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A Remark on Smoothing of Magnetic Schrödinger Semigroups

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Abstract. We prove that, under mild regularity conditions, the magnetic Schrödinger semigroup generated by $H_0(\mathbf{a})=\frac{1}{2}\sum_j(i\partial_j-a_j)^2$ has its range inside the bounded continuous functions. We also give a counterexample for the general case.

1. Definition. Let $a\in L^2_{\mathrm{loc}}(\mathbb{R}^\nu,\mathbb{R}^\nu)$ such that $\operatorname{div}a$ (distributional divergence) is in $L^1_{\mathrm{loc}}(\mathbb{R}^\nu)$. The magnetic Schrödinger semigroup is defined by the Feynman-Kac-Itô formula (see [1, Sects. 14–16] for an early review)

$$(f,\exp(-tH_0(\mathbf{a}))g)=\int_\Omega \exp(F(\omega,t))\overline{f(\omega(0))}g(\omega(t))d\mu_0(\omega),$$

where

$$F(\omega,t)=-i\int_{s=0}^{t} a(\omega(s))\cdot d\omega-(i/2)\int_{s=0}^{t}(\operatorname{div}a)(\omega(s))ds.$$

Here μ_0 is full Wiener measure (the product of Wiener space with Lebesgue measure on the starting points) and the stochastic integral is taken in the sense of Itô.

2. Gauge Invariance. If a is increased by a gradient $\nabla\lambda$, the semigroup and its generator are transformed isometrically via multiplication by $\exp(i\lambda)$. It follows that the spectrum does not change. More precisely, we have

2.1. Theorem [2, p. 168]. *Let $a,b\in L^2_{\mathrm{loc}}(\mathbb{R}^\nu,\mathbb{R}^\nu)$ and suppose that* $\operatorname{curl}a=\operatorname{curl}b$. *Then $H_0(a)$ is closable in L^2 if and only if $H_0(b)$ is. Moreover,*

$$\exp(i\lambda)H_0(a)\exp(-i\lambda)=H_0(b). \quad \square$$

3. Smoothing. We now consider $a\in L^2_{\mathrm{loc}}$ such that $a^2\in K_\nu^{\mathrm{loc}}$ (see e.g. [3, p. 453] for a definition and a review of properties).

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3.1. Theorem. *Let* $a \in L^2_{\mathrm{loc}}$ *such that* $a^2 \in K_\nu^{\mathrm{loc}}$, $\operatorname{div} a = 0$. *Then* $\exp(-tH_0(\mathbf{a}))$ *maps* $L^\infty \cap L^2$ *into* $C_b(\mathbb{R}^\nu)$, *the bounded continuous functions.*

Proof. Let $K \subset \mathbb{R}^\nu$ be compact, $0 < s < t$, $f \in L^\infty \cap L^2$. By the Feynman-Kac-Itô formula,

$$\begin{aligned}
&\|\exp(-tH_0(\mathbf{a}))f - \exp(-sH_0)\exp(-(t-s)H_0(\mathbf{a}))f\|_{K,\infty} \\
&= \sup_{x \in K} \left| \mathbb{E}^x \left[\left(\exp\left(-i \int_{\sigma=0}^{t} a(\omega(\sigma)) \cdot d\omega(\sigma) \right) - \exp\left(-i \int_{\sigma=s}^{t} a(\omega(\sigma)) \cdot d\omega(\sigma) \right) \right) f(\omega(t)) \right] \right| \\
&\leqq \sup_{x \in K} \mathbb{E}^x \left[\left| \int_{\sigma=0}^{s} a(\omega(\sigma)) \cdot d\omega(\sigma) \right| \right] \|f\|_\infty \\
&\leqq \left(\sup_{x \in K} \mathbb{E}^x \left[\left| \int_{\sigma=0}^{s} a(\omega(s)) \cdot d\omega(s) \right|^2 \right] \right)^{1/2} \|f\|_\infty \\
&= \left(\sup_{x \in K} \mathbb{E}^x \left[\int_{\sigma=0}^{s} a^2(\omega(s)) ds \right] \right)^{1/2} \|f\|_\infty \to 0 \quad \text{as} \quad s \to 0,
\end{aligned}$$

since $a^2 \in K_\nu^{\mathrm{loc}}$.

Hence we have established $\exp(-tH_0(\mathbf{a}))f$ to be a limit of continuous functions, uniformly on compact sets.

Note that the restriction $f \in L^2$ is superfluous if the appropriate definition of $\exp(-tH_0(\mathbf{a}))$ in L^∞ as the dual of the strongly continuous L^1-semigroup is adopted. □

3.2. Remark. This is a partial answer to the question on page 491 of [3]. The following example is another part of the answer.

3.3. Counterexample. Consider the discontinuous function

$$\lambda : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto \frac{xyz}{|x|^3 + |y|^3 + |z|^3}.$$

Then $\nabla \lambda \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$, but $\exp(-tH_0(\nabla\lambda)) = \exp(i\lambda)\exp(-tH_0)\exp(-i\lambda)$, so its range contains discontinuous functions.

References

1. Simon, B.: Functional integration and quantum physics. New York: Academic Press 1979
2. Leinfelder, H.: Gauge invariance of Schrödinger operators and related spectral properties. J. Operator Theory **9**, 163–179 (1983)
3. Simon, B.: Schrödinger semigroups. Trans. Am. Math. Soc. **7**, 447–526 (1982)

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