x}\in\mathcal{O}\}$.

Applying Dirichlet-Neumann bracketing, we get

$$N_{-\lambda}(a_1(\varepsilon_1,\varepsilon_2)) \leqq N_{-\lambda}(a_1^+(\Phi_T^*,\varepsilon_1,\varepsilon_2)) + N_0(a_1^+(S_T^*,\varepsilon_1,\varepsilon_2)) \ , \qquad (4.10)_+$$

$$N_{-\lambda}(a_1(-\varepsilon_1,-\varepsilon_2)) \geqq N_{-\lambda}(a_1^-(\Phi_T^*,-\varepsilon_1,-\varepsilon_2)) \ . \qquad (4.10)_-$$

Here the numbers $\lambda>0$, $\varepsilon_1>0$ and $\varepsilon_2\in(0,1)$ are arbitrary and $T>0$ is sufficiently small. Since the function g is strictly positive on $\bar{S}_T$, the set $\mathrm{supp}(V_1)_-\cap\bar{S}_T^*$ is compact. Hence, Lemma 3.2 implies

$$N_0(a_1^+(S_T^*,\varepsilon_1,\varepsilon_2))<\infty \ , \quad \forall\varepsilon_1\in\mathbb{R} \ , \quad \forall\varepsilon_2<1 \ . \qquad (4.11)$$

Next, introduce a finite covering of Φ by open coordinate charts $\{\Phi_j\}$ such that $\Phi_i\cap\Phi_j=\emptyset$ for $i\neq j$, $\cup_j\bar{\Phi}_j=\Phi$, and the boundaries $\partial\Phi_j$ are of Lipschitz class. Set $\Phi_{j,T}=\{\omega\in S^{m-1}: \mathrm{dist}(\omega,\Phi_j)<T\}$, $T>0$. Evidently, the estimates

$$\pm N_{-\lambda}(a_1^\pm(\Phi_T^*,\varepsilon_1,\varepsilon_2)) \leqq \pm\sum_j N_{-\lambda}(a_1^\pm(\Phi_{j,T}^*,\varepsilon_1,\varepsilon_2)) \qquad (4.12)_\pm$$

hold for each $\varepsilon_1\in\mathbb{R}$ and $\varepsilon_2<1$.

Further, on every Φ_j introduce local coordinates ζ so that Φ_j is parametrized by ζ varying over some bounded regular domain $\mathscr{F}_j\subset\mathbb{R}^d$. In the sequel if $\zeta\in\mathscr{F}_j$ and $z=z(\zeta)\in\Phi_j$, we shall write $\mathscr{N}_\zeta$ instead of $\mathscr{N}_z$, $|.|_\zeta$ instead of $|.|_z$, etc.

Assume $T>0$ small enough and introduce on each $\Phi_{j,T}$ nondegenerate coordinates (ζ,y) such that $\Phi_{j,T}$ is parametrized by the values of the variables (ζ,y) over the domain $\mathscr{F}_{j,T}=\{\zeta\in\mathscr{F}_j, y\in\mathscr{N}_\zeta: |y|_\zeta<T\}$; the coordinates (ζ,y) coincide with the ones mentioned in the example in Subsect. 2.1. Then the covariant metric

tensor $\mathfrak{G}$ can be written in the local coordinates (ζ, y) as

$$\mathfrak{G}(\zeta, y)=\begin{pmatrix} \mathfrak{L}(\zeta, y) & 0 \\ 0 & \mathfrak{M}(\zeta, y) \end{pmatrix},$$

where $\mathfrak{L}(\zeta, y)$ is the restriction of $\mathfrak{G}(\zeta, y)$ onto $\mathscr{T}_\zeta \Phi_j \times \mathscr{T}_\zeta \Phi_j$.

Next, for each $\mathscr{F}_j$ choose a point $\zeta_j \in \mathscr{F}_j$, set $|.|_j=|.|_{\zeta_j}$, $\mathscr{K}_j(y)=\mathscr{K}(y/|y|_j; \zeta_j)$, $f_j=f(\zeta, y)_{|\zeta=\zeta_j, y=0}$, and

$$V_2(r, y; \varepsilon, j)=(1-\varepsilon_2)|y|_j^\beta(\mathscr{K}_j(y)-\varepsilon_3)+r^{-\alpha}\chi_0(f_j-\varepsilon_1) ,$$

where $\varepsilon=(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3$, $\varepsilon_2<1$ and $0<\varepsilon_3<\mathscr{K}_{j,0} \equiv \min\limits_{y \in \mathscr{N}_{\zeta_j}} \mathscr{K}_j(y)$. For $\lambda \geqq 0$ and $\varepsilon \in \mathbb{R}^3$ as above, introduce the QFs $a_2^\pm(\lambda, \mathscr{F}_{j,T}, \varepsilon)$ which coincide with the QF $\mathfrak{w}$ (see (3.6)) for $\mathscr{I}=\mathbb{R}_+$, $s=m-1$, $p=1-\varepsilon_2$, $\mathscr{E}=\mathscr{F}_{j,T}$, $\mathscr{W}=(1-\varepsilon_2)r^{-2}\mathfrak{G}_j^{-1}$ where $\mathfrak{G}_j \equiv \mathfrak{G}(\zeta_j, 0)$, and $W=V_2(\varepsilon, j)+(1-\varepsilon_2)\lambda$. At first define the QF $a_2^+(\lambda, \mathscr{F}_{j,T}, \varepsilon)$ on functions $u \in \mathring{C}^\infty(\mathbb{R}_+ \times \mathscr{F}_{j,T})$ which vanish near the origin of the r-semiaxis and then close it in $L^2(\mathbb{R}_+ \times \mathscr{F}_{j,T}; r^{m-1})$. The domain $D[a_2^-]$ is the restriction of $D[a_2^+]$ onto the set of functions which vanish on $\mathbb{R}_+ \times \partial \mathscr{F}_{j,T}$.

Let

$$\varepsilon_1>0,\ \varepsilon_2 \in (0,1) \quad \text{and} \quad \varepsilon' \in \mathbb{R}^3,\ \varepsilon_1'>\varepsilon_1,\ \varepsilon_2' \in (\varepsilon_2, 1),\ \varepsilon_3' \in (0, \mathscr{K}_{j,0}) . \tag{4.13}$$

In the QFs $a_1^+(\Phi_{j,T}^*, \varepsilon_1, \varepsilon_2)$ pass to the spherical coordinates (r, ω) and then change the variables $\omega \to (\zeta, y)$. Choose the numbers $T>0$ and $\mathscr{D}=\max\limits_j \operatorname{diam} \Phi_j$ small enough so that the inequalities

$$\pm(\det \mathfrak{G}_j)^{1/2} a_2^\pm[u; \lambda, \mathscr{F}_{j,T}, \pm\varepsilon'] \leqq \pm\{a_1^\pm[v; \Phi_{j,T}^*, \pm\varepsilon_1, \pm\varepsilon_2]+\lambda \|v\|_{L^2(\Phi_{j,T}^*)}^2\}$$

hold for each $v \in D[a_1^\pm(\Phi_{j,T}^*)]$ and $u(r, \zeta, y)=v(x(r, \zeta, y))$. Hence for every Φ_j we have

$$\pm N_{-\lambda}(a_1^\pm(\Phi_{j,T}^*, \pm\varepsilon_1, \pm\varepsilon_2)) \leqq \pm N_0(a_2^\pm(\lambda, \mathscr{F}_{j,T}, \pm\varepsilon')),\ \forall \lambda>0 , \tag{4.14$_\pm$}$$

where the arbitrary ε_i $(i=1,2)$ and $\varepsilon' \in \mathbb{R}^3$ are as in (4.13).

Combining $(4.10)_\pm$–$(4.12)_\pm$ and $(4.14)_\pm$, we get the following

Lemma 4.2. *For sufficiently small $T>0$ we have*

$$\pm N_{-\lambda}(a_1^\pm(\pm\varepsilon_1, \pm\varepsilon_2)) \leqq \pm\sum_j N_0(a_2^\pm(\lambda, \mathscr{F}_{j,T}, \pm\varepsilon'))+O(1),\ \lambda \downarrow 0 ,$$

with any ε_i $(i=1,2)$ and $\varepsilon' \in \mathbb{R}^3$ as in (4.13).

4.3. For a given integer j set $\mathscr{F}_{j,\infty}=\{\zeta \in \mathscr{F}_j, y \in \mathscr{N}_\zeta\}$ and introduce the QFs $a_3^\pm(\lambda, \varepsilon) \equiv a_3^\pm(\lambda, \varepsilon, j)$ which coincide with the QFs $a_2^\pm(\lambda, \mathscr{F}_{j,T}, \varepsilon)$ but $\mathscr{F}_{j,T}$ is substituted for $\mathscr{F}_{j,\infty}$; the domains of the QFs $a_3^\pm$ are analogous with $D[a_2^\pm]$.

Lemma 4.3. *For each ε, $\varepsilon' \in \mathbb{R}^3$ such that*

$$\varepsilon_1=\varepsilon_1'>0,\ \varepsilon_2 \in (0,1),\ \varepsilon_2' \in (\varepsilon_2, 1),\ \varepsilon_3=\varepsilon_3' \in (0, \mathscr{K}_{j,0}) , \tag{4.15}$$

for every integer j and sufficiently small $T>0$ we have

$$\pm N_0(a_2^\pm(\lambda, \mathscr{F}_{j,T}, \pm\varepsilon)) \leqq \pm N_0(a_3^\pm(\lambda, \pm\varepsilon', j))+O(1),\ \lambda \downarrow 0 . \tag{4.16$_\pm$}$$

Proof. Fix some $R>0$. Introduce the QFs $\tilde{a}_1^{\pm} \equiv \tilde{a}_1^{\pm}(\lambda, \mathscr{F}_{j,T}, R, \varepsilon)$ (respectively $\tilde{a}_2^{\pm} \equiv \tilde{a}_2^{\pm}(\lambda, \mathscr{F}_{j,T}, R, \varepsilon)$) which coincide with the QFs $a_2^{\pm}(\lambda, \mathscr{F}_{j,T}, \varepsilon)$, $0<T\leqq\infty$, but the interval $(0,\infty)$ of integration with respect to r is substituted for $(0,R)$ (respectively (R,∞)). The domain of the QF $\tilde{a}_1^{+}$ (respectively $\tilde{a}_2^{+}$) is defined as the set of restrictions of functions $u\in D[a_2^{+}(\mathscr{F}_{j,T})]$ onto $(0,R)\times\mathscr{F}_{j,T}$ (respectively onto $(R,\infty)\times\mathscr{F}_{j,T}$). The domains of the QFs $\tilde{a}_i^{-}$ $(i=1,2)$ consist of functions $u\in D[\tilde{a}_i^{+}]$ which vanish for $r=R$. Clearly, we have

$$N_0(a_2^{+}(\lambda, \mathscr{F}_{j,T}, \varepsilon)) \leqq N_0(\tilde{a}_1^{+}(0, \mathscr{F}_{j,T}, \varepsilon)) + N_0(\tilde{a}_2^{+}(\lambda, \mathscr{F}_{j,T}, \varepsilon)) \ , \qquad (4.17)_+$$

$$N_0(a_2^{-}(\lambda, \mathscr{F}_{j,T}, -\varepsilon)) \geqq N_0(\tilde{a}_2^{-}(\lambda, \mathscr{F}_{j,T}, -\varepsilon)) \ . \qquad (4.17)_-$$

Note that Lemma 3.2 implies $N_0(\tilde{a}_1^{+}(0, \mathscr{F}_{j,T}, \varepsilon))<\infty$.

Now, set $\delta\mathscr{F}_{j,T}=\{(\zeta,y):\zeta\in\mathscr{F}_j, |y|_\zeta=T\}$, $T<\infty$. Define the QFs $\tilde{a}_3^{\pm}\equiv\tilde{a}_3^{\pm}(\lambda, \mathscr{F}_{j,T}, R, \varepsilon)$ which are the same as the QFs $\tilde{a}_2^{\pm}$ but have different domains: $D[\tilde{a}_3^{+}] =\{u\in D[\tilde{a}_2^{+}]: u_{|(R,\infty)\times\delta\mathscr{F}_{j,T}}=0\}$ and $D[\tilde{a}_3^{-}]$ consists of functions which meet all requirements of $D[\tilde{a}_2^{-}]$ except that they do *not* vanish on $(R,\infty)\times\delta\mathscr{F}_{j,T}$. Applying the truncation trick with respect to y [cf. the derivation of $(4.4)_\pm$], we get

$$\pm N_0(\tilde{a}_2^{\pm}(\lambda, \mathscr{F}_{j,T}, R, \pm\varepsilon)) \leqq \pm N_0(\tilde{a}_3^{\pm}(\lambda, \mathscr{F}_{j,T}, R, \pm\varepsilon')) + O(1), \ \lambda\downarrow 0 \ , \qquad (4.18)_\pm$$

where $\varepsilon, \varepsilon'\in\mathbb{R}^3$ are the same in (4.15). Note that if we have not gone away from the origin of the r-semiaxis, we could not apply the truncation trick because of the factor r^{-2} in front of the term containing the derivatives with respect to y in the QF $\tilde{a}_2^{+}$.

Next, introduce the QF $\hat{a}_3$ which is analogous with the QF $\tilde{a}_2^{+}(0, \mathscr{F}_{j,\infty}, R, \varepsilon)$ but $\mathscr{F}_{j,\infty}$ is replaced by $\hat{\mathscr{F}}_{j,T}=\mathscr{F}_{j,\infty}\backslash\mathscr{F}_{j,T}$, $T<\infty$. The domain of $\hat{a}_3$ consists of restrictions of functions $u\in D[\tilde{a}_2^{+}(\mathscr{F}_{j,\infty})]$ onto $\hat{\mathscr{F}}_{j,T}$. Then we have

$$\begin{aligned} &N_0(\tilde{a}_3^{+}(\lambda, \mathscr{F}_{j,T}, R, \varepsilon)) \leqq N_0(\tilde{a}_3^{+}(\lambda, \mathscr{F}_{j,\infty}, R, \varepsilon)) \ , \\ &N_0(\tilde{a}_3^{-}(\lambda, \mathscr{F}_{j,T}, R, -\varepsilon)) \end{aligned} \qquad (4.19)_+$$

$$\geqq N_0(\tilde{a}_3^{-}(\lambda, \mathscr{F}_{j,\infty}, R, -\varepsilon)) - N_0(\hat{a}_3(0, \hat{\mathscr{F}}_{j,T}, R, -\varepsilon)) \ . \qquad (4.19)_-$$

Note that set $\hat{\mathscr{F}}_{j,T}\cap\operatorname{supp}(V_2)_-$ is bounded. Hence, by Lemma 3.2, the second term at the right-hand side of $(4.19)_-$ is finite.

Finally, employing the truncation trick with respect to r and bearing in mind Lemmas 3.1–3.2, we get

$$\pm N_0(\tilde{a}_3^{\pm}(\lambda, \mathscr{F}_{j,\infty}, R, \pm\varepsilon)) \leqq \pm N_0(a_3^{\pm}(\lambda, \pm\varepsilon', j)) + O(1), \ \lambda\downarrow 0 \ . \qquad (4.20)_\pm$$

The combination of $(4.17)_\pm$–$(4.20)_\pm$ yields $(4.16)_\pm$. □

4.4. Introduce the potential $V_3(\varepsilon)$ which is the same as $V_2(\varepsilon)$, except that the factor $\chi_0 r^{-\alpha}$ is replaced by $r^{-\alpha}$. The hypotheses of Theorem 2.1a)–b) imply $\theta>0$ and, hence, we have

$$(1-\chi_0)V_3\in L^{m/2}(\mathbb{R}_+\times\mathscr{F}_{j,\infty}; r^{m-1}) \ . \qquad (4.21)$$

Define the QFs $a_4^{\pm}$ which are just the same as the QFs $a_3^{\pm}$ but V_2 is substituted for V_3.

Lemma 4.4. *For any j we have*

$$\pm N_0(a_3^{\pm}(\lambda, \pm\varepsilon, j)) \leqq \pm N_0(a_4^{\pm}(\lambda, \pm\varepsilon', j)) + O(1) \ , \qquad \lambda\downarrow 0 \ ,$$

where $\varepsilon, \varepsilon'\in\mathbb{R}^3$ are the same as in (4.15).

The lemma follows easily from (4.21), Lemma 3.3 and a simple variational argument so that we omit the details of the proof.

4.5. In order to complete the proof of Theorem 2.1a)–b) we need one more lemma which is proved in this subsection.

Denote by $a_5^\pm(\lambda, \mathscr{I}, \varepsilon) \equiv a_5^\pm(\lambda, \mathscr{I}, \varepsilon, j)$ the QF $\mathfrak{w}$ (see (3.6)) with

$$\mathscr{I} = (R_1, R_2) \subseteqq \mathbb{R}_+,\ s = m-1,\ p = (1-\varepsilon_2)\lambda^{-2\gamma+2/\beta},\ \mathscr{E} = \mathscr{F}_{j,\infty},$$

$$\mathscr{W} = (1-\varepsilon_2)\begin{pmatrix} \lambda^{-2\gamma+2/\beta} r^{-2}\mathfrak{L}_j^{-1} & 0 \\ 0 & \lambda^{-2\gamma} r^{-2}\mathfrak{M}_j^{-1} \end{pmatrix} \quad \text{and} \quad W = V_3(\varepsilon) + 1 - \varepsilon_2\ .$$

Here we use the notations $\mathfrak{L}_j = \mathfrak{L}(\zeta_j, 0)$ and $\mathfrak{M}_j = \mathfrak{M}(\zeta_j, 0)$. Besides, $\lambda > 0$, and $\varepsilon \in \mathbb{R}^3$ satisfies $\varepsilon_2 < 1$ and $\varepsilon_3 < \mathscr{K}_{j,0}$. Define $D[a_5^+(\mathscr{I})]$ as the set of restrictions of functions $u \in D[a_4^+]$ onto $\mathscr{I} \times \mathscr{F}_{j,\infty}$ and $D[a_5^-(\mathscr{I})]$ as the set of functions $u \in D[a_5^+(\mathscr{I})]$ which vanish for $r = R_1$ (if $R_1 > 0$) and for $r = R_2$ (if $R_2 < \infty$).

Changing the variables $y \to \lambda^{-1/\beta} y$ in the QFs $a_5^\pm(\lambda, (0, R), \varepsilon)$, applying a suitable version of the truncation trick with respect to r, and making use of Lemmas 3.1–3.3, one finds without difficulty that for $\gamma \leqq 0$ and sufficiently small λ we have

$$\begin{aligned} N_0(a_5^-(\lambda, (0, R), \varepsilon)) &\leqq N_0(a_5^+(\lambda, (0, R), \varepsilon)) \\ &\leqq \lambda^{-\theta}(1-\varepsilon_2)^{-m/2} c_6 \left\{ \int_0^R r^{m-1} \|(V_3(\varepsilon))_-\|^{m/2} dr \right. \\ &\quad \left. + \int_{R_1}^R r^{m(m-1)/2} \|(V_3(\varepsilon))_-\|^{m/2} dr \right\}\ , \end{aligned} \tag{4.22}$$

where $\|.\|$ is the norm in $L^{m/2}(\mathscr{F}_{j,\infty})$, $\varepsilon \in \mathbb{R}^3$, $\varepsilon_1 > 0$, $\varepsilon_2 \in (0,1)$, $\varepsilon_3 \in (0, \mathscr{K}_{j,0})$, $0 < R_1 < R < \infty$, and the quantity c_6 does not depend on λ, ε, R_1 and R, provided that R and $1/R_1$ are uniformly bounded.

Set $\pm \mathscr{L}_j^\pm = \limsup_{\varepsilon \to 0} \limsup_{\lambda \downarrow 0} \pm \lambda^\theta N_0(a_3^\pm(\lambda, \pm\varepsilon, j))$.

Lemma 4.5. *Assume that the hypotheses of Theorem 2.1a) (respectively 2.1b)) hold. Then we have*

$$\pm \mathscr{L}_j^\pm \leqq \pm \Xi_j \mathscr{C}_1 (f_j)_-^{m/\alpha} \int_{\mathscr{S}_{\zeta_j}} d\sigma_{\zeta_j}(v) \mathscr{K}^{-t/\beta}(v; \zeta_j) \tag{4.23$_\pm$}$$

(or, respectively,

$$\pm \mathscr{L}_j^\pm \leqq \pm \Xi_j \mathscr{C}_2 \sum_i (f_j + \hat{\Lambda}_i)_-^{(d+1)/\alpha})\ . \tag{4.24$_\pm$}$$

Here $\Xi_j = (\det \mathfrak{L}_j)^{1/2} \operatorname{mes} \mathscr{F}_j$, *the quantities* $\mathscr{C}_i$ $(i = 1, 2)$ *are defined in* (2.1)–(2.2) *and* $\hat{\Lambda}_i$ *are the eigenvalues of the operator which coincides with* $\tilde{\mathscr{A}}(\zeta_j)$ *for* $\mathfrak{K} = \mathscr{K}_j$ *(see* (3.7)*).*

Proof. In the QFs $a_4^\pm$ change the variables $r \to \lambda^{-1/\alpha} r$ and $y \to \lambda^{1/\beta} y$. Hence, we have $a_4^\pm[u; \lambda, \pm\varepsilon] = \lambda^{1-(\theta+m/2)} a_5^\pm[v; \lambda, \mathbb{R}_+, \pm\varepsilon]$ for each $\lambda > 0$, $u \in D[a_4^\pm]$ and $v(r, \zeta, y) = u(\lambda^{-1/\alpha} r, \zeta, \lambda^{1/\beta} y)$. Therefore, the identity

$$N_0(a_4^\pm(\lambda, \pm\varepsilon)) = N_0(a_5^\pm(\lambda, \mathbb{R}_+, \pm\varepsilon)) \tag{4.25$_\pm$}$$

is valid for each $\lambda > 0$ and $\varepsilon \in \mathbb{R}^3$ such that $\varepsilon_2 < 1$ and $\varepsilon_3 < \mathscr{K}_{j,0}$.

Choose $R>0$ great enough so that $r>R$ implies $r^{-\alpha}(f_j-\varepsilon_1)+1-\varepsilon_2>0$ for sufficiently small $\varepsilon_1\in\mathbb{R}$, $\varepsilon_2\in(0,1)$, and, hence, $N_0(a_5^{\pm}(\lambda,(R,\infty),\varepsilon))=0$ for every $\lambda>0$ and appropriate ε. Bearing in mind Lemma 4.4 and the identity $(4.25)_{\pm}$, we obtain the estimates

$$\pm N_0(a_3^{\pm}(\lambda,\pm\varepsilon,j))\leqq\pm N_0(a_5^{\pm}(\lambda,(0,R),\pm\varepsilon',j)) \ , \qquad (4.26)_{\pm}$$

where $\varepsilon,\varepsilon'\in\mathbb{R}^3$ are the same as in (4.15).

Fix a positive integer K and a sequence $\{r_k\}_{k=0}^{K}$ such that $0<r_0<\ldots<r_K=R$. Set $\mathscr{I}_0=(0,r_0)$, $\mathscr{I}_k=(r_{k-1},r_k)$, $k=1,\ldots,K$, and $a_{5,k}^{\pm}(\lambda,\varepsilon)=a_5^{\pm}(\lambda,\mathscr{I}_k,\varepsilon)$, $k=0,\ldots,K$. Then we have

$$N_0(a_5^{+}(\lambda,(0,R),\varepsilon))\leqq\sum_{k=0}^{K}N_0(a_{5,k}^{+}(\lambda,\varepsilon)) \ , \qquad (4.27)_{+}$$

$$N_0(a_5^{-}(\lambda,(0,R),-\varepsilon))\geqq\sum_{k=1}^{K}N_0(a_{5,k}^{-}(\lambda,-\varepsilon)) \ . \qquad (4.27)_{-}$$

Further, introduce the QFs $\mathscr{M}^{\pm}$ which coincide with the QF w (see (3.2)) with $\mathscr{E}=\mathscr{F}_j$, $\mathscr{W}=\mathfrak{M}_j^{-1}$ and $W\equiv 0$. Set $D[\mathscr{M}^{+}]=H^1(\mathscr{F}_j)$, $D[\mathscr{M}^{-}]=H_0^1(\mathscr{F}_j)$. Clearly the QFs $\mathscr{M}^{\pm}$ are closed and nonnegative in $L^2(\mathscr{F}_j)$. Note that Lemmas 3.1–3.2 entail the estimates

$$N_\eta(\mathscr{M}^{\pm})\leqq\eta^{d/2}c_7+1 \ , \qquad \eta>0 \ , \qquad (4.28)_{\pm}$$

where c_7 is independent of η, and the asymptotics

$$\lim_{\eta\to\infty}\eta^{-d/2}N_\eta(\mathscr{M}^{\pm})=\Xi_j/(4\pi)^{d/2}\Gamma(1+d/2) \ . \qquad (4.29)_{\pm}$$

Denote by $\{\eta_q^{\pm}\}_{q\geqq 1}$ the nondecreasing sequence of the eigenvalues of the QFs $\mathscr{M}^{\pm}$ counted with the multiplicities.

Next, put $\mathscr{V}^{\pm}=|y|_j^{\beta}(\mathscr{K}_j(y)\mp\varepsilon_3)$, $\varepsilon_3\in(0,\mathscr{K}_{j,0})$, and denote by $\mathscr{A}^{\pm}\equiv\mathscr{A}^{\pm}(\zeta_j;\varepsilon_3)$ the operator which coincides with $\tilde{\mathscr{A}}(\zeta_j)$ in (3.7) for $\tilde{\mathscr{V}}_{\zeta_j}=\mathscr{V}^{\pm}$. Moreover, $\{\Lambda_i^{\pm}\}_{i\geqq 1}$ are the eigenvalues of $\mathscr{A}^{\pm}$.

Now, fix the integer k, $1\leqq k\leqq K$. Set $r_-=r_{k-1}$, $r_+=r_k$, and

$$J_1^{\pm}=\lambda^{-2\gamma+2/\beta}r_{\pm}^{-2} \ , \qquad J_2^{\pm}=(\lambda^{-\gamma}/r_{\pm})^{\kappa} \ ,$$

$$\pm J_3^{\pm}\equiv\pm J_3^{\pm}(\varepsilon_1,\varepsilon_2)=(1\mp\varepsilon_2)^{-1}\min_{r\in(r_-,r_+)}\pm\{(r/r_{\pm})^{m-1}[r^{-\alpha}(f_j\mp\varepsilon_1)+1\mp\varepsilon_2]\} \ .$$

Besides, we put $\mathfrak{F}^{\pm}(\eta,\Lambda;\lambda)=\eta J_1^{\pm}+\Lambda J_2^{\pm}+J_3^{\pm}$, $\eta\geqq 0$, $\Lambda\geqq 0$, $\lambda>0$, and $b_{q,i}^{\pm}\equiv b_{q,i}^{\pm}(\lambda,k,\pm\varepsilon)=\mathfrak{F}^{\pm}(\eta_q^{\pm},\Lambda_i^{\pm};\lambda)$. Consider the eigenvalue problem

$$-\lambda^{-2\gamma+2/\beta}u_l''(r)+b_{q,i}^{\pm}u_l(r)=\xi_l^{\pm}u_l(r) \ , \qquad r\in\mathscr{I}_k \ , \qquad (4.30)_{\pm}$$

with boundary conditions $u_l(r_-)=u_l(r_+)=0$ in $(4.30)_-$ and $u_l'(r_-)=u_l'(r_+)=0$ in $(4.30)_+$. Set

$$n^{\pm}(\lambda,k,\pm\varepsilon)=\#\{(l,q,i):\xi_l^{\pm}(q,i;\lambda,k,\pm\varepsilon)<0\} \ .$$

Expanding the trial function $u \in D[a^{\pm}_{5,k}]$ in a series with respect to the eigenfunctions of the QF $\mathscr{M}^{\pm}$ and the operator $\mathscr{A}^{\pm}$, we obtain the estimates

$$\pm N_0(a^{\pm}_{5,k}(\lambda, \pm\varepsilon)) \leqq \pm n^{\pm}(\lambda, k, \pm\varepsilon) \ . \tag{4.31$_\pm$}$$

Computing the eigenvalues $\xi^{\pm}_l$ in $(4.30)_{\pm}$ explicitly and bearing in mind the estimates (3.8) and $(4.28)_{\pm}$, we easily get

$$n^{\pm}(\lambda, k, \pm\varepsilon) = (r_+ - r_-)\lambda^{\gamma - 1/\beta} \int_0^\infty \int_0^\infty (\mathfrak{F}^{\pm}(\eta, \Lambda; \lambda))_-^{1/2} dN_\eta(\mathscr{M}^{\pm}) \times dN_\Lambda(\mathscr{A}^{\pm})/\pi + O(\lambda^{-\theta-\gamma+1/\beta}) \ , \quad \lambda \downarrow 0 \ . \tag{4.32$_\pm$}$$

Set $\pm \mathrm{I}^{\pm}_k = \limsup\limits_{\varepsilon \to 0} \limsup\limits_{\lambda \downarrow 0} \pm \lambda^{\theta} n^{\pm}(\lambda, k, \pm\varepsilon)$.

In order to verify $(4.23)_{\pm}$, change the variables $J^{\pm}_1 \eta \to \eta$, $J^{\pm}_2 \Lambda \to \Lambda$ in the integral at the right-hand side of $(4.32)_{\pm}$ and employ the asymptotics (3.9) and $(4.29)_{\pm}$. Thus we obtain

$$\pm \mathrm{I}^{\pm}_k \leqq \pm \mathfrak{C}_j \mathscr{C}^{\pm}_j \frac{t}{\kappa} \int_0^\infty \int_0^\infty (\eta + \Lambda + J^{\pm}_3(0,0))_-^{1/2} \eta^{d/2-1} \Lambda^{t/\kappa - 1} d\eta d\Lambda \ , \tag{4.33$_\pm$}$$

where $\mathfrak{C}_j$ coincides with the quantity $\mathfrak{C}(\mathfrak{R})$ (see (3.10)) for $\mathfrak{R} = \mathscr{K}_j$ and $\mathscr{C}^{\pm}_j = (r_+ - r_-)2d\Xi_j r^{m-1}_{\pm}/(4\pi)^{d/2+1}\Gamma(1+d/2)$. Similarly, under the hypotheses of Theorem 2.1b) (which imply $\alpha = \kappa$) we have

$$\pm \mathrm{I}^{\pm}_k \leqq \pm \mathscr{C}^{\pm}_j r^{-t}_{\pm} \sum_{i \geqq 1} \int_0^\infty (\eta + \hat{\Lambda}_i r^{-\alpha}_{\pm} + J^{\pm}_3(0,0))_-^{1/2} \eta^{d/2-1} d\eta \ . \tag{4.34$_\pm$}$$

Deriving $(4.33)_{\pm}$–$(4.34)_{\pm}$, we have used also the estimates (3.8) and $(4.28)_{\pm}$ in order to take the limit $\lambda \downarrow 0$ under the sign of the Stieltjes integral at the right-hand side of $(4.32)_{\pm}$.

Now, notice that the estimate (4.22) entails

$$\lim_{r_0 \downarrow 0} \limsup_{\varepsilon \to 0} \limsup_{\lambda \downarrow 0} \lambda^{\theta} N_0(a^{+}_{5,0}(\lambda, \varepsilon)) = 0 \ . \tag{4.35}$$

Combine $(4.25)_{\pm}$–$(4.27)_{\pm}$, $(4.31)_{\pm}$ and $(4.33)_{\pm}$–$(4.34)_{\pm}$ and then let $K \to \infty$ bearing in mind (4.35). After some simple calculations we come to $(4.23)_{\pm}$–$(4.24)_{\pm}$. □

The proof of the assertions a)–b) of Theorem 2.1 is completed if we take into account (3.4), put together the results of Lemmas 4.1–4.3 and 4.5, and let $\mathscr{D} \to 0$.

4.6. In this subsection we prove Theorem 2.1c). Note that under its hypotheses Lemmas 4.1–4.2 are valid again.

If $\theta < 0$, then the asymptotic estimate (2.3) follows directly from Lemma 3.1 (see also Remark 2.4). For this reason we restrict our attention to the case $\theta \geqq 0$.

Lemma 4.6. *If $\gamma > 0$, then for sufficiently small $T > 0$ we have*

$$N_0(a^{+}_2(\lambda, \mathscr{F}_{j,T}, \varepsilon)) = O(1) \ , \quad \lambda \downarrow 0 \ , \tag{4.36}$$

where ε is the same as in (4.15).

Proof. Fix some $R>R_0$. Set $\mathcal{I}_1=(R,\infty)$, $\mathcal{I}_2=(0,R)$. Introduce the QFs $a_{2,i}^+(\lambda,\varepsilon)\equiv a_{2,i}^+(\lambda,\varepsilon,R,T)$, $i=1,2$, which are the same as the QFs $a_2^+(\lambda,\mathcal{F}_{j,T},\varepsilon)$, except that the domain of integration $\mathbb{R}_+\times\mathcal{F}_{j,T}$ is replaced by $\mathcal{I}_i\times\mathcal{F}_{j,T}$. The domains $D[a_{2,i}^+]$ are defined as the sets of restrictions of functions $u\in D[a_2^+]$ onto $\mathcal{I}_i\times\mathcal{F}_{j,T}$. Then we have

$$N_0(a_2^+(\lambda,\mathcal{F}_{j,T},\varepsilon))\leqq N_0(a_{2,1}^+(\lambda,\varepsilon))+N_0(a_{2,2}^+(0,\varepsilon))\ . \tag{4.37}$$

Besides, we have $N_0(a_{2,2}^+(0,\varepsilon))<\infty$.

Further, denote by $a_{2,1}^-(\lambda,\varepsilon)$ the restriction of the QF $a_{2,1}^+(\lambda,\varepsilon)$ onto functions which vanish on $\mathcal{I}_1\times\delta\mathcal{F}_{j,T}$. Applying the truncation trick with respect to y and using Lemma 3.2, we get

$$N_0(a_{2,1}^+(\lambda,\varepsilon))\leqq N_0(a_{2,1}^-(\lambda,\varepsilon'))+O(1)\ ,\qquad \lambda\downarrow 0\ , \tag{4.38}$$

where $\varepsilon,\varepsilon'\in\mathbb{R}^3$ are the same as in (4.15).

For $r\in(R,\infty)$ set $\psi_{j,T}(r)=\{(\zeta,x):\zeta\in\mathcal{F}_j, x\in\mathcal{N}_\zeta, 0\leqq|x|_\zeta<\mathrm{T}r^{\kappa/\beta}\}$, $T<\infty$, $\psi_{j,\infty}=\mathcal{F}_{j,\infty}$, and $\Psi_{j,T,R}=\{(r,\zeta,x):r\in(R,\infty),\ (\zeta,x)\in\psi_{j,T}(r)\}$, $T\leqq\infty$. Denote by $a_6(\lambda,\varepsilon)\equiv a_6(\lambda,\varepsilon,R,T,j)$ the QF $\mathfrak{w}$ (see (3.6)) with $\mathcal{I}=\mathcal{I}_1$, $s=\tilde{m}\equiv m-1-\kappa t/\beta$, $p=1-\varepsilon_2$, $\mathcal{E}=\psi_{j,T}(r)$, $\mathcal{W}=(1-\varepsilon_2)\begin{pmatrix} r^{-2}\mathfrak{L}_j^{-1} & 0\\ 0 & r^{-\kappa}\mathfrak{M}_j^{-1}\end{pmatrix}$ and $W=V_4(\varepsilon)$ $\equiv r^{-\alpha}(f_j-\varepsilon_1)+(1-\varepsilon_2)r^{-\kappa}\mathcal{V}^+(\zeta_j;\varepsilon_3)$; here $\varepsilon\in\mathbb{R}^3$ and $\varepsilon_2<1$, $\varepsilon_3<\mathcal{K}_{j,0}$. At first define the QF a_6 on functions $\mathring{C}^\infty(\Psi_{j,T,R})$ which vanish on the set $\{(r,\zeta,x)\in\overline{\Psi_{j,T,R}}:|x|_\zeta=\mathrm{T}r^{\kappa/\beta}\}$ (if $T<\infty$) and close it in $L^2(\Psi_{j,T,R};r^{\tilde{m}})$.

In the QF $a_{2,1}^-$ change the variables $r=\varrho$, $y=\varrho^{-\kappa/\beta}x$, thus mapping $\mathcal{I}_1\times\mathcal{F}_{j,T}$ onto $\Psi_{j,T,R}$. Fix $\varepsilon,\varepsilon'\in\mathbb{R}^3$ as the ones in (4.15). For a given $R>R_0$ choose $T>0$ small enough so that the estimate $a_{2,1}^-[u;\lambda,\varepsilon,R,T]\geqq a_6[v;\lambda,\varepsilon',R,T]$ holds for each $u(r,\zeta,y)\in D[a_{2,1}^-]$ and $v(\varrho,\zeta,x)=u(\varrho,\zeta,\varrho^{-\kappa/\beta}x)$. Therefore, the inequality

$$N_0(a_{2,1}^-(\lambda,\varepsilon,R,T))\leqq N_0(a_6(0,\varepsilon',R,\infty)) \tag{4.39}$$

is valid for each $\lambda>0$ and $R>R_0$.

Expand the trial function $v\in D[a_6(0,\varepsilon,R,\infty)]$ into a series with respect to the eigenfunctions of the operator $\mathcal{A}^\pm(\zeta_j;\varepsilon_3)$. Thus we obtain

$$N_0(a_6(0,\varepsilon',R,\infty))=\sum_{i\geqq 1} N_0(a_{6,i}(\varepsilon,R))\ , \tag{4.40}$$

where the QFs $a_{6,i}$ ($i\in\mathbb{Z}$, $i\geqq 1$) coincide with the QF $\mathfrak{w}$ with $\mathcal{I}=\mathcal{I}_i$, $s=\tilde{m}$, $p=(1-\varepsilon_2)$, $\mathcal{E}=\mathcal{F}_j$, $\mathcal{W}=(1-\varepsilon_2)r^{-2}\mathfrak{L}_j^{-1}$ and $W=V_{4,i}\equiv r^{-\alpha}(f_j-\varepsilon_1)$ $+(1-\varepsilon_2)r^{-\kappa}\Lambda_i^+(\zeta_j;\varepsilon_3)$. The QFs $a_{6,i}$ are defined at first on $\mathring{C}^\infty(\mathcal{I}_1\times\mathcal{F}_j)$ and then are closed in $L^2(\mathcal{I}_1\times\mathcal{F}_j;r^{\tilde{m}})$.

Now, notice the crucial circumstance that $\gamma>0$ implies $\kappa<\alpha$. Hence, we can choose $R>0$ great enough so that $r>R$ entails $V_{4,1}(r)\geqq 0$. Since the sequence $\{\Lambda_i\}$ is nondecreasing, we have

$$N_0(a_{6,i}(\varepsilon,R))=0\ ,\qquad \forall i\geqq 1\ . \tag{4.41}$$

Putting together (4.37)–(4.41), we come to (4.36). □

Now Lemmas 4.1, 4.2 and 4.6 entail immediately (2.3).

5. Proof of Theorems 2.2–2.3

5.1. Theorem 2.2 follows directly from Lemma 3.1 since the hypotheses of this theorem imply $(V-\lambda)_- \in L^{m/2}(\mathbb{R}^m)$ and $(V-\lambda)_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ (see also Remark 2.4).

5.2. Now we pass to the proof of Theorem 2.3. Introduce the QF $a_7^{\pm}(h) \equiv a_7^{\pm}(h, R, \varepsilon, j)$ which is the same as the QF $a_6(R, \varepsilon, \infty, j)$ introduced in the proof of Lemma 4.6 but the factor $(1-\varepsilon_2)$ is replaced by $h^2(1-\varepsilon_2)$, $h>0$, everywhere except in V_4. Set $D[a_7^+] = \{u \in D[a_6] : u_{|r=R} = 0\}$ and denote by $a_7^-(R)$ the restriction of the QF $a_7^+(R)$ onto the set of functions which vanish on $(R, \infty) \times \partial\mathscr{F}_{j,\infty}$.

Lemma 5.1. *Let the hypotheses of Theorem 2.3 hold. Then the asymptotic estimates*

$$\pm N_0(a(h, V)) \leqq \pm \sum_j N_0(a_7^{\pm}(h, R, \pm\varepsilon, j)) + O(h^{-m}) \ , \qquad h \downarrow 0 \ ,$$

are valid for each $\varepsilon \in \mathbb{R}^3$ *such that* $\varepsilon_1 > 0$, $\varepsilon_2 \in (0,1)$ *and* $\varepsilon_3 \in (0, \mathscr{K}_{j,0})$.

The demonstration of the lemma consists of several steps which are quite similar to the proofs of Lemmas 4.1–4.3 and 4.6, so that we omit the details.

5.3. Next, let the QFs $a_8^{\pm}(h, \mathscr{I}) \equiv a_8^{\pm}(h, \mathscr{I}, \varepsilon, j)$ coincide with the QF $\mathfrak{w}$ for some interval $\mathscr{I} \subseteqq \mathbb{R}_+$, $s = \tilde{m}$, $p = h^{2/\beta\gamma}(1-\varepsilon_2)$, $\mathscr{E} = \mathscr{F}_{j,\infty}$, $\mathscr{W} = (1-\varepsilon_2)$ $\cdot \begin{pmatrix} h^{2/\beta\gamma} r^{-2} \mathfrak{M}_j^{-1} & 0 \\ 0 & r^{-\kappa} \mathfrak{L}_j^{-1} \end{pmatrix}$ and $W = V_4(\varepsilon)$; here $\varepsilon \in \mathbb{R}^3$ and $\varepsilon_2 < 1$, $\varepsilon_3 < \mathscr{K}_{j,0}$. Define the QF $a_8^+(\mathscr{I})$ on functions $u \in \mathring{C}^\infty(\mathscr{I} \times \mathscr{F}_{j,\infty})$ and then close it in $L^2(\mathscr{I} \times \mathscr{F}_{j,\infty}; r^{\tilde{m}})$. Respectively, $D[a_8^-] = \{u \in D[a_8^+] : u_{|\mathscr{I} \times \partial\mathscr{F}_{j,\infty}} = 0\}$.

In the QFs $a_7^+(h, \varepsilon, R, j)$ change the variable $r \to h^{-\kappa/(\alpha-\kappa)} r \equiv h^{-1/\alpha\gamma} r$, $y \to h^{\kappa/\beta} y$, and set $R^* \equiv R^*(h) = R h^{1/\alpha\gamma}$. Thus we find that the identity $a_7^{\pm}[u; h, R, \varepsilon, j] = h^{t\kappa/\beta + (1-\tilde{m}/\alpha)/\gamma} a_8^{\pm}[v; h, (R^*, \infty), \varepsilon, j]$ holds for each $u \in D[a_8^{\pm}]$ and $v(r, \zeta, y) = u(h^{-1/\alpha\gamma} r, \zeta, h^{\kappa/\beta} y)$. Hence, we have

$$N_0(a_7^{\pm}(h, R, \varepsilon, j)) = N_0(a_8^{\pm}(h, (R^*, \infty), \varepsilon, j)) \ . \tag{5.1$_\pm$}$$

Next, denote by $\tilde{a}_8(h, \mathscr{I})$ (respectively by $\hat{a}_8(h, \mathscr{I})$) the QF $a_8^+(h, \mathscr{I})$ with domain consisting of functions which meet all the requirements for $D[a_8^+(\mathscr{I})]$ except that they do not vanish on the right end of $\mathscr{I}$, provided that it is finite (respectively on the left end of $\mathscr{I}$, provided that it lies strictly to the right of the origin).

Since the operator $(1-\varepsilon_2)\mathscr{A}^+$ is positively definite in $L^2(\mathscr{N}_{\zeta_j})$ and $\alpha > \kappa$, there exists $\hat{R} = \hat{R}(\varepsilon)$ such that $R > \hat{R}$ implies the nonnegativeness of the QF $\hat{a}_8(h, (R, \infty), \varepsilon, j)$. Moreover, if ε varies over a compact subset of the set $\{\varepsilon \in \mathbb{R}^3 : \varepsilon_2 < 1, \varepsilon_3 < \mathscr{K}_{j,0}\}$, then $\hat{R}(\varepsilon)$ is uniformly bounded. Assume that h is small enough so that $R^* < \hat{R}$. Let $\tilde{R} > \hat{R}$. Evidently, we have

$$\begin{aligned} N_0(a_8^+(h, (R^*, \infty), \varepsilon, j)) &\leqq N_0(\tilde{a}_8(h, (R^*, \tilde{R}), \varepsilon, j)) \\ &+ N_0(\hat{a}_8(h, (\tilde{R}, \infty), \varepsilon, j)) = N_0(\tilde{a}_8(h, (R^*, \tilde{R}), \varepsilon, j)) \ . \end{aligned} \tag{5.2}$$

Applying a suitable version of the truncation trick with respect to r, we easily find that for sufficiently small h we have

$$N_0(\tilde{a}_8(h, (R^*, \tilde{R}), \varepsilon, j)) \leqq N_0(a_8^+(h, (R^*, \tilde{R}), \varepsilon', j)) \ , \tag{5.3}$$

where $\varepsilon, \varepsilon' \in \mathbb{R}^3$ are as in (4.15). Moreover, an elementary variational argument yields

$$N_0(a_8^-(h,(R^*,\infty),-\varepsilon,j)) \geqq N_0(a_8^-(h,(R^*,\tilde{R}),-\varepsilon,j)) \ . \tag{5.4}$$

Combining (5.1)–(5.4), we get the following

Lemma 5.2. *Assume that the hypotheses of Theorem 2.3 hold. Then the inequalities*

$$\pm N_0(a_7^\pm(h,R,\pm\varepsilon,j)) \leqq \pm N_0(a_8^\pm(h,(R^*,\tilde{R}),\pm\varepsilon',j))$$

hold for any $\varepsilon, \varepsilon' \in \mathbb{R}^m$ as in (4.15) *and each R and j provided that h is sufficiently small and $\tilde{R}$ is great enough.*

5.4. Assume that either ϱ is a monotonously increasing function of $h \geqq 0$ such that $\varrho(0)=0$, or ϱ is a nonnegative constant. Let the QFs $a_9^\pm(h,(\varrho,\tilde{R}),\eta,l,\varepsilon,j)$ coincide with the QF $\mathfrak{w}$ with $\mathcal{I}=(\varrho,\tilde{R})$ where $\tilde{R}=\text{const.}<\infty$, $s=\tilde{m}$, $p=(1-\varepsilon_2)h^l$, $l>0$, $\mathscr{E}=\mathscr{F}_j$, $\mathscr{W}=(1-\varepsilon_2)r^{-2}h^l\mathfrak{M}_j^{-1}$, and $W=V_5(\eta,\varepsilon_1)\equiv\eta r^{-\kappa}+(f_j-\varepsilon_1)r^{-\alpha}$, $\eta\geqq 0$. Define the QF a_9^+ on the set $\{u\in\mathring{C}^\infty(\mathcal{I}\times\mathscr{F}_j): u_{|r=\varrho}=u_{|=\tilde{R}}=0\}$ and close it in $L^2(\mathcal{I}\times\mathscr{F}_j;r)$. Respectively, $D[a_9^-]=\{u\in D[a_9^+]: u_{|\partial\mathscr{F}_j}=0\}$.

Lemma 5.3. *Let the hypotheses of Theorem 2.3 hold. Assume that h is sufficiently small, ϱ is a monotonously increasing function of h such that $\varrho(0)=0$, $\eta\geqq\eta_0>0$ and $\tilde{R}$ is great enough so that $r>\tilde{R}$ implies $V_6(r;\eta,\varepsilon_1)\geqq 0$ for each $\eta\geqq\eta_0$.*

a) *The estimates*

$$N_0(a_9^\pm(h,(\varrho,\tilde{R}),\eta,l,\varepsilon,j)) \leqq c_8 h^{-l(d+1)/2}\eta^{-(\mu-d-1)/\kappa} \tag{5.5$_\pm$}$$

hold with $\mu=m+\theta/\gamma$, $\theta\geqq 0$ (cf. *Theorem 2.3*), *$R>0$ and c_8 which is independent of h. Besides, c_8 does not depend on η if $d\geqq 2$, and $c_8=c_8'\eta^\delta$ for $\delta>0$ and some $c_8'(\delta)$ which is independent of η if $d=1$.*

b) *Moreover, we have*

$$\limsup_{\varepsilon\to 0}\ \limsup_{h\downarrow 0} \pm h^{l(d+1)/2} N_0(a_9^\pm(h,(\varrho,\tilde{R}),\eta,l,\varepsilon,j))$$
$$\leqq \pm\mathscr{C}_5\Xi_j(f_j)_-^{\mu/\alpha}\eta^{-(\mu-d-1)/\kappa}\ , \tag{5.6$_\pm$}$$

where Ξ_j is defined in Lemma 4.5, and $\mathscr{C}_5$ – in (2.6).

Proof. Obviously, we have

$$N_0(a_9^+(h,(\varrho,\tilde{R}),\eta,l,\varepsilon,j)) \leqq N_0(a_9^+(h,(0,\tilde{R}),\eta,l,\varepsilon,j)) \ .$$

Then $(5.5)_+$ (and, hence, $(5.5)_-$) as well as $(5.6)_+$ follow easily from Lemma 3.3.
Fix some ϱ_1 and assume that h is small enough so that $\varrho(h)\leqq\varrho_1$ for each $h\geqq 0$. An elementary variational argument implies

$$N_0(a_9^-(h,(\varrho,\tilde{R}),\eta,l,\varepsilon,j)) \geqq N_0(a_9^-(h,(\varrho_1,\tilde{R}),\eta,l,\varepsilon,j)) \ . \tag{5.7}$$

Applying Lemma 3.1 and taking account of (5.7), we get

$$\liminf_{\varepsilon\to 0}\liminf_{h\downarrow 0} h^{l(d+1)/2} N_0(a_9^-(h,(\varrho,\tilde{R}),\eta,l,\varepsilon,j))$$

$$\geqq \Xi_j \int_{\varrho_1}^{\infty} (f_j r^{-\alpha}+\eta r^{-\kappa})_-^{(d+1)/2} r^d dr/(4\pi)^{(d+1)/2}\Gamma(1+(d+1)/2),\ \forall \varrho_1>0\ .$$

Letting $\varrho_1\downarrow 0$, we come to $(5.6)_-$. □

5.5 In this subsection we complete the proof of Theorem 2.3a). Expanding the trial function $u\in D[a_8^\pm]$ into a series with respect to the eigenfunctions of the operator $\mathscr{A}^\pm$, we get

$$N_0(a_8^\pm(h,(R^*,\tilde{R}),\varepsilon,j))$$
$$=\sum_{k\geqq 1} N_0(a_9^\pm(h,(R^*,\tilde{R}),(1\mp\varepsilon_2)\Lambda_k^\pm,2/\beta\gamma,\varepsilon,j))\ . \qquad (5.8)_\pm$$

Note that the hypotheses of Theorem 2.3a) imply $(d+1)/\beta\gamma=m$, so that, by Lemma 5.3, we find that the sum of any finite number of terms at the right-hand side of $(5.8)_\pm$ has order $O(h^{-m})$. Making use of the asymptotics (3.11) and bearing in mind $(5.8)_\pm$, we obtain

$$\pm N_0(a_8^\pm(h,(R^*,\tilde{R}),\varepsilon,j))\leqq$$
$$\pm\sum_{k\geqq 1} N_0(a_9^\pm(h,(R^*,\tilde{R}),(k/\mathfrak{C}^\pm(\varepsilon_2'))^{\kappa/t},2/\beta\gamma,\varepsilon',f))+O(h^{-m})\ , \qquad (5.9)_\pm$$

where $\varepsilon,\varepsilon'\in\mathbb{R}^m$ are as in (4.15) and $\mathfrak{C}^\pm\equiv\mathfrak{C}^\pm(\varepsilon)$ coincides with the quantity $(1\mp\varepsilon_2)^{t/\kappa}\mathfrak{C}(\mathfrak{R})$ (see (3.10)) for $\mathfrak{R}=\mathscr{K}_i\mp\varepsilon_3$. Applying a simple variational argument and bearing in mind Lemma 5.3, we easily check the asymptotic estimate

$$\sum_{k\geqq 1} N_0(a_9^\pm(h,(R^*,\tilde{R}),(k/\mathfrak{C}^\pm(\varepsilon))^{\kappa/t},2/\beta\gamma,\varepsilon,j))=\mathscr{I}^\pm(h)+O(h^{-m})\ , \qquad (5.10)_\pm$$

where

$$\mathscr{I}^\pm(h)=\frac{t}{\kappa}\int_1^\infty X^{t/\kappa-1}N_0(a_9^\pm(h,(R^*,\tilde{R}),X(\mathfrak{C}^\pm(\varepsilon))^{-\kappa/t},2/\beta\gamma,\varepsilon,j))dX\ .$$

Now, note the crucial circumstance that the integrand in $\mathscr{I}^\pm(h)$ vanishes for $X>X_0h^{-\kappa}$ and any $X_0>\max\{0,-(f_j-\varepsilon_1)R^{-\alpha\gamma\kappa}(\mathfrak{C}^\pm)^{\kappa/t}\}$. Change the variable $X=(X_0h^{-\kappa})^Y$, $Y\in(0,1)$, in the integral $\mathscr{I}^\pm(h)$. Next, in the QFs $a_9^\pm(h,(R^*,\tilde{R}),(X_0h^{-\kappa})^Y(\mathfrak{C}^\pm(\varepsilon))^{-\kappa/t},2/\beta\gamma,\varepsilon,j)$ change the variable $r\to h^{Y/\alpha\gamma}r$. Set

$$\tilde{a}_9^\pm(h,Y,\varepsilon)=a_9^\pm(h,(R^{**},\tilde{R}h^{-Y/\alpha\gamma}),X_0^Y(\mathfrak{C}^\pm(\varepsilon))^{\kappa/t},2((1-Y)/\beta\gamma+Y),\varepsilon,j))\ ,$$

where $R^{**}=Rh^{(1-Y)/\alpha\gamma}$. Thus we get

$$\mathscr{I}^\pm(h)=\frac{t}{\kappa}\log(X_0h^{-\kappa})\int_0^1 (X_0h^{-\kappa})^{Yt/\kappa}N_0(\tilde{a}_9^\pm(h,Y,\varepsilon))dY\ . \qquad (5.11)_\pm$$

Assume that $\tilde{R}$ is sufficiently great so that $r>\tilde{R}-1$ entails $r^{-\kappa}X_0^Y(\mathscr{C}^\pm)^{\kappa/t}+r^{-\alpha}(f_j-\varepsilon_1)\geqq 0$ for each $Y\in[0,1]$. Let the QF $\hat{a}_9^\pm(h,Y,\varepsilon)$ be just the same as $\tilde{a}_9^\pm(h,Y,\varepsilon)$ except that $\tilde{R}h^{-Y/\alpha\gamma}$ is substituted for $\tilde{R}$. The truncation trick with respect to r yields

$$\pm N_0(\tilde{a}_9^{\pm}(h, Y, \varepsilon)) \leqq \pm N_0(\hat{a}_9^{\pm}(h, Y, \varepsilon)) , \quad \forall Y \in (0,1) , \qquad (5.12)_{\pm}$$

provided that $\varepsilon, \varepsilon' \in \mathbb{R}^3$ are as in (4.15) and h is sufficiently small. Note that Lemma 5.3 with $l = 2((1-Y)/\beta\gamma + Y))$ and $\varrho = R^{**}$ is applicable in respect to the QFs $\hat{a}_9^{\pm}(h, Y, \varepsilon)$. Thus, putting together $(5.9)_{\pm}$–$(5.12)_{\pm}$, we get

$$\limsup_{\varepsilon \to 0} \limsup_{h \downarrow 0} \pm h^m |\log h|^{-1} N_0(a_8^{\pm}(h, (R^*, \tilde{R}), \varepsilon, j))$$
$$\leqq \pm \mathscr{C}_4(f_j)_-^{m/\alpha} \Xi_j \int_{\mathscr{S}_{\zeta_j}} \mathscr{K}_j^{-t/\beta} d\sigma_{\zeta_j} , \qquad (5.13)_{\pm}$$

where $\mathscr{C}_4$ is defined in (2.5).

Combining the results of Lemmas 5.1–5.2 with $(5.13)_{\pm}$ and then letting $\mathscr{D} \to 0$, we easily come to (2.5).

5.6. At last, we prove Theorem 2.3b). As a matter of fact, the asymptotics (2.6) follow quite straightforwardly from Lemmas 5.1–5.2, the identities $(5.8)_{\pm}$, and Lemma 5.3 with $l = 2/\beta\gamma$ and $\varrho = R^*$ since $(d+1)/\beta\gamma \equiv \mu$. We only point out that the series $\sum_{k \geqq 1} [(1 \mp \varepsilon_2)\Lambda_k^{\pm}]^{\delta - (\mu - d - 1)/\kappa}$ with $\delta = 0$ if $d \geqq 2$ and $\delta > 0$ small enough if $d = 1$, is convergent (see also Remark 2.1). Thus the identities $(5.8)_{\pm}$ together with Lemma 5.3a) for $l = 2/\beta\gamma$ and $\eta = (1 \mp \varepsilon_2)\Lambda_k^{\pm}$ entail the estimate

$$N_0(a_8^{\pm}(h, (R^*, \tilde{R}), \varepsilon, j)) \leqq c_9 h^{-\mu} ,$$

where c_9 is independent of h. Consequently, if we multiply both sides of $(5.8)_{\pm}$ by h^{μ}, we can take the limit $h \downarrow 0$ under the summation sign at the right-hand side of $(5.8)_{\pm}$, and use $(5.6)_{\pm}$.

References

1. Alsholm, P.K., Kato, T.: Scattering with long range potentials. Proc. Symp. Pure Math. **23**, 393–399 (1973)
2. Birman, M.Sh., Solomjak, M.Z.: Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory. AMS Translations, Series 2, Providence, R.I., 1980
3. Davies, E.B., Simon, B.: Scattering theory for systems with different spatial asymptotics on the left and right. Commun. Math. Phys. **63**, 277–301 (1978)
4. Gurarie, D.: Nonclassical eigenvalue asymptotics for operators of Schrödinger type. Bull. AMS **15**, 233–237 (1986)
5. Helffer, B., Mohamed, A.: Caractérisation du spectre essentiel de l'opérateur de Schrödinger avec un champ magnétique. Ann. Inst. Fourier **38**, 95–112 (1988)
6. Ivrii, V. Ya.: Asymptotics of the discrete spectrum for some operators in $\mathbb{R}^d$. Funkc. Anal. i Prilozhen. **19**, 73–74 (1985) (Russian)
7. Klein, O.: Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac. Z. Physik **53**, 157–165 (1929)
8. Levendorskii, S.Z.: Spectral asymptotics of differential operators with operator coefficients and applications. Dokl. Akad. Nauk SSSR **280**, 1303–1306 (1985) (Russian)
9. Raikov, G.D.: Spectral asymptotics for the Schrödinger operator with a stabilizing potential. In: Proceedings of Symp. Part. Diff. Equ., Holzhau 1988. Teubner-Texte zur Mathematik. **112**, 223–228 (1989)

10. Reed, M., Simon, B.: Methods of modern mathematical physics. IV. Analysis of operators. New York: Academic Press 1978
11. Robert, D.: Comportement asymptotique des valeurs propres d'opérateurs de Schrödinger à potentiel "dégénéré". J. Math. Pures Appl. **61**, 275–300 (1982)
12. Ruijenaars, S., Bongaarts, P.: Scattering theory for one-dimensional step potentials. Ann. Inst. Henri Poincaré **26A**, 1–17 (1977)
13. Simon, B.: Nonclassical eigenvalue asymptotics. J. Funct. Anal. **53**, 84–98 (1983)
14. Solomjak, M.Z.: Spectral asymptotics of Schrödinger operators with non-regular homogeneous potential. Mat. Sb. **127**, 21–39 (1985) (Russian)
15. Tamura, H.: Asymptotic formulas with sharp remainder estimates for bound states for Schrödinger operators. J. Analys. Math., I: **40**, 166–182 (1981); II: **41**, 85–108 (1982)
16. Tamura, H.: The asymptotic formulas for the number of bounded states in the strong coupling limit. J. Math. Soc. Jpn. **36**, 355–374 (1984)

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