Commun. Math. Phys. 124, 57–78 (1989)

Communications in
Mathematical
Physics

Algebraic Study on the Super-KP Hierarchy and the Ortho-Symplectic Super-KP Hierarchy

Kimio Uneo[1], Hirofumi Yamada[2] and Kaoru Ikeda[2]

[1] Department of Mathematics, Waseda University, 3-4-1 Ohkubo, Shinjuku-Ku, Tokyo 160, Japan

[2] Department of Mathematics, Tokyo Metropolitan University, 2-1-1 Fukasawa, Setagaya-Ku, Tokyo 158, Japan

Abstract. Bilinear residue formulas are established for the super-KP hierarchy and the ortho-symplectic super-KP hierarchy. Furthermore, superframes corresponding to the ortho-symplectic super-KP hierarchy are completely characterized. Soliton solutions to the super-KP hierarchy are given.

1. Introduction

This paper is devoted to algebraic study of super-wave functions and soliton solutions of the super Kadomtsev–Petviashvili (SKP) hierarchy and the ortho-symplectic (OSp) SKP hierarchy.

The SKP hierarchy was first introduced by Manin–Rudal [12] and was extensively studied by Ueno–Yamada [17–20], Yamada [21], Mulase [13], Ikeda [9] and Radul [14]. Especially, in [19] we proved that the SKP hierarchy equivalently leads to the super-Grassmann equation that connects a point in the universal super-Grassmann manifold *USGM* with an initial data of a solution. In that argument, the Birkhoff (Riemann–Hilbert) decomposition in the group of super-microdifferential operators plays a key role. However this operator formalism is rather inconvenient for treating geometrical solutions such as soliton solutions and super-quasi-periodic solutions. We therefore require a super-wave function, as in the case of the ordinary soliton theory.

The theory of the KP hierarchy itself is explained as follows [2, 6, 15, 16]: Let $\mathscr{R}$ be the ring of formal power series over $\mathbf{C}$, $\mathscr{R} = \mathbf{C}[[x, t]]$ (x is a space variable and $t = (t_1, t_2, t_3, \ldots)$ an infinite number of time variables.). The algebra $\mathscr{R}$ is a differential algebra with a derivation $\partial_x = \partial/\partial x$. By $\mathscr{E}_{\mathscr{R}}$ we denote the ring of microdifferential operators over $\mathscr{R}$,

$$\mathscr{E}_{\mathscr{R}} = \mathscr{R}((\partial_x{}^{-1})) = \left\{ \sum_{-\infty < \nu \ll +\infty} p_\nu(x, t)\partial_x{}^\nu \,|\, p_\nu(x, t) \in \mathscr{R} \right\}.$$

A wave operator

$$W = W(x, t, \partial_x) = \sum_{j=0}^{\infty} w_j(x, t)\partial_x{}^{-j} \quad (w_0 = 1) \tag{1.1}$$

is a monic element in $\mathscr{E}_{\mathscr{R}}$ of 0-th order satisfying the Sato equations

$$\frac{\partial W}{\partial t_n} = B_n W - W \partial_x{}^n. \tag{1.2}$$

where $B_n = (W\partial_x{}^n W^{-1})_+$ (= the differential operator part of $W\partial_x{}^n W^{-1}$). The compatibility conditions for (1.2) give rise to the Lax or the Zakharov–Shabat representations of the KP hierarchy. A wave function and its dual version are introduced by

$$w(x,t,\lambda) = W(x,t,\partial_x)\left(\exp\left(x\lambda + \sum_{n=1}^{\infty} t_n \lambda^n\right)\right), \tag{1.3}$$

$$w^*(x,t,\lambda) = (W^*(x,t,\partial_x))^{-1}\left(\exp\left(-x\lambda - \sum_{n=1}^{\infty} t_n \lambda^n\right)\right), \tag{1.4}$$

where $W^* = \sum_{j=0}^{\infty} (-\partial_x)^{-j} w_j(x,t)$ is the formal adjoint operator of W. (In general, for $P \in \mathscr{E}_{\mathscr{R}}$, P^* stands for the formal adjoint operator of P.) A wave function for the KP hierarchy and its dual are completely characterized by the following bilinear residue formula (BRF):

$$\mathrm{Res}_{\lambda=\infty}(d\lambda\, w(x,t,\lambda) w^*(x',t',\lambda)) = 0. \tag{1.5}$$

This BRF is obtained through consideration on the duality of the Laplace transform [a primitive communication with M. Noumi]. In the definition of the BKP and CKP hierarchies [3–5], the even time evolutions are suppressed. Hence $t = (t_1, t_3, \ldots)$. We further impose some additional conditions on a wave operator:

$$\text{(BKP)} \quad W^{-1} = \partial_x{}^{-1} W^* \partial_x, \tag{1.6}$$

$$\text{(CKP)} \quad W^{-1} = W^*. \tag{1.7}$$

The BRF for these hierarchies are as follows:

$$\text{(BKP)} \quad \mathrm{Res}_{\lambda=\infty}(d\lambda/\lambda\, w(x,t,\lambda) w(x',t',-\lambda)) = 1, \tag{1.8}$$

$$\text{(CKP)} \quad \mathrm{Res}_{\lambda=\infty}(d\lambda\, w(x,t,\lambda) w(x',t',-\lambda)) = 0. \tag{1.9}$$

A supersymmetric extension of differential calculus on $\mathscr{R}$ are accomplished by replacing ∂_x by $D = \partial_\theta + \theta\partial_x$, where θ is an abstract Grassmann variable; $\theta^2 = 0$. The operator D is a square root of ∂_x.

The SKP hierarchy is described by the Sato equations:

$$D_n(W) = \varepsilon_n(B_n W - W D^n), \quad B_n = (W D^n W^{-1})_+, \quad n = 1,2,3,\ldots, \tag{1.10}$$

where $W = \sum_{j=0}^{\infty} w_j(x,\theta,t) D^{-j}$ is a monic super-microdifferential operator (a super-wave operator), D_n are super-vector fields with the parity $\underline{n}$ and $\varepsilon_n = (-)^{n(n+1)/2}$. (For the precise definition, see Sect. 2.) The main results in [19] are that the SKP hierarchy can be interpreted as a dynamical system on $USGM$, the Lie superalgebra $\mathfrak{gl}(\infty|\infty)$ appears as the infinitesimal transformation group on the solution space of the SKP hierarchy. As for the super-Fock representation of $\mathfrak{gl}(\infty|\infty)$, see [1.10].

Using the so-called "2-spinor representation" of super-microdifferential operators, we furthermore show that there is a natural projection map from the solution space of the SKP hierarchy to the direct product of two copies of the solution space of the KP hierarchy.

We define a super-wave function associated with a super-wave operator W by

$$w(x,\theta,t,\lambda,\xi)=W(x,\theta,t,D)(\exp H(x,\theta,t,\lambda,\xi))$$

with an appropriate phase factor $H(x,\theta,t,\lambda,\xi)$. where (λ,ξ) are $(1|1)$-dimensional spectral parameters (λ is even, ξ is odd). One of the main results in this paper is the characterization of a super-wave function and its dual of the SKP hierarchy by the following BRF:

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)w(x,\theta,t,\lambda,\xi)w^*(x',\theta',t',\lambda,\xi))=0, \tag{1.11}$$

where $\Delta(d\lambda/d\xi)$ is the super-volume form on the (λ,ξ)-space (odd quantity). To show this BRF, we establish the theory of the super-Laplace transform and its duality.

By adding a symmetry condition for a super-wave operator of the SKP hierarchy, one obtains the OSp-SKP hierarchy which is related with the infinite dimensional Lie superalgebra $\mathfrak{osp}(\infty|\infty)$. As in the case of the SKP hierarchy, there is a projection map from the OSp-SKP hierarchy to the direct product of the BKP and the CKP hierarchies. The BRF for the OSp-SKP hierarchy is also obtained.

This paper is organized as follows. Section 2 outlines the theory of the SKP hierarchy [19], including some new results: We establish the one-to-one correspondence between formally regular solutions to the hierarchy and points in the biggest cell of *USGM*. Furthermore, we describe the hierarchy in the 2-spinor picture. In Sect. 3, we will introduce a super-wave function and its dual for the SKP hierarchy. Through analysis of the super-Laplace transform, we prove the BRF for super-microdifferential operators (Theorem 3.6), and for a super-wave function and its dual (Theorem 3.7). Section 4 is devoted to a study of the OSp-SKP hierarchy, especially the BRF (Theorem 4.1). We also give a characterization of the OSp-SKP hierarchy by superframes in the biggest cell of *USGM* (Theorem 4.6). In Sect. 5, we construct soliton solutions to the SKP hierarchy by means of the so-called direct method.

2. The SKP Hierarchy and the Universal Super-Grassmann Manifold

In this section we review the theory of the super-KP hierarchy developed in [17–19]. We will omit proofs of the propositions except for Proposition 2.3 and Proposition 2.5. For the details, see [19].

Let $\mathscr{A}$ be a Grassmann algebra of finite or infinite dimensions over $\mathbf{C}$, and $t=(t_1,t_2,\dots)$ super-time variables (t_{2k} are even, t_{2k-1} are odd). The supercommutative algebra $\mathscr{S}$ of superfields is, by definition

$$\mathscr{S}=\mathbf{C}[[x,\theta,t]]\otimes\mathscr{A}.$$

We introduce naturally the $\mathbf{Z}_2$-gradation of $\mathscr{S}$, $\mathscr{S}=\mathscr{S}_{\underline{0}}\oplus\mathscr{S}_{\underline{1}}$ and define the body map ε by the canonical projection

$$\varepsilon: \mathscr{S} \to \mathscr{R} = \mathscr{S}/(\mathscr{S}_{\underline{1}}) = \mathbf{C}[[x, t_2, t_4, \dots]],$$

where $(\mathscr{S}_{\underline{1}})$ is the ideal generated by the subspace $\mathscr{S}_{\underline{1}}$. A super-differential operator $D = (\partial/\partial\theta) + \theta(\partial/\partial x)$ and super-vector fields

$$D_{2l} = \frac{\partial}{\partial t_{2l}}, \quad D_{2l-1} = \frac{\partial}{\partial t_{2l-1}} + \sum_{k=1}^{\infty} t_{2k-1} \frac{\partial}{\partial t_{2l+2k-2}}$$

act on $\mathscr{S}$. They satisfy the following commutation and anti-commutation relations [12]:

$$[D, D_l]_{(-)^{l-1}} = 0, \quad [D_{2l}, D_{2k}] = [D_{2l}, D_{2k-1}] = 0,$$

$$[D_{2l-1}, D_{2k-1}]_+ = 2D_{2l+2k-2}.$$

We define the algebra $\mathscr{D}$ of super-differential operators by $\mathscr{D} = \mathscr{S}[D]$. Adding the formal inverse element $D^{-1} = \theta + (\partial/\partial\theta)(\partial/\partial x)^{-1}$ to $\mathscr{D}$, we obtain the algebra of super-microdifferential operators. Precisely,

$$\mathscr{E} = \mathbf{C}[[x, \theta, t]]((D^{-1})) \otimes \mathscr{A}.$$

The algebra structure of $\mathscr{E}$ is prescribed by the generalized super-Leibniz rule [12]:

$$D^{2k} \cdot f = \sum_{j=0}^{\infty} \binom{k}{j} D^{2j}(f) D^{2k-2j},$$

$$D^{2k+1} \cdot f = \sum_{j=0}^{\infty} \binom{k}{j} D^{2j+1}(f) D^{2k-2j} + (-)^a \sum_{j=0}^{\infty} \binom{k}{j} 2^{2j}(f) D^{2k-2j+1},$$

for any integer k and $f \in \mathscr{S}_{\underline{a}}$. The algebra $\mathscr{E}$ is endowed with a natural $\mathbf{Z}_2$-gradation, $\mathscr{E} = \mathscr{E}_{\underline{0}} \oplus \mathscr{E}_{\underline{1}}$. Namely an operator $P = \sum_{-\infty < j \ll \infty} p_j(x, \theta, t) D^j \in \mathscr{E}_{\underline{a}}$ $(a = 0, 1)$ if and only if $p_j(x, \theta, t) \in \mathscr{S}_{\underline{a+j}}$ for any j. Moreover we define the body part $\varepsilon(P)$ (we use the same notation as the body map on $\mathscr{S}$) by

$$\varepsilon(P) = \sum_{j:\mathrm{even}} \varepsilon(p_j(x, \theta, t)) \partial_x^{j/2},$$

which is a microdifferential operator with coefficients in $\mathscr{R}$.

Now we introduce the SKP hierarchy [12, 17–19]. Let L be a super-micro-differential operator

$$L = \sum_{i=0}^{\infty} u_i D^{1-i} \in \mathscr{E}_{\underline{1}}$$

with $u_0 = 1, D(u_1) + 2u_2 = 0$. The SKP hierarchy is a system of the Lax equations:

$$D_{2l}(L) = (-)^l [B_{2l}, L],$$

$$D_{2l-1}(L) = (-)^l \{[B_{2l-1}, L]_+ - 2L^{2l}\}, \quad l = 1, 2, \dots, \tag{2.1}$$

where $B_l = (L^l)_+$ (= the super-differential operator part of L^l), and $D_l(L) = \Sigma D_l(u_i) D^{1-i}$. The system (2.1) is equivalent to a system of the Zakharov–Shabat equations:

$$(-)^k D_{2k}(B_{2l}) - (-)^l D_{2l}(B_{2k}) + [B_{2l}, B_{2k}] = 0,$$
$$(-)^k D_{2k}(B_{2l-1}) - (-)^l D_{2l-1}(B_{2k}) + [B_{2l-1}, B_{2k}] = 0,$$
$$(-)^k D_{2k-1}(B_{2l-1}) + (-)^l D_{2l-1}(B_{2k-1}) - [B_{2l-1}, B_{2k-1}]_+ + 2B_{2l+2k-2} = 0,$$
$$k, l = 1, 2, \dots . \tag{2.2}$$

The first equation in (2.2) with $k=2$, $l=3$ gives rise to the SKP equation, which is regarded as a supersymmetric extension of the single KP equation: Set

$$B_4 = D^4 + 2v_3 D + 2v_4,$$
$$B_6 = D^6 + 3v_3 D^3 + 3v_4 D^2 + v_5 D + v_6 \quad (v_j \in \mathcal{S}_{\underline{j}}).$$

Then the SKP equation reads

$$3D_4(v_3) = -3v_{3,xx} + 2v_{5,x},$$
$$3D_4(v_4) = -3v_{4,xx} + 6v_3 v_{3,x} - 4v_3 v_5 + 2v_{6,x},$$
$$D_4(v_5) + D_6(v_3) = v_{5,xx} - 2v_{3,xxx} - 6v_3 D(v_{3,x}) - 6(v_3 v_4)_x - 2D(v_3 v_5),$$
$$D_4(v_6) + D_6(v_4) = v_{6,xx} + 2v_3 D(v_6) - 2v_{4,xxx} - 6v_3 D(v_{4,x}) - 6v_4 v_{4,x} + 2D(v_4) v_5.$$

Before describing the procedure of integrating the SKP hierarchy, we consider a matrix representation of the algebra $\mathcal{E}$. Let

$$\psi: \mathcal{E} \to Mat(\mathbf{Z}; \mathcal{S})$$

be an algebra homomorphism defined by $\psi(P) = (\psi(P)_{\mu\nu})_{\mu,\nu\in\mathbf{Z}} (P \in \mathcal{E}, \psi(P)_{\mu\nu} \in \mathcal{S})$ with the matrix entries prescribed by

$$D^\mu \cdot P = \sum_{j\in\mathbf{Z}} \psi(P)_{\mu\nu} D^\nu. \tag{2.3}$$

More precisely, letting $P = \sum_{j\in\mathbf{Z}} p_j(x, \theta, t) D^j$,

$$\psi(P)_{2\mu,\nu} = \sum_{k=0}^{\infty} \binom{\mu}{k} D^{2k}(p_{\nu-2\mu+2k}),$$
$$\psi(P)_{2\mu+1,\nu} = \sum_{k=0}^{\infty} \binom{\mu}{k} \{D^{2k+1}(p_{\nu-2\mu+2k}) - (-)^\nu D^{2k}(p_{\nu-2\mu-1+2k})\}.$$

From the definition (2.3) and the associativity of the multiplication in $\mathcal{E}$, it is easy to see that ψ is actually an injective algebra homomorphism. (Furthermore ψ becomes a superalgebra homomorphism under an appropriate $\mathbf{Z}_2$-gradation of $Mat(\mathbf{Z}; \mathcal{S})$.)

Now let us integrate the SKP hierarchy. One first finds a monic super-micro-differential operator (a super-wave operator)

$$W = \sum_{j=0}^{\infty} w_j(x, \theta, t) D^{-j} \in \mathcal{E}_{\underline{0}}$$

satisfying

$$L = WDW^{-1},$$
$$D_n(W) = \varepsilon_n(B_nW - WD^n), \quad n = 1, 2, \dots, \tag{2.4}$$

where $\varepsilon_n = (-)^{n(n+1)/2}$. Equations (2.4) are referred to as the Sato equations for the SKP hierarchy. Introducing

$$\Psi = \exp\left(\sum_{n=1}^{\infty} \varepsilon_n t_n D^n\right),$$

one readily sees that the operator $\tilde{W} = W \cdot \Psi$ solves

$$D_n(\tilde{W}) = \varepsilon_n B_n \tilde{W}, \quad n = 1, 2, \dots .$$

Apart from this, consider the following equations:

$$D_n(Y) = \varepsilon_n B_n Y,$$

where Y is a super-differential operator of the infinite order

$$Y = \sum_{j=0}^{\infty} y_j(x, \theta, t) D^j \quad (y_j \in \mathscr{S}_{\underline{j}}),$$

with an initial condition $Y|_{t=0} = 1$. Putting $U = \tilde{W}^{-1} Y$, one sees that the coefficients of U are independent of t, and that

$$Y = WZ. \tag{2.5}$$

Here the operator Z is defined by

$$Z = \Psi U = \sum_{j \in \mathbf{Z}} z_j(x, \theta, t) D^j.$$

Taking the $(-)$ part of (2.5) (for $P \in \mathscr{E}$, $(P)_- = P - (P)_+$) yields the following equation:

$$(WZ)_- = 0. \tag{2.6}$$

Introduce a $\mathbf{Z} \times \mathbf{N}^c$ matrix $\mathscr{Z}$ by

$$\mathscr{Z} = (\psi(Z)_{\mu\nu})_{\mu \in \mathbf{Z}, \nu \in \mathbf{N}^c}.$$

Then Eq. (2.6) reads

$${}^t\vec{w}\mathscr{Z} = 0, \tag{2.7}$$

where $\vec{w} = (w_{-j})_{j \in \mathbf{Z}}$, $w_j = w_j(x, \theta, t)$ for $j \geqq 0$, $w_j = 0$ for $j < 0$. The matrix $\mathscr{Z}$ solves

$$D_n(\mathscr{Z}) = \Gamma^n \mathscr{Z}, \quad n = 1, 2, \dots, \tag{2.8}$$
$$D(\mathscr{Z}) = \Lambda \mathscr{Z} - \mathscr{Z}^\dagger \Lambda_{\mathbf{N}^c}, \tag{2.9}$$

where $\Lambda = (\delta_{\mu+1,\nu})_{\mu,\nu \in \mathbf{Z}}$, $\Gamma = ((-)^\nu \delta_{\mu+1,\nu})_{\mu,\nu \in \mathbf{Z}}$, $\Lambda_{\mathbf{N}^c} = (\delta_{\mu+1,\nu})_{\mu,\nu \in \mathbf{N}^c}$ and $\mathscr{Z}^\dagger = (\psi(Z)^\dagger_{\mu\nu})$ (for $f = f_0 + f_1 \in \mathscr{S} = \mathscr{S}_{\underline{0}} \oplus \mathscr{S}_{\underline{1}}$, we set $f^\dagger = f_0 - f_1$). From these equations, the matrix $\mathscr{Z}$ is represented as

$$\mathscr{Z} = \Phi \cdot \Xi \cdot \exp(-\theta \Lambda_{\mathbf{N}^c} - x(\Lambda_{\mathbf{N}^c})^2),$$

where

$$\Phi=\exp\left(\theta\Lambda+x\Lambda^2+\sum_{n=1}^{\infty}t_n\Gamma^n\right),\tag{2.10}$$

and Ξ is a constant $\mathbf{Z}\times\mathbf{N}^c$ matrix

$$\Xi=(\xi_{\mu\nu})_{\mu\in\mathbf{Z},\nu\in\mathbf{N}^c}\in Mat(\mathbf{Z}\times\mathbf{N}^c;\mathscr{A})\quad\text{with}\quad\xi_{\mu\nu}\in\mathscr{A}_{\underline{\mu+\nu}}.$$

One can see that $\xi_{\mu\nu}=\delta_{\mu\nu}$ for $\mu\leqq\nu$. Therefore we have the following proposition.

Proposition 2.1. *The coefficients $w_j(x,\theta,t)\in\mathscr{S}_{\underline{j}}$ $(j\geqq 1)$ of a super-wave operator $W\in\mathscr{E}_{\underline{0}}^{\text{monic}}$ solve a system of an infinite number of linear algebraic equations*

$${}^t\vec{w}\Phi\Xi=0.\tag{2.11}$$

Equation (2.11) is referred to as the Grassmann equation for the SKP hierarchy. The Grassmann equation has a unique solution for matrix Ξ in the set of superframes:

$$\begin{aligned}SFR(\mathbf{N}^c;\mathscr{A})=\{&\Xi=(\xi_{\mu\nu})_{\mu\in\mathbf{Z},\nu\in\mathbf{N}^c}\in Mat(\mathbf{Z}\times\mathbf{N}^c;\mathscr{A})\,|\,\xi_{\mu\nu}\in\mathscr{A}_{\underline{\mu+\nu}},\\&\exists m\in\mathbf{N}\ \textit{such that}\ \xi_{\mu\nu}=\delta_{\mu\nu}\ \textit{for}\ \mu<-m,\ \mu\leqq\nu,\\&\xi_{\mu\nu}=0\ \textit{for}\ -m\leqq\nu<0,\ \mu\leqq-m,\ \textit{and}\ \varepsilon(\Xi)\ \textit{is of maximal rank}\}.\end{aligned}$$

The resulting solutions w_j belong to the quotient algebra $\mathscr{Q}$ of $\mathscr{S}$. Let $\mathscr{E}_{\mathscr{Q}}$ be the superalgebra of super-microdifferential operators with coefficients in $\mathscr{Q}$.

Proposition 2.2. *For a solution ${}^t\vec{w}$ to the Grassmann equation with $\Xi\in SFR(N^c;\mathscr{A})$, set $W=\sum_{j=0}^{\infty}w_jD^{-j}\in(\mathscr{E}_{\mathscr{Q}}^{\text{monic}})_{\underline{0}}$. Then the operator W solves the Sato equations (2.3) for the SKP hierarchy with $B_n=(WD^nW^{-1})_+$.*

We introduce the supergroup $SGL(\mathbf{N}^c;\mathscr{A})$ by

$$\begin{aligned}SGL(\mathbf{N}^c;\mathscr{A})=\{&g=(g_{\mu\nu})_{\mu,\nu\in\mathbf{N}^c}\in Mat(\mathbf{N}^c;\mathscr{A})\,|\,g_{\mu\nu}\in\mathscr{A}_{\underline{\mu+\nu}},\\&\exists m\in\mathbf{N}\ \textit{such that}\ g_{\mu\nu}=\delta_{\mu\nu}\ \textit{for}\ \mu\leqq\nu,\ \mu<-m,\\&g_{\mu\nu}=0\ \textit{for}\ -m\leqq\nu<0,\ \mu\leqq-m,\ \textit{and}\ (\varepsilon(g_{\mu\nu}))_{-m\leqq\mu,\nu<0}\ \textit{is invertible}\}.\end{aligned}$$

This supergroup acts on the space $SFR(\mathbf{N}^c;\mathscr{A})$ from the right. The universal super-Grassmann manifold $USGM$ is by definition, the quotient space of $SFR(\mathbf{N}^c;\mathscr{A})$:

$$USGM=SFR(\mathbf{N}^c;\mathscr{A})/SGL(\mathbf{N}^c;\mathscr{A}).$$

From the formula of solutions to the Grassmann equation (Theorem 2.4), we can see that the biggest cell of $USGM$,

$$USGM^{\phi}=\{\Xi=(\xi_{\mu\nu})\in SFR(\mathbf{N}^c;\mathscr{A})\,|\,\xi_{\mu\nu}=\delta_{\mu\nu}\ \text{for}\ \mu\leqq\nu\}/SGL(\mathbf{N}^c;\mathscr{A})$$

provides super-wave operators with coefficients in $\mathscr{S}$. We denote by $W(\Xi)\in\mathscr{E}_{\mathscr{Q}}$ the super-wave operator associated with a superframe Ξ in Proposition 2.2. It is obvious that, if two superframes Ξ,Ξ' determine the same point in $USGM$, the associated super-wave operators $W(\Xi)$ and $W(\Xi')$ coincide. There arises a natural question whether, if two super-wave operators $W(\Xi)$, $W(\Xi')$ coincide, the superframes Ξ and Ξ' determine the same point, namely $\Xi = \Xi' \bmod SGL(\mathbf{N}^c; \mathscr{A})$. The answer is "yes," at least for superframes that belong to $USGM^{\Phi}$.

Proposition 2.3. *Suppose* $\Xi, \Xi' \in USGM^{\Phi}$, *and* $W(\Xi) = W(\Xi')$. *Then* $\Xi = \Xi' \bmod SGL(\mathbf{N}^c; \mathscr{A})$.

Proof. We note that, for a superframe Ξ in $USGM^{\Phi}$, there exists a matrix $g \in SGL(\mathbf{N}^c; \mathscr{A})$ such that $(\Xi \cdot g)_{\mu\nu} = \tilde{\xi}_{\mu\nu} (\mu \in \mathbf{Z}, \nu \in \mathbf{N}^c)$ with $\tilde{\xi}_{\mu\nu} = \delta_{\mu\nu}$ if $\mu \in \mathbf{N}^c$. We call such a superframe $\tilde{\Xi} = (\tilde{\xi}_{\mu\nu})_{\mu \in \mathbf{Z}, \nu \in \mathbf{N}^c}$ normalized. Set $W(\Xi) = \sum_{j=0}^{\infty} w_j(x, \theta, t) D^{-j}$ $(w_j(x, \theta, t) \in \mathscr{S}_{\underline{j}})$. What we have to show is that a normalized superframe $\tilde{\Xi}$ is uniquely determined from the Grassmann equation

$$ {}^t\vec{w} \Phi \tilde{\Xi} = 0 \tag{2.12}$$

$(\vec{w} = (w_{-j})_{j \in \mathbf{Z}},\ w_j = w_j(x, \theta, t)$ for $j \geqq 0$, $w_j = 0$ for $j < 0)$. Applying D^n to (2.12) and setting $x = \theta = t = 0$, we have the following equations:

$$ {}^t\vec{w}[n] \tilde{\Xi} = 0 \quad (n = 0, 1, 2, \ldots), \tag{2.13}$$

where ${}^t\vec{w}[n] = D^n({}^t\vec{w}\Phi)|_{x=\theta=t=0} = (w[n]_j)_{j \in \mathbf{Z}}$ with $w[n]_j \in \mathscr{A}_{j+n}$, $w[n]_n = 1$ and $w[n]_j = 0$ for $j > n$. Equations (2.13) imply the orthogonality relations between the vectors $\vec{w}[n]$ and the superframe $\tilde{\Xi}$. It is easy to see that each column vector of $\tilde{\Xi}$ is uniquely determined from this orthogonality. ■

Thus super-wave operators in $\mathscr{E}$ correspond one-to-one to points in $USGM^{\Phi}$.

To study the time evolution of solutions to the SKP hierarchy, introduce an infinite number of supersymmetric derivations;

$$\bar{D} = \partial_\theta - \theta \partial_x \quad (\bar{D}^2 = -\partial_x),$$

$$\bar{D}_{2l} = \frac{\partial}{\partial t_{2l}}, \quad \bar{D}_{2l-1} = \frac{\partial}{\partial t_{2l-1}} - \sum_{k=1}^{\infty} t_{2k-1} \frac{\partial}{\partial t_{2l+2k-2}}.$$

Consider an even derivation

$$X = a \frac{\partial}{\partial x} + \zeta \bar{D} + \sum_{n=1}^{\infty} c_n \bar{D}_n,$$

where $a \in \mathscr{A}_0, \zeta \in \mathscr{A}_{\underline{1}}, c_n \in \mathscr{A}_{\underline{n}}$. X commutes with the derivations D and D_n so that it acts infinitesimally on the solution space of the SKP hierarchy. For a superfield $f \in \mathscr{S}$, one has

$$(e^X f)(x, \theta, t) = f(x', \theta', t'),$$

where $x' = x + a + \theta\zeta$, $\theta' = \theta + \zeta$, $t'_{2l-1} = t_{2l-1} + c_{2l-1}$ and $t'_{2l} = t_{2l} + c_{2l} + \sum_{k=1}^{l} t_{2k-1} c_{2l-2k+1}$. Since the fundamental solution matrix Φ (2.10) has the multiplicative property with respect to the time evolution, i.e.,

$$(e^X \Phi)(x, \theta, t) = \Phi(x, \theta, t) \Phi(a, \zeta, c),$$

the SKP hierarchy is translated to a dynamical system on $USGM$ with the time evolution

$$\Xi \bmod SGL(\mathbf{N}^c; \mathscr{A}) \to \Phi(x, \theta, t) \cdot \Xi \bmod SGL(\mathbf{N}^c; \mathscr{A}).$$

In order to solve the Grassmann equation explicitly, we need some algebraic concepts. With a matrix $X = (x_{ij})_{i,j \in \mathbf{Z}}$,

$$\check{X} = (X_{\alpha\beta})_{\alpha,\beta=0,1}$$

is associated, where the blocks are put as $X_{\alpha\beta} = (x_{ij})_{i \in 2\mathbf{Z}+\alpha, j \in 2\mathbf{Z}+\beta}$. Applying this rearrangement to the Grassmann equation, it is rewritten into the form

$$(\ldots, w_4, w_2, 1, 0, \ldots; \ldots, w_3, w_1, 0, \ldots) \cdot \check{\Phi} \cdot \check{\Xi} = 0. \tag{2.14}$$

Let $A = (A_{\alpha\beta})_{\alpha,\beta=0,1}$ be an invertible matrix with $A_{\alpha\beta} \in Mat(m_\alpha \times m_\beta; \mathscr{A}_{\underline{\alpha+\beta}})$. The invertibility of such a matrix is equivalent to that of the matrices $\varepsilon(A_{00})$ and $\varepsilon(A_{11})$. A superdeterminant (or the Berezinian) [11] of the matrix A is, by definition,

$$\operatorname{sdet} A = \det(A_{00} - A_{01} A_{11}{}^{-1} A_{10}) / \det A_{11}.$$

The inverse of the superdeterminant is given by

$$\mathrm{s}^{-1}\!\det A = \det(A_{11} - A_{10} A_{00}{}^{-1} A_{01}) / \det A_{00}.$$

We should remark that a superdeterminant is multiplicative with respect to the product of matrices. By virture of Cramer's formula in linear algebra, one sees that the even unknowns w_{2j} in (2.14) are expressed in the form of a quotient of superdeterminants. To get the formulas representing the odd unknowns w_{2j+1}, we first look for the formula for w_1, and consider the first Sato equation $D_1(W) = -(B_1 W - WD)$. Then we obtain the following theorem.

Theorem 2.4. *The coefficients of a super-wave operator attached to a superframe $\Xi \in SFR(\mathbf{N}^c; \mathscr{A})$ are given by*

$$w_1 = D\{\log(\operatorname{sdet}({}^t\check{\Xi}_0 \cdot \check{\Phi} \cdot \check{\Xi}))\} = D_1\{\log(\operatorname{sdet}({}^t\check{\Xi}_0 \cdot \check{\Phi} \cdot \check{\Xi}))\},$$

and

$$w_{2j} = (-)^j \operatorname{sdet}({}^t\check{\Xi}_{2j} \cdot \check{\Phi} \cdot \check{\Xi}) / \operatorname{sdet}({}^t\check{\Xi}_0 \cdot \check{\Phi} \cdot \check{\Xi}),$$
$$w_{2j+1} = (-)^j \{(D + D_1)(\operatorname{sdet}({}^t\check{\Xi}_{2j} \cdot \check{\Phi} \cdot \check{\Xi}))\} / 2 \operatorname{sdet}({}^t\check{\Xi}_0 \cdot \check{\Phi} \cdot \check{\Xi}),$$

for $j = 0, 1, 2, \ldots$. Here the frame $\check{\Xi}_{2j}$ is defined by

$$\check{\Xi}_{2j} = \begin{pmatrix} \Xi_j & 0 \\ 0 & \Xi_0 \end{pmatrix},$$

where $\Xi_j = (\delta_{\mu\nu}(\mu \in \mathbf{Z}; \mu < -j) | \delta_{\mu,\nu+1}(\mu \in \mathbf{Z}; -j \leqq \nu < 0))$.

Finally we describe the 2-spinor picture of the SKP hierarchy. Let $\tilde{\mathscr{S}} = \mathbf{C}[[x, t]] \otimes \mathscr{A}$ and $\mathscr{E}_{\tilde{\mathscr{S}}} = \tilde{\mathscr{S}}((\partial_x{}^{-1}))$ be the algebra of microdifferential operators with coefficients in $\tilde{\mathscr{S}}$. Put

$$\mathscr{L} = Mat(1|1; \mathbf{C}) \otimes \mathscr{E}_{\tilde{\mathscr{S}}},$$

whose $\mathbf{Z}_2$-gradation $\mathscr{L} = \mathscr{L}_{\underline{0}} \oplus \mathscr{L}_{\underline{1}}$ is naturally introduced. We denote by $\tilde{\varepsilon}$ the body map, $\tilde{\varepsilon}: \tilde{\mathscr{S}} \to \mathscr{R}$, which is defined in the same way as before. The same notation

$\tilde{\varepsilon}$ expresses the body map $\mathscr{E}_{\tilde{\mathscr{S}}} \to \mathscr{E}_{\mathscr{P}}$, which further extends to the body map $\tilde{\varepsilon}_{\mathscr{L}}: \mathscr{L} \to \mathscr{E}_{\mathscr{R}} \oplus \mathscr{E}_{\mathscr{R}}$. The 2-spinor representation is a superalgebra homomorphism $\pi: \mathscr{E} \to \mathscr{L}$ defined by

$$\pi(\theta) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi(D) = \begin{pmatrix} 0 & 1 \\ \partial_x & 0 \end{pmatrix},$$

$$\pi(f) = \operatorname{diag}(f, (-)^a f) \quad \text{for} \quad f \in \tilde{\mathscr{S}}_{\underline{a}}.$$

We consider the Sato equation (2.4) with $n = 2l$ in the 2-spinor representation. Let W be a super-wave operator in $\mathscr{E}$ and $\pi(W) = (W_{ij})_{i,j=0,1}$. Each entry W_{ij} belongs to $(\mathscr{E}_{\tilde{\mathscr{S}}})_{\underline{i+j}}$, and the body part of the diagonal entries satisfy

$$\frac{\partial}{\partial t_{2l}} \tilde{\varepsilon}(W_{ii}) = (-)^l \{\tilde{\varepsilon}(B_{2l,ii}) \tilde{\varepsilon}(W_{ii}) - \tilde{\varepsilon}(W_{ii}) \partial_x{}^l\},$$

where $\pi(B_{2l}) = (B_{2l,ij})_{i,j=0,1}$. These are nothing but the Sato equations for the KP hierarchy. Therefore we have the following proposition.

Proposition 2.5. *Let $\mathscr{W}^{\phi}_{\mathrm{SKP}}$ be the space of super-wave operators in $\mathscr{E}$ of the* SKP *hierarchy, and $\mathscr{W}^{\phi}_{\mathrm{KP}}$ be the space of wave operators in $\mathscr{E}_{\mathscr{R}}$ of the* KP *hierarchy. Then we have the projection $\rho = \tilde{\varepsilon}_{\mathscr{L}} \circ \pi$,*

$$\rho: \mathscr{W}^{\phi}_{\mathrm{SKP}} \to \mathscr{W}^{\phi}_{\mathrm{KP}} \times \mathscr{W}^{\phi}_{\mathrm{KP}}.$$

3. Super-Laplace Transform and Bilinear Residue Formula

First we discuss the concept of the "formal adjoint" in $\mathscr{E}_{\mathbb{C}[[x,\theta]] \otimes \mathscr{A}}$. The super-integration of a superfield $f(x, \theta) = u(x) + \theta v(x)$ is, by definition,

$$\int \Delta(dx/d\theta) f(x, \theta) = \int dx\, v(x),$$

where $\Delta(dx/d\theta)$ is the $(1\,|\,1)$-dimensional volume form (an odd quantity). For a given $P = P(x, \theta, D) \in \mathscr{E}_{\mathbb{C}[[x,\theta]] \otimes \mathscr{A}}$, the formal adjoint operator $P^* = P^*(x, \theta, D) \in \mathscr{E}_{\mathbb{C}[[x,\theta]] \otimes \mathscr{A}}$ is introduced through

$$\begin{aligned} &\int \Delta(dx/d\theta) P(x, \theta, D)(f(x, \theta)) \cdot g(x, \theta) \\ &\quad = \int \Delta(dx/d\theta) f(x, \theta) \cdot P^*(x, \theta, D)(g(x, \theta)), \end{aligned}$$

for $f(x, \theta)$, $g(x, \theta) \in (\mathbb{C}[[x, \theta]] \otimes \mathscr{A})_{\underline{0}}$. Then we have

$$(wD^n)^* = (-)^{an} \varepsilon_n D^n \cdot w \quad (w \in (\mathbb{C}[[x, \theta]] \otimes \mathscr{A})_{\underline{a}}, n \in \mathbf{Z}),$$

and, in general,

$$(P_1 P_2)^* = (-)^{a_1 a_2} P_2{}^* P_1{}^* \quad (P_j \in (\mathscr{E}_{\mathbb{C}[[x,\theta]] \otimes \mathscr{A}})_{\underline{a_j}}).$$

We introduce a super-wave function and its dual version. Let

$$H(x, \theta, t, \lambda, \xi) = x\lambda + \sum_{l=1}^{\infty} (-)^l t_{2l} \lambda^l + (\xi + h(t, \xi))(\theta + \lambda^{-1} h(t, \xi)),$$

where

$$h(t, \lambda) = \sum_{l=1}^{\infty} (-)^l t_{2l-1} \lambda^l.$$

Here (λ,ξ) are regarded as $(1|1)$-dimensional spectral parameters. For a super-wave operator $W = W(x,\theta,t,D) = \sum_{j=0}^{\infty} w_j(x,\theta,t)D^{-j}$ $(w_0 = 1)$, define a super-wave function and its dual by

$$w(x,\theta,t,\lambda,\xi) = W(x,\theta,t,D)(\exp(H(x,\theta,t,\lambda,\xi))), \tag{3.1}$$

$$w^*(x,\theta,t,\lambda,\xi) = W^*(x,\theta,t,D)^{-1}(\exp(-H(x,\theta,t,\lambda,\xi))). \tag{3.2}$$

Note that

$$D^2(\exp H) = \lambda \exp H, \quad D_n(\exp H) = \varepsilon_n D^n(\exp H),$$
$$D^{-2\mu}(\exp H) = \lambda^{-\mu}\exp H, \quad D^{-2\mu+1}(\exp H) = \lambda^{-\mu}(\lambda\theta - \xi - h)\exp H.$$

By the Sato equations (2.4), a super-wave function and its dual satisfy the linear equations

$$D_n(w) = \varepsilon_n B_n(w), \quad D_n(w^*) = -\varepsilon_n B_n{}^*(w^*). \tag{3.3}$$

We consider the duality of the super-Laplace transform. Let $\mathbf{V} = \mathbf{V}_{\mathbf{C}} \otimes \mathscr{A}$ be an $\mathscr{A}$-module, where

$$\mathbf{V}_{\mathbf{C}} = \left\{ \sum_{-\infty \ll \mu < \infty} e_\mu c_\mu \,|\, c_\mu \in \mathbf{C} \right\} = (\mathbf{V}_{\mathbf{C}})_{\underline{0}} \oplus (\mathbf{V}_{\mathbf{C}})_{\underline{1}}$$

with basis elements $e_\mu \in (\mathbf{V}_{\mathbf{C}})_{\underline{\mu}}$. The $\mathscr{A}$-module $\mathbf{V}$ has a natural pairing $\langle\,,\rangle : \mathbf{V}\otimes\mathbf{V} \to \mathscr{A}$ defined by

$$\langle e_\mu, e_{-\nu-1}\rangle = \delta_{\mu\nu}, \quad \langle ua, v\rangle = \langle u, av\rangle \quad \text{and} \quad \langle u, va\rangle = \langle u, v\rangle a,$$

for $u, v \in \mathbf{V}$, $a \in \mathscr{A}$. We identify an element $u = \sum_{-\infty \ll \mu < \infty} e_\mu a_\mu$ with a super-microfunction $u(x,\theta) = \sum_{-\infty \ll \mu < \infty} \delta^{(\mu)}(x,\theta) a_{-\mu-1}$, where we have defined the super-delta function by $\delta(x,\theta) = \theta\delta(x)$, and $\delta^{(\mu)}(x,\theta) = D^\mu(\delta(x,\theta))$ $(\mu \in \mathbf{N})$. More precisely, one has

$$\delta^{(2\mu)}(x,\theta) = \theta\partial_x{}^\mu(\delta(x)), \quad \delta^{(2\mu+1)}(x,\theta) = \partial_x{}^\mu(\delta(x)),$$
$$\delta^{(-2\mu-2)}(x,\theta) = \theta x^\mu Y(x)/\mu!, \quad \delta^{(-2\mu-1)}(x,\theta) = x^\mu Y(x)/\mu!,$$

for $\mu \in \mathbf{N}$, where $Y(x)$ is the Heaviside function. Set

$$\mathbf{V}^\phi = \left\{ u = \sum_{-\infty \ll \mu < 0} e_\mu a_\mu \right\}$$
$$= \left\{ u(x,\theta) = \sum_{0 \leqq \mu \ll \infty} \delta^{(\mu)}(x,\theta) a_{-\mu-1} \right\}.$$

Define the super-Laplace transform of $\delta^{(\mu)}(x,\theta)$ by

$$\int \Delta(dx/d\theta)\exp(-\lambda x - \xi\theta)\delta^{(\mu)}(x,\theta) = \left(\xi + \lambda\frac{\partial}{\partial\xi}\right)^\mu (1) \quad (\mu \in \mathbf{Z}). \tag{3.4}$$

Note that $(\xi + \lambda(\partial/\partial\xi))^2 = \lambda$. Hence we can rewrite (3.4) as

$$\hat{e}_\mu(\lambda,\xi) = \int \Delta(dx/d\theta)\exp(-\lambda x - \xi\theta)e_\mu = \begin{cases} \xi\lambda^{-\mu/2-1} & (\mu\text{:even}) \\ \lambda^{-(\mu+1)/2} & (\mu\text{:odd}). \end{cases}$$

For a general element $u=\sum e_\mu a_\mu$, we set $\hat{u}(\lambda,\xi)=\sum \hat{e}_\mu(\lambda,\xi)a_\mu$. By the super-Laplace transform we get the identification

$$\mathbf{V}\ni u(x,\theta)\mapsto \hat{u}(\lambda,\xi)\in\mathbf{C}((\lambda^{-1},\xi))\otimes\mathscr{A}. \tag{3.5}$$

For $\hat{u}(\lambda,\xi)=\sum\lambda^\mu a_\mu+\xi\sum\lambda^\mu b_\mu\in\mathbf{C}((\lambda^{-1},\xi))\otimes\mathscr{A}$, set

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\hat{u}(\lambda,\xi))=b_{-1}.$$

To show the bilinear residue formula, we have to present some lemmas on the residue calculus, the super-Laplace inverse transform and the formal adjoint of operators.

Lemma 3.1. *For $u,v\in\mathbf{V}$, we have*

$$\langle u,v\rangle=\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\hat{u}(\lambda,\xi)\hat{v}(\lambda,\xi)).$$

Proof. It is easy to see that

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\hat{e}_\mu(\lambda,\xi)\hat{e}_{-\nu-1}(\lambda,\xi))=0$$

if $\mu-\nu$ is odd. If $\mu-\nu$ is even, then

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\hat{e}_\mu(\lambda,\xi)\hat{e}_{-\nu-1}(\lambda,\xi))=\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\xi\lambda^{-1+(\mu-\nu)/2})=\delta_{\mu\nu}. \quad\blacksquare$$

For a super-microdifferential operator $P\in\mathscr{E}_{\mathbf{C}[[x,\theta]]\otimes\mathscr{A}}$, we define a super-differential operator of infinite order $\hat{P}\in\mathscr{D}^\infty_{\mathbf{C}((\lambda^{-1},\xi))\otimes\mathscr{A}}$ through the super-Laplace transform: $\hat{P}(\hat{u}(\lambda,\xi))=(Pu)\hat{}(\lambda,\xi)$. For example, $(D^\mu)\hat{}=(\hat{D})^\mu=(\xi+\lambda(\partial/\partial\xi))^\mu$ for $\mu\in\mathbf{Z}$, and $(\partial/\partial\theta)\hat{}=\xi$. And $\hat{f}=f(-d/d\lambda,(-)^{a-1}d/d\xi)$ for a superfield $f=f(x,\theta)\in(\mathbf{C}[[x,\theta]]\otimes\mathscr{A})_a$.

For a column vector $(a_\mu)_{\mu\in\mathbf{Z}}$ that corresponds to an element $u=\sum e_\mu a_\mu$ of $\mathbf{V}$, set $(a_\mu(x,\theta))_{\mu\in\mathbf{Z}}=\exp(\theta\Lambda+x\Lambda^2)(a_\mu)_{\mu\in\mathbf{Z}}$. Then one sees that

$$a_\mu(x,\theta)=\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\exp(\lambda x+\xi\theta)\hat{D}^\mu(\hat{u}(\lambda,\xi))), \tag{3.6}$$

$$D(a_\mu(x,\theta))=a_{\mu+1}(x,\theta). \tag{3.7}$$

Lemma 3.2. *An element $u\in\mathbf{V}$ belongs to $\mathbf{V}^\Phi$ if and only if*

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)\exp(\lambda x+\xi\theta)\hat{u}(\lambda,\xi))=0.$$

Proof. A direct consequence of (3.6) and (3.7). $\blacksquare$

Lemma 3.3. *Let $p(x,\theta)\in(\mathbf{C}[[x,\theta]]\otimes\mathscr{A})_\nu$. If $\mu-\nu$ is even, then*

$$(\hat{p}\hat{D}^\mu)^*=(\hat{D}^\mu)^*p\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\xi}\right).$$

Proof. Let both μ and ν be odd. For even superfields $f(\lambda,\xi)$ and $g(\lambda,\xi)$,

$$\begin{aligned}\int\Delta(d\lambda/d\xi)((\hat{p}\hat{D}^\mu)(f))g&=\int\Delta(d\lambda/d\xi)(\hat{D}^\mu(f))p\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\xi}\right)(g)\\&=\int\Delta(d\lambda/d\xi)f(\hat{D}^\mu)^*p\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\xi}\right)(g).\end{aligned}$$

The other case (μ,ν are even) is similarly checked. $\blacksquare$

Lemma 3.4. $(\hat{D}^{\mu})^*(\exp(\lambda x+\xi\theta))=(-D)^{\mu}(\exp(\lambda x+\xi\theta)),$

$$(\hat{D}^{\mu})(\exp(-\lambda x-\xi\theta))=(-)^{\mu}(D^{\mu})^*(\exp(-\lambda x-\xi\theta)).$$

Proof is straightforward. ■

Lemma 3.5. *Let* $P\in\mathscr{E}_{\mathrm{C}[[x,\theta]]\otimes\mathscr{A}}$ *is an even operator. Then*

$$P(\exp(\lambda x+\xi\theta))=\hat{P}^*(\exp(\lambda x+\xi\theta)),$$
$$P^*(\exp(-\lambda x-x\theta))=\hat{P}(\exp(-\lambda x-\xi\theta)).$$

Proof. Without loss of generality, we can set $P=p(x,\theta)D^{\mu}$, where $p(x,\theta)$ has the parity μ.

$$\begin{aligned}(\hat{p}\hat{D}^{\mu})^*(\exp(\lambda x+\xi\theta))&=(\hat{D}^{\mu})^*p\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\xi}\right)(\exp(\lambda x+\xi\theta))\\&=(\hat{D}^{\mu})^*(p(x,\theta)(\exp(\lambda x+\xi\theta)))\\&=p(x,\theta)(-)^{\mu}(\hat{D}^{\mu})^*(\exp(\lambda x+\xi\theta))\\&=p(x,\theta)D^{\mu}(\exp(\lambda x+\xi\theta)).\end{aligned}$$

The other one is similarly checked. ■

Now we can state the bilinear residue formula (BRF) in $\mathscr{E}_{\mathrm{C}[[x,\theta]]\otimes\mathscr{A}}$.

Theorem 3.6. *Let* $P,Q\in\mathscr{E}_{\mathrm{C}[[x,\theta]]\otimes\mathscr{A}}$ *are even operators. Then* $PQ\in\mathscr{D}_{\mathrm{C}[[x,\theta]]\otimes\mathscr{A}}$ *if and only if the* BRF

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)P(\exp(\lambda x+\xi\theta))Q^*(\exp(-\lambda x'-\xi\theta'))=0$$

holds for any (x,θ), (x',θ').

Proof. The condition $PQ\in\mathscr{D}$ is equivalent to $PQ(e_{-k-1})\in\mathbf{V}^{\phi}$ for all $k\in\mathbf{N}$, and also to

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)((PQ)\hat{})^*(\exp(\lambda x+\xi\theta)\hat{e}_{-k-1}(\lambda,\xi))=0 \tag{3.8}$$

for all $k\in\mathbf{N}$. Here we recall that $\hat{e}_{-k-1}(\lambda,\xi)=\lambda^{k/2}$ (k:even), $=\xi\lambda^{(k-1)/2}$ (k:odd). Multiplying $(1/k!)(-x')^{k/2}$ (k:even), $((-\theta')/(k-1)!)(-x')^{(k-1)/2}$($k$:odd) from the right of (3.8), and summing up over $k\in\mathbf{N}$, we have a generating function expression

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)((PQ)\hat{})^*(\exp(\lambda x+\xi\theta))\exp(-\lambda x'-\xi\theta'))=0.$$

This and Lemma 3.5 complete the proof. ■

We are in the position to state one of our main results in this paper.

Theorem 3.7. *Formal superfields* $w(x,\theta,t,\lambda,\xi)$ *and* $w^*(x,\theta,t,\lambda,\xi)$ *of the form (3.1) and (3.2) are a super-wave function and its dual for the* $\dot{\mathrm{S}}$KP *hierarchy if and only if they satisfy the* BRF

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)w(x',\theta',t',\lambda,\xi)w^*(x,\theta,t,\lambda,\xi))=0 \tag{3.9}$$

for any (x,θ,t) *and* (x',θ',t').

Proof. From Theorem 3.6 it follows that

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)(D')^{\mu}(w(x',\theta',t,\lambda,\xi))w^*(x,\theta,t,\lambda,\xi))=0. \tag{3.10}$$

Equations (3.3) show that, for any multi-index $\alpha=(\alpha_1,\alpha_2,\dots)$,

$$\operatorname{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)(D_t)^\alpha(w(x',\theta',t,\lambda,\xi))w^*(x,\theta,t,\lambda,\xi))=0,$$

where we have put $(D_t)^\alpha=D_{t_1}{}^{\alpha_1}D_{t_2}{}^{\alpha_2}\cdots$. The BRF (3.9) follows as a generating function expression of (3.10). Conversely, if (3.9) is satisfied, we have

$$\operatorname{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)(D_n-\varepsilon_n B_n)(w(x,\theta,t,\lambda,\xi))w^*(x',\theta',t,\lambda,\xi))=0.$$

Note that

$$(D_n-\varepsilon_n B_n)(w(x,\theta,t,\lambda,\xi))=(D_n(W)W^{-1}-\varepsilon_n B_n^c)W(\exp(H(x,\theta,t,\lambda,\xi))),$$

where $B_n^c=-(WD^nW^{-1})_-$. Then Theorem 3.6 implies that $D_n(W)W^{-1}-\varepsilon_n B_n^c\in\mathscr{D}$, however, which should be of negative order, by definition. Thus we get $D_n(W)=\varepsilon_n B_n^c W$, that are equivalent to the Sato equations. ■

4. Ortho-Symplectic SKP Hierarchy

In this section we discuss the OSp-SKP hierarchy. Let W be a super-wave operator in $\mathscr{E}$ for the SKP hierarchy. The OSp-SKP hierarchy is defined by the condition:

$$D^{-1}W^*D=W^{-1} \tag{4.1}$$

in the OSp-sector $t_{4n+1}=t_{4n+4}=0$ for $n=0,1,2,\dots$. The Sato equations read

$$D_{4n+2}(W)=-(B_{4n+2}W-WD^{4n+2}), \tag{4.2}$$

$$D_{4n+3}(W)=B_{4n+3}W-WD^{4n+3},\quad n=0,1,2,\dots \tag{4.3}$$

with the symmetries

$$D^{-1}(B_{4n+2})^*D=-B_{4n+2},\quad D^{-1}(B_{4n+3})^*D=B_{4n+3},$$

in the OSp-sector. In this section the time variables t are supposed to be restricted in the OSp-sector.

We define a super-wave function for a solution $W=\sum_{j=0}^{\infty}w_j(x,\theta,t)D^{-j}$ to the OSp-SKP hierarchy by

$$w(x,\theta,t,\lambda,\xi)=W(\exp\tilde{H}), \tag{4.4}$$

where

$$\tilde{H}=\tilde{H}(x,\theta,t,\lambda,\xi)=x\lambda-\sum_{n=0}^{\infty}t_{4n+2}\lambda^{2n+1}$$
$$+\left(\xi+\sum_{n=0}^{\infty}t_{4n+3}\lambda^{2n+2}\right)\left(\theta+\sum_{n=0}^{\infty}t_{4n+3}\lambda^{2n+1}\right).$$

We also put

$$v(x,\theta,t,\lambda,\xi)=WD^{-1}(\exp(-\tilde{H})). \tag{4.5}$$

Theorem 4.1. *The superfields of the form (4.4) and (4.5) are super-wave functions of* *the* OSp-SKP *hierarchy if and only if they enjoy*

$$\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)w(x,\theta,t,\lambda,\xi)v(x',\theta',t',\lambda,\xi))=1 \tag{4.6}$$

for any (x,θ,t) *and* (x',θ',t').

Proof. Suppose that W is a solution to the OSp-SKP hierarchy. From the BRF for the SKP hierarchy, we get

$$\begin{aligned} 0 &= \mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)W(e^{\tilde{H}(x,\theta,t,\lambda,\xi)})(W^*)^{-1}(e^{-\tilde{H}(x',\theta',t',\lambda,\xi)})) \\ &= \mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)W(e^{\tilde{H}(x,\theta,t,\lambda,\xi)})D'WD'^{-1}(e^{-\tilde{H}(x',\theta',t',\lambda,\xi)})) \\ &= \mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)W(e^{\tilde{H}(x,\theta,t,\lambda,\xi)})D'(v(x',\theta',t',\lambda,\xi)). \end{aligned} \tag{4.7}$$

The superfield $v(x',\theta',t',\lambda,\xi)$ solves the linear differential equation $D_n'(v)=\varepsilon_n B_n'(v)$ so that one has

$$D_n'(\mathrm{Res}_{\lambda=\infty}(\Delta(d\lambda/d\xi)w(x,\theta,t,\lambda,\xi)v(x',\theta',t',\lambda,\xi))=0.$$

Namely, the left-hand side of (4.6) is independent of t'. Putting $x=x'$, $\theta=\theta'$ and $t=t'$ therein one gets the equality (4.6). Conversely suppose that (4.6) holds. Then the second equation in (4.7), we get $W\cdot(DWD^{-1})^*=1$ by means of Theorem 3.6. This completes the proof. ■

Now we discuss the 2-spinor representation of the OSp-SKP hierarchy. We introduce the super-adjoint in $\mathscr{L}$ by, for $P=(P_{ij})_{i,j=0,1}$,

$$P^{\#}=\begin{pmatrix} P_{00}^* & (-)^a P_{10}^* \\ (-)^{a+1}P_{01}^* & P_{11}^* \end{pmatrix} \quad (P\in\mathscr{L}_{\underline{a}}),$$

where P_{ij}^* is the formal adjoint operator of P_{ij} in $\mathscr{E}_{\mathscr{G}}$. We define Lie superalgebra $\mathfrak{osp}(\mathscr{E}_{\mathscr{G}})$ by

$$\begin{aligned} \mathfrak{osp}(\mathscr{E}_{\mathscr{G}}) &= \mathfrak{osp}(\mathscr{E}_{\mathscr{G}})_{\underline{0}}\oplus\mathfrak{osp}(\mathscr{E}_{\mathscr{G}})_{\underline{1}}, \\ \mathfrak{osp}(\mathscr{E}_{\mathscr{G}})_{\underline{a}} &= \{P\in\mathscr{L}_{\underline{a}}\,|\,M^{-1}P^{\#}M=(-)^aP\}, \end{aligned}$$

where $M=\mathrm{diag}(\partial_x,1)$. The corresponding Lie supergroup $\mathrm{OSp}(\mathscr{E}_{\mathscr{G}})$ is introduced by

$$OSp(\mathscr{E}_{\mathscr{G}})=\{P\in\mathscr{L}_{\underline{0}}\,|\,P \text{ is invertible and } M^{-1}P^{\#}M=P^{-1}\}.$$

Proposition 4.2. *Let* $\mathscr{W}^{\phi}_{\mathrm{OSp}}$ *be the space of super-wave operators in* $\mathscr{E}$ *of the* OSp-SKP *hierarchy, and* $\mathscr{W}^{\phi}_{\mathrm{OSp}}$ *(respectively* $\mathscr{W}^{\phi}_{\mathrm{CKP}}$*) be the space of wave operators in* $\mathscr{E}_{\mathscr{R}}$ *of the* BKP *(respectively* CKP*) hierarchy. Then we have the projection*

$$\rho|_{\mathscr{W}^{\phi}_{\mathrm{OSp}}}:\mathscr{W}^{\phi}_{\mathrm{OSp}}\to\mathscr{W}^{\phi}_{\mathrm{BKP}}\times\mathscr{W}^{\phi}_{\mathrm{CKP}},$$

where the map ρ *was introduced in Proposition 2.5.*

Proof. First we note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\pi(X)^{\#}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=(-)^a\pi(X^*) \quad (X\in\mathscr{E}_{\underline{a}})$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\pi(D)=M.$$

Let $W \in \mathscr{W}^{\phi}_{\mathrm{OSp}}$. Then the condition (4.1) reads in the 2-spinor representation

$$M^{-1}\pi(W)^{\#}M = \pi(W)^{-1}, \tag{4.8}$$

namely, $\pi(W) \in OSp(\mathscr{E}_{\tilde{\mathscr{G}}})$. Let $\pi(W) = (W_{ij})_{i,j=0,1}$. Applying the body map $\tilde{\varepsilon}_{\mathscr{L}}$ to the both sides of (4.8), we see that $\tilde{\varepsilon}(W_{00})$ (respectively $\tilde{\varepsilon}(W_{11})$) satisfies the BKP (respectively CKP) condition. ■

In the rest of this section we characterize solutions to the OSp-SKP hierarchy in terms of superframes in $USGM^{\phi}$.

Proposition 4.3. *Let P be an operator in $\mathscr{E}_{\underline{a}}$. Then it follows that*

$$\psi^{\vee}(P^*) = (-)^a \operatorname{offdiag}({}^tK, {}^tK)^{st}\psi^{\vee}(P) \operatorname{offdiag}(K, K),$$

where offdiag(A,B) *stands for* $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ *with* $A, B \in Mat(\mathbf{Z}\times\mathbf{Z})$, *and* $K = \Lambda J$, $J = ((-)^{\mu}\delta_{\mu,-\nu})_{\mu,\nu\in\mathbf{Z}}$. *The symbol $\psi^{\vee}(P)$ means the "check" of the matrix $\psi(P)$* (cf. *Sect. 2*), *and "st" is the supertransposition of a matrix* (cf. [7]).

Proof. We only have to show the claim in the case of $P = uD^j$ $u \in \mathscr{S}_{\underline{a}}$. For $u = f(x,t) + \theta g(x,t)$,

$$\psi^{\vee}(u) = \begin{pmatrix} \sigma(f) + \theta\sigma(g) & 0 \\ \theta\sigma(f_x) + \sigma(g) & (-)^a(\sigma(f) + \theta\sigma(g)) \end{pmatrix},$$

where $\sigma(f) = \left(\begin{pmatrix} i \\ i-k \end{pmatrix} f^{(i-k)}(x,t)\right)_{i,k\in\mathbf{Z}}$, $\begin{pmatrix} m \\ n \end{pmatrix} = 0$ for $n < 0$. By a simple calculation we have

$$\operatorname{offdiag}({}^tK, {}^tK)^{st}\psi^{\vee}(u) \operatorname{offdiag}(K, K) = (-)^a\psi^{\vee}(u).$$

Since $\psi^{\vee}(D) = \begin{pmatrix} 0 & 1 \\ \Lambda & 0 \end{pmatrix}$, it follows that

$$\psi^{\vee}((D^j)^*) = (-)^j \operatorname{offdiag}({}^tK, {}^tK)^{st}\psi^{\vee}(D^j) \operatorname{offdiag}(K, K).$$

Then we have

$$\psi^{\vee}((uD^j)^*) = (-)^{a+j} \operatorname{offdiag}({}^tK, {}^tK)^{st}\psi^{\vee}(uD^j) \operatorname{offdiag}(K, K). \quad ■$$

Now we introduce the Lie supergroup $OSp(\mathscr{S})$ [9],

$$OSp(\mathscr{S}) = \{A = (A_{\alpha\beta})_{\alpha,\beta=0,1} \mid A_{\alpha\beta} \in Mat(\mathbf{Z}\times\mathbf{Z}, \mathscr{S}_{\underline{\alpha+\beta}}) \text{ and } \varepsilon(A) \text{ is invertible and } \operatorname{diag}(J, -{}^tK)^{st}\check{A} \operatorname{diag}(J, -K) = \check{A}^{-1}.\}.$$

Notice that for an operator U in $\mathscr{E}_{\underline{0}}$, the condition

$$D^{-1}U^*D = U^{-1}, \tag{4.9}$$

is equivalent to that $\psi^{\vee}(U) \in OSp(\mathscr{S})$. We introduce the following inner products $\langle\ ,\ \rangle_B \langle\ ,\ \rangle_C$,

$$\langle \vec{f}, \vec{g} \rangle_B = \sum_{j\in\mathbf{Z}} (-)^j f_j g_{-j}, \quad \langle \vec{f}, \vec{g} \rangle_C = \sum_{j\in\mathbf{Z}} (-)^{j+1} f_j g_{-j-1},$$

for column vectors $\vec{f}=(f_j)_{j\in\mathbf{Z}}$, $\vec{g}=(g_j)_{j\in\mathbf{Z}}$. Put $\psi^\vee(U)=((u_{ij}^{\alpha\beta})_{i,j\in\mathbf{Z}})_{\alpha,\beta=0,1}$ and $\vec{u}_j^{\alpha\beta}=(u_{i,j}^{\alpha\beta})_{i\in\mathbf{Z}}$. If $\psi^\vee(U)\in\mathrm{OSp}(\mathscr{S})$, i.e., $D^{-1}U^*D=U^{-1}$, we have the following relation:

$$\begin{aligned}
\langle\vec{u}_{-i}^{00},\vec{u}_j^{00}\rangle_B-\langle\vec{u}_{-i}^{10},\vec{u}_{u_j}^{10}\rangle_C&=(-)^i\delta_{i,j},\\
\langle\vec{u}_{-i}^{00},\vec{u}_j^{01}\rangle_B-\langle\vec{u}_{-i}^{10},\vec{u}_j^{11}\rangle_C&=0,\\
\langle\vec{u}_{-i-1}^{01},\vec{u}_j^{00}\rangle_B+\langle\vec{u}_{-i-1}^{11},\vec{u}_j^{10}\rangle_C&=0,\\
\langle\vec{u}_{-i-1}^{01},\vec{u}_j^{01}\rangle_B+\langle\vec{u}_{-i-1}^{11},\vec{u}_j^{11}\rangle_C&=(-)^i\delta_{i,j},\quad i,j\in\mathbf{Z}.
\end{aligned}\tag{4.10}$$

Let W be a supper-wave operator in $\mathscr{E}$ of the OSp-SKP hierarchy and Ξ be the superframe $\Xi=\psi(W)\Xi_0|_{x=\theta=t=0}$. Put the "check" of $\Xi,\check{\Xi}=((\vec{\xi}_i^{\alpha\beta})_{i<0})_{\alpha,\beta=0,1}$. We note that the entries of $\vec{\xi}_i^{\alpha\beta}$ belong to $\mathscr{A}_{\underline{\alpha+\beta}}$. From $D^{-1}W^*D|_{x=\theta=t=0}=W^{-1}|_{x=\theta=t=0}$, we obtain

$$\begin{aligned}
\langle\vec{\xi}_i^{00},\vec{\xi}_j^{00}\rangle_B-\langle\vec{\xi}_i^{10},\vec{\xi}_j^{10}\rangle_C&=0,\\
\langle\vec{\xi}_i^{00},\vec{\xi}_j^{01}\rangle_B-\langle\vec{\xi}_i^{10},\vec{\xi}_j^{11}\rangle_C&=0,\\
\langle\vec{\xi}_i^{01},\vec{\xi}_j^{00}\rangle_B+\langle\vec{\xi}_i^{11},\vec{\xi}_j^{10}\rangle_C&=0,\\
\langle\vec{\xi}_i^{01},\vec{\xi}_j^{01}\rangle_B+\langle\vec{\xi}_i^{11},\vec{\xi}_j^{11}\rangle_C&=0,\quad i,j<0.
\end{aligned}\tag{4.11}$$

Simple computations show that the above condition is invariant under the right action of $SGL(\mathbf{N}^c;\mathscr{A})$ on the superframe Ξ. We refer to (4.11) as the orthogonality condition. Conversely, let Ξ be a superframe of $USGM^\Phi$ satisfying the orthogonality condition. Solve the following Grassmann equation

$${}^t\vec{w}\exp\left(\theta\Lambda+x\Lambda^2+\sum_{j\equiv 2,3(\mathrm{mod}\,4)}t_j\Gamma^j\right)\Xi=0.\tag{4.12}$$

The solution to (4.12) ${}^t\vec{w}=(\ldots,w_2,w_1,1,0,\ldots)$, $w_j\in\mathscr{S}_{\underline{j}}$ determines a super-wave operator $W=\sum_{j=0}^{\infty}w_jD^{-j}$ of the SKP hierarchy. Put $W_0=W|_{t=0}$.

Proposition 4.4. *Let Ξ be a superframe satisfying the orthogonality condition and W be the super-wave operator determined by Ξ via (4.12). Then the superframe $\psi^\vee(W_0^{-1})\cdot\check{\Xi}_0$ also satisfies the orthogonality condition.*

Proof. From $\psi^\vee(W_0)\psi^\vee(W_0^{-1})=1$, we obtain the linear equation

$${}^t\vec{w}_0\psi(W_0^{-1})\Xi_0=0,\tag{4.13}$$

where ${}^t\vec{w}_0$ is the 0-th row vector of $\psi(W_0)$. Due to the arguments in Sect. 2, we see that $\psi(W_0^{-1})\Xi_0=\exp(\theta\Lambda+x\Lambda^2)\tilde{\Xi}\exp(-\theta\Lambda_{\mathbf{N}^c}-x\Lambda_{\mathbf{N}^c}^2)$, where $\tilde{\Xi}$ is a superframe of $USGM^\Phi$. Then we have

$${}^t\vec{w}_0\exp(\theta\Lambda+x\Lambda^2)\tilde{\Xi}=0.\tag{4.14}$$

Since $(4.12)|_{t=0}$ and (4.14) yield the same solution, $\tilde{\Xi}=\Xi g$ for some $g\in SGL(\mathbf{N}^c;\mathscr{A})$ (see Proposition 2.3.). Hence $\tilde{\Xi}$ satisfies the orthogonality condition. Moreover observing that $\exp(\theta\Lambda+x\Lambda^2)\in OSp(\mathscr{S})$, we get the conclusion. ■

Proposition 4.5. *Let U be a monic operator in $\mathscr{E}_{\underline{0}}$ of order 0. Put $\psi^\vee(U)=$*

$((\vec{u}_i^{\alpha\beta})_{i\in\mathbf{Z}})_{\alpha,\beta=0,1}$, and suppose that $\vec{u}_i^{\alpha\beta}(i<0)$ satisfy the orthogonality condition. Then $\psi^\vee(U)$ belongs to $OSp(\mathscr{S})$, that is U enjoys (4.9).

Proof. Put $A=(a_{ij})_{i,j\in\mathbf{Z}}=\psi(D^{-1}U^*DU)$. The entries are given by

$$\begin{aligned}
a_{2i,2j}&=(-)^i\{\langle\vec{u}_{-i}^{00},\vec{u}_j^{00}\rangle_B-\langle\vec{u}_{-i}^{10},\vec{u}_j^{10}\rangle_C\},\\
a_{2i,2j+1}&=(-)^i\{\langle\vec{u}_{-i}^{00},\vec{u}_j^{01}\rangle_B-\langle\vec{u}_{-1}^{10},\vec{u}_j^{11}\rangle_C\},\\
a_{2i+1,2j}&=(-)^i\{\langle\vec{u}_{-i-1}^{01},\vec{u}_j^{00}\rangle_B+\langle\vec{u}_{-i-1}^{11},\vec{u}_j^{10}\rangle_C\},\\
a_{2i+1,2j+1}&=(-)^i\{\langle\vec{u}_{-i-1}^{01},\vec{u}_j^{01}\rangle_B+\langle\vec{u}_{-i-1}^{11},\vec{u}_j^{11}\rangle_C\}.
\end{aligned}$$

Because of the assumption on the orthogonality condition and $D^{-1}U^*DU\in\mathscr{E}_{\underline{0}}^{\mathrm{monic}}$, we can easily see that $a_{ij}=0$ for $i<0$, $j>0$, and that $a_{ii}=1$ for $i\in\mathbf{Z}$. To show that $D^{-1}U^*DU=1$, we have to verify $a_{0,-j}=0$ for $j\geqq 1$, because $D^{-1}U^*DU=\sum_{j=0}^{\infty}a_{0,-j}D^{-j}$. We use the induction. Put $U=1+uD^{-1}+$ lower order terms. Then we see that $a_{10}=\langle\vec{u}_{-1}^{01},\vec{u}_0^{00}\rangle_B+\langle\vec{u}_{-1}^{11},\vec{u}_0^{10}\rangle_C=u-u=0$. Since the recursive relation $a_{0,j-1}=(-)^{j-1}\{D(a_{0j})-a_{1j}\}$ follows from $D(A)=\Lambda A-A^\dagger\Lambda$ (see (2.9)), we get $a_{0,-1}=D(a_{00})-a_{10}=0$. By the induction on j, we can show that $a_{0,-j}=0$ for $j>0$. ∎

Combining Propositions 4.4 and 4.5, we obtain the following corollary.

Corollary 4.6. *Let Ξ be a superframe satisfying the orthogonality condition, and $W(\Xi)$ be the super-wave operator associated with Ξ. Then, for the initial value $W_0=W(\Xi)|_{t=0}$, $\psi^\vee(W_0)$ belongs to $OSp(\mathscr{S})$.*

Let π be the set of multi-indices $\alpha=(\alpha_i)_{i=0}^n$ $(n\in\mathbf{N})$. We denote $|\alpha|=\sum_{i=0}^n\alpha_i$ for $\alpha\in\pi$. The void index is denoted by ϕ. We write $t_{\mathrm{odd}}^\alpha=t_3^{\alpha_0}t_7^{\alpha_1}\cdots t_{4n+3}^{\alpha_n}$ and $t_{\mathrm{even}}^\beta=t_2^{\beta_0}t_6^{\beta_1}\cdots t_{4n+2}^{\beta_n}$.

Now we state the main theorem in this section.

Theorem 4.7. *Let W be a super-wave operator for the* SKP *hierarchy associated with the superframe $\Xi\in USGM^\phi$. If Ξ satisfies the orthogonality condition (4.11), then $W|_{t_{4n}=t_{4n+1}=0}(n\in\mathbf{N})$ is a super-wave operator for the* OSp-SKP *hierarchy.*

Proof. In this proof we set $U=W|_{t_{4n}=t_{4n+1}=0}$. Expand U^* and U^{-1} to the formal power series in $(t_{4n+2},t_{4n+3})_{n=0}^\infty$:

$$U^*=\sum_{\alpha,\beta\in\pi}t_{\mathrm{odd}}^\alpha t_{\mathrm{even}}^\beta(U^*)_{\alpha\beta},\quad U^{-1}=\sum_{\alpha,\beta\in\pi}t_{\mathrm{odd}}^\alpha t_{\mathrm{even}}^\beta(U^{-1})_{\alpha\beta}.$$

It is enough to show that

$$D^{-1}(U^*)_{\alpha\beta}D=(-)^{|\alpha|}(U^{-1})_{\alpha\beta}. \tag{4.20}$$

We prove (4.20) by the double induction on $|\alpha|$ and $|\beta|$. From Proposition 4.4 and Corollary 4.6, (4.20) holds for $\alpha=\beta=\phi$. Suppose that (4.20) holds for $\alpha=\phi$ and β with $|\beta|\leqq m$ $(m\in\mathbf{N})$. Put $\tilde\beta=(\tilde\beta_i)_{i=0}^n\in\pi$, where $|\tilde\beta|=m+1$ and $\tilde\beta_n\neq 0$. From the equations

$$D_{4n+2}U^*=-U^*B_{4n+2}^*-D^{4n+2}U^* \tag{4.21}$$

$$D_{4n+2}U^{-1}=U^{-1}B_{4n+2}+D^{4n+2}U^{-1}, \tag{4.22}$$

which can be deduced from (4.2), we obtain

$$\tilde{\beta}_n(U^*)_{\phi,\tilde{\beta}} = -\sum_{\beta'+\beta''=\tilde{\beta}-e_n}(U^*)_{\phi,\beta'}(B^*_{4n+2})_{\phi,\beta''} - D^{4n+2}(U^*)_{\phi,\tilde{\beta}-e_n},$$

where $e_j = (\delta_{ij})_{i=0}^n \in \pi$. By (4.20) with $\alpha = \phi$ and $\beta = \beta', \tilde{\beta} - e_n$, and by $D^{-1}(B^*_{4n+2})_{\phi,\phi}D = -(B_{4n+2})_{\phi,\phi}$, we see that

$$D^{-1}(B^*_{4n+2})_{\phi,\beta''}D = -(B_{4n+2})_{\phi,\beta''}.$$

Therefore we obtain

$$\begin{aligned}\tilde{\beta}_n D^{-1}(U^*)_{\phi,\tilde{\beta}}D &= \sum_{\beta'+\beta''=\tilde{\beta}-e_n}(U^{-1})_{\phi,\beta'}(B_{4n+2})_{\phi,\beta''} - (U^{-1})_{\phi,\tilde{\beta}-\varepsilon_n}D^{4n+2}\\ &= \tilde{\beta}_n(U^{-1})_{\phi,\tilde{\beta}} \quad \text{(by (4.22))}.\end{aligned}$$

Next, suppose that (4.20) holds for α with $|\alpha| \leqq 2m$ ($m \in \mathbf{N}$) and arbitrary β. Let $\tilde{\alpha} = (\tilde{\alpha}_i)_{i=0}^n$, where $\tilde{\alpha}_i = 0$ or $1, \tilde{\alpha} = 1$ and $|\tilde{\alpha}| = 2m+1$. From the equations

$$D_{4n+3}(U^*) = U^*B^*_{4n+3} - D^{4n+3}U^*, \tag{4.24}$$

$$D_{4n+3}(U^{-1}) = -U^{-1}B_{4n+3} + D^{4n+3}U^{-1}, \tag{4.25}$$

we obtain

$$\begin{aligned}(U^*)_{\tilde{\alpha},\beta} = &\sum_{0\leqq i\leqq n-1}\tilde{\alpha}_i\,\mathrm{sgn}\,(i)D_{4i+4n+6}(U^*)_{\tilde{\alpha}-e_i-e_n,\beta}\\ &+\sum_{\substack{\alpha'+\alpha''=\tilde{\alpha}-e_n\\ \beta'+\beta''=\beta}}\mathrm{sgn}\,(\alpha',\alpha'',\alpha')(U^*)_{\alpha'\beta'}(B^*_{4n+3})_{\alpha''\beta''} - D^{4n+3}(U^*)_{\tilde{\alpha}-e_n,\beta}.\end{aligned}$$

Here we have defined $\mathrm{sgn}\,(i)$ and $\mathrm{sgn}\,(\alpha',\alpha'',\alpha')$ through

$$\begin{aligned}t_{4i+3}^{\tilde{\alpha}_i e_i}t_{\mathrm{odd}}^{\tilde{\alpha}-\tilde{\alpha}_i e_i - e_n} &= \mathrm{sgn}\,(i)t_{\mathrm{odd}}^{\tilde{\alpha}-e_n},\\ t_{\mathrm{odd}}^{\alpha'}P_\gamma t_{\mathrm{odd}}^{\alpha''} &= \mathrm{sgn}\,(\alpha',\alpha'',\gamma)t_{\mathrm{odd}}^{\tilde{\alpha}-\tilde{e}_n}P_\gamma,\end{aligned}$$

where P_γ is an arbitrary monomial of parity $|\gamma| \bmod 2$, $\gamma \in \pi$. By the induction hypothesis, we see that

$$\begin{aligned}D^{-1}(U^*)_{\tilde{\alpha}-e_n-\tilde{\alpha}_i e_i,\beta}D &= -(U^{-1})_{\tilde{\alpha}-e_n-\tilde{\alpha}_i e_i,\beta} \quad \text{if} \quad \tilde{\alpha}_i \neq 0,\\ D^{-1}(U^*)_{\alpha'\beta'}D &= (-)^{|\alpha'|}(U^{-1})_{\alpha'\beta'},\\ D^{-1}(U^*)_{\tilde{\alpha}-e_n,\beta}D &= (U^{-1})_{\tilde{\alpha}-e_n,\beta}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&D^{-1}(t_{\mathrm{odd}}^{\alpha''}t_{\mathrm{even}}^{\beta''}(UD^{4n+3}U^{-1})_{\alpha''\beta''})^*D\\ &= (-)^{|\alpha''|}t_{\mathrm{odd}}^{\alpha''}t_{\mathrm{even}}^{\beta''}\sum_{\substack{\gamma'+\gamma''=\alpha''\\ \delta'+\delta''=\beta''}}\mathrm{sgn}\,(\gamma',\gamma'',\gamma'+1)D^{-1}((U^{-1})^*)_{\gamma'\delta'}D^{4n+3}(U^*)_{\gamma''\delta''}D.\end{aligned}$$

From $D^{-1}((U^{-1})^*)_{\phi\phi}D = (U)_{\phi\phi}$ and the induction hypothesis, we obtain

$$D^{-1}((U^{-1})^*)_{\gamma'\delta'}D = (-)^{|\gamma'|}(U)_{\gamma'\delta'},$$

and

$$D^{-1}(B^*_{4n+3})_{\alpha'',\beta''}D = (-)^{|\alpha''|}(B_{4n+3})_{\alpha'',\beta''}.$$

Hence we get

$$D^{-1}(U^*)_{\tilde{\alpha}\beta}D = \sum_{0\leqq i\leqq n-1} \tilde{\alpha}_i \operatorname{sgn}(i) D_{4i+4n+6}(U^{-1})_{\tilde{\alpha}-e_i-e_n,\beta}$$
$$+ \sum_{\substack{\alpha'+\alpha''=\tilde{\alpha}-e_n \\ \beta'+\beta''=\beta}} \operatorname{sgn}(\alpha',\alpha'',\alpha')(U^{-1})_{\alpha'\beta'}(B_{4n+3})_{\alpha''\beta''} - D^{4n+3}(U^{-1})_{\tilde{\alpha}-e_n,\beta}.$$

Comparing the right-hand side and the coefficient of $t_{\text{odd}}^{\tilde{\alpha}} t_{\text{even}}^{\beta}$ of U^{-1} in (4.25), we see that

$$D^{-1}(U^*)_{\tilde{\alpha},\beta}D = -(U^{-1})_{\tilde{\alpha},\beta}.$$

One can show similarly that $D^{-1}(U^*)_{\tilde{\alpha}\beta}D = (U^{-1})_{\tilde{\alpha}\beta}$ for the case that $|\alpha|$ is even. ■

5. Soliton Solutions to the SKP Hierarchy

We proceed to the construction of soliton solutions. Let $\alpha_\nu, \beta_\nu, c_\nu$ be even generic elements in $\mathscr{A}$ and η_ν, ω_ν be odd ones $(-2N \leqq \nu \leqq -1)$. Consider the following condition on a super-wave function:

$$w(x,\theta,t,\lambda,\xi) = \left(\sum_{j=0}^{2N} w_j(x,\theta,t)D^{-j}\right)(\exp H(x,\theta,t,\lambda,\xi))$$

and

$$w(x,\theta,t,\alpha_\nu,\eta_\nu) = c_\nu w(x,\theta,t,\beta_\nu,\omega_\nu) \quad \text{for even } \nu, \tag{5.1}$$
$$((\hat{D}^{-1})^* w)(x,\theta,t,\alpha_\nu,\eta_\nu) = c_\nu((\hat{D}^{-1})^* w)(x,\theta,t,\beta_\nu,\omega_\nu) \quad \text{for odd } \nu.$$

The operator $(\hat{D}^{-1})^*$ is the formal adjoint operator of $\hat{D}^{-1}$

$$(\hat{D}^{-1})^* = -\frac{\partial}{\partial\xi} + \lambda^{-1}\xi.$$

A superanalogue of Cauchy's residue formula reads [8]

$$\operatorname{Res}_{\lambda=\alpha}\left\{\Delta(d\lambda/d\xi)\frac{\xi-\eta}{(\lambda-\alpha-\xi\eta)^{n+1}}f(\xi,\eta)\right\} = \frac{1}{n!}(D_{\lambda,\xi}^{2n}f)(\alpha,\eta),$$
$$\operatorname{Res}_{\lambda=\alpha}\left\{\Delta(d\lambda/d\xi)\frac{1}{(\lambda-\alpha-\xi\eta)^{n+1}}f(\lambda,\xi)\right\} = \frac{1}{n!}(D_{\lambda,\xi}^{2n+1}f)(\alpha,\eta),$$

where $D_{\lambda,\xi} = (\partial/\partial\xi) + \xi(\partial/\partial\lambda)$, and α is an even constant, η an odd constant. We remark that

$$((\hat{D}^{-1})^* w)(x,\theta,t,\alpha,\eta) = \operatorname{Res}_{\lambda=\alpha}\left(\Delta(d\lambda/d\xi)W(x,\theta,t,D)D^{-1}(\exp H)\frac{\xi-\eta}{\lambda-\alpha-\xi\eta}\right).$$

The condition (5.1) implies the following linear equation:

$$(w_1,\ldots,w_{2N})[(\phi_{j,-2\nu})_{\substack{1\leqq j\leqq 2N\\1\leqq\nu\leqq N}} | (\phi_{j+1,-2\nu+1})_{\substack{1\leqq j\leqq 2N\\1\leqq\nu\leqq N}}]$$
$$= -(\phi_{0,-2\nu})_{1\leqq\nu\leqq N} | (\phi_{1,-2\nu+1})_{1\leqq\nu\leqq N}), \tag{5.2}$$

where

$$\phi_{j,\nu} = (D^{-j}\exp H)(\alpha_\nu, \eta_\nu) - c_\nu(D^{-j}\exp H)(\beta_\nu, \omega_\nu).$$

Solving (5.2), one gets an N-soliton to the SKP hierarchy. We can rewrite (5.2) into the super-Grassmann equation:

$$ {}^t\vec{w}\Phi\Xi = 0, \tag{5.3}$$

where $\vec{w} = (w_{-j})_{j\in\mathbf{Z}}(w_{-j} = 0 \text{ for } j \geqq 1)$,

$$\Phi = \exp\left(\theta\Lambda + x\Lambda^2 + \sum_{n=1}^{\infty} t_n \Gamma^n\right),$$

$$\Lambda = (\delta_{\mu+1,\nu})_{\mu,\nu\in\mathbf{Z}}, \quad \Gamma = ((-)^\nu \delta_{\mu+1,\nu})_{\mu,\nu\in\mathbf{Z}}$$

$$\Xi = (\Xi_{\mu,\nu})_{\mu\in\mathbf{Z},\nu\in\mathbf{N}^c} \quad \text{with}$$

$$\Xi_{\mu,\nu} = \begin{cases} \alpha_\nu^{\mu/2} - c_\nu \beta_\nu^{\mu/2} & (\mu\text{:even}) \\ -\eta_\nu \alpha_\nu^{(\mu-1)/2} + c_\nu \omega_\nu \beta_\nu^{(\mu-1)/2} & (\mu\text{:odd}) \end{cases} \quad \text{for} \quad \begin{array}{l} -2N \leqq \nu < -1, \\ \nu\text{:even} \end{array}$$

$$\Xi_{\mu,\nu} = \begin{cases} -\eta_\nu \alpha_\nu^{-1+\mu/2} + c_\nu \omega_\nu \beta_\nu^{-1+\mu/2} & (\mu\text{:even}) \\ \alpha_n^{(\mu-1)/2} - c_n \beta_\nu^{(\mu-1)/2} & (\mu\text{:odd}) \end{cases} \quad \text{for} \quad \begin{array}{l} -2N < \nu \leqq -1, \\ \nu\text{:odd} \end{array}$$

$$\Xi_{\mu,\nu} = \delta_{\mu,\nu} \quad \text{for } -\infty < \nu < -2N.$$

Acknowledgement. The authors thank Dr. M. Noumi for stimulating discussions on the basis of the KP hierarchy.

References

1. Bergvelt, M. J.: A note on super Fock space, preprint, Max–Planck Institut 1988
2. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. Proc. RIMS Symp. "Nonlinear Integrable Systems—Classical Theory and Quantum Theory—Miwa, T., Jimbo, M. (eds.), pp. 39–119, Singapore: World Scientific 1983
3. ———: J. Phys. Soc. Jpn. **40**, 3806–3812 (1981)
4. ———: J. Phys. Soc. Jpn. **40**, 3813–3818 (1981)
5. ———: Physica **4D**, 343–365 (1982)
6. ———: Publ. RIMS, Kyoto Univ. **18**, 1077–1111 (1982)
7. DeWitt, B.: Supermanifolds. Cambridge: Cambridge Univ. Press 1984
8. Friedan, D., Martinec, E., Shenker, S.: Nucl. Phys. **B271**, 93–165 (1986)
9. Ikeda, K.: A supersymmetric extension of the Toda lattice hierarchy. Lett. Math. Phys. **14**, 321–328 (1987)
10. Kac, V. G., van de Leur, J. W.: Super boson–fermion correspondence. Ann. Inst. Fourier **37**, 99–137 (1987)
11. Leites, D. A.: Introduction to the theory of supermanifolds. Russ. Math. Surv. **35**:1, 1–64 (1980)
12. Manin, Yu. I., Radul, A. O.: A supersymmetric extension of the Kadomtsev–Petviashvili hierarchy. Commun. Math. Phys. **98**, 65–77 (1985)
13. Mulase, M.: Solvability of the super KP equation and a generalization of the Birkhoff decomposition. Invent. Math. **92**, 1–46 (1988)
14. Radul, A. O.: Algebra-geometric solutions to the super Kadomtsev–Petviashvili hierarchy, preprint, 1988
15. Sato, M.: Soliton equations and the universal Grassmann manifold, Lecture Note Series, Sophia Univ. No. 18 (1984), (Notes by M. Noumi in Japanese)
16. Sato, M., Sato, Y.: Soliton equations as dynamical systems on an infinite dimensional Grassmann

manifold. In: Proc. U.S.–Japan Seminar "Nonlinear Partial Differential Equations in Applied Science". Fujita, H., Lax, P. D., Strang, G. (eds.) pp. 259–271. Kinokuniya: North-Holland 1982
17. Ueno, K., Yamada, H.: A supersymmetric extension of nonlinear integrable systems. In: Proc. Conf. "Topological and Geometrical Methods in Field Theory. Westerholm, J., Hietarinta, J. (eds.), pp. 59–72. Singapore: World Scientific 1987
18. ———,: Super Kadomtsev–Petviashvili hierarchy and super Grassmann manifold. Lett. Math. Phys. **13**, 59–68 (1987)
19. ———,: Supersymmetric extension of the Kadomtsev–Petviashvili hierarchy and the universal super Grassmann manifold. Adv. Stud. Pure Math. **16**, Conformal Field Theory and Solvable Lattice Model (1988), pp. 373–426
20. ———,: Soliton solutions and bilinear residue formula for the super Kadomtsev–Petviashvili hierarchy. To appear in Proc. 16-th Int'l Colloq., "Group Theoretical Methods in Physics"
21. Yamada, H.: Super Grassmann hierarchies—A multicomponent theory—. Hiroshima Math. J. **17**, 373–394 (1987)

Communicated by H. Araki

Received November 28, 1988