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New Weyl Groups for $A_1^{(1)}$ and Characters of Singular Highest Weight Modules

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Abstract. We consider singular Verma modules over $A_1^{(1)}$, i.e., Verma modules for which the central charge is equal to minus the dual Coxeter number. We calculate the characters of certain factor modules of these Verma modules. In one class of cases we are able to prove that these factor modules are actually the irreducible highest modules for those highest weights. We introduce new Weyl groups which are infinitely generated abelian groups and are proper subgroups or isomorphic between themselves. Using these Weyl groups we can rewrite the character formulae obtained in the paper in the form of the classical Weyl character formula for the finite-dimensional irreducible representations of semisimple Lie algebras (respectively Weyl-Kac character formula for the integrable highest weight modules over affine Kac-Moody algebras) so that the new Weyl groups play the role of the usual Weyl group (respectively affine Weyl group).

0. Introduction

The notion of a Weyl group is very essential for the representation theory of semisimple Lie algebras and groups. It allows the nice classical formula of Weyl for the characters of the finite-dimensional irreducible representations L of the semisimple Lie algebras. Connectedly it permutes the weights of the finite-dimensional irreducible representations L of the semisimple Lie algebras and determines the embedding pattern of reducible Verma modules over such algebras. Later the notion of a Weyl group was generalized for affine Kac-Moody algebras [1] and for finite-dimensional Lie superalgebras [2]. For affine Kac-Moody algebras the Weyl character formula holds for the integrable highest weight modules L by replacing the Weyl group with the affine Weyl group [3]. The affine Weyl group or Weyl-Kac group determines the embedding patter of Verma modules [4, 5] except for the so-called *singular Verma modules*. The latter were introduced in [6] by the

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property that the central charge is equal to the minus dual Coxeter number or equivalently that they are reducible with respect to every imaginary root (see below for definitions).

In [6–8] in order to describe embeddings between singular Verma modules we introduced reflections (or translations) corresponding to imaginary roots with nontrivial action on the highest weights of the singular Verma modules. In the present paper we make the next step, i.e., we introduce a new Weyl group, denoted by W_a, and several of its extensions.

Our criterion is the following. The new Weyl group should be such so that the formula for the characters of the analogues of L should look the same as for L by replacing the Weyl or Weyl-Kac group by W_a. Thus we have first to calculate the appropriate characters.

We use results of Malikov, Feigin, and Fuchs [9] for the singular vectors of singular Verma modules to calculate the characters of the highest weight modules $F_\Lambda = \mathscr{M}(\Lambda)/I(\Lambda)$, where $\mathscr{M}(\Lambda)$ is a singular Verma module and $I(\Lambda)$ is the submodule of $\mathscr{M}(\Lambda)$ generated by these singular vectors. First we consider singular Verma modules $\mathscr{M}(\Lambda_0)$ which are irreducible with respect to real roots. In this case our result for the character of F_{Λ_0}, $ch F_{\Lambda_0}$ coincides with $ch\, V(\Lambda_0)$, where $V(\Lambda_0)$ is a highest weight module (HWM) constructed by Wakimoto [10]; thus $F_{\Lambda_0} \cong V(\Lambda_0)$. Then we use a result by Rao [11] that $V(\Lambda_0) \cong L(\Lambda_0)$, the irreducible HWM with weight Λ_0, thus $F_{\Lambda_0} \cong L(\Lambda_0)$. We further calculate $ch F_{\Lambda_0^\pm}$ when $\mathscr{M}(\Lambda_0^\pm)$ are reducible also with respect to some real roots. These calculations are done in Sect. 2.

We use our calculations for $ch L(\Lambda_0)$, $ch F_{\Lambda_0^\pm}$ and we introduce new Weyl groups W_a, $W_a^\pm \supset W_a$, $(W_a^+ \cong W_a^-)$ which are infinitely generated abelian groups so that $ch L(\Lambda_0)$, $ch F_{\Lambda_0^\pm}$ look as the classical Weyl character formula with W replaced by W_a, $W_a^\pm$.

Finally we look for the connection of W_a, $W_a^\pm$ with the Weyl-Kac group W. If we require *only* coincidence with the action of the groups in the highest weights we can embed W in other infinite abelian groups $W_e^\pm$ $(W_e^+ \cong W_e^-)$, so that $W_e^\pm \supset W_a^\pm \supset W_a$. These notions and results are contained in Sect. 3.

In Sect. 4 we give some discussion on the application of the character calculation to modular invariance and on the generalization of these ideas for other affine Kac-Moody algebras, for the (super) Virasoro algebras and for other superalgebras.

1. Reducibility of Verma Modules

Let $\mathfrak{G}$ be the affine Lie algebra $A_1^{(1)}$. Let $\mathfrak{H}$ be the Cartan subalgebra of $\mathfrak{G}$ and $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{H} \oplus \mathfrak{G}_-$ be the standard decomposition of $\mathfrak{G}$ with $\mathfrak{G}_\pm$ diagonalized by $\mathfrak{H}$. Let $e_1, e_2 \in \mathfrak{G}_+$, $f_1, f_2 \in \mathfrak{G}_-$, $h_1, h_2 \in \mathfrak{H}$ be the canonical generators so that $[e_i, f_j] = \delta_{ij} h_i$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$, with $(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ the Cartan matrix of $\mathfrak{G}$.

Let $\Delta = \Delta^+ \cup \Delta^-$ be the root system of $\mathfrak{G}, \Delta^+ (\Delta^-)$ be the set of positive (negative) roots. (For unexplained notation see [12].)

Let $\Lambda \in \mathfrak{H}^*$ and let $\mathcal{M}(\Lambda)$ be the Verma module with highest weight Λ. We recall the Kac-Kazhdan criterion [4] according to which $\mathcal{M}(\Lambda)$ is reducible iff the following condition is fulfilled

$$2(\Lambda+\varrho,\beta)=m(\beta,\beta) \tag{1.1}$$

for some $m\in\mathbb{N}$, $\beta\in\Delta^+$, in (1.1); $(\cdot,\cdot)$ is the standard scalar product on $\mathfrak{H}^*$; $\varrho\in\mathfrak{H}^*$ is defined by $(\varrho,\alpha_i^\vee)=1$, $\alpha_i^\vee\equiv 2\alpha_i/(\alpha_i,\alpha_i)$ for every simple root α_i. We recall that any $\beta\in\Delta^+$ can be writtenn as

$$\beta=p\alpha_1+(p+\delta)\alpha_2 \qquad p\in\mathbb{Z}_+,\ \delta\in\{0,\pm1\},\ 2p+\delta>0, \tag{1.2}$$

where α_1,α_2 are the simple roots so that $(\alpha_1,\alpha_1)=(\alpha_2,\alpha_2)=2=-(\alpha_1,\alpha_2)$.

The roots with $\delta=0$, $\beta=n\bar{d}$, $\bar{d}\equiv\alpha_1+\alpha_2$ are called imaginary roots and those with $\delta\neq0$-real roots. We have $\Delta=\Delta_R\cup\Delta_I$, where $\Delta_R(\Delta_I)$ is the set of real (imaginary) roots: $\Delta_R^\pm\equiv\Delta^\pm\cap\Delta_R$, $\Delta_I^\pm=\Delta^\pm\cap\Delta_I$. Clearly we have

$$(\beta,\beta')=\delta\delta'(\alpha_2,\alpha_2)=2\delta\delta'. \tag{1.3}$$

Thus the product $(\beta,\cdot)$ of an imaginary root β with any other root is zero.

Setting $m_i\equiv(\Lambda+\varrho_1,\alpha_i)$, $i=1,2$ we rewrite (1.1) as

$$(m_1+m_2)p+\delta m_2=\delta^2 m \tag{1.1'}$$

with β as in (1.2). Then for imaginary roots we obtain

$$m_1+m_2=0=c+2, \tag{1.4a}$$

where we have also introduced the central charge $c\equiv(\Lambda,\alpha_1+\alpha_2)$, [note $(\varrho,\alpha_1)=(\varrho,\alpha_2)=1$]; while for real roots (1.1) goes to

$$p(m_1+m_2)+\delta m_2=p(c+2)+\delta m_2=m\in\mathbb{N}. \tag{1.4b}$$

We shall mostly discuss reducibility of Verma modules with respect to (w.r.t.) the imaginary roots. From (1.1) it is clear that $\mathcal{M}(\Lambda)$ is reducible or irreducible simultaneously for all imaginary roots. As in [6, 8] we shall call such modules *singular Verma modules.* Furthermore from (1.4) it is clear that a singular Verma module is either irreducible w.r.t. the real roots, $m_i\in\pm\mathbb{N}$; or is reducible w.r.t. to *all real roots* with $\delta=1$, $m_2=m\in\mathbb{N}$, so that (1.1) holds with the same m for all these roots; or is reducible w.r.t. to *all real roots* with $\delta=-1$, $m_2=-m$, with one and the same $m\in\mathbb{N}$.

Let $\Lambda_1,\Lambda_2\in\mathfrak{H}^*$ be such that $(\Lambda_i,\alpha_j^\vee)=\delta_{ij}$, thus $(\varrho=\Lambda_1+\Lambda_2)$. Then any weight $\Lambda\in h^*$ can be written as

$$\Lambda=(m_1-1)\Lambda_1+(m_2-1)\Lambda_2. \tag{1.5a}$$

Thus for a singular Verma module we have

$$\Lambda=(m_0-1)\Lambda_1-(m_0+1)\Lambda_2, \qquad m_0=m_1=-m_2. \tag{1.5b}$$

We find it useful to make the following statements for an arbitrary affine Lie algebra $\mathfrak{G}$. The Kac-Kazhdan [4] condition (1.1) looks exactly the same. It is known [4] that if (1.1) holds then $\mathcal{M}(\Lambda-m\beta)$ is isomorphic to one (or more) submodules of $\mathcal{M}(\Lambda)$. Such an embedding is realized as follows.

Let v_0 be the highest weight vector of $\mathcal{M}(\Lambda)$, i.e. $Xv_0=0$, $X\in\mathfrak{G}_+$, $Hv_0=\Lambda(H)v_0$, $H\in\mathfrak{H}$, $\mathcal{M}(\Lambda)\cong\mathcal{U}(\mathfrak{G}_-)\otimes v_0$, with $\mathcal{U}(\mathfrak{G}_-)$ the universal enveloping algebra of $\mathfrak{G}_-$. The embedding of $\mathcal{M}(\Lambda-m\beta)$ in $\mathcal{M}(\Lambda)$ is equivalent to the existence in $\mathcal{M}(\Lambda)$ of at least one vector $v_s\neq v_0$ having the properties of the highest weight vector of $\mathcal{M}(\Lambda-m\beta)$ [4]. Each vector v_s, called singular vector may be expressed as follows [6]:

$$v_s=\mathcal{P}(f_1,\dots,f_n)\otimes v_0\,,\qquad n=\operatorname{rank}\mathfrak{G}\,, \tag{1.6}$$

where $\mathcal{P}$ is a homogeneous polynomial of degrees $mk_1,\dots,m_{k_n}$, and k_i come from the simple root decomposition of

$$\beta=k_1\alpha_1+\dots+k_n\alpha_n\,.$$

Thus if β is a simple root we have (further we omit the symbol $\otimes$)

$$v_s=\mathrm{const}\cdot f_i^m v_0\,,\quad \text{for}\quad \beta=\alpha_i\,. \tag{1.6$'$}$$

Malikov, Feigin, and Fuchs [9] have found a formulae for the singular vectors for nonsingular Verma modules. (In this situation v_s is unique for a fixed m and β.)

For the case of a singular Verma module $\mathcal{M}(\Lambda)$ it is clear that $\mathcal{M}(\Lambda-n\bar{d})$, $\forall n\in\mathbb{N}$ is isomorphic to a submodule of $\mathcal{M}(\Lambda)$. More than this $\mathcal{M}(\Lambda+n'\bar{d})$, $n'\in\mathbb{Z}$ is isomorphic to a submodule of $\mathcal{M}(\Lambda+n\bar{d})$, $n\in\mathbb{Z}$, iff $n'<n$. (Here, $\bar{d}=\alpha_0+\dots+\alpha_l$ for $A_l^{(1)}$; for the general case see [12].)

We return to the case of $\mathfrak{G}=A_1^{(1)}$. For that in [9] there are exhaustive results also for the singular vectors of singular Verma modules. It is shown that the number of singular vectors for fixed $\beta=m\bar{d}$ is equal to the number of partitions of m and all singular vectors are given explicitly. To formulate these results of [9] one needs some notation.

Let

$$f_3=[f_1,f_2]\,,\quad f_4=-[f_1,f_3]\,,\quad f_5=[f_2,f_3],\dots,f_{3k}=[f_1,f_{3k-1}]\,,$$
$$f_{3k+1}=-[f_1,f_{3k}]\,,\quad f_{3k+2}=[f_2,f_{3k}]\,.$$

f_i form a basis of $\mathfrak{G}_-$, $[f_i,f_j]=\alpha_{ij}f_{i+j}$, where $\alpha_{ij}=-1,0,1$, $\alpha_{ij}\equiv(j-i)\bmod 3$. One also has

$$[h_1,f_j]=-2\alpha_{0j}f_j\,,\qquad [h_2,f_j]=2\alpha_{0j}f_j\,,$$
$$[e_1,f_j]=2\alpha_{-1j}f_{j-1}\,,\qquad [e_2,f_j]=2\alpha_{-2,j}f_{j-2}\,.$$

The singular vectors are given by the following:

Theorem 1 [9]. *Let $\mathcal{M}(\Lambda)$ be reducible w.r.t. the imaginary roots, i.e., $\Lambda=(m_0-1)\Lambda_1-(m_0+1)\Lambda_2$, (cf.* (1.5b)*). Let*

$$F_k=f_1f_{3k-1}+f_2f_{3k-2}+\dots+f_{3k-1}f_1-m_0f_{3k}\,,\qquad k\in\mathbb{N}\,. \tag{1.7}$$

For all $k_1,\dots,k_r\in\mathbb{N}$ the vector $F_{k_1}\dots F_{k_r}v_0$ is a singular vector of degree $k=k_1+\dots+k_r$ of $\mathcal{M}(\Lambda)$. This vector does not depend on the permutation of the numbers $k_1,\dots,k_r$. The vectors $F_{k_1}\dots F_{k_r}v_0$ are linearly independent.

Corollary. $F_kF_n=F_nF_k$, $\forall n,k$.

Remark. The simplest singular vector

$$F_1v_0=(f_1f_2+f_2f_1-m_0f_3)v_0=[(1-m_0)f_1f_2+(m_0+1)f_2f_1]v_0$$

was given in [6], formula (37).

Let us denote by I_Λ the submodule of $\mathcal{M}(\Lambda)$ generated by the singular vectors of Theorem 1, and by F_Λ the factor module of $\mathcal{M}(\Lambda)$ by I_Λ

$$F_\Lambda = \mathcal{M}(\Lambda)/I_\Lambda . \tag{1.8}$$

Malikov, Feigin, and Fuchs [9] made the hypothesis that F_Λ is irreducible if $\mathcal{M}(\Lambda)$ is not reducible w.r.t. the real roots. We shall prove this fact in the next section by calculating first the character of F_Λ, then using a construction of Wakimoto and a result of Rao.

2. Calculation of Characters

Our results will rely on some calculations of characters so we recall the basic facts [12].

Let Γ, (respectively Γ_+) be the set of all integer (respectively integer dominant) elements of $\mathfrak{H}^*$, i.e. $\lambda \in \mathfrak{H}^*$ such that $(\lambda, \alpha_i) \in \mathbb{Z}$, (respectively $\mathbb{Z}_+$), $i = 1, 2$.

We recall that for each invariant subspace $V \subset \mathcal{U}(\mathfrak{G}_-)v_0 \cong \mathcal{M}(\Lambda)$ we have the following decomposition:

$$V = \bigoplus_{\mu \in \Gamma_+} V^\mu , \tag{2.1}$$

$$V^\mu = \{u \in V | h_k \cdot u = (\Lambda - \mu)(h_k)u,\ k = 1, 2\} . \tag{2.2}$$

(Note that $V^0 = \mathbb{C}v_0$.) Following [12] let $E(\mathfrak{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu \in \mathfrak{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathfrak{H}^* | \mu \leqq \lambda\}$, using any ordering of $\mathfrak{H}^*$; the formal exponents $e(\mu)$ have the properties $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$. Then we define

$$ch\, V = \sum_{\mu \in \Gamma_+} (\dim V^\mu) e(\Lambda - \mu) = e(\Lambda) \sum_{\mu \in \Gamma_+} (\dim V^\mu) e(-\mu) . \tag{2.3}$$

For the Verma module $\mathcal{M}(\Lambda)$, $\dim V^\mu = P(\mu)$, [3, 12] where $P(\mu)$ is defined as the number of partitions of $\mu \in \Gamma_+$ into a sum of positive roots, where each root is counted with its multiplicity; $P(0) = 1$. We recall several ways to write $ch\,\mathcal{M}(\Lambda)$ [3, 12]:

$$ch\,\mathcal{M}(\Lambda) = e(\Lambda) \sum_{\mu \in \Gamma_+} P(\mu) e(-\mu) = e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{-\mathrm{mult}\,\alpha} , \tag{2.4}$$

or more concretely, for the $A_1^{(1)}$ case setting $e(-\bar{d}) \equiv q$, $e(-\alpha_1) \equiv z$, [then $e(-\alpha_2) = qz^{-1}$], and noting that $\mathrm{mult}\,\alpha = 1$, $\forall \alpha \in \Delta^+$:

$$ch\,\mathcal{M}(\Lambda) = e(\Lambda) \Big/ \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n z^{-1})(1 - q^{n-1} z) . \tag{2.5}$$

Let us set

$$V_{j_1 \dots j_m} \equiv \mathcal{U}(\mathfrak{G}_-) F_{j_1} \dots F_{j_m} v_0 , \tag{2.6}$$

where $F_{j_1} \dots F_{j_m} v_0$ is a singular vector from Theorem 1. Clearly $V_{j_1 \dots j_m}$ is symmetric in its indices and $V_{j_1 \dots j_m} \subset V_{k_1 \dots k_n}$ if

$$\{k_1, \dots, k_n\} \subset \{j_1, \dots, j_m\} , \tag{2.7}$$

the inclusion being proper unless $m=n$. Further we have

$$ch\,V_{j_1\dots j_m}=q^{j_1+\dots+j_m}ch\mathscr{M}(\Lambda_0)\,,\qquad \Lambda_0=(m_0-1)\Lambda_1-(m_0+1)\Lambda_1\,. \tag{2.8}$$

Following [12] we introduce in the ring $\mathbb{C}[(q,z)]$ of formal power series in q,z a partial ordering by putting $f=\sum f_\lambda e(\lambda)\leqq g=\sum g_\lambda e(\lambda)$ iff $f_\lambda\leqq g_\lambda$ for every λ. Then because of (2.7, 8) we have

$$ch\,V_{j_1\dots j_m}<ch\,V_{k_1\dots k_n}\,,\quad \text{if}\quad \{k_1,\dots,k_n\}\subset\{j_1\dots j_m\}\quad\text{and}\quad n<m\,. \tag{2.9}$$

The first crucial fact is that we can compute the character of I_{Λ_0} in terms of the characters of $V_{j_1\dots j_n}$.

Proposition 1. *The following formula holds:*

$$ch I_{\Lambda_0}=\sum_{n=1}^{\infty}(-1)^{n+1}\sum_{\substack{j_1,\dots,j_n\in\mathbb{N}\\ j_1<j_2<\dots<j_n}} ch\,V_{j_1\dots j_n}\,,\qquad \Lambda_0=(m_0-1)\Lambda_1-(m_0+1)\Lambda_2\,. \tag{2.10}$$

Proof. We shall obtain the formula by an induction procedure.

First we note that the first term ($n=1$) in (2.10), i.e. $\sum_{j_1\in\mathbb{N}} ch\,V_{j_1}$ should be present in order to account for every possible element of I_{Λ_0}. However, it is clear that

$$ch I_{\Lambda_0}<\sum_{j_1\in\mathbb{N}} ch\,V_{j_1} \tag{2.11}$$

because of the overlaps between the V_{j_1}. In particular, each $V_{j_1j_2}$ $(j_1<j_2)$ is contained in both V_{j_1} and V_{j_2}. This means that the sum $\sum_{j_1,j_2\in\mathbb{N},\,j_1<j_2} ch\,V_{j_1j_2}$ is contained twice in the right-hand side of (2.11). Thus, it should be subtracted in order to avoid this overcounting; this is the term with $n=2$ in (2.10). [Analogously, $V_{j_1j_2j_3}$ is contained in V_{j_1}, V_{j_2}, V_{j_3}, $V_{j_1j_2}$, $V_{j_1j_3}$, $V_{j_2j_3}$, thus $ch\,V_{j_1j_2j_3}$ is not contained in the two terms $n=1,2$ of (2.10). That is why it should be added which gives the third term in (2.10)]. Further, we proceed by induction. Suppose the right-hand side of (2.10) describes correctly the contribution of all $V_{j_1\dots j_n}$ for $n\leqq p-1$, $p>2$. Consider now a term $V_{j_1\dots j_p}$. It is contained in $V_{j_1},\dots,V_{j_p}$, i.e., in $p=\binom{p}{1}$ terms; in $V_{j_1j_2}$, $V_{j_1j_3}\dots V_{j_{p-1}j_p}$, i.e., $\binom{p}{2}$ terms; and more generally in $\binom{p}{k}$ terms $V_{j_{i_1}}\dots j_{i_k}$, $k<p$, $\{j_{i_1},\dots,j_{i_k}\}\subset\{j_1,\dots,j_p\}$. Thus it is contained in $2^p-2=\sum_{k=1}^{p-1}\binom{p}{k}$ terms. Next note that the terms with k *odd* are with sign plus, in (2.10), while those with k *even* are with sign minus. To calculate how many there are from each sign we note that in $-(a-b)^p=\sum_{k=0}^{p}\binom{p}{k}(-1)^{k+1}a^{p-k}b^k$, where $a,b>0$, the terms with plus and minus sign are equal in number. $\left(\text{Indeed, } -(1-1)^p=\sum_{k=0}^{p}\binom{p}{k}(-1)^{k+1}=0.\right)$

The subsum $\sum_{n=1}^{p-1}$ in (2.10) corresponds to $-(a-b)^p$ without the two terms $k=0$, $p\left(\binom{p}{0}=\binom{p}{p}=1\right)$. Thus if p is odd the two missing terms are with opposite signs, and if p is even the two missing terms are with minus sign. Thus if p is *odd* $V_{j_1\dots j_p}$ is contained in $2^{p-1}-1$ terms for which the character enters (2.10) with plus sign and $2^{p-1}-1$ term for which the character enters (2.10) with minus sign. Thus $V_{j_1\dots j_p}$ is not taken into account in the first $p-1$ terms of (2.10); thus $ch\,V_{j_1\dots j_p}$ should be *added* with sign plus. Consider now p *even*, then $V_{j_1\dots j_p}$ is contained in 2^{p-1} terms for which the character enters (2.10) with sign plus, and 2^{p-1} terms for which the character enters (2.10) with sign minus. Thus the net result is that $V_{j_1\dots j_p}$ is accounted for *twice* and $ch\,V_{j_1\dots j_p}$ should be *subtracted.* So the right-hand side of (2.14) describes correctly all terms with $n=p$ if the terms of with $n<p$ are described correctly. □

Proposition 2. *The characters of the submodule* I_{Λ_0},

$$\Lambda_0=(m_0-1)\Lambda_1-(m_0+1)\Lambda_2$$

and of the factor module $F_{\Lambda_0}\equiv\mathcal{M}(\Lambda_0)/I_{\Lambda_0}$ *are given by*

$$chI_{\Lambda_0}=ch\mathcal{M}(\Lambda_0)\left[1-\prod_{n=1}^{\infty}(1-q^n)\right], \tag{2.12}$$

$$\begin{aligned} chF_{\Lambda_0}&=ch\mathcal{M}(\Lambda_0)\prod_{n=1}^{\infty}(1-q^n)=e(\Lambda_0)\Big/\prod_{n=1}^{\infty}(1-q^{n-1}z)(1-q^nz^{-1}) \\ &=e(\Lambda_0)\Big/\prod_{\alpha\in\Delta_R^+}(1-e(-\alpha)), \end{aligned} \tag{2.13}$$

where Δ_R^+ *is the set of positive real roots.*

Proof. We have

$$chF_{\Lambda_0}=ch\mathcal{M}(\Lambda_0)-chI_{\Lambda_0}, \tag{2.14}$$

thus (2.13) follows from (2.12), (2.4), and (2.5). For chI_{Λ_0} we have substituting (2.8) in (2.10)

$$chI_{\Lambda_0}=ch\mathcal{M}(\Lambda_0)\sum_{n=1}^{\infty}(-1)^{n+1}\sum_{\substack{j_1,\dots,j_n\in\mathbb{N}\\ j_1<\dots<j_n}}q^{j_1+\dots+j_n} \tag{2.15}$$

$$=ch\mathcal{M}(\Lambda_0)\left[1-\sum_{n=0}^{\infty}(-1)^n\sum_{\substack{j_1\dots j_n\in\mathbb{N}\\ j_1<\dots<j_n}}q^{j_1+\dots+j_n}\right], \tag{2.16}$$

which is equal to (2.12). □

In [10] Wakimoto constructed a family of highest weight modules $\pi_{\mu\nu}$ over $A_1^{(1)}$ parametrized by $(\mu,\nu)\in\mathbb{C}^2$. We shall not repeat the construction of the $\pi_{\mu\nu}$ representation spaces $V(\mu,\nu)$ from [10]. We need only two facts. The highest weight module $V(\mu,0)$ has weight $\Lambda_{\mu,0}=-(1+\mu)\Lambda_1+(\mu-1)\Lambda_1$, i.e. the highest weight Λ_0 of a singular Verma module [cf. (1,56) with $\mu=-m_0$]. Furthermore it is shown that [10]

$$ch\,V(\mu,0)=e(\Lambda_{\mu,0})\Big/\prod_{\alpha\in\Delta_R^+}(1-e(-\alpha)), \tag{2.17}$$

which coincides with (2.13). Consequently

$$F_{\Lambda_0}\cong V(\mu,0),\qquad \Lambda_0=\Lambda_{\mu,0}. \tag{2.18}$$

Next we use an observation due to Rao [11]:

Theorem 2 [11]. *The modules $V(\mu,0)$ constructed by Wakimoto are irreducible if $\mu\notin\mathbb{Z}\backslash\{0\}$.*

A necessary condition for the irreducibility of $V(\mu,0)$ was obtained in [13] (Theorem B). Combining Proposition 2, (2.18) and Theorem 2 we have proved:

Theorem 3. *Let $L(\Lambda_0)$ be the irreducible highest weight module with highest weight*

$$\Lambda_0=(m_0+1)\Lambda_1-(m_0+1)\Lambda_2,\ m_0\notin\mathbb{Z}\backslash\{0\},$$

such that the Verma module $\mathscr{M}(\Lambda_0)$ is reducible with respect to the imaginary roots and irreducible with respect to the real roots. Then

$$L(\Lambda_0)\cong F_{\Lambda_0}\cong\mathscr{M}(\Lambda_0)/I_{\Lambda_0}\cong V(-m_0,0), \tag{2.19}$$

where I_{Λ_0} is the submodule of $\mathscr{M}(\Lambda_0)$ generated by the singular vectors of Malikov, Feigin, Fuchs from Theorem 1, and $V(-m_0,0)$ is the highest weight module with highest weight Λ_0, constructed by Wakimoto. In particular,

$$ch L(\Lambda_0)=e(\Lambda_0)\Big/\prod_{\alpha\in\Lambda_R^+}(1-e(-\alpha))=e(\Lambda_0)\Big/\prod_{k=1}^{\infty}(1-q^kz^{-1})(1-q^{k-1}z). \tag{2.20}$$

For $\Lambda_0=-\varrho=-\Lambda_1-\Lambda_2$, i.e., $m_0=0$, formula (2.20) was conjectured in [4].

Next we consider the case when $\mathscr{M}(\Lambda_0)$ is reducible w.r.t. to the imaginary roots and to some real roots. Consider first the case when $\mathscr{M}(\Lambda_0)$ is singular and reducible also w.r.t. the real roots with $\delta=-1$ [cf. (1.2)], i.e. (1.1) holds for a fixed $m\in\mathbb{N}$, and $\beta=k\bar{d}+\alpha_1$, $k=0,1,\ldots$. Then according to (1.4b), (1.5b) we have $m_2=-m$, $\Lambda_0\to\Lambda_0^+=(m-1)\Lambda_1-(m+1)\Lambda_2$, $m\in\mathbb{N}$. Thus all Verma modules

$$\mathscr{M}(\Lambda_0^+-m\beta)=\mathscr{M}(\Lambda_0^+-m(p-1)\bar{d}-m\alpha_1),\qquad p\geqq 1,$$

are isomorphic to submodules of $\mathscr{M}(\Lambda_0^+)$. More than this all these are singular Verma modules and for $p>1$ all are isomorphic to submodules of $\mathscr{M}(\Lambda_0^+-m\alpha_1)$. $\mathscr{M}_n=\mathscr{M}(\Lambda_0^+-n\bar{d}-\alpha_1)$ are isomorphic to submodules of $\mathscr{M}(\Lambda_0^+-m\alpha_1)$ and thus to submodules of $\mathscr{M}(\Lambda_0^+)$. The most general statement is the following:

Proposition 3. *Let $\mathscr{M}(\Lambda_0^+)$ be a singular Verma module reducible with respect to the positive roots*

$$\beta=p\alpha_1+(p-1)\alpha_2,$$

$$\Lambda_0^+=(m-1)\Lambda_1-(m+1)\Lambda_2,\qquad m\in\mathbb{N}.$$

Then there are the following invariant embeddings of Verma modules:

$$\mathscr{M}(\Lambda_0^+-n\bar{d}-\varepsilon m\alpha_1)\supset\mathscr{M}(\Lambda_0^+-n'\bar{d}-\varepsilon' m\alpha_1), \tag{2.21}$$

$n,n'\in\mathbb{Z}$; $\varepsilon,\varepsilon'=0,1$; $n\leqq n'$; $n=n'\Rightarrow\varepsilon=0$, $\varepsilon'=1$; $\varepsilon=1$, $\varepsilon'=0\Rightarrow n'-n\geqq m$.

Proof. First we note that $\mathscr{M}(\Lambda_0^+-n\bar{d}-\varepsilon m\alpha_1)$ is reducible w.r.t. the imaginary roots for $\forall n\in\mathbb{Z}$, $\varepsilon=0,1$; $\mathscr{M}(\Lambda_0^+-n\bar{d})$ is reducible w.r.t. all positive roots $\beta=p\alpha_1+(p-1)\alpha_2$, $p\geqq 1$, so that (1.1) holds with the same m from Λ_0^+; $\mathscr{M}(\Lambda_0^+-n\bar{d}-m\alpha_1)$ is reducible w.r.t. all positive roots $\beta=(p-1)\alpha_1+p\alpha_2$, $p\geqq 1$, so that (1.1) holds with the same m from Λ_0^+.

For $\varepsilon=\varepsilon'=0,1$ the embeddings in (2.21) are described by the singular vectors of degree $n'-n(>0)$ of Theorem 1. For $\varepsilon=0$, $\varepsilon'=1$ and $n'-n=km$, $k=0,1,\ldots$ the embeddings in (2.21) are described by the singular vectors $F(m,p,t)v_0$ of Theorem 3.2 of [9], which are valid also for nonsingular Verma modules $(t\equiv m_1+m_2)$. In our case $t=m_1+m_2=0$ and we have

$$v_s=v_s^{m\beta}=F_1(m,k)v_0\,,\qquad \beta=k\bar{d}+\alpha_1\,,\qquad k=\frac{n'-n}{m}=0,1,\ldots,\tag{2.22a}$$

$$F_1(m,k)\equiv F(m,k+1,0)=(f_1^m f_2^m)^k f_1^m\,.$$

[Note that $F_1(m,0)=f_1^m$ as it should – cf. (1.6').] For $\varepsilon=0$, $\varepsilon'=1$ and $n'-n\neq km$, $k=0,1,\ldots$ the embeddings in (2.21) are described by composition of embeddings of the two types described so far. Let $n'-n=km+l$, $k=0,1$, $l=1,2,\ldots,m-1$; then (2.21) is described, e.g., by

$$v_s=F_{l_1}\ldots F_{l_s}F_1(m,k)v_0\,,\qquad l_1+\ldots+l_s=l\,,\tag{2.23}$$

F_{l_i} are given by formula (1.7). For $\varepsilon=1$, $\varepsilon'=0$, $n'-n=km$, $k=1,2,\ldots$ the embeddings are described again by the singular vectors of Theorem 3.2 of [9], however, with f_1,f_2 interchanged, i.e.,

$$v_s=F_2(m,k)v_0=(f_2^m f_1^m)^k f_2^m\,.\tag{2.24}$$

For $\varepsilon=1$, $\varepsilon'=0$, $n'-n=km+l$, $k=1,2,\ldots,l=1,2,\ldots,m-1$ the embeddings are described, e.g., by

$$v_s=F_{l_1}\ldots F_{l_s}F_2(m,k)v_0\,,\qquad l_1+\ldots+l_s=l\,.\quad\square\tag{2.25}$$

Remark. The Verma modules $\mathcal{M}(\Lambda_0^+-n\bar{d}-\varepsilon m\alpha_1)$, $\Lambda_0^+=(m-1)\Lambda_1-(m+1)\Lambda_2$, $m\in\mathbb{N}$, $n\in\mathbb{Z}$, $\varepsilon=0,1$ were shown in [6] to form a multiplet, i.e. a set $\mathcal{N}$ of Verma modules $V'\neq V$ such that 1) if $V\in\mathcal{N}$, then $\mathcal{N}\supset\mathcal{N}_V$, where $\mathcal{N}_V$ is the set of all Verma modules such that $V'\in\mathcal{N}_V\Leftrightarrow V\supset V'$ or $V'\subset V$; 2) $\mathcal{N}$ does not contain a proper subset with property 1). These multiplets are parametrized by $m\in\mathbb{N}$ (we disregard an arbitrary complex parameter which is the eigenvalue of $\bar{d}$). Thus all Verma modules which are reducible w.r.t. the imaginary roots and w.r.t. the positive roots with one and the same m in (1.1) are in one multiplet. (cf. [6], (35), (36)).

It is clear that the structure of $\mathcal{M}(\Lambda_0^+)$ is determined only by the noncomposition embeddings described in Proposition 3. Some further compositions are obvious, i.e. from (2.21) we see

$$\mathcal{M}(\Lambda_0^+)\supset\mathcal{M}(\Lambda_0^+-m\alpha_1)\supset\ldots\supset\mathcal{M}(\Lambda_0^+-k\bar{d}-m\alpha_1)\supset\ldots,\tag{2.26a}$$

and moreover the singular vectors in (2.22a) have the property

$$v_s=F_1(m,k)v_0=(f_1^m f_2^m)^k\,F_1(m,0)v_0\,.\tag{2.26b}$$

Thus we have proved:

Proposition 4. *Let $\mathcal{M}(\Lambda_0^+)$ be a singular Verma module reducible w.r.t. the positive roots $\beta=p\alpha_1+(p-1)\alpha_2$, $p\geqq 1$, $\Lambda_0^+=(m-1)\Lambda_1-(m+1)\Lambda_2$, $m\in\mathbb{N}$. Then the only noncomposition embeddings of Verma modules in $\mathcal{M}(\Lambda_0^+)$ are those generated by the singular vectors corresponding to reducibility w.r.t. the imaginary roots and w.r.t. the simple root $\beta=\alpha_1$, $(p=1)$, i.e., $v_s=f_1^m v_0$ (cf. (1.6')).*

Now we can prove the following:

Proposition 5. *Let $\tilde{I}_{\Lambda_0^+}$ be the submodule of $\mathscr{M}(\Lambda_0^+)$ generated by the singular vectors of Theorem 1 and $v_s = f_1^m v_0$.*

Let

$$\tilde{F}_{\Lambda_0^+} = \mathscr{M}(\Lambda_0^+)/\tilde{I}_{\Lambda_0^+}\,. \tag{2.27}$$

Then

$$\begin{aligned} ch\tilde{F}_{\Lambda^+} &= (1-z^m)chF_{\Lambda^+} \\ &= (1-z^m)e(\Lambda_0^+)\Big/\prod_{n=1}^{\infty}(1-q^n z^{-1})(1-q^{n-1}z) \\ &= e(\Lambda_0^+)(1+z+z^2+\ldots+z^{m-1})\Big/\prod_{n=1}^{\infty}(1-q^n z^{-1})(1-q^n z)\,, \end{aligned} \tag{2.28}$$

where chF_{Λ^+} is from (2.13).

Proof. Let $V^m = \mathscr{U}(\mathfrak{G}_-)f_1^m v_0$, $\tilde{I}_{\Lambda_0^+} = V^m \cup I_{\Lambda_0^+}$, where $I_{\Lambda_0^+}$ is the submodule generated by the singular vectors of Theorem 1. We can also write

$$\tilde{I}_{\Lambda_0^+} = V^m \cup (I_{\Lambda_0^+} \backslash V^m)\,, \tag{2.29}$$

then

$$ch\tilde{I}_{\Lambda_0^+} = ch\,V^m + ch(I_{\Lambda_0^+}\backslash V^m)\,. \tag{2.30}$$

We use $f_1 \leftrightarrow e(-\alpha_1) = z$ and

$$ch\,V^m = z^m ch\mathscr{M}(\Lambda_0^+)\,, \tag{2.31a}$$

$$ch(I_{\Lambda_0^+}\backslash V^m) = (1-z^m)chI_{\Lambda_0^+}\,, \tag{2.31b}$$

to obtain taking into account (2.12), (2.13),

$$ch\tilde{F}_{\Lambda_0^+} = ch\mathscr{M}(\Lambda_0^+) - ch\tilde{I}_{\Lambda_0^+} = (1-z^m)chF_{\Lambda_0^+}\,. \quad \square \tag{2.32}$$

Analogously we can consider the singular Verma modules $\mathscr{M}(\Lambda_0^-)$ reducible w.r.t. the real roots $\beta = (p-1)\alpha_1 + p\alpha_2$ [$p \geqq 1$, $\delta = 1$ in (1.2)]. In this case [cf. (1.5b)] $\Lambda_0^- = -(\mathrm{m}+1)\Lambda_1 + (\mathrm{m}-1)\Lambda_2$, $\mathrm{m} \in \mathbb{N}$.

Proposition 6. *Let $\mathscr{M}(\Lambda_0^-)$ be a singular Verma module, reducible w.r.t. the positive roots $\beta = (p-1)\alpha_1 + p\alpha_2$, $\Lambda_0^- = -(m+1)\Lambda_1 + (m-1)\Lambda_2 m \in \mathbb{N}$. Then the only non-composition embeddings of Verma modules in $\mathscr{M}(\Lambda_0^-)$ are those generated by the singular vectors corresponding to reducibility w.r.t. the imaginary roots and w.r.t. the simple root $\beta = \alpha_2$, $(p=1)$, i.e. $v_s = f_2^m v_0$ (cf. (1.6′)). Let $\tilde{I}_{\Lambda_0^-}$ be the submodule of $\mathscr{M}(\Lambda_0^-)$ generated by the above singular vectors. Let*

$$\tilde{F}_{\Lambda_0^-} = \mathscr{M}(\Lambda_0^-)/\tilde{I}_{\Lambda_0^-}\,. \tag{2.33}$$

Then

$$\begin{aligned} ch\tilde{F}_{\Lambda_0^-} &= (1-q^m z^{-m})chF_{\Lambda_0^-} \\ &= (1-q^m z^{-m})e(\Lambda_0^-)\Big/\prod_{n=1}^{\infty}(1-q^n z^{-1})(1-q^{n-1}z) \\ &= e(\Lambda_0^-)(1+qz^{-1}+q^2z^{-2}+\ldots \\ &\quad + q^{m-1}z^{-m+1})\Big/\prod_{n=1}^{\infty}(1-q^{n+1}z^{-1})(1-q^{n-1}z)\,. \end{aligned} \tag{2.34}$$

Proof. The proof is analogous to the proof of Propositions 4, 5. Note that $f_2 \leftrightarrow e(-\alpha_2) = e(-\bar{d} + \alpha_1) = qz^{-1}$. $\square$

Corollary. *Let $\tilde{I}^n_{\Lambda_0^+}$ ($\tilde{I}^n_{\Lambda_0^-}$) be the submodule of $\mathcal{M}(\Lambda_0^+ - n\bar{d})$ (respectively $\mathcal{M}(\Lambda_0^- - n\bar{d})$) generated by the singular vectors of Theorem 1, and $v_s = f_1^m v_0$, (respectively $v_s = f_2^m v_0$) and let*

$$\tilde{F}^n_{\Lambda_0^\pm} = \mathcal{M}(\Lambda_0^\pm - n\bar{d}) / \tilde{I}^n_{\Lambda_0^\pm}. \tag{2.35}$$

Then

$$ch \tilde{F}^n_{\Lambda_0^+} = (1 - z^m) e(\Lambda_0^+ - n\bar{d}) \Big/ \prod_{n=1}^{\infty} (1 - q^n z^{-1} z)(1 - q^{n-1} z), \tag{2.36}$$

$$ch \tilde{F}^n_{\Lambda_0^-} = (1 - q^m z^{-m}) e(\Lambda_0^- - n\bar{d}) \Big/ \prod_{n=1}^{\infty} (1 - q^n z^{-1})(1 - q^{n-1} z). \tag{2.37}$$

3. Weyl Groups for the Singular Highest Weight Modules

As we have noted earlier [6, 7, 8] the Weyl-Kac group W [1] which is generated by the reflections s_{α_i}, where α_i ($i = 0, \ldots, l$) are the simple roots for an affine Kac-Moody algebra $\mathfrak{G}$, is not adequate for the description of the singular highest weight modules (HWM).

The problem is not only in the fact that the usual formula for the Weyl-Kac reflections

$$s_\alpha \cdot (\Lambda + \varrho) = \Lambda + \varrho - \frac{2(\Lambda + \varrho, \alpha)}{(\alpha, \alpha)} \alpha \tag{3.1a}$$

is not well defined for $(\alpha, \alpha) = 0$ since one may try to use the endomorphism t_α of $\mathfrak{H}^*$ for $\alpha \in \sum_{i=0}^{l} \mathbb{C}\alpha_i$ (cf. [12]):

$$t_\alpha(\lambda) = \lambda + (\lambda | \bar{d})\alpha - [(\lambda, \alpha) + \tfrac{1}{2}(\alpha, \alpha)(\lambda, \bar{d})]\bar{d}. \tag{3.1b}$$

However if $\alpha = n\bar{d}$, $t_\alpha(\lambda) = \lambda$, i.e. the action is trivial.

In [6, 7, 8] we introduced the notion of *imaginary* "*reflections*" or "*translations*" so that s_β, β imaginary, is defined and acts nontrivially on the highest weight Λ of a singular HWM, by the formula

$$s_\beta \cdot \Lambda \equiv \Lambda - \beta, \quad \beta = n\bar{d}, \quad (\beta, \beta) = 0, \quad (\Lambda + \varrho, \beta) = 0. \tag{3.2}$$

The idea of these translations was to obtain a description of the embedding pattern of submodules of the singular Verma modules in the same way as the Weyl-Kac group describes the embedding pattern of Verma modules with dominant integral highest weight [4]. Equivalently, these translations describe the possible multiplets of singular Verma modules [6, 8].

Here we develop further these ideas to introduce a Weyl group suitable for the character formula for $L(\Lambda_0)$, Λ_0 singular highest weight.

Consider the infinite abelian multiplicative group W_a generated by the symbols $w(n)$, $n \in \mathbb{N}$, with the properties $w(n)^2 = 1$, $w(n)w(n') = w(n')w(n)$. Such a group is called a torsion group or a p-group [14] (here $p = 2$). The group W_a may be parametrized as follows:

$$W_a = \{w = w(n_1) \ldots w(n_k) | k \in \mathbb{Z}_+ ; \text{ for } k = 0 \ w = 1; \text{ for } k > 0, \ n_i \in \mathbb{N}, \ n_1 < \ldots < n_k\}. \tag{3.3}$$

If $w=1$, or $w=w(n_1)\dots w(n_k)$, $n_i\in\mathbb{N}$, $n_i\neq n_j$, $i\neq j$, we shall say that w is given in a *reduced form*. It is clear that every element of W_a has a reduced form.

We define the action of $w\in W_a$ on $\mathfrak{H}^*$ analogously to the action of $s_{n\bar{d}}$ in (2.2). For $\lambda\in h^*$ and w in a reduced form we set

$$w\cdot\lambda=w(n_1)\dots w(n_k)\cdot\lambda\equiv\lambda-(n_1+\dots+n_k)\bar{d}\,,$$
$$n_i\in\mathbb{N}\,,\qquad n_i\neq n_j\,,\qquad 1\cdot\lambda=\lambda\,. \tag{3.4}$$

This action is non-associative, i.e. it may happen that $(w_1w_2)\cdot\lambda\neq w_1\cdot(w_2\cdot\lambda)$, but this is not essential for our purposes. (The action is associative if we consider W_a as a semi-group [i.e. if we drop the relation $w(n)^2=1$].)

Analogously to the Weyl(-Kac) group we introduce the *length* of $w\in W_a$, denoted $l(w)$, for w given in a reduced form:

$$l(w)=k\,,\qquad w=w(n_1)\dots w(n_k)\,,\qquad n_i\in\mathbb{N}\,,$$
$$n_i\neq n_j\,,\qquad i\neq j\,,\qquad l(1)=0\,. \tag{3.5}$$

The length of an arbitrary element W_a is defined to be equal to the length of its reduced form. (Note that an element of length k has $k!$ reduced forms.) Now we can prove the following:

Proposition 7. *Let $\mathcal{M}(\Lambda_0)$, F_{Λ_0}, and $L(\Lambda_0)$ be as in Proposition 2, $\Lambda_0=(m-1)\Lambda_1-(m+1)\Lambda_2$. Let W_a be the Weyl group in* (3.2). *Then we have*

$$chF_{\Lambda_0}=ch\mathcal{M}(\Lambda_0)\sum_{w\in W_a}(-1)^{l(w)}e(w\cdot(\Lambda_0+\varrho)-\Lambda_0-\varrho)\,,$$
$$(\Lambda_0+\varrho,\bar{d})=0\,, \tag{3.6a}$$
$$chL(\Lambda_0)=chF_{\Lambda_0}\,,\quad \textit{if}\quad m\notin\mathbb{Z}\backslash\{0\}\,. \tag{3.6b}$$

Proof. Consider the sum in the right-hand side of (3.6a)

$$\sum_{w\in W_a}(-1)^{l(w)}e(w\cdot(\Lambda_0+\varrho)-\Lambda_0-\varrho) \tag{3.7a}$$
$$=\sum_{k=0}^{\infty}\sum_{\substack{n_1<\dots<n_k\\ n_1,\dots,n_k\in\mathbb{N}}}(-1)^{l(w(n_1)\dots w(n_k))}e((w(n_1)\dots w(n_k))\cdot(\Lambda_0+\varrho)-\Lambda_0-\varrho) \tag{3.7b}$$
$$=\sum_{k=0}^{\infty}(-1)^k\sum_{\substack{n_1,\dots,n_k\in\mathbb{N}\\ n_1<\dots<n_k}}e(-n_1\bar{d}-n_2\bar{d}\dots n_k\bar{d}) \tag{3.7c}$$
$$=\sum_{k=0}^{\infty}(-1)^k\sum_{\substack{n_1,\dots,n_k\in\mathbb{N}\\ n_1<\dots<n_k}}q^{n_1+\dots+n_k}=\prod_{n=1}^{\infty}(1-q^n)\,. \tag{3.7d}$$

Comparing with (2.13) and (2.20) we see that the proof is finished. □

Consider now the subgroups S^+,S^- of the Weyl-Kac group W so that $S^+=\{1,s_1\}$, $S^-=\{1,s_2\}$, and $s_a=s_{\alpha_a}$ is given in (3.1a). Note that each one of S^+,S^- is isomorphic to the Weyl group W_0 of the $A_1=sl(2,\mathbb{C})$, the finite-dimensional algebra underlying $A_1^{(1)}$. Consider further the infinite abelian groups $W_a^{\pm}$ with generators s_1 or s_2 and the generators of W_a, i.e.

$$W_a^{\pm}=\{w=w's=sw'|w'\in W_a,\, s\in S^{\pm}\}\,. \tag{3.8a}$$

This definition is meaningful since

$$s_\alpha w(n)\cdot\Lambda=s_\alpha\cdot(\Lambda-n\bar{d})=\Lambda-n\bar{d}-2\frac{(\Lambda-n\bar{d},\alpha)}{(\alpha,\alpha)}\alpha=\Lambda-\frac{2(\Lambda,\alpha)}{(\alpha,\alpha)}\alpha-n\bar{d}=w(n)s_\alpha\cdot\Lambda\,.$$

Thus

$$W_a^\pm\cong W_a\times W_0\,. \tag{3.8b}$$

We set

$$\begin{gathered} l(w)=l_a(w')+l_0(s)\,,\qquad w=w's\,,\qquad w'\in W_a\,,\\ s\in S^\pm\,,\qquad l_0(s_\alpha)=1\,,\qquad l_0(1)=0\,. \end{gathered} \tag{3.9}$$

Next we prove the analogue of Proposition 7 for $F_{\Lambda_0^\pm}$:

Proposition 8. *Let* $\mathcal{M}(\Lambda_0^\pm)$, $\tilde{F}_{\Lambda_0^\pm}$ *be as in Propositions 3, 4, 5,*

$$\Lambda_0^\pm=(\pm m-1)\Lambda_1-(\pm m+1)\Lambda_2\,,\qquad m\in\mathbb{N}\,.$$

Let $W_a^\pm$ *be the Weyl groups in* (3.8). *Then we have*

$$ch\tilde{F}_{\Lambda_0^\pm}=ch\mathcal{M}(\Lambda_0^\pm)\sum_{w\in W_a^\pm}(-1)^{l(w)}e(w\cdot(\Lambda_0^\pm+\varrho)-\Lambda_0^\pm-\varrho)\,. \tag{3.10}$$

Proof. Consider the sum in the right-hand side of (3.10) for sign "+":

$$\begin{aligned} &\sum_{w\in W_a^+}(-1)^{l(w)}e(w\cdot(\Lambda_0^++\varrho)-\Lambda_0^+-\varrho)\\ &=\sum_{w\in W_a}(-1)^{l(w)}e(w\cdot(\Lambda_0^++\varrho)-\Lambda_0^+-\varrho)\\ &\quad-\sum_{w\in W_a}(-1)^{l(w)}e(w\cdot(s_1\cdot(\Lambda_0^++\varrho)-\Lambda_0^+-\varrho))\,. \end{aligned} \tag{3.11a}$$

By the definition of Λ_0^+, $s_1\cdot(\Lambda_0^++\varrho)=\Lambda_0^++\varrho-m\alpha_1$, i.e., then we have

$$\begin{aligned} w\cdot(s_1\cdot(\Lambda_0^++\varrho))&=w\cdot(\Lambda_0^++\varrho-m\alpha_1)=w\cdot(\Lambda_0^++\varrho)-m\alpha_1\,,\\ e(w\cdot(s_1\cdot(\Lambda_0^++\varrho))-\Lambda_0^+-\varrho)&=e(w\cdot(\Lambda_0^++\varrho)-\Lambda_0^+-\varrho-m\alpha_1)\\ &=e(w\cdot(\Lambda_0^++\varrho)-\Lambda_0^+-\varrho)e(-m\alpha_1)\,, \end{aligned} \tag{3.11b}$$

so the right-hand side of (3.11a) becomes [using also (3.6a)]

$$\begin{aligned} &(1-e(-m\alpha_1))\sum_{w\in W_a}(-1)^{l(w)}e(w\cdot(\Lambda_0^++\varrho)-\Lambda_0^+-\varrho)\\ &=(1-z^m)chF_{\Lambda_0^+}/ch\mathcal{M}(\Lambda_0^+)\,, \end{aligned} \tag{3.11c}$$

which proves (3.10) for sign "+". For sign "−", we use that by definition $s_2(\Lambda_0^-+\varrho)=\Lambda_0^-+\varrho-m\alpha_2$. □

We note that (3.6), (3.10) have exactly the form of the Weyl (respectively Weyl-Kac) character formula for the irreducible finite-dimensional representations of semi-simple Lie algebras (respectively for the integrable representations of affine Kac-Moody algebras); one should replace W_a or $W_a^\pm$ with the Weyl group W_0 (respectively Weyl-Kac group W). One interesting question is what is the relation between the Weyl-Kac group W to the new Weyl groups W_a, $W_a^\pm$. We need to recall some properties of W.

The Weyl group W is generated by the two simple reflections s_1, s_2 [cf. (3.1a)]. It is known that [15]

$$W=\{(s_1s_2)^k,\ (s_2s_1)^l,\ (s_1s_2)^m s_1,\ (s_2s_1)^n s_2 | k, l, m, n\in\mathbb{Z}_+\}\,. \tag{3.12}$$

We note that (cf. [6], formula (28)):

$$(s_1s_2)^k s_1 = s_\beta\,, \qquad \beta=(k+1)\alpha_1+k\alpha_2\,, \qquad k\geqq 0\,, \tag{3.13}$$

$$(s_2s_1)^k s_2 = s_\beta\,, \qquad \beta=k\alpha_1+(k+1)\alpha_2\,, \qquad k\geqq 0\,. \tag{3.14}$$

We recall the action of W on the weights $\Lambda\in\mathfrak{H}$. For $\Lambda=d\bar{d}+\lambda_1\alpha_1+c\bar{c}$, $d, \lambda_1, c\in\mathbb{C}$, let us denote

$$\begin{gathered}\chi(\Lambda)\equiv[d, m_1, m_2]\,, \qquad m_i\equiv(\Lambda+\varrho, \alpha_i)\,, \qquad i=1,2\,,\\ m_1=2\lambda_1+1\,, \qquad m_2=c-2\lambda_1+1\,.\end{gathered} \tag{2.15}$$

We also introduce as in [6], formula (27)):

$$\Lambda_{1k}\equiv(s_2s_1)^k\cdot(\Lambda+\varrho)-\varrho\,, \qquad \Lambda_{2k}=(s_1s_2)^k\cdot(\Lambda+\varrho)-\varrho\,, \tag{3.16}$$

$$\Lambda'_{1k}=s_1(s_2s_1)^k\cdot(\Lambda+\varrho)-\varrho\,, \qquad \Lambda'_{2k}=s_2(s_1s_2)^k\cdot(\Lambda+\varrho)-\varrho\,. \tag{3.17}$$

Then in terms of (3.16) we have (cf. [6], formulae (26) with all subscripts 0 replaced by 2):

$$\begin{aligned}\chi(\Lambda_{1k})=[&d-k(m_1+k(m_1+m_2)),\ m_1+2k(m_1+m_2),\\ &m_2-2k(m_1+m_2)]\,,\end{aligned} \tag{3.18a}$$

$$\begin{aligned}\chi(\Lambda_{2k})=[&d-k(-m_1+k(m_1+m_2)),\ m_1-2k(m_1+m_2),\\ &m_2+2k(m_1+m_2)]\,,\end{aligned} \tag{3.18b}$$

$$\begin{aligned}\chi(\Lambda'_{1k})=[&d-k(m_1+k(m_1+m_2)),\ m_2-(2k+1)(m_1+m_2),\\ &m_1+(2k+1)(m_1+m_2)]\,,\end{aligned} \tag{3.18c}$$

$$\begin{aligned}\chi(\Lambda'_{2k})=[&d-(k+1)(-m_1+(k+1)(m_1+m_2)),\ m_2+(2k+1)(m_1+m_2),\\ &m_1-(2k+1)(m_1+m_2)]\,,\end{aligned} \tag{3.18d}$$

Formulae (3.18) describe embeddings of Verma modules in the following cases: 1) $m_1, m_2\in\mathbb{N}$; in this case the irreducible factor modules $\mathcal{M}(\Lambda)/I(\Lambda)$ describe the integrable highest weight modules L_Λ of $A_1^{(1)}$ [4, 12, 15, 6]; 2) $m_1, m_2\in-\mathbb{N}$; in this case $\mathcal{M}(\Lambda)$ is embedded in all $\mathcal{M}(\Lambda^*)$ given in (3.16, 3.17) and is irreducible [4, 6]; 3) $m_1\in\mathbb{N}$, $m_2=0$ (or equivalently $m_2\in\mathbb{N}$, $m_1=0$); in this case $\Lambda_{2,k+1}=\Lambda'_{1k}$ and $\Lambda'_{2k}=\Lambda_{1k}$, then we have the embeddings

$$\mathcal{M}(\Lambda)\supset\mathcal{M}(\Lambda'_{10})\supset\mathcal{M}(\Lambda_{11})\supset\ldots\supset\mathcal{M}(\Lambda_{1k})\supset\mathcal{M}(\Lambda'_{1k})\supset\ldots$$

[6]; 4) $m_1\in-\mathbb{N}$, $m_2=0$ (or $m_2\in-\mathbb{N}$, $m_1=0$); we use the coincidence between Verma modules as in 3) but the embeddings in the chain are in the opposite direction; 5) $m_1\in\pm\mathbb{N}$, $m_2\notin\mathbb{Z}$ (or $m_2\in\pm\mathbb{N}$, $m_1\notin\mathbb{Z}$); in this case we have $\mathcal{M}(\Lambda)\supset\mathcal{M}(\Lambda-m_i\alpha_i)$ if $m_i\in\mathbb{N}$ and $\mathcal{M}(\Lambda)\subset\mathcal{M}(\Lambda-m_i\alpha_i)$ if $m_i\in-\mathbb{N}$; 6) $m_1+m_2=0$,

$m_1 \notin \mathbb{Z}\backslash\{0\}$; 7) $m_1+m_2=0$, $m_1 \in \mathbb{Z}\backslash\{0\}$. The last two cases are actually the cases we consider. Thus we can see the following correspondences. Let $\Lambda=\Lambda_0^+=(m-1)\Lambda_1 -(m+1)\Lambda_2$, $m\in\mathbb{N}$,

$$\Lambda_{1k}=(s_2s_1)^k\cdot(\Lambda_0^+ +\varrho)-\varrho=\Lambda_0^+ -km\bar{d}\,, \tag{3.19a}$$

$$\Lambda_{2k}=(s_1s_2)^k\cdot(\Lambda_0^+ +\varrho)-\varrho=\Lambda_0^+ +km\bar{d}\,, \tag{3.19b}$$

$$\Lambda'_{1k}=s_1(s_2s_1)^k\cdot(\Lambda_0^+ +\varrho)-\varrho=\Lambda_0^+ -km\bar{d}-m\alpha_1\,, \tag{3.19c}$$

$$\Lambda'_{2k}=s_2(s_1s_2)^k\cdot(\Lambda_0^+ +\varrho)-\varrho=\Lambda_0^+ +km\bar{d}+m\alpha_2\,. \tag{3.19d}$$

Thus every element of the Weyl-Kac group has a counterpart in a subset of the multiplet which we consider in the cases $\Lambda=\Lambda_0^+$, i.e., when we start with a singular Verma module reducible also w.r.t. positive roots. The correspondence

$$W_a \ni W(mk)\mapsto(s_2s_1)^k\,,\qquad k\in\mathbb{Z}_+\,, \tag{3.20}$$

although with coinciding action on Λ_0^+ is not isomorphism even between semigroups because, e.g., $w(m(k_1+k_2))$ and $w(mk_1)w(mk_2)$ are mapped to $(s_2s_1)^{k_1+k_2}$. The same applies for the correspondence

$$W_a^+ \ni sw(mk)\mapsto s_1(s_2s_1)^k\in W,\qquad k\in\mathbb{Z}_+\,, \tag{3.21}$$

with coinciding action on Λ_0^+, because $s_1(s_2,s_1)^k$ do not form a subgroup of W.

The case $\Lambda=\Lambda_0^-$ is analogous; in (3.19) the quantities added to Λ_0^+ will change signs [e.g., $\Lambda_0^+ -km\bar{d}\to\Lambda_0^- +km\bar{d}$ in (3.19a)]; further $s_1\leftrightarrow s_2$, $W_a^+\to W_a^-$.

Everything above can be summarized by the following diagram which appeared in [6] [cf. (35)]

$$\begin{array}{ccccccccccc}
\overset{w}{\to} & \Lambda+m\bar{d} & \overset{w}{\to}\dots\overset{w}{\to} & \Lambda+\bar{d} & \overset{w}{\to} & \Lambda & \overset{w}{\to} & \Lambda-\bar{d} & \overset{w}{\to}\dots\overset{w}{\to} & \Lambda-m\bar{d} \\
 & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s_1} \\
\overset{w}{\to} & \Lambda+m\bar{d}-m\alpha_1 & \overset{w}{\to}\dots\overset{w}{\to} & \Lambda+\bar{d}-m\alpha_1 & \overset{w}{\to} & \Lambda-m\alpha_1 & \overset{w}{\to} & \Lambda-\bar{d}-m\alpha_1 & \overset{w}{\to}\dots\overset{w}{\to} & \Lambda-m\bar{d}-m\alpha_1
\end{array} \tag{3.22}$$

where $\Lambda=\Lambda_0^+$, $w=w(1)$. In [6, 7] we considered an extended Weyl group W_e which in the language of this paper should be described as an infinite abelian group with generators $w(n)s=sw(n)$, $n\in\mathbb{Z}$, $s\in S^+$ (or S^-), i.e.

$$\begin{aligned} W_e^\pm=\{w'=ws|w=w(n_1)\dots w(n_k),\quad & k\in\mathbb{Z}_+, \\ w=1\quad\text{if}\quad k=0,\quad n_i\in\mathbb{Z}\backslash\{0\},\quad & w(n)^2=1\,, \\ w(n)w(n')=w(n')w(n),\quad s\in S^\pm,\quad & sw=ws\}\,. \end{aligned} \tag{3.23}$$

Thus we have the following inclusions of groups

$$W_a\subset W_a^\pm\subset W_e^\pm\,. \tag{3.24}$$

If we consider the action of the Weyl-Kac group W on Λ_0^+, besides (3.20) and (3.21) we have the following correspondences:

$$W\ni(s_1s_2)^k\mapsto w(-mk)\in W_e^+\,,\qquad k=0,1,\dots, \tag{3.25}$$

$$W\ni s_2(s_1s_2)^k\mapsto sw(-m(k+1))\in W_e^+\,,\qquad k=0,1,\dots, \tag{3.26}$$

however, again these maps give rise to no isomorphism.

Finally we introduce a notion which will uniformize all formulas for the characters. Let W^* be any of the Weyl groups we considered, let $\Lambda \in \mathfrak{H}^*$ and V be a HWM with highest weight Λ. Then we shall say that $w \in W^*$ is *V-active* if there exists a HWM V' with highest weight $w(\Lambda+\varrho)-\varrho$ which is isomorphic to a submodule of V. Further we shall restrict this definition to the category of Verma modules, i.e., V and V' above should be Verma modules. Thus we have:

Proposition 9. *Let Λ_0 be as in Proposition 7, $m \notin \mathbb{Z}\backslash\{0\}$, $\Lambda_0^{\pm}$ as in Proposition 8. Then we have*

$$ch L(\Lambda_0) = ch \mathcal{M}(\Lambda_0) \sum_{\substack{w \in W_a^{\pm} \\ w-\mathcal{M}(\Lambda_0)\text{-active}}} (-1)^{l(w)} e(w \cdot (\Lambda_0+\varrho)-\Lambda_0-\varrho) \tag{3.27a}$$

$$= ch \mathcal{M}(\Lambda_0) \sum_{\substack{w \in W_e^{\pm} \\ w-\mathcal{M}(\Lambda_0)\text{-active}}} (-1)^{l(w)} e(w(\Lambda_0+\varrho)-\Lambda_0-\varrho), \tag{3.27b}$$

$$ch \tilde{F}_{\Lambda_0^{\pm}} = ch \mathcal{M}(\Lambda_0^{\pm}) \sum_{\substack{w \in W_e^{\pm} \\ w-\mathcal{M}(\Lambda_0^{\pm})\text{-active}}} (-1)^{l(w)} e(w(\Lambda_0^{\pm}+\varrho)-\Lambda_0^{\pm}-\varrho). \tag{3.28}$$

4. Discussion

Several paths of applications and generalizations are available. First one should look for new modular invariants connected with the characters calculated in this paper. For this we note that all characters [cf. e.g. (2.13), (2.20)] contain the factor

$$\psi(q,z) = 1 \Big/ \prod_{n=1}^{\infty} (1-q^n z^{-1})(1-q^{n-1}z) = \frac{q^{1/12} z^{-1/2} \eta(\tau)}{\theta_{1,2}(\zeta,\tau)-\theta_{-1,2}(\zeta,\tau)}, \tag{4.1}$$

where $q = e^{2\pi i\tau}$, $z = e^{2\pi i\zeta}$, $\eta(\tau)$ is the Dedekind function, and $\theta_{n,m}(\zeta,\tau) = \theta_{n,m}(\zeta,\tau,0)$ where the latter is the classical theta function associated with $A_1^{(1)}$ so that [12]

$$\theta_{n,m}(\zeta,\tau,u) = e^{-2\pi i m u} \sum_{k \in \mathbb{Z}+n/2m} e^{2\pi i k(k^2\tau - k\zeta)}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z} \bmod 2m\mathbb{Z}. \tag{4.2}$$

Taking into account the transformations of η and θ under the generators S and T of the modular group (cf. [12] (13.6.8))

$$S(\zeta,\tau) = (\zeta/\tau, -1/\tau), \qquad T(\zeta,\tau) = (\zeta, \tau+1), \tag{4.3}$$

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n \in \mathbb{N}} (1-q^n), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau),$$
$$\eta(\tau+1) = e^{\pi i/12} \eta(\tau), \tag{4.4}$$

$$\begin{aligned} (\theta_{1,2}-\theta_{-1,2})(\zeta/\tau, -1/\tau) &= -i\sqrt{-i\tau}(\theta_{1,2}-\theta_{-1,2})(\zeta,\tau), \\ (\theta_{1,2}-\theta_{-1,2})(\zeta,\tau+1) &= e^{\pi i/4}(\theta_{1,2}-\theta_{-1,2})(\zeta,\tau), \end{aligned} \tag{4.5}$$

we can easily see that

$$\mathcal{M}(\zeta,\tau) \equiv \left| \frac{\eta(\tau)}{\theta_{1,2}(\zeta,\tau)-\theta_{-1,2}(\zeta,\tau)} \right|^2 \tag{4.6}$$

is a modular invariant for the case $m_0 = 0$, $\Lambda_0 = -\varrho$ [cf. (2.20), (3.15)].

Next one should try to generalize the Weyl group W_a (and then $W_a^{\pm}$, $W_e^{\pm}$) for an arbitrary affine Kac-Moody algebra $\mathfrak{G}$. From the paper of Malikov, Feigin, Fuchs [9] we know that the singular vectors of Theorem 1 are related to the second order Casimir operator Ω of $sl(2,\mathbb{C})$ by renormalizing the Virasoro algebra Sugawara-type construction of [5, 12]. The operator Ω for the finite-dimensional algebra $\mathfrak{G}_0$ underlying $\mathfrak{G}$ will give rise to similar singular vectors and we should be able to define the group W_a in an analogous way. However there should be similar singular vectors [9] related to the higher order Casimir operator Ω^i (whose number together with Ω is equal to $l \equiv \operatorname{rank} \mathfrak{G}_0$). Thus we should end up with Weyl groups W_a^i, $i=1,\ldots,l$, each of which is isomorphic to W_a. Moreover they should be mutually commuting since the Casimir operators commute. Work in this direction is in progress.

Another path is pursued in [16]. There we show that the group W_a is also a Weyl group for the Virasoro and $N=1$ super-Virasoro algebras. This is not trivial since these algebras do not possess a (generalized) Cartan matrix. The notion of V-active elements of W_a plays a crucial role in [16].

Finally one should try to apply these ideas to superalgebras with associated (generalized) Cartan matrices. We have already noted [17, 8] that if a Verma module over such a superalgebra is reducible w.r.t. some odd root α [such that $(\alpha,\alpha)=0$] then the corresponding homomorphisms between Verma modules cannot be described by the Weyl-Kac group for the same reasons as in the case of singular Verma modules over affine Lie algebras. In [17, 8] we have introduced the so-called *odd reflections* (or translations) by a formula looking exactly as (3.2). It is natural to try to continue this development by combining it with the ideas of the present paper. Work along these lines is in progress.

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