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Existence, Regularity, and Asymptotic Behavior of the Solutions to the Ginzburg-Landau Equations on $\mathbb{R}^3$

Yisong Yang

Institute for Mathematics and its Applications, University of Minnesota, Room 514, Vincent Hall, 206 Church Street SE, Minneapolis, MN 55455, USA

Abstract. This paper studies the solutions of the Ginzburg-Landau equations on $\mathbb{R}^3$ in the presence of an arbitrarily distributed external magnetic field. The existence and regularity of the solutions at the lowest energy level are established. The solutions found are in the Coulomb gauge. If the external field is sufficiently regular, the solutions are shown to have nice asymptotic decay properties at infinity.

1. Introduction

In the Ginzburg-Landau semi-quantum mechanical theory of superconductivity the behavior of a superconductor cooled below the transition temperature in the absence of an external magnetic field is described by the equations

$$\left.\begin{aligned} D_A^2\phi+\frac{\lambda}{2}(1-|\phi|^2)\phi=0\,, \\ \operatorname{curl}^2\mathbf{A}+\frac{i}{2}(\phi^*D_A\phi-\phi(D_A\phi)^*)=0\,, \end{aligned}\right\} \tag{1.1}$$

which are the equations of motion of the free energy density

$$\mathscr{E}=\frac{1}{2}|\operatorname{curl}\mathbf{A}|^2+\frac{1}{2}|D_A\phi|^2+\frac{\lambda}{8}(|\phi|^2-1)^2\,. \tag{1.2}$$

Here the complex scalar field ϕ is an order parameter so that $|\phi|^2$ gives the relative density of the superconducting condensed electron pairs, called the Cooper pairs which behave like charged bosonic particles, $\mathbf{A}$ is a gauge photon field, and $D_A\phi=\nabla\phi-i\mathbf{A}\phi$. In this model, $\lambda>0$ is a dimensionless coupling constant with $\lambda<1$ and $\lambda>1$ describing type I and type II superconductors respectively, the electric field is absent, the magnetic field is determined through $\mathbf{H}=\operatorname{curl}\mathbf{A}$, and the ground states (or the superconducting vacua) are given by $\mathbf{A}=0$, $\phi=e^{i\theta}$, $\theta\in\mathbb{R}^1$. The Ginzburg-Landau equations (1.1), which have been accepted as the fundamental equations for low-temperature superconductivity theory, were first introduced by Ginzburg and Landau in 1950 in their phenomenological approach to superconductivity (cf. Ginzburg [6]) and later derived by Gorkov [7] theoretically from his formulation of the Bardeen-Cooper-Schrieffer theory. The relativistic generalization of the above model in the context of quantum field theories is recognized as the abelian Higgs model which has shed great light on many aspects in particle physics.

It is well-known, that, when an external magnetic field $\mathbf{H}_{\mathrm{ext}}$ is applied, various distinguished phenomena such as the Meissner-Ochsenfeld effect, surface currents, and the Abrikosov mixed state occur in superconductors cooled below a critical temperature. In order to use the Ginzburg-Landau equations to obtain an appropriate description of these phenomena, one needs to study the solutions of the equations with full nonlinearity under the influence of an external field. Unfortunately, in this direction, mathematical results are still fragmentary. For example, Carroll and Glick [4] proved an existence and uniqueness theorem for a weak solution of Eqs. (1.1) under the condition that both λ and $\mathbf{H}_{\mathrm{ext}}$ are sufficiently small and $\mathbf{H}_{\mathrm{ext}}$ is a constant field; Odeh [13] considered the existence of periodic weak solutions of (1.1) on $\mathbb{R}^2$ simulating the lattice structure of the Abrikosov mixed states with the assumption that the external field was absent and $\lambda >$ some critical value; Klimov [10] studied the existence of multiple weak solutions of Eqs. (1.1) over a bounded domain in $\mathbb{R}^3$ also assuming that no external field was present.

Besides the above restrictions, the regularity of these solutions has never been analyzed. The difficulty lies in the fact that, in order to study Eqs. (1.1), one has always to choose the function space of the gauge potential $\mathbf{A}$ as the set of all vector fields with zero divergence (i.e. in the Coulomb gauge) to make (1.1) a nondegenerate elliptic system. This choice may render the regularity study of the weak solutions almost impossible if no symmetry assumption is made: as in the case of the Navier-Stokes equations, one will have an extra "pressure term" ∇p which lies in the orthogonal complement of the subspace of divergence-free vector fields in L^2 in the equation for $\mathbf{A}$. In the context of fluid dynamics, this pressure term is natural but it will be a nuisance in the Ginzburg-Landau equations. In other words, the full Ginzburg-Landau equations have not really been solved.

In this paper we prove the existence of *regular* solutions of the Ginzburg-Landau equations on $\mathbb{R}^3$ in the presence of an arbitrarily distributed external magnetic field. The solutions found are in the Coulomb gauge and stay at the lowest energy level. These solutions are physically most interesting because they are energetically stable. Asymptotic decay properties of the solutions will also be established under some additional regularity assumptions on the external field.

It should be noted that if the external field is absent, the Ginzburg-Landau equations (1.1) over $\mathbb{R}^2$ and $\mathbb{R}^3$ have extensively been studied in recent years since they give static solutions of the abelian Higgs model in particle physics. On $\mathbb{R}^3$, an argument based on some topological considerations shows that all finite energy solutions of Eq. (1.1) are superconducting vacua (cf. Felsager [5]). On $\mathbb{R}^2$, there have been a lot of interesting contributions. In the work of Nielsen and Olesen [12] the finite energy solutions, now called vortices, are explained as string-like field configurations in three dimensions of the Ginzburg-Landau theory. For the critical choice of the coupling constant $\lambda = 1$ (the intermediate phase between type I and type II superconductors) the second order Ginzburg-Landau equations can be solved by the first order Bogomol'nyi equations [3] and the prescribed vortex problem is completely settled (Jacobs and Rebbi [8, 15], Weinberg [18], Taubes [16, 17], Jaffe and Taubes [9]). For arbitrary $\lambda > 0$, it has been shown that the finite energy solutions of the system (1.1) have exponential decay properties due to the broken $U(1)$ symmetry (Jaffe and Taubes [9]), that (1.1) possesses a family of topologically nontrivial radial-symmetric finite energy solutions (Plohr [14]), and that the nonlinear desingularization phenomenon (Berger and Fraenkel [2]) occurs for these solutions as $\lambda \to \infty$ (Berger and Chen [1]).

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2. Existence of Regular Solutions

Let $\mathbf{H}_{\text{ext}}$ be an external magnetic field. In the presence of this external field, the Ginzburg-Landau energy density becomes

$$\mathscr{E} = \frac{1}{2}|\operatorname{curl}\mathbf{A}|^2 + \frac{1}{2}|D_A\phi|^2 + \frac{\lambda}{8}(|\phi|^2 - 1)^2 - \operatorname{curl}\mathbf{A}\cdot\mathbf{H}_{\text{ext}}$$

and the corresponding equations of motion are in the form

$$\left.\begin{aligned} D_A^2\phi + \frac{\lambda}{2}(1 - |\phi|^2)\phi &= 0, \\ \operatorname{curl}^2\mathbf{A} + \frac{i}{2}(\phi^* D_A\phi - \phi(D_A\phi)^*) &= \operatorname{curl}\mathbf{H}_{\text{ext}}. \end{aligned}\right\} \tag{2.1}$$

For convenience, we shall use the notation

$$L^p = L^p(\mathbb{R}^3), \qquad W^{k,p} = W^{k,p}(\mathbb{R}^3),$$

$$\mathring{W}^{1,2} = \text{the completion of the set } C_0^\infty(\mathbb{R}^3) \text{ under the norm } \|\mathbf{A}\|^2_{\mathring{W}^{1,2}} = \int_{\mathbb{R}^3} |\nabla\mathbf{A}|^2 d^3x,$$

where $p \geqq 1$, $k = 1, 2, \ldots$, and if $\mathbf{A} = (A_j)$ is a vector field, then $\nabla\mathbf{A} = (\partial_j A_k)$ and $|\nabla\mathbf{A}|^2 = \operatorname{tr}[(\nabla\mathbf{A})\cdot(\nabla\mathbf{A})^\tau] = \sum(\partial_j A_k)^2$. Define

$$K = \{\mathbf{A} \in \mathring{W}^{1,2} | \nabla\cdot\mathbf{A} = 0\} \subset \mathring{W}^{1,2}.$$

Namely, K consists of those vector fields in $\mathring{W}^{1,2}$ satisfying the Coulomb gauge condition.

For $(\mathbf{A}, \phi) \in K \times W^{1,2}_{\text{loc}}$, the total energy is given by

$$E(\mathbf{A}, \phi) = \int_{\mathbb{R}^3} \mathscr{E}(\mathbf{A}, \phi) d^3x.$$

The functional E is not finite at every point in $K \times W^{1,2}_{\text{loc}}$ but it is bounded from below if $\mathbf{H}_{\text{ext}} \in L^2$:

$$E(\mathbf{A}, \phi) \geqq -\frac{1}{2}\|\mathbf{H}_{\text{ext}}\|^2_{L^2}.$$

Now we are ready to state our existence and regularity theorem for solutions of the Ginzburg-Landau equations (2.1).

Theorem 2.1. *For* $\mathbf{H}_{\text{ext}} \in L^2 \cap W^{1,2}_{\text{loc}}$, *Eqs.* (2.1) *have a solution* $(\mathbf{A}, \phi) \in (K \cap W^{2,2}_{\text{loc}}) \times W^{4,2}_{\text{loc}}$. *This solution is of finite energy and solves the minimization problem*

$$E_m = \min\{E(\mathbf{A}, \phi) | (\mathbf{A}, \phi) \in K \times W^{1,2}_{\text{loc}}\}. \tag{2.2}$$

Moreover, if $\mathbf{H}_{\text{ext}}$ *is smooth, so is* $(\mathbf{A}, \phi)$.

Proof. The key point in our approach to the above problem is that we will not try to solve (2.2) directly in the space $K \times W^{1,2}_{\text{loc}}$, otherwise, we shall still end up with the unwanted extra "pressure term". Instead, we will consider the minimization problem

$$I_m = \min\{I(\mathbf{A}, \phi) | (\mathbf{A}, \phi) \in \mathring{W}^{1,2} \times W^{1,2}_{\text{loc}}\}, \tag{2.3}$$

where

$$I(\mathbf{A}, \phi) = \frac{1}{2} \|\nabla \mathbf{A}\|^2_{L^2} + \frac{1}{2} \|D_A \phi\|^2_{L^2} + \frac{\lambda}{8} \|(|\phi|^2 - 1)\|^2_{L^2} - \int_{\mathbb{R}^3} \operatorname{curl} \mathbf{A} \cdot \mathbf{H}_{\text{ext}} d^3x.$$

It is easy to see that I is bounded from below on $\mathring{W}^{1,2} \times W^{1,2}_{\text{loc}}$, so I_m is a finite number.

We need the inequality (cf. Ladyzhenskaya [11])

$$\|\mathbf{A}\|^6_{L^6} \leqq c \|\nabla \mathbf{A}\|^6_{L^2}, \qquad \mathbf{A} \in \mathring{W}^{1,2}. \tag{2.4}$$

Let $\{(\mathbf{A}_j, \phi_j)\}$ be a minimizing sequence of the problem (2.3). From a simple interpolation inequality, we find

$$\frac{1}{4} \|\nabla \mathbf{A}_j\|^2_{L^2} + \frac{1}{2} \|D_{A_j} \phi_j\|^2_{L^2} + \frac{\lambda}{8} \|(|\phi_j|^2 - 1)\|^2_{L^2} \leqq \sup_j I(\mathbf{A}_j, \phi_j) + \|\mathbf{H}_{\text{ext}}\|^2_{L^2}, \tag{2.5}$$

therefore, using (2.4) and (2.5), we see that $\{\mathbf{A}_j\}$ is a bounded sequence in L^6 and $\mathring{W}^{1,2}$. For simplicity, we may assume there exists $\mathbf{A} \in L^6 \cap \mathring{W}^{1,2}$ such that $\mathbf{A}_j \xrightarrow{w} \mathbf{A}$ in L^6 and $\mathring{W}^{1,2}$. From the compact embedding $W^{1,2}(\Omega) \to L^p(\Omega)$ (where $1 \leqq p < 6$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain) we may further assume

$$\mathbf{A}_j \xrightarrow{s} \mathbf{A} \quad \text{in} \quad L^p(\Omega)\,(1 \leqq p < 6).$$

From

$$\begin{aligned} |D_{A_j} \phi_j|^2 &\geqq \frac{1}{2} |\nabla \phi_j|^2 - 3|\mathbf{A}_j \phi_j|^2 \\ &\geqq \frac{1}{2} |\nabla \phi_j|^2 - \frac{3}{2} (|\mathbf{A}_j|^4 + (|\phi_j|^2 - 1)^2) - 3|\mathbf{A}_j|^2 \end{aligned}$$

and (2.5) we conclude that $\{\nabla \phi_j\}$ is bounded in $L^2(\Omega)$. Moreover, it follows from $|\phi_j|^4 \leqq 2(|\phi_j|^2 - 1)^2 + 1$ and (2.5) that $\{\phi_j\}$ is bounded in $L^4(\Omega)$. Hence, we may assume $\{\phi_j\}$ is weakly convergent in $W^{1,2}(\Omega)$. From a diagonal subsequence argument, we are easily convinced that one can find some $\phi \in W^{1,2}_{\text{loc}} \cap L^4_{\text{loc}}$ and assume $\phi_j \xrightarrow{w} \phi$ in $W^{1,2}(\Omega)$ for every bounded domain $\Omega \subset \mathbb{R}^3$. In particular, $\phi_j \xrightarrow{s} \phi$ in $L^p(\Omega)$ $(1 \leqq p < 6)$.

Consequently, for any bounded domain Ω,

$$I^{\Omega}(\mathbf{A}, \phi) \leqq \liminf_{j\to\infty} I^{\Omega}(\phi_j, \mathbf{A}_j),$$

where

$$I^{\Omega}(\mathbf{A}, \phi) = \frac{1}{2} \|\nabla \mathbf{A}\|^2_{L^2(\Omega)} + \frac{1}{2} \|D_A\phi\|^2_{L^2(\Omega)} + \frac{\lambda}{8} \|(|\phi|^2 - 1)\|^2_{L^2(\Omega)} - \int_{\Omega} \operatorname{curl}\mathbf{A} \cdot \mathbf{H}_{\text{ext}} d^3x .$$

On the other hand, $\forall \varepsilon > 0$, there is a bounded domain $\Omega_\varepsilon \subset \mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3 - \Omega} |\mathbf{H}_{\text{ext}}|^2 d^3x < \varepsilon, \qquad \Omega \supset \Omega_\varepsilon .$$

This implies

$$I^{\Omega}(\mathbf{A}_j, \phi_j) \leqq I(\mathbf{A}_j, \phi_j) + \varepsilon \|\nabla \mathbf{A}_j\|_{L^2},$$

and therefore,

$$I^{\Omega}(\mathbf{A}, \phi) \leqq I_m + \varepsilon M, \qquad \Omega \supset \Omega_\varepsilon, \tag{2.6}$$

where $M \equiv \sup_j \|\nabla \mathbf{A}_j\|_{L^2}$. But $\|\nabla \mathbf{A}\|_{L^2} \leqq \liminf_{j\to\infty} \|\nabla \mathbf{A}_j\|_{L^2}$, hence, (2.6) becomes

$$\frac{1}{2} \|\nabla \mathbf{A}\|^2_{L^2(\Omega)} + \frac{1}{2} \|D_A\phi\|^2_{L^2(\Omega)} + \frac{\lambda}{8} \|(|\phi|^2 - 1)\|^2_{L^2(\Omega)} \leqq I_m + \int_{\mathbb{R}^3} \operatorname{curl}\mathbf{A} \cdot \mathbf{H}_{\text{ext}} d^3x + 2\varepsilon M, \qquad \Omega \supset \Omega_\varepsilon .$$

Letting $\Omega \to \mathbb{R}^3$, we get, by the arbitrariness of $\varepsilon > 0$, the inequality $I(\mathbf{A}, \phi) \leqq I_m$. Therefore, the minimizer of I is found. $(\mathbf{A}, \phi)$ is the solution of the equations

$$\left.\begin{aligned} D_A^2\phi + \frac{\lambda}{2}(1 - |\phi|^2)\phi = 0, \\ -\nabla^2\mathbf{A} + \frac{i}{2}(\phi^* D_A\phi - \phi(D_A\phi)^*) = \operatorname{curl}\mathbf{H}_{\text{ext}} . \end{aligned}\right\} \tag{2.7}$$

Standard elliptic regularity argument proves that $(\mathbf{A}, \phi) \in [W^{2,2}_{\text{loc}} \cap \mathring{W}^{1,2}] \times W^{4,2}_{\text{loc}}$. If $\mathbf{H}_{\text{ext}}$ is smooth, so is $(\mathbf{A}, \phi)$.

Since for $\mathbf{A} \in K$, $\|\nabla \mathbf{A}\|_{L^2} = \|\operatorname{curl}\mathbf{A}\|_{L^2}$, so $I_m \leqq E_m$. In order to show $(\mathbf{A}, \phi)$ is a solution to both the Eqs. (2.1) and the minimization problem (2.2), we have only to verify that $\mathbf{A} \in K$.

Indeed, let $u \in C_0^\infty(\mathbb{R}^3)$ be an arbitrary scalar test function. We have, using (2.7b), the identity $\nabla \cdot (\psi^* D_A\phi) = (D_A\psi)^* \cdot (D_A\phi) + \psi^* D_A^2\phi$, and then Eq. (2.7a),

$$\int_{\mathbb{R}^3} \nabla(\nabla \cdot \mathbf{A}) \cdot \nabla u d^3x = \int_{\mathbb{R}^3} (\nabla^2\mathbf{A}) \cdot \nabla u d^3x$$

$$= \int_{\mathbb{R}^3} \left\{ \frac{i}{2} (\nabla \cdot [\phi(D_A\phi)^* - \phi^* D_A\phi]) u - \mathbf{H}_{\text{ext}} \cdot (\operatorname{curl} \nabla u) \right\} d^3x = 0 .$$

Hence $\nabla \cdot \mathbf{A}$ is a harmonic function on $\mathbb{R}^3$. But $\nabla \cdot \mathbf{A} \in L^2$, therefore we must have $\nabla \cdot \mathbf{A} \equiv 0$, namely, $\mathbf{A}$ is globally in the Coulomb gauge. This proves Theorem 2.1.

If the external field is absent, the energy minimizing solutions are no other than the superconducting vacua. However, it can be observed that the presence of a small external field will change this situation: the lowest energy level will no longer contain the vacuum solutions.

In fact, let $\Omega \subset \mathbb{R}^3$ be a small bounded domain so that the first eigenvalue λ_1 of the problem $\Delta u + \lambda u = 0$, $u|_{\partial\Omega} = 0$ satisfies $\lambda_1 > 1$. Suppose $\mathbf{H}_{\text{ext}}$ is produced from a vector potential $\mathbf{A}_{\text{ext}} \in C_0^\infty(\Omega)$: $\mathbf{H}_{\text{ext}} = \operatorname{curl} \mathbf{A}_{\text{ext}}$, where $\nabla \cdot \mathbf{A}_{\text{ext}} = 0$ but $\mathbf{A}_{\text{ext}} \not\equiv 0$. For $\mathbf{A} = \mathbf{A}_{\text{ext}}$, $\phi = 1$ we have

$$E(\mathbf{A}, \phi) = \frac{1}{2}(\|\mathbf{A}_{\text{ext}}\|_{L^2}^2 - \|\operatorname{curl} \mathbf{A}_{\text{ext}}\|_{L^2}^2) < 0 .$$

Therefore $E_m < 0$ and the lowest energy level does not contain the vacuum solutions.

In the subsequent sections, we will not restrict ourselves to the energy minimizing solutions obtained in Theorem 2.1. The results concerning asymptotic decay and so on are proved for finite energy solutions. To simplify the statements, we shall always assume the solutions are sufficiently smooth. This assumption can be ensured by requiring that the external magnetic field $\mathbf{H}_{\text{ext}}$ be sufficiently smooth.

3. Boundedness of the Order Parameter

Our asymptotic decay results depend on the following pointwise boundedness of the order parameter ϕ. The approach here follows the main line of Taubes [17].

Lemma 3.1. *If* $(\mathbf{A}, \phi)$ *is a finite energy solution of the Ginzburg-Landau equations* (2.1), *then* $|\phi| \leqq 1$ *on* $\mathbb{R}^3$.

Proof. Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$. Then from Eq. (2.1a), for any $\psi \in W_0^{1,2}(\Omega)$, we have

$$\operatorname{Re} \int_{\mathbb{R}^3} d^3x \left\{ D_A\psi \cdot (D_A\phi)^* + \frac{\lambda}{2}(|\phi|^2 - 1)\psi\phi^* \right\} = 0 . \tag{3.1}$$

Define a function $\eta \in C_0^\infty(\mathbb{R}^1)$ with the properties

$$0 \leqq \eta \leqq 1 , \qquad \eta(s) = \begin{cases} 1 , & |s| \leqq 2 , \\ 0 , & |s| \geqq 3 . \end{cases}$$

Introduce a family of cutoff functions $\eta_\varrho(x) = \eta(|x|/\varrho)$, $x \in \mathbb{R}^3$, $\varrho > 0$. Set $\Omega_\varrho = \{x \in \mathbb{R}^3 \,|\, |x| < \varrho\}$.

Suppose $\Omega_\varrho^+ = \{x \in \Omega_\varrho \,|\, |\phi(x)| > 1\} \neq \emptyset$ if $\varrho \geqq$ some ϱ_0. For $\varrho \geqq \varrho_0$ define $\psi_\varrho \in W_0^{1,2}(\Omega_{3\varrho})$ by

$$\psi_\varrho(x) = \eta_\varrho(x)(|\phi(x)| - 1)^+ \frac{\phi(x)}{|\phi(x)|} ,$$

where $(|\phi(x)|-1)^+ \equiv \max\{|\phi(x)|-1, 0\}$. Define $f=\phi/|\phi|$ on $(\mathbb{R}^3)^+ \equiv \{x\in\mathbb{R}^3 \,|\, |\phi(x)|>1\}$. Then $f^*f=1$ and on $\Omega_{3\varrho}^+$,

$$D_A\psi_\varrho = \nabla\eta_\varrho(|\phi|-1)f + ((\nabla|\phi|)f + (|\phi|-1)D_Af)\eta_\varrho .$$

Replacing ψ in (3.1) by ψ_ϱ, we have, by using the simple relations $D_A\phi=(\nabla|\phi|)f + |\phi|D_Af$, $\mathrm{Re}[f(D_Af)^*]=0$,

$$\int_{\Omega_{3\varrho}^+} d^3x\{|\nabla|\phi||^2\eta_\varrho + (|\phi|-1)\nabla\eta_\varrho\cdot\nabla|\phi| + |\phi|(|\phi|-1)\eta_\varrho|D_Af|^2 + \frac{\lambda}{2}(|\phi|-1)^2(|\phi|+1)|\phi|\eta_\varrho\}=0 . \tag{3.2}$$

From $(|\phi|-1)\leqq(|\phi|^2-1)$ (on $\Omega_{3\varrho}^+$) and the Schwarz inequality we have

$$\left|\int_{\Omega_{3\varrho}^+}(|\phi|-1)\nabla\eta_\varrho\cdot\nabla|\phi|d^3x\right| \leqq \left(\int_{\mathbb{R}^3}(|\phi|^2-1)^2d^3x\right)^{1/2}\left(\int_{\Omega_{3\varrho}^+}|\nabla\eta_\varrho\cdot\nabla|\phi||^2d^3x\right)^{1/2} . \tag{3.3}$$

But, away from the zeros of ϕ,

$$\nabla|\phi| = \frac{1}{2|\phi|}(\phi^*D_A\phi + \phi(D_A\phi)^*) .$$

Hence

$$|\nabla|\phi|| \leqq |D_A\phi| . \tag{3.4}$$

From the definition of η_ϱ, we have

$$|\nabla\eta_\varrho| \leqq \frac{C}{\varrho} ,$$

where $C>0$ is a constant independent of $\varrho>0$. Inserting the above inequality into (3.3) and using (3.4) we find

$$\left|\int_{\Omega_{3\varrho}^+}(|\phi|-1)\nabla\eta_\varrho\cdot\nabla|\phi|d^3x\right| \leqq \frac{C_1}{\varrho}\|(|\phi|^2-1)\|_{L^2}\|D_A\phi\|_{L^2} . \tag{3.5}$$

Combining (3.2), (3.5), and the inequality

$$\frac{1}{4}\|\mathbf{H}\|_{L^2}^2 + \frac{1}{2}\|D_A\phi\|_{L^2}^2 + \frac{\lambda}{8}\|(|\phi|^2-1)\|_{L^2}^2 \leqq E(\mathbf{A},\phi) + \|\mathbf{H}_{\mathrm{ext}}\|_{L^2}^2 , \tag{3.6}$$

we obtain

$$\int_{\Omega_{3\varrho}^+} d^3x\{|\nabla|\phi||^2 + (|\phi|-1)|\phi||D_Af|^2 + \frac{\lambda}{2}(|\phi|-1)^2(|\phi|+1)|\phi|\}\eta_\varrho \leqq \frac{4C_1}{\varrho}(E(\mathbf{A},\phi) + \|\mathbf{H}_{\mathrm{ext}}\|_{L^2}^2) .$$

Letting $\varrho\to\infty$ one finds $\mathrm{mes}((\mathbb{R}^3)^+)=0$. Hence $|\phi|\leqq 1$ on $\mathbb{R}^3$. This contradiction shows that $(\mathbb{R}^3)^+=\emptyset$. Hence the proof of Lemma 3.1 is complete.

4. Decay of $1-|\phi|^2$

Let $D_k\phi=\partial_k\phi-iA_k\phi$. We have the identity

$$D_kD_j\phi-D_jD_k\phi=-iF_{kj}\phi\,,\qquad F_{kj}=\partial_kA_j-\partial_jA_k\,. \tag{4.1}$$

Define $g=D_A\phi$, $g_k=D_k\phi$. Using (4.1) and (2.1) one finds

$$\begin{aligned}D_A^2g_l=D_kD_kg_l=&-\frac{\lambda}{2}(1-|\phi|^2)g_l+\frac{\lambda+1}{2}|\phi|^2g_l\\&+\frac{\lambda-1}{2}\phi^2g_l^*-2iF_{kl}g_k+i(\operatorname{curl}\mathbf{H}_{\text{ext}})_l\phi\,,\end{aligned} \tag{4.2}$$

where the summation convention over repeated indices has been observed.

Lemma 4.1. *If* $(\mathbf{A},\phi)$ *is a finite energy solution of Eqs.* (2.1) *with* $\mathbf{H}_{\text{ext}}\in W^{1,2}$, *then* $|g|\in W^{1,2}$. *Hence, as an immediate consequence of* (2.4) *and a standard interpolation inequality, we have* $|g|\in L^p(2\leqq p\leqq 6)$.

Proof. Let η_ϱ be the cutoff function defined in Sect. 3. Multiplying both sides of (4.2) by $\eta_\varrho^2g_l^*$ and integrating by parts, we have

$$\begin{aligned}\text{the left-hand-side}&=\int_{\mathbb{R}^3}\{\eta_\varrho^2g_l^*D_kD_kg_l\}\,d^3x\\&=-\int_{\mathbb{R}^3}\eta_\varrho^2|D_kg_l|^2d^3x-\frac{2}{\varrho}\int_{\mathbb{R}^3}(\partial_k\eta)\left(\frac{x}{\varrho}\right)g_l^*(\eta_\varrho D_kg_l)\,d^3x\,,\end{aligned}$$

$$\begin{aligned}|\text{the right-hand-side}|\leqq&\frac{3\lambda+2}{2}\|g\|_{L^2}^2+2\int_{\mathbb{R}^3}|F_{kl}|\,|\eta_\varrho^2g_kg_l|\,d^3x\\&+\int_{\mathbb{R}^3}|(\operatorname{curl}\mathbf{H}_{\text{ext}})_l|\,|g_l|\,d^3x\,.\end{aligned}$$

Therefore, by virtue of (3.6) and a simple interpolation inequality, we obtain the bound

$$\int_{\mathbb{R}^3}\eta_\varrho^2|D_kg_l|^2d^3x\leqq C_1+2\int_{\mathbb{R}^3}|F_{kl}|\,|\eta_\varrho^2g_kg_l|\,d^3x\,, \tag{4.3}$$

where $C_1>0$ is a constant depending on $E(\mathbf{A},\phi)$, $\|\mathbf{H}_{\text{ext}}\|_{W^{1,2}}$, and λ but independent of $\varrho\geqq 1$.

From the Hölder inequality we have

$$\int_{\mathbb{R}^3}|F_{kl}|\eta_\varrho^2|g_kg_l|\,d^3x\leqq 2\int_{\mathbb{R}^3}|\mathbf{H}|\,|g|^{1/2}|\eta_\varrho g|^{3/2}d^3x\leqq 2\|\mathbf{H}\|_{L^2}\|g\|_{L^2}^{1/2}\|\eta_\varrho g\|_{L^6}^{3/2}; \tag{4.4}$$

from (2.4) we have

$$\|\eta_\varrho g\|_{L^6}\leqq C\|\nabla\eta_\varrho|g|\,\|_{L^2}\leqq C[\|\eta_\varrho\nabla|g|\,\|_{L^2}+\|\nabla\eta_\varrho\|_{L^\infty}\|g\|_{L^2}]\,. \tag{4.5}$$

Away from the zeros of g_l, for fixed $l=1,2,3$,

$$|\partial_k|g_l||=\frac{1}{2|g_l|}|g_l^*D_kg_l+g_l(D_kg_l)^*|\leqq|D_kg_l|\,. \tag{4.6}$$

Combining (4.4), (4.5), and (4.6) we find

$$\int_{\mathbb{R}^3}|F_{kl}|\eta_\varrho^2|g_kg_l|\,d^3x\leqq C_2+C_3\|\eta_\varrho D_Ag\|_{L^2}^{3/2}\,, \tag{4.7}$$

where $C_2, C_3 > 0$ depend on $E(\mathbf{A}, \phi)$, $\|\mathbf{H}_{\text{ext}}\|_{W^{1,2}}$, and λ but are independent of $\varrho \geqq 1$.

Inserting (4.7) into (4.3) and using a simple interpolation inequality, we have

$$\int_{\mathbb{R}^3} \eta_\varrho^2 |D_A g|^2 d^3x \leqq C_4,$$

where $C_4 > 0$ only depends on C_2, C_3. Letting $\varrho \to \infty$ in the above inequality one obtains $|D_A g| \in L^2$. Using (4.6) again we conclude that $|g| \in W^{1,2}$. This proves Lemma 4.1.

Lemma 4.2. *For any $u \in W^{1,p}$, $p > 3$, we have $u \to 0$ as $|x| \to \infty$.*

For a proof of this lemma, see, for example, Jaffe and Taubes [9].

Theorem 4.3. *If $(\mathbf{A}, \phi)$ is a finite energy solution of the Ginzburg-Landau equations* (2.1) *with the external field satisfying $\mathbf{H}_{\text{ext}} \in W^{1,2}$, then $1 - |\phi|^2 \to 0$ as $|x| \to \infty$ and $|\phi| < 1$ on $\mathbb{R}^3$ or otherwise $|\phi| \equiv 1$.*

Proof. Set $w = 1 - |\phi|^2$. Then, by virtue of Lemma 3.1,

$$|\nabla w| = |\phi^* g + \phi g^*| \leqq 2|g|.$$

Hence, applying Lemma 4.1, we see that $\nabla w \in L^6 \cap L^2$. From (2.4) we obtain $w \in W^{1,6}$. Hence $w \to 0$ as $|x| \to \infty$ (Lemma 4.2).

Finally, from Eq. (2.1a) and the relation

$$\nabla^2 |\phi|^2 = \phi (D_A^2 \phi)^* + \phi^* D_A^2 \phi + 2|D_A \phi|^2,$$

we get $\nabla^2 w \leqq \lambda |\phi|^2 w$. Since $w \to 0$ as $|x| \to \infty$, so, using the maximum principle, we have $w > 0$ on $\mathbb{R}^3$ or otherwise $w \equiv 0$. This completes the proof of Theorem 4.3.

5. Decay of H and $D_A\phi$

We shall put some additional assumptions on the external field $\mathbf{H}_{\text{ext}}$ to ensure the decay of $\mathbf{H}$ and $D_A \phi$.

Theorem 5.1. *Suppose $\mathbf{H}_{\text{ext}} \in W^{2,2}$. If $(\mathbf{A}, \phi)$ is a finite energy solution of Eqs.* (2.1), *then $\mathbf{H} = \operatorname{curl} \mathbf{A} \in W^{2,2}$. In particular, $\mathbf{H} \to 0$ as $|x| \to \infty$.*

Proof. Since $\operatorname{curl}^2 \mathbf{A} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$, we have $\operatorname{curl}^3 \mathbf{A} = -\nabla^2 \mathbf{H}$. As a consequence, applying the operator curl to Eq. (2.1b) one finds

$$\nabla^2 \mathbf{H} = |\phi|^2 \mathbf{H} + i(D_A \phi)^* \times D_A \phi - \operatorname{curl}^2 \mathbf{H}_{\text{ext}}. \tag{5.1}$$

Let η_ϱ be the cutoff function defined in Sect. 3. Then for fixed $j = 1, 2, 3$,

$$\int_{\mathbb{R}^3} \eta_\varrho^2 H_j \nabla^2 H_j d^3x = -2 \int_{\mathbb{R}^3} (\nabla \eta_p \cdot \nabla H_j) \eta_\varrho H_j d^3x - \int_{\mathbb{R}^3} \eta_\varrho^2 |\nabla H_j|^2 d^3x.$$

Hence, by the Schwarz inequality,

$$\int_{\mathbb{R}^3} \eta_\varrho^2 |\nabla H_j|^2 d^3x \leqq C \|H_j\|_{L^2}^2 + \|H_j\|_{L^2} \|\nabla^2 H_j\|_{L^2}, \tag{5.2}$$

where C is a constant independent of $\varrho \geqq 1$. From (3.6), Lemma 4.1, and (5.1) we see that $\nabla^2 \mathbf{H} \in L^2$. Letting $\varrho \to \infty$ in (5.2) we get $\nabla H_j \in L^2$. Hence $\mathbf{H} \in W^{1,2}$.

On the other hand, since $\eta_\varrho \mathbf{H} \in W^{2,2}$, by the well-known L^2 estimates (cf. Ladyzhenskaya [11]) we obtain

$$\|\eta_\varrho \mathbf{H}\|_{W^{2,2}} \leqq C_1\{\|\nabla^2 \eta_\varrho \mathbf{H}\|_{L^2} + \|\mathbf{H}\|_{L^2}\} \leqq C_2\{\|\nabla^2 \mathbf{H}\|_{L^2} + \|\mathbf{H}\|_{W^{1,2}}\},$$

where $C_1, C_2 > 0$ are constants independent of $\varrho \geqq 1$. Letting $\varrho \to \infty$ in the above inequality one finds $\mathbf{H} \in W^{2,2}$.

From the Sobolev embedding

$$W^{2,2} \to W^{1,p}, \quad 1 < p < 6 \tag{5.3}$$

and Lemma 4.2, we have $\mathbf{H} \to 0$ as $|x| \to \infty$. This proves the theorem.

Theorem 5.2. *Under the assumption of Theorem 5.1, if, moreover, $\mathbf{A} \in K$, namely, the solution is in the Coulomb gauge, then $D_A\phi \in W^{2,2}$ and hence $D_A\phi \to 0$ as $|x| \to \infty$.*

Proof. From (2.4) we see that $\mathbf{A} \in L^6$. Now rewrite (4.2) as follows:

$$\begin{aligned}\nabla^2 g_l &= 2i\mathbf{A} \cdot \nabla g_l + |\mathbf{A}|^2 g_l + \frac{\lambda}{2}(|\phi|^2 - 1)g_l \\ &\quad + \frac{\lambda+1}{2}|\phi|^2 g_l + \frac{\lambda-1}{2}\phi^2 g_l^* - 2iF_{kl}g_k + i(\operatorname{curl}\mathbf{H}_{\text{ext}})_l\phi\,.\end{aligned} \tag{5.4}$$

Let η_ϱ be the cutoff function introduced in Sect. 3. Multiplying both sides of (5.4) by $\eta_\varrho^2 g_l^*$ and integrating by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \eta_\varrho^2 |\nabla g_l|^2 d^3x &\leqq 2\int_{\mathbb{R}^3} \eta_\varrho |\nabla \eta_\varrho \cdot \nabla g_l|\,|g_l| d^3x \\ &\quad + 2\int_{\mathbb{R}^3} \eta_\varrho^2 |\mathbf{A} \cdot \nabla g_l|\,|g_l| d^3x + \|\,|\mathbf{A}|g_l\|_{L^2}^2 \\ &\quad + (2\lambda+1)\|g_l\|_{L^2}^2 + 2\|\mathbf{H}|g|^2\|_{L^1} + \|\mathbf{H}_{\text{ext}}\|_{W^{1,2}}\|g\|_{L^2}\,.\end{aligned} \tag{5.5}$$

Using the inequalities

$$\int_{\mathbb{R}^3} \eta_\varrho^2 |\mathbf{A} \cdot \nabla g_l|\,|g_l| d^3x \leqq \|\mathbf{A}\|_{L^6} \|g_l\|_{L^3} \|\eta_\varrho |\nabla g_l|\,\|_{L^2},$$

$$\|\,|\mathbf{A}|g_l\|_{L^2}^2 \leqq \|\mathbf{A}\|_{L^6}^2 \|g\|_{L^3}^2,$$

$$\|\mathbf{H}|g|^2\|_{L^1} \leqq \|\mathbf{H}\|_{L^2} \|g\|_{L^4}^2,$$

and the fact that $\mathbf{A} \in L^6$, $g \in L^p (2 \leqq p \leqq 6)$ in (5.5) we can find a constant C independent of $\varrho \geqq 1$ such that $\|\eta_\varrho |\nabla g_l|\,\|_{L^2} \leqq C$. Letting $\varrho \to \infty$ one gets $\nabla g_l \in L^2$. This proves $g \in W^{1,2}$.

Now, since $\eta_\varrho g_l$ is of compact support, we can find an absolute constant $C_1 > 0$ such that

$$\begin{aligned}\|\eta_\varrho g_l\|_{W^{2,2}} &\leqq C_1\{\|\nabla^2 \eta_\varrho g_l\|_{L^2} + \|\eta_\varrho g_l\|_{L^2}\} \\ &\leqq C_2\{\|\eta_\varrho \nabla^2 g_l\|_{L^2} + \|g_l\|_{W^{1,2}}\},\end{aligned} \tag{5.6}$$

where C_2 is a constant independent of $\varrho \geqq 1$.

Except for the first term on the right-hand-side of (5.4), all other terms belong to L^2. Hence, using the decomposition

$$\eta_\varrho \mathbf{A} \cdot \nabla g_l = \mathbf{A} \cdot \nabla(\eta_\varrho g_l) - g_l \mathbf{A} \cdot \nabla \eta_\varrho$$

and (5.6) and (5.4), we obtain the bound

$$\|\eta_\varrho g_l\|_{W^{2,2}} \leqq C_3 \|\mathbf{A}\cdot\nabla(\eta_\varrho g_l)\|_{L^2} + C_4, \tag{5.7}$$

where $C_3, C_4 > 0$ are constants depending on $\|g_l\|_{W^{1,2}}$, $\|\mathbf{A}\|_{L^6}$, $\|\mathbf{H}\|_{L^2}$, $\|\mathbf{H}_{\text{ext}}\|_{W^{1,2}}$ but not on $\varrho \geqq 1$.

On the other hand, using the Hölder inequality, we have

$$\|\mathbf{A}\cdot\nabla(\eta_\varrho g_l)\|_{L^2} \leqq \|\mathbf{A}\|_{L^6}\|\nabla(\eta_\varrho g_l)\|_{L^3},$$
$$\|\nabla(\eta_\varrho g_l)\|_{L^3} \leqq \|\nabla(\eta_\varrho g_l)\|_{L^2}^{1/3}\|\nabla(\eta_\varrho g_l)\|_{L^4}^{2/3};$$

using the Sobolev embedding inequality, we have

$$\|\nabla(\eta_\varrho g_l)\|_{L^4} \leqq \|\eta_\varrho g_l\|_{W^{1,4}} \leqq C_5\|\eta_\varrho g_l\|_{W^{2,2}},$$

where $C_5 > 0$ is an absolute constant. As a consequence, one finds, after inserting the above estimates into (5.7), the inequality

$$\|\eta_\varrho g_l\|_{W^{2,2}} \leqq C_6\|\mathbf{A}\|_{L^6}\|g_l\|_{W^{1,2}}^{1/3}\|\eta_\varrho g_l\|_{W^{2,2}}^{2/3} + C_4, \tag{5.8}$$

where C_6 is independent of $\varrho \geqq 1$. It then yields from using a simple interpolation inequality and letting $\varrho \to \infty$ that $g_l \in W^{2,2}$.

The behavior $g \to 0$ as $|x| \to \infty$ follows from (5.3) and Lemma 4.2.

6. Exponential Decay Estimates

We shall show in this section if $\mathbf{H}_{\text{ext}}$ decays exponentially fast at infinity in a sense to be made precise shortly (in particular, if $\mathbf{H}_{\text{ext}}$ is compactly supported), so do $\mathbf{H}$, $D_A\phi$, and $1-|\phi|^2$. This means the interaction is now of a local character.

Let $(\mathbf{A}, \phi)$ be a finite energy solution of the Ginzburg-Landau equations (2.1) with $\mathbf{H}_{\text{ext}} \in W^{2,2}$ and $\mathbf{A} \in K$ to ensure the decay properties established in Sect. 5.

From the identity

$$\nabla^2|g|^2 = 2|D_A g|^2 + g^* \cdot D_A^2 g + g\cdot(D_A^2 g)^*$$

and Eq. (4.2) we obtain

$$\begin{aligned}
\nabla^2|g|^2 &= 2|D_A g|^2 - \lambda(1-|\phi|^2)|g|^2 + (\lambda+1)|\phi|^2|g|^2 \\
&\quad + \frac{\lambda-1}{2}(\phi^2 g^*\cdot g^* + (\phi^*)^2 g\cdot g) - 2iF_{kl}(g_l^* g_k - g_l g_k^*) + i(\operatorname{curl}\mathbf{H}_{\text{ext}})_l(g_l^* - g_l)\phi \\
&\geqq [(\lambda+1) - |\lambda-1|]\,|\phi|^2|g|^2 - \{\lambda(1-|\phi|^2) + 4|\mathbf{H}|\}\,|g|^2 - 2|\operatorname{curl}\mathbf{H}_{\text{ext}}|\,|g| \\
&\geqq 2\min\{\lambda,1\}\left(|\phi|^2 - \frac{\varepsilon}{3}\right)|g|^2 - \{\lambda(1-|\phi|^2) \\
&\quad + 4|\mathbf{H}|\}\,|g|^2 - C(\varepsilon,\lambda)\,|\operatorname{curl}\mathbf{H}_{\text{ext}}|^2,
\end{aligned} \tag{6.1}$$

where $\varepsilon \in (0,1)$ is arbitrary.

Since $1-|\phi|^2$, $\mathbf{H} \to 0$ as $|x| \to \infty$, from (6.1) we see that a sufficiently large $\varrho > 0$ can be chosen to make

$$\nabla^2|g|^2 \geqq 2\min\{\lambda,1\}(1-\varepsilon)\,|g|^2 - C(\varepsilon,\lambda)\,|\operatorname{curl}\mathbf{H}_{\text{ext}}|^2 \tag{6.2}$$

for $x \in \mathbb{R}^3 - \Omega_\varrho$.

Assume now $\operatorname{curl}\mathbf{H}_{\text{ext}}$ decays exponentially:

$$|\operatorname{curl}\mathbf{H}_{\text{ext}}| \leqq C_1 e^{-\mu|x|}, \quad x \in \mathbb{R}^3, \quad C_1 > 0, \quad \mu > 0. \tag{6.3}$$

Set

$$\sigma(x) = C_2 e^{-(1-\varepsilon)^{1/2} m|x|},$$

where C_2, $m > 0$ are to be determined. We have, using (6.2),

$$\begin{aligned} \nabla^2(\sigma - |g|^2) &\leqq m^2\sigma - 2(1-\varepsilon)\min\{\lambda, 1\}\,|g|^2 \\ &\quad - m^2\varepsilon C_2 e^{-(1-\varepsilon)^{1/2} m|x|} + C_3 e^{-2\mu|x|} \\ &\leqq 2(1-\varepsilon)\min\{\lambda, 1\}\,(\sigma - |g|^2) - (m^2\varepsilon C_2 - C_3) e^{-(1-\varepsilon)^{1/2} m|x|}, \end{aligned} \tag{6.4}$$

provided we choose

$$m = \min\{\sqrt{2}(1-\varepsilon)^{1/2}\min\{\lambda^{1/2}, 1\}, 2(1-\varepsilon)^{-1/2}\mu\}. \tag{6.5}$$

Take $C_2 > 0$ sufficiently large to make $m^2\varepsilon C_2 - C_3 \geqq 0$. Then (6.4) becomes

$$\nabla^2(\sigma - |g|^2) \leqq 2(1-\varepsilon)\min\{\lambda, 1\}\,(\sigma - |g|^2). \tag{6.6}$$

Also, we may choose C_2 large enough to make

$$(\sigma - |g|^2)|_{|x| = \varrho} \geqq 0.$$

Since $\sigma - |g|^2 \to 0$ as $|x| \to \infty$, applying the maximum principle in (6.6) we have

$$|g|^2 \leqq \sigma = C_2 e^{-(1-\varepsilon)^{1/2} m|x|}, \quad |x| > \varrho, \tag{6.7}$$

where m is determined through (6.5).

From Eq. (2.1a), we easily find

$$\nabla^2 w = \lambda|\phi|^2 w - 2|g|^2,$$

where $w = 1 - |\phi|^2$. We can use a similar argument as that in the derivation of the exponential decay estimate for $|g|$ to obtain the bound

$$w \leqq C_3 e^{-(1-\varepsilon)^{1/2}\tilde{m}|x|}, \quad x \in \mathbb{R}^3,$$

where $C_3 > 0$ is a constant and $\tilde{m} = (1-\varepsilon)^{1/2}\min\{\lambda^{1/2}, m\}$.

The exponential decay of $\mathbf{H}$ can be deduced from (5.1) under the additional assumption

$$|\operatorname{curl}^2\mathbf{H}_{\text{ext}}| \leqq C_4 e^{-\gamma|x|}, \quad x \in \mathbb{R}^3, \tag{6.8}$$

where C_4, $\gamma > 0$ are constants.

Indeed, (5.1) gives us the inequality

$$\begin{aligned} \nabla^2|\mathbf{H}|^2 &\geqq 2|\phi|^2|\mathbf{H}|^2 - 2|g|^2|\mathbf{H}| - 2|\mathbf{H}|\,|\operatorname{curl}^2\mathbf{H}_{\text{ext}}| \\ &\geqq (2|\phi|^2 - \varepsilon)|\mathbf{H}|^2 - \frac{2}{\varepsilon}(|g|^4 + |\operatorname{curl}^2\mathbf{H}_{\text{ext}}|^2). \end{aligned} \tag{6.9}$$

Consequently, using (6.7) and (6.8) in (6.9) and arguing as before we can obtain the estimate

$$|\mathbf{H}|^2 \leqq C_5 e^{-(1-\varepsilon)^{1/2}\hat{m}|x|}, \quad x \in \mathbb{R}^3,$$

where $\hat{m} = 2\min\{(1-\varepsilon), (1-\varepsilon)^{-1/2}\gamma, m\}$ and $C_5 > 0$ is a constant depending on $\varepsilon \in (0,1)$.

In summary, we have

Theorem 6.1. *Let* $(\mathbf{A}, \phi)$ *be a finite energy solution of Eqs.* (2.1) *with* $\mathbf{A} \in K$ and $\mathbf{H}_{\text{ext}} \in W^{2,2}$.

(a) *If* $\mathbf{H}_{\text{ext}}$ *decays according to* (6.3), *then for any* $\varepsilon \in (0,1)$ *there is a constant* $C(\varepsilon) > 0$ *such that*

$$0 \leqq 1 - |\phi|^2 \leqq C(\varepsilon) e^{-(1-\varepsilon)m_1|x|}, \qquad |D_A \phi| \leqq C(\varepsilon) e^{-(1-\varepsilon)m_2|x|},$$

where $x \in \mathbb{R}^3$, $m_1 = \min\{\lambda^{1/2}, 2m_2\}$, *and* $m_2 = \min\{\lambda^{1/2} 2^{-1/2}, 2^{-1/2}, \mu\}$.

(b) *If, in addition,* $\mathbf{H}_{\text{ext}}$ *satisfies the decay property* (6.8), *then for any* $\varepsilon \in (0,1)$ *there is a constant* $C(\varepsilon) > 0$ *such that*

$$|\mathbf{H}| \leqq C(\varepsilon) e^{-(1-\varepsilon)m_3|x|}, \qquad x \in \mathbb{R}^3,$$

where $m_3 = \min\{1, 2m_2, \gamma\}$.

Note. (a) If $\mathbf{H}_{\text{ext}}$ is of compact support, then, in the above decay estimates, $m_2 = 2^{-1/2}\min\{\lambda^{1/2}, 1\}$ and $m_3 = \min\{1, 2m_2\}$.

(b) If $\mathbf{H}_{\text{ext}}$ satisfies both (6.3) and (6.8), then the solutions produced by Theorem 2.1 enjoy the above exponential decay property.

In the following we make a brief discussion about the flux quantization problem typical in superconductivity theory.

Suppose that $\mathbf{H}_{\text{ext}}$ decays according to (6.3). Let $(\mathbf{A}, \phi)$ be a finite energy solution satisfying the general assumption in Theorem 6.1. Let M be a surface in $\mathbb{R}^3$. We shall call M an extended surface, if M is noncompact, orientable, without boundary, and there is a sufficiently large number $\varrho_0 > 0$ such that $M \cap B_\varrho$ is a 2-manifold with boundary $\partial(M \cap B_\varrho) = M \cap \partial B_\varrho$ for all $\varrho \geqq \varrho_0$ and the total length of the curve $\partial(M \cap B_\varrho)$ does not grow faster than the exponential functions $e^{\delta\varrho} (\delta > 0)$ as $\varrho \to \infty$, where $B_\varrho = \{x \in \mathbb{R}^3 \,|\, |x| < \varrho\}$.

Theorem 6.2. *Let* M *be an extended surface in* $\mathbb{R}^3$ *and consider the normalized excited magnetic flux passing through* M *defined by the integral*

$$\Phi_M = \frac{1}{2\pi} \int_M \mathbf{H} \cdot d\mathbf{S}.$$

(a) Φ_M *is an integer.*

(b) *If* M *can continuously be deformed into a plane, then* $\Phi_M = 0$.

Proof. Assume $\varrho \geqq \varrho_0$ is sufficiently large so that $|\phi| > \frac{1}{2}$ for $|x| = \varrho$. Let $\partial(M \cap B_\varrho)$ take the inherited orientation from M. Using the Gauss formula and Theorem 6.1 (a) we obtain

$$\begin{aligned} \left| \int_{M \cap B_\varrho} \mathbf{H} \cdot d\mathbf{S} + i \int_{\partial(M \cap B_\varrho)} d \ln \phi \right| &\leqq \left| \int_{\partial(M \cap B_\varrho)} \phi^{-1} D_A \phi \cdot d\mathbf{x} \right| \\ &\leqq 2C(\varepsilon) \left| \int_{\partial(M \cap B_\varrho)} dl \right| e^{-(1-\varepsilon)m_2\varrho}. \end{aligned} \tag{6.10}$$

On the other hand

$$-i \int_{\partial(M \cap B_\varrho)} d \ln \phi = \int_{\partial(M \cap B_\varrho)} d \arg \phi = 2\pi N,$$

where N is an integer. Letting $\varrho\to\infty$ in (6.10) we see that the proof for part (a) is complete.

Finally, since Φ_M continuously depends on M and Φ_M is an integer, therefore Φ_M is invariant under any continuous deformation of M. If M is a plane, we can rotate M to obtain M^-, namely, the same plane with opposite orientation. Hence $\Phi_M=\Phi_{M^-}$. This implies $\Phi_M=0$. Part (b) is proved.

Remarks. (a) Theorem 6.2 tells us that although the external flux passing through an extended surface M can take any value, the excited flux through M may only attain a number of quanta.

(b) The zero net flux property stated in Theorem 6.2(b) appears to be a special feature of the Ginzburg-Landau theory on $\mathbb{R}^3$. It may imply that magnetic strings in $\mathbb{R}^3$ are closed and vortices living on any cross section of $\mathbb{R}^3$ appear in pairs with opposite local winding numbers or topological charges.

7. The Case of an Arbitrary Source Current

In the presence of an arbitrary external source 3-current $\mathbf{J}_{\text{ext}}$, the Ginzburg-Landau equations become

$$\left.\begin{aligned} &D_A^2\phi+\frac{\lambda}{2}(1-|\phi|^2)\phi=0,\\ &\operatorname{curl}^2\mathbf{A}+\frac{i}{2}(\phi^*D_A\phi-\phi(D_A\phi)^*)=-\mathbf{J}_{\text{ext}}\end{aligned}\right\}\tag{7.1}$$

which are the equations of motion of the Lagrangian density

$$\mathscr{L}=\frac{1}{2}|\operatorname{curl}\mathbf{A}|^2+\frac{1}{2}|D_A\phi|^2+\frac{\lambda}{8}(|\phi|^2-1)^2+\mathbf{A}\cdot\mathbf{J}_{\text{ext}}.$$

Note that the left-hand-side of (7.1b) is divergence-free by virtue of (7.1a), therefore we must assume $\mathbf{J}_{\text{ext}}$ satisfies the natural consistency constraint

$$\nabla\cdot\mathbf{J}_{\text{ext}}=0.\tag{7.2}$$

Using (2.4) and the Hölder inequality, we easily see that the action $L=\int_{\mathbb{R}^3}\mathscr{L}d^3x$ is bounded from below on the space $K\times W^{1,2}_{\text{loc}}$ if $\mathbf{J}_{\text{ext}}\in L^{6/5}$. By a similar approach as that in the presence of an external magnetic field, one obtains

Theorem 7.1. *Suppose $\mathbf{J}_{\text{ext}}\in L^{6/5}\cap L^2_{\text{loc}}$ and satisfies the consistency condition (7.2) in the following weak sense:*

$$\int_{\mathbb{R}^3}\mathbf{J}_{\text{ext}}\cdot\nabla u d^3x=0,\qquad \forall u\in C_0^\infty(\mathbb{R}^3).$$

Then

(a) *Eqs. (7.1) have a least action solution $(\mathbf{A},\phi)$ in the space $K\times W^{1,2}_{\text{loc}}$; this solution is regular, that is, $(\mathbf{A},\phi)\in W^{2,2}_{\text{loc}}\times W^{4,2}_{\text{loc}}$. If $\mathbf{J}_{\text{ext}}$ is smooth, so is $(\mathbf{A},\phi)$.*

(b) *If $\mathbf{J}_{\text{ext}}\in L^2$, then $1-|\phi|^2\to0$ as $|x|\to\infty$ and $|\phi|<1$ on $\mathbb{R}^3$ or otherwise $|\phi|\equiv1$.*

(c) *If $\mathbf{J}_{\text{ext}}\in W^{1,2}$, then $\mathbf{H}=\operatorname{curl}\mathbf{A}$, $D_A\phi\in W^{2,2}$ and $\mathbf{H}$, $D_A\phi\to0$ as $|x|\to\infty$.*

(d) *Assume* $\mathbf{J}_{\mathrm{ext}} \in W^{1,2}$. *If* $|\mathbf{J}_{\mathrm{ext}}| \leqq C_2 e^{-\mu|x|}$, *then the exponential decay estimates in Theorem 6.1(a) hold for* $1-|\phi|^2$, $D_A\phi$. *Also, Theorem 6.2 holds here for the excited flux passing through extended surfaces in* $\mathbb{R}^3$. *In addition, if* $|\mathrm{curl}\,\mathbf{J}_{\mathrm{ext}}| \leqq C_2 e^{-\gamma|x|}$, *the decay estimate in Theorem 6.1(b) holds for* $\mathbf{H}$.

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