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The Hannay Angles: Geometry, Adiabaticity, and an Example

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Abstract. The Hannay angles were introduced by Hannay as a means of measuring a holonomy effect in classical mechanics closely corresponding to the Berry phase in quantum mechanics. Using parameter-dependent momentum mappings we show that the Hannay angles are the holonomy of a natural connection. We generalize this effect to non-Abelian group actions and discuss non-integrable Hamiltonian systems. We prove an averaging theorem for phase space functions in the case of general multi-frequency dynamical systems which allows us to establish the almost adiabatic invariance of the Hannay angles. We conclude by giving an application to celestial mechanics.

1. Introduction

Consider a classical system whose Hamiltonian $H(r)$ depends smoothly on a set of time-dependent parameters r. Hannay [21] and Berry [8] have shown that, under a closed adiabatic loop in the space of classically integrable Hamiltonians, the angle variables pick up extra angles, the *Hannay angles*, in addition to the time integral of the instantaneous frequencies. (Here the term *adiabatic* means that the time dependence of the parameters is assumed to be slow.) Hannay explains these angles by the fact that the action-angle coordinates $(J, \varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are parameter-dependent so that the canonical transformation to these coordinates produces an additional term in the Hamiltonian. More explicitly, let $H(p, q, r)$ be an integrable Hamiltonian for all fixed values of the parameters. When the parameters $r = r(\varepsilon t)$ change in time, $r(s+T) = r(s)$, dynamics is given by the time-dependent Hamiltonian

$$h = h_0(J, \varepsilon t) + \varepsilon h_1(J, \varphi, \varepsilon t), \tag{1}$$

where h_0 is just the original Hamiltonian expressed in action variables, whereas εh_1 arises from the time-derivative of the generating function of the canonical

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transformation to action-angle variables. As is well known, the actions are first integrals of the *averaged* system. Then the "adiabatic principle" asserts that in the adiabatic limit $\varepsilon \to 0$ the actions of (1) are constant. Taking this principle for granted, Hannay then analyses the additional angle shifts and discusses some examples of such non-trivial angles.

The Berry phase [7] is the quantum analogue of the Hannay angles. Soon after its theoretical discovery the quantum mechanical effect found experimental verification and a formal mathematical setting [31]. It should not come as too big a surprise that the situation is slightly different in the case of the Hannay angles:

1. Experimental verification of the Hannay angles involves, in general, high precision measurements because these angles are typically small compared to the purely dynamical angle shifts. In contrast to this, interference experiments can be performed for the Berry phase. Recently, however, Kugler [25] has given a demonstration consisting of a steel wire set at an angle with respect to (w.r.t.) the axis of a rotating base. If the wire vibrates and the base is slowly rotated one observes a change of the direction of vibration. As Kugler remarks this is essentially a Foucault pendulum in disguise. What is special about this example is that here the two degrees of freedom are decoupled (cf. Montgomery [28]).

2. Concerning the "adiabatic principle" Hannay observed that it is "difficult to eliminate the mathematical loopholes which prevent the simple statement that it holds rigorously in the limit of slow change" [21].

Indeed the adiabatic theorem is easier to prove in quantum mechanics than in classical mechanics, at least for the case of a non-degenerate eigenvalue. See, e.g., Kato [23].

3. The Berry phase can be defined for all parameter-dependent Hamiltonians, integrable or not. In contrast to this the Hannay angles are only defined for integrable systems (which are non-generic for more than one freedom).
Yet from semiclassical considerations one expects that a generalized Hannay effect exists for all classical Hamiltonians.

In this article we answer some of the problems formulated above. To clarify the subject we distinguish between:

– the *geometrical Hannay effect* (called "non-adiabatic angles" by Berry and Hannay in [9]) which is the holonomy of a non-trivial connection in the bundle $\pi_R : M \times R \to R$ with parameter space R as base space and the phase space M as fibre.

– the *adiabatic Hannay angles* which are defined for small $\varepsilon > 0$ by

$$\Delta\varphi := \varphi(T/\varepsilon) - \varphi(0) - \int_0^{T/\varepsilon} \omega(J(t), \varepsilon t)\, dt, \tag{2}$$

where $\omega(J, \varepsilon t) = \partial h_0(J, \varepsilon t)/\partial J$.

In Sect. 2 we study the geometry of the Hannay angles, using an intrinsic (coordinate-free) approach and formulate the generalization to non-integrable Hamiltonian systems.

A natural connection is defined using the parameter-dependent symplectic (canonical) action of a group G on the phase space M. In the case of an integrable system studied by Hannay one can use the torus $G=\mathbb{T}^n$, n denoting the number of degrees of freedom. Sometimes one has a non-Abelian symmetry group (e.g. the group $G=SO(3)$ of rotations), and one can study the effect of such a symmetry. Apart from applications, the study of such group actions should be an interesting question in symplectic geometry. In fact, the geometrical Hannay effect can be interpreted as a natural law of motion for parameter-dependent phase space symmetries. We also refer to the independent work of Montgomery [28] which will be remarked upon in Sect. 2.

In the case of integrable motion the *geometrical* Hannay angles are smooth functions on phase space, once a parameter variation is given. The smoothness of the angles is due to the fact that the symmetry group $\mathbb{T}^n$ is compact.

So only for the *adiabatic* Hannay angles analytical problems arise, due to resonances. Since resonances are of measure zero, the geometrical angles are a good idealisation of the dynamical effect.

The situation changes if one studies a family of ergodic Hamiltonians. There the geometrical Hannay effect is defined using the group $G=\mathbb{R}$ of parameter-dependent Hamiltonian motions. Since $\mathbb{R}$ is non-compact even the *geometrical* effect will be discontinuous in the initial conditions in general so that a clear-cut separation between geometrical and analytical questions does not exist anymore. It is important to keep in mind that this sensitive dependence on initial conditions is of *physical* nature.

In Sect. 3 we study the adiabatic Hannay angles. For time-dependent Hamiltonians KAM theory is not applicable in general, since a submanifold of constant actions typically contains resonant as well as non-resonant tori. So our starting point is the work of Neishtadt [29] and Bakhtin [6], who, developing an idea of Kasuga [22], invented a method of showing adiabatic invariance of the actions controlling the influence of the resonances on the motion by means of a measure-theoretical approach. We also present a complete proof of a fundamental lemma (see Appendix) which was not given explicitly in these papers. We proceed then by extending Neishtadt's theorem so as to control the behaviour of the angles (indeed we prove a theorem for general phase space functions). Our theorems are formulated for general, not necessarily Hamiltonian, systems. We calculate explicit upper bounds for the deviation of the time dynamics from the averaged dynamics. We can prove that for every $\varepsilon>0$ the Hannay angles are given by

$$\Delta\varphi=\int_0^T\left[\frac{\partial}{\partial J}\int_{\mathbb{T}^n} h_1(J,\varphi,\tau)\frac{d\varphi}{(2\pi)^n}\right]_{J=J(0)} d\tau+\mathcal{O}(\varepsilon^b)\,, \tag{3}$$

where $b\in[0,1/2)$, for all initial conditions which do not belong to a set of small measure; i.e. the difference between the geometrical and adiabatic angles is typically small, as asserted by Hannay. Regarding the behaviour in the perturbation parameter ε, our results are optimal, since "examples show that for a set of initial conditions of measure of order 1, an almost adiabatic invariant can undergo a variation of order 1 over time $1/\varepsilon^{3/2}$ due to temporary captures into resonances" (Arnol'd et al. [5]).

In the fourth and last section we study the motion of a satellite around a slowly rotating oblate (or prolate) planet, and we prove that a non-trivial Hannay angle appears and that it is of the order of magnitude of the square of the oblateness parameter. Clearly such a non-relativistic effect only exists if the symmetry axis of the planet does not coincide with the rotation axis. In the case of smaller planets (such as Mars) the irregularities in the solid crust should give rise to an observable effect on co-orbiting satellites. We believe that this example illustrates that the Hannay angles might quite naturally be of relevance in celestial mechanics. We also want to point out that in many cases the symmetry which gives integrability also implies zero Hannay angles [19].

Some of the computational details (including a harmonic oscillator example for Sect. 2 and the evaluation of the elliptic integrals in Sect. 4) have been skipped for brevity of presentation. We will send them to any reader upon request.

2. On the Geometry of Parameter Dependent Phase Space Symmetries

The purpose of this section is to interpret the Hannay angles in terms of the holonomy of a connection on a principal bundle and to give a generalization to non-integrable systems.

As in the case of Simon's [31] interpretation of Berry's phase in terms of the holonomy of a Hermitian line bundle, this connection is *natural*, thus in particular defined independently of coordinates.

Furthermore, this connection does not depend on the precise form of the dynamics generated by the parameter-dependent Hamiltonian function but only on its symmetries. In the case treated by Hannay these symmetries are the canonical automorphisms of the invariant tori in phase space. It is this concept of symmetries which lends itself to generalisations.

In this section all mappings are assumed to be smooth unless we state the contrary explicitly. We assume that the $2n$-dimensional phase space (M, ω) is exact symplectic, i.e. that the symplectic two-form ω on the manifold M is exact: $\omega = -d\theta$ $\Big($for the important case of the cotangent bundle $M := T^*N$ of a configuration manifold N we may use the canonical forms, which can be written locally in canonical coordinates: $\theta = \sum_{i=1}^{n} p_i dq^i$ and $\omega = \sum_{i=1}^{n} dq^i \wedge dp_i\Big)$.

Let R denote the l-dimensional manifold of parameter values. Without loss of generality M and R are assumed to be connected.

We look at the product manifold $M \times R$ with the canonical projections $\pi_R : M \times R \to R$ and $\pi_M : M \times R \to M$ on the factors. The connection describing the generalized Hannay effect shall be an Ehresmann connection on the bundle π_R, i.e. a smooth assignment $\overline{\mathrm{hor}}$ of horizontal subspaces (cf., e.g., Dubrovin et al. [13, Sect. 24]). Up to now we have only the trivial connection arising from the canonical decomposition of tangent vector fields $X : M \times R \to T(M \times R)$ into the sum $X = \mathrm{ver}\, X + \mathrm{hor}\, X$ of its (vertical) M- and (horizontal) R-components, since $T(M \times R) \cong TM \times TR$.

We shall work with the exterior derivative d on $M \times R$ and sometimes with the partial exterior derivatives d_M, d_R on $M \times R$. Clearly, one has the relations $d = d_M + d_R$, $d_M \circ d_M = d_R \circ d_R = 0$ and $d_M \circ d_R = -d_R \circ d_M$. But observe that the partial exterior derivatives are not natural w.r.t. mappings. Since later on we define connections and curvatures on principal bundles, we shall also use d for exterior derivatives on vector-valued forms.

The Hamiltonian function $H: M \times R \to \mathbb{R}$ is assumed to be invariant under the action

$$\Phi: G \times M \times R \to M \times R,$$

$$\Phi_g: M \times R \to M \times R, \qquad \Phi_g(m, r) := \Phi(g, m, r)$$

of a k-dimensional compact connected Lie group G on $M \times R: H \circ \Phi_g = H$ (for an introduction to group actions and the momentum mapping cf., e.g., Appendix 5 of Arnol'd [3]). In the case treated by Hannay $G = \mathbb{T}^n$, the n-dimensional torus.

We assume the group properties

$$\Phi_e = \mathrm{id}, \qquad \Phi_{g_1} \circ \Phi_{g_2} = \Phi_{g_1 g_2}$$

of a left action, and invariance of the parameters:

$$\pi_R \circ \Phi_g = \pi_R .$$

Furthermore we assume the group action to be symplectic in the sense that for all $r \in R$, $g \in G$,

$$\Phi_g^{r*} \omega = \omega$$

for the diffeomorphisms $\Phi_g^r: M \to M$ defined by

$$\Phi_g^r(m) := \pi_M \Phi_g(m, r) .$$

Note that this does *not* imply that $\Phi_g^* \tilde{\omega} = \tilde{\omega}$ for $\tilde{\omega} := \pi_M^* \omega$. In fact, it will turn out that the violation of this equality is responsible for the geometrical effect.

We are to describe the Ehresmann connection on the bundle $\pi_R: M \times R \to R$ by a two-form $\hat{\omega}$ of rank $2n$ on $M \times R$ which is derived from $\tilde{\omega}$. Then the *characteristic bundle* $R_{\hat{\omega}}$ of $\hat{\omega}$, i.e. the subbundle of $T(M \times R)$ spanned by the *characteristic vector fields* $Y: M \times R \to T(M \times R)$ being defined by $\mathbf{i}_Y \hat{\omega} = 0$ (that is, $\hat{\omega}(Y, X) = 0$ for all vector fields X) produces the horizontal subspace (see, e.g. Abraham and Marsden [1]). Since $\hat{\omega}$ will not be closed in general ($d\hat{\omega} \neq 0$), $R_{\hat{\omega}}$ is not integrable in the sense of Frobenius which means that the commutator $[Y_1, Y_2]$ of two characteristic vector fields is not a characteristic vector field (see Proposition 5.1.2 of [1]).

There is always a natural law of motion in phase space M under variation of the parameters R. But this law depends on the amount of knowledge one has about the symmetries Φ of the problem.

If one does not know any symmetries (i.e. the symmetry group G is trivial) then the most natural motion in M under parameter variation is *no* motion. In that case one would simply consider the characteristic bundle $R_{\tilde{\omega}}$ of $\tilde{\omega}$ which is clearly spanned by the horizontal tangent vectors $X = \mathrm{hor}\, X$. In this trivial example no holonomy effect arises since $\tilde{\omega}$ is closed: $d\tilde{\omega} = d\pi_M^* \omega = \pi_M^* d\omega = 0$.

The most natural way to take into account the symmetry Φ of the problem is to average the symplectic structure $\tilde{\omega}$ w.r.t. the symmetry group G (whose Lie algebra we denote by $\mathbf{g}$ with dual $\mathbf{g}^*$).

Therefore we define the average $\langle\delta\rangle$ of an exterior form δ on $M\times R$ by

$$\langle\delta\rangle := \int_G \Phi_g^*\delta dg,$$

where dg denotes the normalized Haar measure on G.

The average has the properties $d\langle\delta\rangle = \langle d\delta\rangle$,

$$\langle\pi_R^*\gamma\rangle = \int_G \Phi_g^*\pi_R^*\gamma dg = \int \pi_R^*\gamma dg = \pi_R^*\gamma,$$

for an exterior form γ on R, and $\mathbf{i}_X\langle\delta\rangle = \langle\mathbf{i}_X\delta\rangle$ for invariant vector fields X (a vector field X is called invariant if it equals its average $\langle X\rangle := \int_G \Phi_g^* X dg$).

We shall use the averaged forms $\bar\theta := \langle\tilde\theta\rangle$ for $\tilde\theta := \pi_M^*\theta$ and $\bar\omega := \langle\tilde\omega\rangle$ which are related by $\bar\omega = -d\bar\theta$. On vertical vector fields $\bar\omega$ coincides with $\tilde\omega$ since we assumed $\Phi_g^{p*}\omega = \omega$.

Averaging of the one-form $\tilde\theta$ is useful even independently of the question of parameters because in general $\Phi_g^{p*}\theta \neq \theta$.

If the parameter space R is one-dimensional ($l=1$), then rank $\bar\omega = 2n$, since odd-dimensional antisymmetric matrices have even rank. In that case we can directly look at the characteristic vector fields of $\bar\omega$.

But for $l>1$ the rank of $\bar\omega$ is greater than $2n$ in general, so that we cannot obtain l independent characteristic vector fields of $\bar\omega$ itself. This is not an annoying technical complication but a sign of the non-vanishing curvature of the connection we are looking for. Our strategy will be to split $\bar\omega$ into two Φ_g-invariant pieces

$$\bar\omega = \hat\omega + \Omega \tag{4}$$

such that rank $\hat\omega = 2n$.

Before doing so we define the parameter-dependent momentum mapping

$$J: M\times R \to \mathbf{g}^*$$

by

$$J(y)\cdot\xi := \mathbf{i}_{\xi_L}\bar\theta(y), \qquad \xi\in\mathbf{g}, \qquad y\in M\times R$$

with the lifted vector field ξ_L on $M\times R$ defined by

$$\xi_L(y) := \frac{d}{dt}\Phi_{\exp t\xi}(y)\bigg|_{t=0}$$

(ξ_L is called the *infinitesimal generator* of the action corresponding to ξ).

ξ_L is the unique vertical vector field which has the action variable

$$\hat J(\xi): M\times R \to \mathbb{R}, \qquad \hat J(\xi)(y) := J(y)\cdot\xi \tag{5}$$

as "Hamiltonian function" w.r.t. the averaged two-form $\bar\omega$:

$$d\hat J(\xi) = d\mathbf{i}_{\xi_L}\bar\theta = -\mathbf{i}_{\xi_L}d\bar\theta = \mathbf{i}_{\xi_L}\bar\omega, \tag{6}$$

since the Lie-derivative w.r.t. the infinitesimal generator ξ_L,

$$L_{\xi_L}\langle\delta\rangle = (\mathbf{i}_{\xi_L}d + d\mathbf{i}_{\xi_L})\langle\delta\rangle = \frac{d}{dt}\Phi_{\exp t\xi}^*\langle\delta\rangle\bigg|_{t=0}$$

of an averaged form $\langle\delta\rangle$ vanishes.

The momentum mapping $J: M\times R \to \mathbf{g}^*$ is Ad*-equivariant, i.e. the diagram

$$\begin{array}{ccc} M\times R & \xrightarrow{\Phi_g} & M\times R \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathbf{g}^* & \xrightarrow{\mathrm{Ad}^*_{g^{-1}}} & \mathbf{g}^* \end{array}$$

commutes, or $\hat{J}(\xi)(\Phi_g(y)) = \hat{J}(\mathrm{Ad}_{g^{-1}}\xi)(y)$. By def. (5), this is equivalent to the identity

$$(\mathbf{i}_{\xi_L}\bar{\theta})(\Phi_g(y)) = \mathbf{i}_{(\mathrm{Ad}_{g^{-1}}\xi)_L}\bar{\theta}(y)$$

which follows from the general identity $(\mathrm{Ad}_{g^{-1}}\xi)_L = \Phi_g^*\xi_L$ for group actions derived in Proposition 4.1.26 of [1].

Lemma 1. *The averaged one-form $\bar{\theta}$ admits the (non-unique) decomposition*

$$\bar{\theta} = \tilde{\theta} - \sigma + dF$$

with a one-form σ which is horizontal, i.e. $\sigma(\mathrm{ver}X)=0$ for all vector fields X.

Proof. We show that for all $g\in G$,

$$\Phi_g^*\tilde{\theta} = \tilde{\theta} - \sigma_g + dF_g \text{ with } \sigma_g \text{ horizontal}.$$

Then the lemma follows with $\sigma := \int_G \sigma_g dg$ and $F := \int_G F_g dg$. For compact connected Lie groups G the exponential mapping $\exp: \mathbf{g}\to G$ is surjective. Let $\exp\xi = g$. Then

$$\begin{aligned}\Phi_g^*\tilde{\theta} - \tilde{\theta} &= \int_0^1 \frac{\partial}{\partial t}\Phi^*_{\exp t\xi}\tilde{\theta}dt = \int_0^1 \Phi^*_{\exp t\xi}L_{\xi_L}\tilde{\theta}dt \\ &= \int_0^1 \Phi^*_{\exp t\xi}(d\mathbf{i}_{\xi_L}\tilde{\theta} - \mathbf{i}_{\xi_L}\tilde{\omega})dt = -\mathbf{i}_{\xi_L}\int_0^1 \Phi^*_{\exp t\xi}\tilde{\omega}dt + d\int_0^1 \Phi^*_{\exp t\xi}\mathbf{i}_{\xi_L}\tilde{\theta}dt,\end{aligned}$$

since ξ_L is $\Phi_{\exp t\xi}$-invariant. Now

$$\mathbf{i}_{\xi_L}\int_0^1 \Phi^*_{\exp t\xi}\tilde{\omega}dt = \mathbf{i}_{\xi_L}\int_0^1 \Phi^*_{\exp t\xi}(\tilde{\omega}-\bar{\omega})dt + \mathbf{i}_{\xi_L}\bar{\omega} = \sigma_g + d\hat{J}(\xi),$$

by Eq. (6), since $\bar{\omega}$ coincides with $\tilde{\omega}$ on vertical vector fields and since ξ_L is vertical. □

To understand the structure of the averaged symplectic form $\bar{\omega} = \tilde{\omega} + d\sigma$ more clearly, it seems appropriate to use local coordinates r^i, $i=1,\dots,l$, in the parameter space R. Then the one-form σ of Lemma 1 can be written as

$$\sigma = \sigma\left(\frac{\partial}{\partial r^i}\right)dr^i = H_i dr^i,$$

and thus

$$\bar{\omega} = \tilde{\omega} + dH_i \wedge dr^i.$$

To state our theorem we assume that the group action Φ_g is *free*, i.e. that for each $y\in M\times R$ the mapping $g\mapsto \Phi_g(y)$ is one-to-one. Thus a fortiori the momentum mapping $J: M \times R \to \mathbf{g}^*$ is a submersion (that is: regular everywhere) and for every $j \in J(M \times R)$ the level set $S_j := J^{-1}(j)$ of constant actions j is a $2n+l-k$-dimensional submanifold of $M \times R$, by the implicit mapping theorem. The above assumption may seem to be rather restrictive since in many applications the symmetries have fixed points etc. But observe that in applications it is sufficient to know that the group action is free in some tubular neighbourhood of S_j where j describes the initial momenta of the system.

We could equally well start with the weaker assumption that J is a submersion, thus being lead to the study of *locally* free actions on S_j and thus to V-manifolds (orbifolds). See for comparison Duistermaat and Heckman [15].

By the *Hamiltonian vector field* X_H of a parameter-dependent Hamilton function $H: M \times R \to \mathbb{R}$ we mean the unique vertical vector field on $M \times R$ with $\mathbf{i}_{X_H}\bar{\omega} = d_M H$, i.e. we consider the parameter as being fixed.

For vector fields Y on $M \times R$ which are horizontal lifts of vector fields on parameter space we define the vector fields

$$\bar{Y} := Y + X_{\sigma(Y)} .$$

Localizing in parameter space it is sufficient to consider $Y_i := \dfrac{\partial}{\partial r^i}$ so that

$$\bar{Y}_i := \frac{\partial}{\partial r^i} + X_{H_i} . \tag{7}$$

The intuition behind this definition is that one has to correct the pure parameter variation $\partial/\partial r^i$ by the Hamiltonian vector field of H_i in order to preserve the parameter-dependent symmetry. Observe that although σ and thus the functions H_i are not uniquely defined, the X_{H_i} *are*. Using these vector fields we can define a new connection on $\pi_R: M \times R \to R$ by saying that the horizontal subspace of this connection is spanned by these vector fields $\bar{Y}_i$. Then we have a new splitting $Z = \overline{\mathrm{hor}}\, Z + \overline{\mathrm{ver}}\, Z$ of general vector fields Z w.r.t. this connection. Accordingly we define the two-form Ω on $M \times R$ used in Eq. (4) by $\Omega := \overline{\mathrm{hor}}\, \bar{\omega}$ (i.e. $\Omega(V, W) = \bar{\omega}(\overline{\mathrm{hor}}\, V, \overline{\mathrm{hor}}\, W)$) which locally equals

$$\Omega = \frac{1}{2} dr^k \wedge \mathbf{i}_{\bar{Y}_k}\bar{\omega} = \frac{1}{2}\left[\{H_k, H_l\} + \frac{\partial H_l}{\partial r^k} - \frac{\partial H_k}{\partial r^l}\right] dr^k \wedge dr^l , \tag{8}$$

where the Poisson bracket $\{H_k, H_l\} := \bar{\omega}(X_{H_k}, X_{H_l})$. Then the splitting $\bar{\omega} = \hat{\omega} + \Omega$ of the averaged symplectic form $\bar{\omega}$ defines the two-form $\hat{\omega}$. Note that in general $d\hat{\omega} = -d\Omega \neq 0$.

Theorem 2. *The vector fields $\bar{Y} = Y + X_{\sigma(Y)}$ are the averages of the vector fields Y, i.e.*

$$\bar{Y} = \langle Y \rangle . \tag{9}$$

Therefore the splitting $\bar{\omega} = \hat{\omega} + \Omega$ is Φ_g-invariant. Furthermore, $\operatorname{rank}\hat{\omega} = \operatorname{rank}\omega = 2n$ *and the vector fields $\bar{Y}$ span the characteristic bundle $R_{\hat{\omega}}$ of $\hat{\omega}$. Regarding these characteristic vector fields as horizontal, we obtain a connection on $M \times R$ which is tangential to the level sets $S_j = J^{-1}(j)$ of the momentum mapping.*

For the case $G = \mathbb{T}^n$ this defines a connection on the principal $\mathbb{T}^n$-bundle

$$\pi_j : S_j \to B_j := S_j/\mathbb{T}^n$$

whose **g**-*valued one-form A is given by*

$$A(\eta) := -I^* \mathbf{i}_{\eta_L} \hat{\omega} ,$$

for any vertical vector field η_L with $\mathbf{i}_{\eta_L} dJ = \eta$ and $I : S_j \to M \times R$ being the inclusion of the submanifold S_j.

The curvature F of this connection is given by

$$F(\eta) := I^* L_{\eta_L} \Omega .$$

The holonomy of the connection A gives the Hannay angles.

If the fibration of the phase space M into invariant tori is topologically trivial, then B_j is diffeomorphic to the parameter manifold R, the diffeomorphism

$$\tau_j : B_j \to R$$

being defined by $\tau_j := \pi_R \circ \pi_j^{-1}$. So on the bundle $\pi_R : S_j \to R$ we can define a natural connection as well.

Proof. First we show that $\bar{Y}$ equals the average of the vector field Y. For this it suffices to localize the question and show that $\langle \partial/\partial r^k \rangle$ equals $\bar{Y}_k$, as defined in Eq. (7). Clearly $\operatorname{hor}\langle \partial/\partial r^k \rangle = \partial/\partial r^k = \operatorname{hor} \bar{Y}_k$.

To show that $\operatorname{ver}\langle \partial/\partial r^k \rangle = \operatorname{ver} \bar{Y}_k$ we test the one-form $\mathbf{i}_{\langle \partial/\partial r^k \rangle} \bar{\omega}$ on vertical vector fields Z. This is sufficient since $\bar{\omega}$ is of maximal rank on the vertical subspace.

Locally in parameter-space R,

$$\mathbf{i}_{\bar{Y}_k} \bar{\omega} = \mathbf{i}_{\bar{Y}_k}(\tilde{\omega} + dH_i \wedge dr^i) = \left[\{H_i, H_k\} + \frac{\partial H_i}{\partial r^k} - \frac{\partial H_k}{\partial r^i} \right] dr^i . \tag{10}$$

Thus $\mathbf{i}_Z \mathbf{i}_{\bar{Y}_k} \bar{\omega} = 0$. On the other hand,

$$\begin{aligned} \mathbf{i}_Z \mathbf{i}_{\langle \frac{\partial}{\partial r^k} \rangle} \bar{\omega} &= \mathbf{i}_Z \int_G \mathbf{i}_{\left(\Phi_g^* \frac{\partial}{\partial r^k}\right)} \bar{\omega} dg = \mathbf{i}_Z \int_G \Phi_g^* \mathbf{i}_{\frac{\partial}{\partial r^k}} \bar{\omega} dg \\ &= \mathbf{i}_Z \int_G \Phi_g^* dH_k dg = \mathbf{i}_Z \langle dH_k \rangle = \mathbf{i}_Z d_M \langle H_k \rangle = 0 , \end{aligned}$$

since $0 = \int_G \Phi_g^*(\bar{\omega} - \tilde{\omega}) dg = \int_G \Phi_g^*(dH_i \wedge dr^i) dg = d\langle H_i \rangle \wedge dr^i$, and thus $d_M \langle H_i \rangle = 0$, by independence of the forms dr^i.

Thus we have shown Eq. (9). By definition (8) of Ω, the splitting (4) is Φ_g-invariant. $\operatorname{rank} \hat{\omega}$ must be $\geqq 2n$ since $\hat{\omega}$ coincides with $\bar{\omega}$ on vertical vector fields. On the other hand, by definition (8) of Ω and by Eq. (10), the l independent (local) vector fields $\bar{Y}_k$ are characteristic vector fields of $\hat{\omega}$. So $\operatorname{rank} \hat{\omega} \leqq 2n$.

To show that the vector fields $\bar{Y}_k$ are tangential to the level sets $S_j = J^{-1}(j)$ we observe that for all Lie algebra elements $\xi \in \mathbf{g}$,

$$\mathbf{i}_{\bar{Y}_k} d\hat{J}(\xi) = \mathbf{i}_{\bar{Y}_k} \mathbf{i}_{\xi_L} \bar{\omega} = \mathbf{i}_{\bar{Y}_k} \mathbf{i}_{\xi_L} \hat{\omega} = -\mathbf{i}_{\xi_L} \mathbf{i}_{\bar{Y}_k} \hat{\omega} = 0 ,$$

by Eq. (6) and the fact that $\mathbf{i}_{\xi_L} \bar{\omega} = \mathbf{i}_{\xi_L}(\hat{\omega} + \Omega) = \mathbf{i}_{\xi_L} \hat{\omega}$.

Clearly, the one-forms $-\mathbf{i}_{\eta_L}\hat{\omega}$ vanish on characteristic vector fields $\bar{Y}_k$ of $\hat{\omega}$. On the other hand,

$$\mathbf{i}_{\xi_L}(-\mathbf{i}_{\eta_L}\hat{\omega})=\mathbf{i}_{\eta_L}\mathbf{i}_{\xi_L}\hat{\omega}=\mathbf{i}_{\eta_L}\mathbf{i}_{\xi_L}\bar{\omega}=\mathbf{i}_{\eta_L}d\hat{J}(\xi)=\mathbf{i}_{\eta_L}dJ\cdot\xi=\eta\cdot\xi\,.$$

$A(\eta)$ does not depend on the representative η_L of $\eta\in\mathbf{g}^*$ (which is not unique).

To show that F is the curvature of A we remark that $\mathbf{i}_{\eta_L}\Omega=0$ and thus

$$L_{\eta_L}\Omega=\mathbf{i}_{\eta_L}d\Omega=-\mathbf{i}_{\eta_L}d\hat{\omega}\,.$$

Then the claim follows from Cartan's structure equation $F=dA+\frac{1}{2}[A,A]$ and the fact that the Lie algebra $\mathbf{g}$ of the torus has vanishing structure constants.

We still have to check regularity properties of the bundle $\pi_j:S_j\to B_j$. Since, by assumption, the action of Φ_g, restricted to S_j, is free and since the group G is compact, by Proposition 4.1.23 of [1], $\pi_j:S_j\to B_j$ is a bundle with smooth base manifold B_j. Since the bundle is locally trivial, it is a principal fibre bundle with group $G=\mathbb{T}^n$ (for the definition cf., e.g., Chapter I.5 of Kobayashi and Nomizu [24]).

In the definition of the Hannay angles the true parameter variation (given by the vector fields $Y=\text{hor}\,Y$) is replaced by the averaged parameter variation (given by the vector fields $\bar{Y}=\overline{\text{hor}}\,\bar{Y}$). By Eq. (9) this leads to the horizontal vector fields $\bar{Y}$ of the connection A.

To show that $\tau_j:B_j\to R$ is a diffeomorphism we remark that we already proved that locally τ_j is a diffeomorphism onto its image. $S_j\cap\pi_R^{-1}(r)$ consists of at most one torus, since the parameter-dependent mapping is regular. Therefore τ_j is injective.

It is sufficient to show that $\pi_R:S_j\to R$ and thus τ_j are surjective for $j\in J(M\times R)$, or that the set of values of the momentum mapping is parameter-independent.

We remark that for all $\xi\in\mathbf{g}$,

$$\left\langle\frac{\partial}{\partial r^k}\hat{J}(\xi)\right\rangle=\left\langle\mathbf{i}_{\frac{\partial}{\partial r^k}}d\hat{J}(\xi)\right\rangle=\mathbf{i}_{\left\langle\frac{\partial}{\partial r^k}\right\rangle}d\hat{J}(\xi)=\mathbf{i}_{\bar{Y}_k}d\hat{J}(\xi)=0\,,$$

i.e., the mean variation of the action variables $\hat{J}(\xi)$ w.r.t. the parameters r^k vanishes. Therefore the invariant tori with constant value j of the actions intersect in M for *nearby* parameter values. In fact by a generalization of Poincaré's geometric theorem there are many such intersections (as indicated in Appendix 9 of [3]).

But here we need a *global* property, namely we show the following: Let the momentum mapping $J_r:M\to\mathbf{g}^*$ be defined by $J_r(m):=J(m,r)$, for fixed parameter $r\in R$, and let $j\in\mathbf{g}^*$ be in the range of J_{r_0} for some $r_0\in R$. Then for *all* $r_1\in R$ the intersection of the torus $J_{r_0}^{-1}(j)\subset M$ with $J_{r_1}^{-1}(j)$ is not empty (so trivially j must be in the range of J_{r_1}).

To this end we connect the two parameter values with a path $\gamma:[0,1]\to R$, $\gamma(0)=r_0$, $\gamma(1)=r_1$. Using the natural connection defined above one has a parameter-dependent symplectic transformation $\tilde{\Psi}_t:M\to M$ of the phase space given by the time dependent Hamiltonian $\tilde{H}_t:M\to\mathbb{R}$,

$$\tilde{H}_t(y):=\sigma(Z(t))\,(y)\,,$$

where $Z(t)$ is the tangent vector of the curve γ at time t.

We must exclude the possibility that an orbit $\tilde{\psi}_t(m)$ starting at a point $m \in J_{r_0}^{-1}(j)$ on the torus with action j leaves M at a time $t_0 < 1$, i.e. that $\tilde{\psi}_t(m) \in M$ for $0 \leqq t < t_0$, $\tilde{\psi}_{t_0}(m)$ is undefined.

Assume that this is the case. Then there must be a smaller time $t_1 < t_0$ such that the torus $J_{\gamma(t_1)}^{-1}(j)$ has empty intersection with the initial torus $J_{r_0}^{-1}(j)$ (otherwise we could find an accumulation point a on the torus $J_{r_0}^{-1}(j)$ with $J_{\gamma(t_0)}(a) = j$).

Now there is an $\varepsilon > 0$ and a time $t_2 \in (t_1, t_0)$ such that $J_{\gamma(t)}^{-1}(k) \subset M$ for all $k \in U_\varepsilon := \{k \in \mathbf{g}^* \mid \|k - j\| \leqq \varepsilon\}$ in some norm on the dual Lie algebra $\mathbf{g}^*$ and for all times $t \in [0, t_2]$.

Clearly the set $V_\varepsilon \subset M$ given by

$$V_\varepsilon := \{m \in M \mid \exists t \in [0, t_2] \text{ such that } J_{\gamma(t)}(m) \in U_\varepsilon\}$$

is compact.

By Theorem 2.2 of Duistermaat [14] topological triviality of the bundle $J_{r_0}: M \to \mathbb{R}^n$ implies that M has the form $M \cong \mathbb{T}^n \times B$, with B n-dimensional and that the symplectic two-form $\omega = d\varphi^k \wedge dJ_k$ in terms of global action-angle variables. Therefore we can find a symplectic embedding $E: M \to T^*\mathbb{T}^n$ such that the torus $J_{r_0}^{-1}(j)$ is mapped to the zero section N of $T^*\mathbb{T}^n$.

$V_{\varepsilon/2}$ is in the interior of $V_\varepsilon \subset M$. So there is a time-dependent Hamiltonian H_t on $T^*\mathbb{T}^n$ of t-independent compact support $E(V_\varepsilon)$ with $H_t(E(m)) = \tilde{H}_t(m)$ for $m \in V_{\varepsilon/2}$. The time-dependent flow on $T^*\mathbb{T}^n$ generated by H_t is called ψ_t.

Generalizing the celebrated theorem of Conley and Zehnder [12], Chaperon [11] showed that for such Hamiltonian isotopies Ψ_t on $T^*\mathbb{T}^n$, the number of intersections $\#(\Psi_1(N) \cap N) \geqq n+1$. So the intersection of the tori $J_{r_0}^{-1}(j) \cap J_{\gamma(t_2)}^{-1}(j)$ contains at least $n+1$ points, contrary to the assumption. So for any parameter value r the intersection of the tori $J_{r_0}^{-1}(j) \cap J_{r_1}^{-1}(j)$ is not empty, which shows that the range of values of the momentum mapping is parameter-independent. □

Remarks. 1. We have shown the assertion of Hannay concerning the existence of a natural curvature two-form on *parameter* space. As the reader will have noticed, this proof is not trivial, using a global theorem in symplectic topology which was conjectured by Arnol'd in the sixties. If one drops the assumption that the group action is free, the mapping $\tau_j: B_j \to R$ is in general *not* a diffeomorphism. A simple example is the motion of a particle in a one-dimensional periodic potential which is shifted by one period during one revolution in the parameter space $R \cong S^1$. In this case $B_j \cong \mathbb{R}$ in the low energy region, i.e. it is the universal covering space of the parameter space R.

2. In our definition of the two-form $\bar{\omega} = \langle \tilde{\omega} \rangle$ we have averaged over the whole group G. For applications this is not the only sensible definition.

In principle the best way to exploit the symmetries of a parameter-dependent Hamiltonian function H is to average over the flow generated by H (for fixed parameters). But in this case the symmetry group $\mathbb{R}$ is not compact and the result is not smooth in general. Nevertheless, for non-integrable systems this approach is probably the only possible.

In the general case one could average $\tilde{\omega}$ over the isotropy group of G instead of over G itself, i.e. over the subgroup which leaves the momenta invariant.

For the case of a central potential in $\mathbb{R}^3$ the isotropy group of $G = SO(3)$ is isomorphic to $SO(2)$ and consists of the rotations around the angular momentum vector. In this example from the point of view of applications it would be best not to average over the three-dimensional group or the one-dimensional isotropy group but over the two-dimensional invariant tori of constant energy and angular momentum.

3. Disregarding the analytic problem for a moment, we see that it is at least conceptually clear how to generalize the notion of the Hannay angles to non-integrable systems. The two-form $\hat{\omega}$ still defines a natural connection on $M \times R$, and going around a closed loop in parameter space, one has a net motion in the phase space M.

Concerning the analytic problem, in the ergodic case one has a chance to smooth the averaged two-form, thus disregarding sets of measure zero (e.g. closed orbits for $n \geqq 2$). A good starting point seems to be the work of Kasuga [22], who studied the adiabatic invariants of such systems. Nevertheless, even the existence of the limit which defines the averaged two-form $\bar{\omega}$ is not at all obvious in that case, due to the fact divergence of nearby initial points.

4. The material of this paper was first presented at a conference in Bologna in May 1988 [18]. Independently, Montgomery [28] considered the geometry of parameter-dependent symmetries, the constructions being very similar to ours. However, we want to point out some major differences. First, we assume from the beginning that the symplectic two-form ω is exact. Under this assumption it is possible to *show* the existence of a unique equivariant momentum mapping J with mean zero parameter variation. Unlike Montgomery, we use the invariant splitting $\bar{\omega} = \hat{\omega} + \Omega$ to show that the connection is given by the characteristic bundle $R_{\hat{\omega}}$.

On the other hand, Montgomery considers cases where only local group actions exist. Furthermore, he discusses the very interesting examples of the Foucault pendulum (based on the work of Koiller, cf. [28]), and coupled harmonic oscillators, where the degeneracies of the frequency set play an important rôle.

The reader is invited to test the above concepts by working out Hannay's example of a one-dimensional generalized harmonic oscillator $H := (Xq^2 + 2Ypq + Zp^2)/2$ with variable parameters X, Y, Z.

3. Averaging of Phase Space Functions

In this section we want to show how the holonomy described in the previous section manifests in the dynamics of a parameter-dependent integrable Hamiltonian system when the parameters undergo a closed loop adiabatically. From the Hamiltonian (1) it follows that the equations of the motion are

$$\dot{J} = -\varepsilon \frac{\partial h_1}{\partial \varphi}(J, \varphi, \varepsilon t), \tag{11}$$

$$\dot{\varphi} = \omega(J, \varepsilon t) + \varepsilon \frac{\partial h_1}{\partial J}(J, \varphi, \varepsilon t). \tag{12}$$

If the number of degrees of freedom is one then one knows from KAM theory [2] that the actions are adiabatic invariants for all times (provided $\omega \neq 0$), and the existence of the Hannay angle for all initial conditions can be proven by standard methods [17].

When the number of degrees of freedom exceeds one, in general KAM theory cannot be used to prove adiabatic invariance of the actions. Adding the slow variable $\tau := \varepsilon t$ to the actions J one obtains an autonomous system of differential equations on $\mathbb{R}^{n+1} \times \mathbb{T}^n$. This system is no longer Hamiltonian, but it remains in the standard form of averaging theory [4], to which an approach by Neishtadt [29] may be applied.

Therefore we are led to consider a standard type multi-frequency system

$$\dot{I} = \varepsilon f(I, \varphi, \varepsilon), \tag{13}$$

$$\dot{\varphi} = \omega(I) + \varepsilon g(I, \varphi, \varepsilon), \tag{14}$$

and the corresponding averaged system

$$\dot{\bar{I}} = \varepsilon f_0(\bar{I}), \tag{15}$$

where $I \in \mathbb{R}^m$, $\varphi \in \mathbb{T}^n$, $m \geqq n$, $\varepsilon \geqq 0$, and $f_0(I) := \int_0^{2\pi} \dots \int_0^{2\pi} f(I, \varphi, 0) d\varphi/(2\pi)^n$. The overdot denotes the derivative w.r.t. the time t. Let $I(t) = I(t; I^0, \varphi^0, \varepsilon)$ and $\bar{I}(t) = \bar{I}(t; I^0, \varepsilon)$ be the solutions of Eqs. (13)–(15), respectively, but with identical initial value I^0. Let $U \subset \mathbb{R}^m \times \mathbb{T}^n$ be a bounded set such that $I(t)$ and $\bar{I}(t)$ are well-defined up to time T/ε for all $(I^0, \varphi^0) \in U$, and let $\varrho > 0$. For $(I^0, \varphi^0) \in U$ define

$$\Delta(I^0, \varphi^0, \varepsilon) := \max_{t \in [0, T/\varepsilon]} |I(t) - \bar{I}(t)|, \tag{16}$$

$$U(\varrho, \varepsilon) := \{(I^0, \varphi^0) \in U | \Delta(I^0, \varphi^0, \varepsilon) \leqq \varrho\}, \tag{17}$$

$$D(\varrho, \varepsilon) := U \backslash U(\varrho, \varepsilon). \tag{18}$$

The set $D(\varrho, \varepsilon)$, contains the initial conditions for which the solutions of (13) and (15) deviate by more than ϱ in the time interval $[0, T/\varepsilon]$.

For simplicity of presentation we will assume in the sequel that the functions f and g in (13), (14) are ε-independent. If they are $\mathscr{C}^1$ w.r.t. ε this assumption can be made without loss of generality.

Assumptions. (A1) Let $G \subset \mathbb{R}^m$ be a bounded open set and suppose that f and g are $\mathscr{C}^1$ on $\bar{G} \times \mathbb{T}^n$.

(A2) Suppose that both $f(I, \cdot)$ and $\partial f(I, \cdot)/\partial I_j$ are $\mathscr{C}^{n+2}$ (w.r.t. φ). We remark that f and $\partial f/\partial I$ of class $\mathscr{C}^{n+1}$ would suffice [29]. However our slightly stronger assumption allows us to give a clearer exposition, especially for dimensional analysis purposes, as no logarithmic terms appear in the estimates.

(A3) Let $G^> \subset \mathbb{R}^m$ be an open bounded set with $\bar{G} \subset G^>$. We assume that $\omega \in \mathscr{C}^1(G^>)$ and rank $(\partial\omega(I)/\partial I) = n$ for all $I \in G^>$.

Notation. We denote by $|\cdot|$ the Euclidean norm (whatever the finite-dimensional Euclidean space may be). For $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$, $|v|_1 := \sum_{i=1}^n |v_i|$. If f is a k times continuously differentiable function on $\bar{G}\times\mathbb{T}^n$, then we define the seminorms

$$\|f\|_k^I := \max_{|\beta|_1=k} \max_{(I,\varphi)\in\bar{G}\times\mathbb{T}^n} \left|\frac{\partial^\beta f}{\partial I^\beta}(I,\varphi)\right|,$$

where $\beta\in\mathbb{N}_0^m$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. $\|f\|_k^\varphi$ is defined analogously, except that the derivatives are taken w.r.t. φ. The Lebesgue measure on Euclidean spaces, the Haar measure on $\mathbb{T}^n$, as well as the product measure, will all be denoted by "meas."

All the constants appearing in the propositions of this section will be stated explicitly. In order to make our account of averaging as transparent as possible we define a few constants beforehand.

As $\operatorname{rank}(\partial\omega(I)/\partial I)=n$ and $\bar{G}$ is compact, there exists a finite open covering $\{G_\alpha\}_{\alpha=1}^l$ of $\bar{G}$ with $G_\alpha\subset G^>$ such that on each (bounded) set G_α one can choose a permutation $(I_1^{(\alpha)},\dots,I_m^{(\alpha)})$ of $(I_1,\dots,I_m)$ so that $\operatorname{rank}\dfrac{\partial(\omega_1,\dots,\omega_n)}{\partial(I_1^{(\alpha)},\dots,I_n^{(\alpha)})}=n$ on G_α. I.e., there are bounded $\mathscr{C}^1$-diffeomorphisms

$$\gamma_\alpha: G_\alpha\to\Gamma_\alpha\subset\mathbb{R}^m,$$

$$(I_1^{(\alpha)},\dots,I_m^{(\alpha)})\mapsto(\omega_1,\dots,\omega_n,I_{n+1}^{(\alpha)},\dots,I_m^{(\alpha)}),$$

where $\Gamma_\alpha:=\gamma_\alpha(G_\alpha)$.

Let

$$d_\omega:=2\max_{\alpha=1,\dots,l}\max_{j=1,\dots,n}\sup_{\Gamma_\alpha}|\omega_j|, \tag{19}$$

$$d_I:=2\max_{j=1,\dots,m}\sup_{G^>}|I_j|, \tag{20}$$

$$M:=(2\pi)^n\sum_{\alpha=1}^{l}\sup_{\Gamma_\alpha}\left|\frac{\partial(I_1^{(\alpha)},\dots,I_n^{(\alpha)})}{\partial(\omega_1,\dots,\omega_n)}\right| d_\omega^n d_I^{m-n}. \tag{21}$$

Then $M\geqq\operatorname{meas}(G\times\mathbb{T}^n)$.

Let $\varrho_1>0$. We define the set U referred to in (17) and (18) as the set of initial conditions $(I^0,\varphi^0)\in G\times\mathbb{T}^n$ such that for all $t\in[0,T/\varepsilon]$ one has $\operatorname{dist}(I(t;I^0,\varphi^0,\varepsilon),\partial G)>\varrho_1$ and $\operatorname{dist}(\bar{I}(t;I^0,\varepsilon),\partial G)>\varrho_1$, where $I(t;I^0,\varphi^0,\varepsilon)$ and $\bar{I}(t;I^0,\varepsilon)$ denote, respectively, the solutions of (13) and (15). Note that if $\operatorname{dist}(I^0,\partial G)>\varrho_1+T\|f\|_0$, then $(I^0,\varphi^0)\in U$. Because of the regular dependence of the flow on initial conditions, U is open.

Now we give a version of a theorem by Neishtadt [29] (and Bakhtin [6]) but state all constants explicitly.

Theorem 3. *Let $\varepsilon\geqq0$ and $\varrho>0$. Then*

$$\int_U \Delta(I^0,\varphi^0,\varepsilon)\,dI^0 d\varphi^0\leqq Mk_1\sqrt{\varepsilon}, \tag{22}$$

$$\operatorname{meas}D(\varrho,\varepsilon)\leqq Mk_1\frac{\sqrt{\varepsilon}}{\varrho}, \tag{23}$$

where

$$k_1:=e^{m\|f_0\|_1^I T}\left(\frac{2c_1}{d_\omega}+k_2\right), \tag{24}$$

$$k_2:=Te^{m(\|f\|_1^I+\|g\|_1^\varphi)T}\left[c_4 d_\omega+\frac{m}{d_\omega}(c_3\|f\|_0+c_2\|g\|_0)\right]. \tag{25}$$

The constants c_1–c_4 will be defined by Eqs. (30)–(32), *respectively.*

Note that Neishtadt's result is optimal as examples [29] show that the estimate $\sqrt{\varepsilon}/\varrho$ is the best possible power law estimate. Before giving the proof we state a lemma that will be the essential ingredient in proving Theorem 3. In the standard approach to averaging (cf., e.g., [4]) one exactly solves the homological equation $\omega \cdot \partial u/\partial \varphi = \tilde{f}$, where $\tilde{f}(I,\varphi) := f(I,\varphi) - f_0(I)$. The function u is singular on the resonance surfaces. Following an idea of Kasuga [22] one may give estimates in averaging theory by using a smooth auxiliary function $w(I,\varphi)$ such that $|\tilde{f} - \omega \cdot \partial w/\partial \varphi|$ is small only in an averaged sense. Lemma 4 asserts the existence and properties of the function w; the proof can be found in the appendix.

Lemma 4. *For any $\chi > 0$ there exists a $\mathscr{C}^1$-function $w : \bar{G} \times \mathbb{T}^n \to \mathbb{R}^m$ (depending on χ) satisfying*

$$\text{(i)}\quad \|w\|_0 \leqq \frac{c_1}{\chi}, \tag{26}$$

$$\text{(ii)}\quad \left\|\frac{\partial w}{\partial \varphi_j}\right\|_0 \leqq \frac{c_2}{\chi} \qquad (j = 1, \ldots, n), \tag{27}$$

$$\text{(iii)}\quad \int_{G \times \mathbb{T}^n} \left|\frac{\partial w}{\partial I_j}\right| dI d\varphi \leqq \frac{M c_3}{\chi} \qquad (j = 1, \ldots, m), \tag{28}$$

$$\text{(iv)}\quad \int_{G \times \mathbb{T}^n} \left|\tilde{f} - \frac{\partial w}{\partial \varphi}\omega\right| dI d\varphi \leqq M c_4 \chi, \tag{29}$$

where

$$c_1 := c_2 := \frac{3^{n+1}}{2} \|\tilde{f}\|_{n+2}^{\varphi}, \tag{30}$$

$$c_3 := \frac{3^{n+1}}{2}\left[\max_{j=1,\ldots,m} \left\|\frac{\partial \tilde{f}}{\partial I_j}\right\|_{n+2}^{\varphi} + 4\|\tilde{f}\|_{n+2}^{\varphi}\|\omega\|_1^I d_\omega^{-1}\right], \tag{31}$$

$$c_4 := \frac{3^{n+1}}{d_\omega} \|\tilde{f}\|_{n+2}^{\varphi}. \tag{32}$$

Proof of Theorem 3. Let $(I^0, \varphi^0) \in U$ and $t \in [0, T/\varepsilon]$. Then (13) and (15) imply that

$$\begin{aligned} I(t) - \bar{I}(t) &= \varepsilon \int_0^t [f(I(u), \varphi(u)) - f_0(\bar{I}(u))] du \\ &= \varepsilon \int_0^t [f_0(I(u)) - f_0(\bar{I}(u))] du + \varepsilon[w(I(t), \varphi(t)) - w(I^0, \varphi^0)] \\ &\quad + \varepsilon \int_0^t \psi(I(u), \varphi(u), \varepsilon) du, \end{aligned} \tag{33}$$

where

$$\psi(I, \varphi, \varepsilon) := \tilde{f} - \frac{\partial w}{\partial \varphi}\omega - \varepsilon\left(\frac{\partial w}{\partial \varphi} g + \frac{\partial w}{\partial I} f\right),$$

and $w(I, \varphi)$ is the phase space function constructed in Lemma 4.

As a result of Lemma 4,

$$|I(t)-\bar{I}(t)| \leqq \varepsilon m \|f_0\|_1^I \int_0^t |I(u)-\bar{I}(u)|du + \varepsilon\left[\frac{2c_1}{\chi} + \int_0^{t/\varepsilon} |\psi(u)|du\right],$$

where $\psi(u) \equiv \psi(I(u), \varphi(u), \varepsilon)$. The application of Gronwall's lemma yields

$$\int_U \Delta(I^0, \varphi^0, \varepsilon) dI^0 d\varphi^0 \leqq e^{m\|f_0\|_1^I T}\left[\frac{2\varepsilon c_1 M}{\chi} + \varepsilon \int_0^{T/\varepsilon} \int_U |\psi(u)| dI^0 d\varphi^0 du\right]. \tag{34}$$

To obtain an estimate on the last integral a change of variables from (I^0, φ^0) to $(I(\tau), \varphi(\tau))$ will be carried out. By explicit differentiation of the Jacobian one obtains a differential equation which leads to the estimate

$$\left|\det \frac{\partial(I(t), \varphi(t))}{\partial(I^0, \varphi^0)}\right| \geqq e^{-m(\|f\|_1^I + \|g\|_1^\varphi)T}.$$

Therefore Lemma 4 yields

$$\int_U |\psi(u)| dI^0 d\varphi^0 \leqq M e^{m(\|f\|_1^I + \|g\|_1^\varphi)T}\left[c_4\chi + \varepsilon m\left(\frac{c_2\|g\|_0}{\chi} + \frac{c_3\|f\|_0}{\chi}\right)\right]. \tag{35}$$

Combining this result with (34) and choosing $\chi = d_\omega\sqrt{\varepsilon}$, we obtain the first statement of Theorem 3.

Since $\Delta(I^0, \varphi^0, \varepsilon) > \varrho$ when $(I^0, \varphi^0) \in D(\varrho, \varepsilon)$ one gets the second assertion, ϱ meas $(D(\varrho, \varepsilon)) \leqq \int_U \Delta(I^0, \varphi^0, \varepsilon) dI^0 d\varphi^0 \leqq M k_1 \sqrt{\varepsilon}$. $\square$

Remark. When combining (34) and (35) one can improve the value of the constant k_1 by optimizing in χ.

We want now to show that the domain of validity of the Hannay angles (Formula (3)) is the same as the domain of validity of almost adiabatic invariance. To this end we extend Neishtadt's theorem to averaging of phase space functions.

Let $(I^0, \varphi^0) \in U$, $t \in [0, T/\varepsilon]$ and define

$$A(t) := \frac{\varepsilon}{T} \int_0^t a(I(u), \varphi(u), \varepsilon) du, \tag{36}$$

$$\bar{A}(t) := \frac{\varepsilon}{T} \int_0^t a_0(\bar{I}(u)) du, \tag{37}$$

where, as before, $a_0(I) := \int_0^{2\pi} \dots \int_0^{2\pi} a(I, \varphi, 0) d\varphi/(2\pi)^n$. When $t = T/\varepsilon$, $A(t)$ and $\bar{A}(t)$ denote respectively the time average of the phase space function $a(I, \varphi, \varepsilon)$ and of its average over the (fast) angular variables. Again, to simplify the presentation we will consider only phase space functions independent of ε. We assume that

(A5) $a \in \mathscr{C}^{n+3}(\bar{G} \times \mathbb{T}^n)$.

Moreover, we define for $(I^0, \varphi^0) \in U$,

$$\hat{\Delta}(I^0, \varphi^0, \varepsilon) := \max_{t \in [0, T/\varepsilon]} |A(t) - \bar{A}(t)|. \tag{38}$$

$$\hat{D}(\varrho, \varepsilon) := \{(I^0, \varphi^0) \in U | \hat{\Delta}(I^0, \varphi^0, \varepsilon) > \varrho\}. \tag{39}$$

Theorem 5. *Let $\varepsilon \geqq 0$ and $\varrho > 0$. Then*

$$\int_U \hat{\Delta}(I^0, \varphi^0, \varepsilon) dI^0 d\varphi^0 \leqq Mk_3 \sqrt{\varepsilon}, \tag{40}$$

$$\operatorname{meas} \hat{D}(\varrho, \varepsilon) \leqq Mk_3 \frac{\sqrt{\varepsilon}}{\varrho}, \tag{41}$$

where

$$k_3 := mk_1 \|a_0\|_1^I + \frac{2\hat{c}_1}{d_\omega T} + \hat{k}_2, \tag{42}$$

$$\hat{k}_2 := e^{m(\|f\|_1^I + \|g\|_1^\varphi)T} \left[\hat{c}_4 d_\omega + \frac{m}{d_\omega} (\hat{c}_3 \|f\|_0 + \hat{c}_2 \|g\|_0) \right]. \tag{43}$$

Here the constants $\hat{c}_1$–$\hat{c}_4$ are obtained from c_1–c_4 upon replacement of $\tilde{f}$ by $\tilde{a}$ in the definition.

Proof. Let $(I^0, \varphi^0) \in U$ and $t \in [0, T/\varepsilon]$. Then

$$A(t) - \bar{A}(t) = \frac{\varepsilon}{T} \int_0^t [a_0(I(u)) - a_0(\bar{I}(u))] du + \frac{\varepsilon}{T} \int_0^t \tilde{a}(I(\tau), \varphi(\tau)) du,$$

where $\tilde{a}(I, \varphi) := a(I, \varphi) - a_0(I)$.

Replace in Lemma 4 the phase space function $\tilde{f}$ by $\tilde{a}$ and consider the corresponding $\hat{w}$ whose existence is asserted in this lemma. We now proceed in analogy with the proof of Theorem 3.

$$A(t) - \bar{A}(t) = \frac{\varepsilon}{T} \int_0^t [a_0(I(u)) - a_0(\bar{I}(u))] du$$
$$+ \frac{\varepsilon}{T} [\hat{w}(I(t), \varphi(t)) - \hat{w}(I^0, \varphi^0)] + \frac{\varepsilon}{T} \int_0^t \hat{\psi}(u) du,$$

where

$$\hat{\psi}(I, \varphi, \varepsilon) := \tilde{a} - \frac{\partial \hat{w}}{\partial \varphi} \omega - \varepsilon \left[\frac{\partial \hat{w}}{\partial \varphi} g + \frac{\partial \hat{w}}{\partial I} f \right]$$

and $\hat{\psi}(u) \equiv \hat{\psi}(I(u), \varphi(u), \varepsilon)$. Then

$$\hat{\Delta}(I^0, \varphi^0, \varepsilon) \leqq m \|a_0\|_1^I \Delta(I^0, \varphi^0, \varepsilon) + \frac{\varepsilon}{T} \left[\frac{2\hat{c}_1}{\chi} + \int_0^{T/\varepsilon} |\hat{\psi}(u)| du \right], \tag{44}$$

and as before one may obtain the bound

$$\int_U |\hat{\psi}(u)| dI^0 d\varphi^0 \leqq M e^{m(\|f\|_1^I + \|g\|_1^\varphi)T} \left[\hat{c}_4 \chi + \varepsilon m \left(\frac{\hat{c}_2 \|g\|_0}{\chi} + \frac{\hat{c}_3 \|f\|_0}{\chi} \right) \right]. \tag{45}$$

Choosing $\chi = d_\omega \sqrt{\varepsilon}$, this yields the first estimate of the theorem. As for the second, one has ϱ meas $(\hat{D}(\varrho, \varepsilon)) \leqq \int_U \hat{\Delta}(I^0, \varphi^0, \varepsilon) dI^0 d\varphi^0 \leqq Mk_3 \sqrt{\varepsilon}$. □

Applying these averaging results we can prove Eq. (3) concerning the Hannay angles. In fact, suppose that $h(J, \varphi, \tau)$ is a time-dependent periodic Hamiltonian of the form $h(J, \varphi, \tau) = h_0(J, \tau) + \varepsilon h_1(J, \varphi, \tau)$, where $\varepsilon \geqq 0$, $\tau := \varepsilon t$, $h(J, \varphi, \tau + T)$

$=h(J, \varphi, \tau)$ and $(J, \varphi, \tau) \in G \times \mathbb{T}^n \times \mathbb{R}$. $G \subset \mathbb{R}^n$ is assumed to be a bounded open set and we suppose that h_1 is of class $\mathscr{C}^{n+4}$ on $G \times \mathbb{T}^n \times \mathbb{R}$, h_0 is $\mathscr{C}^2$ on $\bar{G} \times \mathbb{R}$ and that h_0 is non-degenerate, i.e. rank $(\partial^2 h_0(J, \tau)/\partial J_i \partial J_k) = n$ for all $(J, \tau) \in G \times \mathbb{R}$. The equations of motion are (11) and (12). Let $\varrho_1 > 0$ and define $U_0 := \{(J^0, \varphi^0) \in G \times \mathbb{T}^n | \operatorname{dist}(J(t), \partial G) > \varrho_1$ for all $t \in [0, T/\varepsilon]\}$, where $J(t)$ denotes the solution of (11), (12) with initial conditions (J^0, φ^0). Take now $I := (J, \tau)$ with $\dot{\tau} = \varepsilon$ and $\tau^0 = \tau(0) = 0$. Then (11), (12) become equivalent to (13), (14). The adiabatic Hannay angles [cf. (2)] are given by $\Delta\varphi = A(T/\varepsilon)$, where we have now set

$$A(t) := \varepsilon \int_0^t \frac{\partial h_1}{\partial J} (J(u), \varphi(u), \tau(u)) du. \tag{46}$$

Of course, the geometrical Hannay angles are then given by $\bar{A}(T/\varepsilon)$, where

$$\bar{A}(t) = \int_0^{\varepsilon t} \int_{\mathbb{T}^n} \frac{\partial h_1}{\partial J} (J^0, \varphi, \tau) \frac{d\varphi}{(2\pi)^n} d\tau. \tag{47}$$

Let $M_0 := (2\pi)^n d_\omega^n \left| \frac{\partial(J_1, \ldots, J_n)}{\partial(\omega_1, \ldots, \omega_n)} \right|$. With the assumptions made above we can prove the following:

Theorem 6. *Let $\varepsilon > 0$ and $\varrho > 0$. Then:*

$$\text{(i) meas} \left\{ (J^0, \varphi^0) \in U_0 \middle| \max_{t \in [0, T/\varepsilon]} |J(t) - J^0| > \varrho \right\} \leqq M_0 k_1' \frac{\sqrt{\varepsilon}}{\varrho}, \tag{48}$$

i.e. the actions are almost adiabatic invariants,

$$\text{(ii) meas} \left\{ (J^0, \varphi^0) \in U_0 \middle| \max_{t \in [0, T/\varepsilon]} |A(t) - \bar{A}(t)| > \varrho \right\} \leqq M_0 k_3' \frac{\sqrt{\varepsilon}}{\varrho}, \tag{49}$$

in particular,

$$\left| \Delta\varphi - \int_0^T \left[\int_{\mathbb{T}^n} \frac{\partial}{\partial J} h_1(J^0, \varphi, \tau) \frac{d\varphi}{(2\pi)^n} \right] d\tau \right| < \varrho \tag{50}$$

for all initial conditions outside a set of measure $\mathcal{O}(\sqrt{\varepsilon}/\varrho)$.

The constants are given by

$$k_1' := 2\left(\frac{c_1'}{d_\omega} + k_2' T \right), \tag{51}$$

$$k_2' := \frac{n}{d_\omega} \left(c_2' \left\| \frac{\partial h_1}{\partial J} \right\|_0 + c_3' \left\| \frac{\partial h_1}{\partial \varphi} \right\|_0 \right) + c_4' d_\omega, \tag{52}$$

$$k_3' := 2\left(n k_1' \left\| \left(\frac{\partial h_1}{\partial J} \right)_0 \right\|_1^J + \frac{c_1'}{d_\omega T} + k_2' \right), \tag{53}$$

where c_1'–c_4' are obtained from c_1–c_4 by replacing $\tilde{f}$ by $\partial h_1/\partial\varphi$.

Remark. Note that all the constants considered are independent of both ε and ϱ, so that we have proven that the error term in (50) is in fact $\mathcal{O}(\varepsilon^b)$, for any $b \in [0, 1/2)$.

Proof. On $G \times \mathbb{T}^n \times [-T, 2T]$ we consider the time evolution $J(t; J^0, \varphi^0, \tau^0)$ given by the Hamiltonian h_0, if $\tau \in [-T, 0)$, and $h_0 + \varepsilon h_1$, if $\tau \in [0, 2T]$, and define

$$U := \{(J^0, \varphi^0, \tau^0) \in G \times \mathbb{T}^n \times [-T, 0] \,|\, \mathrm{dist}(J(t|; J^0, \varphi^0, \tau^0), \partial G) > \varrho_1 \text{ for all } t \in [0, 2T/\varepsilon]\} .$$

These definitions are motivated by the fact that

$$\operatorname{meas}\left\{(J^0, \varphi^0) \in U_0 \middle| \max_{t \in [0, T/\varepsilon]} |J(t) - J^0| > \varrho\right\} \leqq \frac{1}{T} \operatorname{meas} D(\varrho, \varepsilon), \tag{54}$$

$$\operatorname{meas}\left\{(J^0, \varphi^0) \in U_0 \middle| \max_{t \in [0, T/\varepsilon]} |A(t) - \bar{A}(t)| > \varrho\right\} \leqq \frac{1}{T} \operatorname{meas} \hat{D}(\varrho, \varepsilon), \tag{55}$$

where

$$D(\varrho, \varepsilon) := \left\{(J^0, \varphi^0, \tau^0) \in U \middle| \max_{t \in [0, 2T/\varepsilon]} |J(t) - J^0| > \varrho\right\}$$

and

$$\hat{D}(\varrho, \varepsilon) := \left\{(J^0, \varphi^0, \tau^0) \in U \middle| \max_{t \in [0, 2T/\varepsilon]} |A(t) - \bar{A}(t)| > \varrho\right\}.$$

The trivial evolution of the τ-variable manifests itself in $\bar{f}$ having a vanishing last component. For this reason we may assume the auxiliary phase space functions w and $\hat{w}$ (cf. the proofs of Theorems 3 and 5) to have values in $\mathbb{R}^n$ (rather than $\mathbb{R}^{n+1}$). Due to the Hamiltonian origin of the system the exponential factors appearing in the constants are equal to one so that [by mimicking the proofs of Eqs. (34), (35), (44), and (45)]

$$\int_U \max_{t \in [0, 2T/\varepsilon]} |J(t) - J^0| dI^0 d\varphi^0 \leqq M \left[\frac{2c_1'}{d_\omega} + \frac{2T}{d_\omega}\left(n\left(c_2'\left\|\frac{\partial h_1}{\partial J}\right\|_0 + c_3'\left\|\frac{\partial h_1}{\partial \varphi}\right\|_0\right) + c_4' d_\omega\right)\right]\sqrt{\varepsilon}, \tag{56}$$

$$\int_U \max_{t \in [0, 2T/\varepsilon]} |A(t) - \bar{A}(t)| dI^0 d\varphi^0 \leqq 2M\left(nk_1'\left\|\left(\frac{\partial h_1}{\partial J}\right)_0\right\|_1^J + \frac{c_1'}{d_\omega T} + k_2'\right)\sqrt{\varepsilon}. \tag{57}$$

Noting that $M = TM_0$ the theorem follows from Eqs. (54), (55), and (56), (57). □

4. On the Motion of a Satellite Around an Oblate (or Prolate) Planet

In this section we are concerned with the effect of a non-spherical mass distribution of a rotating planet on a coorbiting satellite.

Let $\varrho(q'), q' \in \mathbb{R}^3$, denote the mass density of a planet with total mass $M := \int \varrho(q') d^3q'$. We assume that the centre of mass $\frac{1}{M}\int \varrho(q') q' d^3q'$ is situated at the origin of the coordinate system and that the mass distribution is symmetric w.r.t. the q_1-axis (the polar axis). The gravitational potential is (G being the gravitational constant)

$$V(q) := -G \int \frac{\varrho(q')}{|q' - q|} d^3q' = -\frac{GM}{|q|}\left(1 + \sum_{l=2}^{\infty} J_l |q|^{-l} P_l(\cos\theta)\right), \tag{58}$$

with $\cos\theta = q_1/|q|$, P_l being the Legendre polynomial, and the coefficients given by

$$J_l := \frac{1}{M} \int \varrho(q') P_l\left(\frac{q'_1}{|q'|}\right) |q'|^l d^3 q' .$$

We consider only values of $|q|$ which are larger than the maximal radius $R := \sup_{q' \in \mathbb{R}^3} \{|q'| \,|\, \varrho(q') > 0\}$ of the planet.

A satellite orbiting around the planet is described by the Hamiltonian function $H := \frac{1}{2}p^2 + V(q)$ which has the first integral $L_1 := q_2 p_3 - q_3 p_2$, by axial symmetry of the potential. In general the motion of the satellite will be non-integrable, i.e. there exists no third independent commuting integral. On the other hand Kyner [26] showed that for distant orbits KAM theory applies, i.e. there exist invariant tori in phase space. Observe that this implies orbital stability for all distant orbits since the existence of the integral L_1 makes Arnol'd diffusion impossible.

Clearly, the $l=2$ perturbation dominates for large distances if $J_2 \neq 0$. Therefore it may seem to be appropriate to set $J_3 = J_4 = \ldots = 0$ so as to study the effect of the J_2-term, but the motion in this potential is non-integrable. So we prefer to study the so-called *Vinti-potential* with coefficients $J_{2k} := (J_2)^k$ and $J_{2k+1} := 0$, which leads to integrable motion [26].

For $J_2 > 0$ and $d := +J_2^{1/2}$,

$$V(q) = -\frac{GM}{|q|}\left(1 + \sum_{k=1}^{\infty} \left(\frac{J_2}{|q|}\right)^{2k} P_{2k}\left(\frac{q_1}{|q|}\right)\right) = -\frac{GM}{2}\left(\frac{1}{|q-z|} + \frac{1}{|q+z|}\right)$$

with $z := (d, 0, 0)$, i.e. the Vinti potential is the gravitational potential of two equal masses at $\pm z$. For $J_2 < 0$ (oblate planets) one can use analytical continuation (see [3], Chap. 47).

Now we shall study the effect of a slow rotation of the planet on a coorbiting satellite. If the rotation-axis coincides with the q_1-axis there exists no non-relativistic effect. Let us now consider the case where we let the planet rotate around the q_3-axis with constant angular frequency ω and investigate the impact of the rotation on the satellite.

In the co-rotating frame one obtains the new Hamiltonian

$$\tilde{H}_\omega(P, Q) := \frac{1}{2} P^2 - \frac{\alpha}{2}\left(\frac{1}{|Q-z|} + \frac{1}{|Q+z|}\right) - \omega(Q_1 P_2 - Q_2 P_1),$$

on the cotangent bundle T^*M_0 of $M_0 := \mathbb{R}^3 \setminus \{-z, z\}$, with $\alpha := GM$. We switch to prolate ellipsoidal coordinates $(\xi_1, \xi_2, \varphi) \in (0, +\infty) \times [0, \pi) \times [0, 2\pi)$ given by

$$Q_1 = d\cosh\xi_1 \cos\xi_2, \qquad Q_2 = d\sinh\xi_1 \sin\xi_2 \cos\varphi, \qquad Q_3 = d\sinh\xi_1 \sin\xi_2 \sin\varphi .$$

We then proceed by going to extended phase space and using a new time parameter s defined by

$$\frac{dt}{ds} = 2d^2(\cosh^2\xi_1 - \cos^2\xi_2) > 0 . \tag{59}$$

Then

$$\mathscr{H}_\omega := \frac{dt}{ds}(\tilde{H}_\omega - E) = H_1 + H_2 + \omega\hat{H} , \tag{60}$$

where

$$H_1 := p_{\xi_1}^2 + \frac{p_\varphi^2}{\sinh^2 \xi_1} - 2Ed^2 \cosh^2 \xi_1 - 2d\alpha \cosh \xi_1 ,$$

$$H_2 := p_{\xi_2}^2 + \frac{p_\varphi^2}{\sin^2 \xi_2} + 2Ed^2 \cos^2 \xi_2 ,$$

$$\hat{H} := -d^2 \left(p_{\xi_2} \sinh 2\xi_1 + p_{\xi_1} \sin 2\xi_2 - p_\varphi \frac{\sinh^2 \xi_1 + \sin^2 \xi_2}{d \sinh \xi_1 \sin \xi_2} \tan \varphi \right) \cos \varphi .$$

$\mathscr{H}_\omega$ generates on the submanifold $\mathscr{H}_\omega = 0$ the time evolution (of the original Hamiltonian $\tilde{H}_\omega$) in the new parameter s (see [33, Chaps. 3.2 and 4.3]). For $\omega = 0$ the Hamiltonian equations separate and are therefore integrable, with constants of motion $\mathscr{H}_0 = 0$ and H_1.

To compute the Hannay angle we will average $\hat{H}$ over the three-dimensional invariant tori. By averaging w.r.t. the cyclic angle φ, one observes that the result is zero. So seemingly there is no geometrical effect of the rotation of the planet on the satellite.

If one compares averaging results with the true dynamics it is important to consider time scales. For a satellite whose mean distance $(r_1 + r_2)/2 = d \cosh \xi_1$ from the centres of attraction is large compared to their mutual distance $2d$, the three frequencies describing the motion on the invariant torus must be nearly in resonance, since the Kepler ellipses are completely resonant.

This means that for the physical system averaging w.r.t. the angle variable conjugate to p_φ happens only on long time scales.

For polar orbits with initial condition $\varphi = 0$, $p_\varphi = 0$ one has even exact resonance, so that partial averaging (see [5]) has to be applied. We now consider this special case. We use the coordinates $(\xi_1, \xi_2) \in (0, +\infty) \times [0, 2\pi)$ for the two-

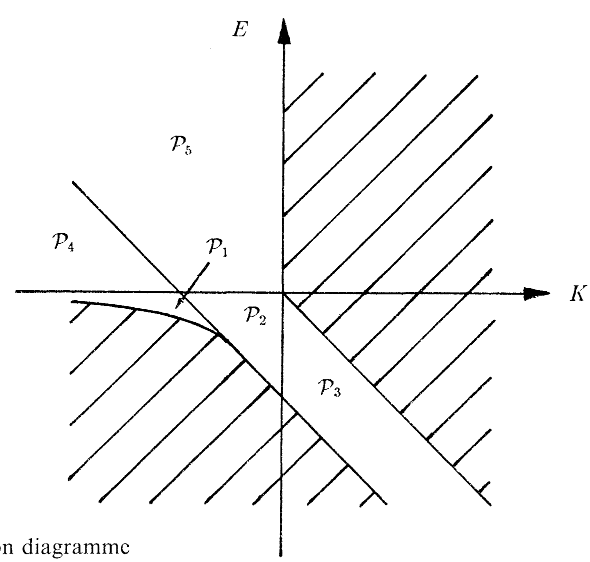

Fig. 1. Bifurcation diagramme

dimensional configuration space $M_1 := \{q \in M_0 | q_3 = 0\}$ and denote the restrictions of phase space functions to T^*M_1 with the same letter as before.

We start by describing the *bifurcation set* for the constants $H_0 := \tilde{H}_{\omega=0} = E$ and $H_1 = -H_2 = K$. So we determine the set of image points where the mapping $H_0 \times H_1 : T^*M_1 \to \mathbb{R}^2$ is not locally trivial and where the topology of the invariant manifolds (possibly) changes. The image $(H_0 \times H_1)(T^*M_1)$ is the region in $\mathbb{R}^2$ bounded by the curves

$$K(E) = \begin{cases} -2Ed^2 & \text{if} \quad E < 0 \\ 0 & \text{if} \quad E \geqq 0 \end{cases}$$

and

$$K(E) = \begin{cases} -2d(\alpha + Ed) & \text{if} \quad E < -\dfrac{\alpha}{2d} \\ \dfrac{\alpha^2}{2E} & \text{if} \quad -\dfrac{\alpha}{2d} \leqq E < 0 . \end{cases}$$

The bifurcation set of $H_0 \times H_1$ then consists of these curves and the intersections of the three lines $E = 0$, $K = 0$ and $K(E) = -2d(\alpha + Ed)$ with the image of $H_0 \times H_1$ and describes the values of the constants of motion where the phase portrait (possibly) changes. See [1], Chap. 4.5, as a general reference of the notion of bifurcation sets and Strand and Reinhardt [32] for the analysis of the two-centre problem.

The bifurcation set bounds five subregions of the image of $H_0 \times H_1$, which we denote by $\mathscr{P}_1, \ldots, \mathscr{P}_5$ (see Fig. 1). We shall only be concerned with orbits whose (H_0, H_1) values lie in the region $\mathscr{P}_1$. These orbits are characterized by the fact that they stay in a bounded phase-space region (since their energy H_0 is negative), and that they never meet the segment in configuration space connecting the two centers [since $H_1 < -2d(\alpha + H_0 d)$ and thus $d \cosh \xi_1 = (r_1 + r_2)/2 > d$].

As a next step we determine the action variables for this region,

$$I_{\xi_2} := \frac{1}{2\pi} \oint p_{\xi_2} d\xi_2 = \frac{\sqrt{|K| + 2|E| d^2}}{2\pi} 4\mathbf{E}(\sqrt{C}),$$

with $\mathbf{E}(\cdot)$ denoting the complete elliptic integral of the second kind (cf., e.g., Byrd and Friedman [10]) and $C := 2Ed^2/(K + 2Ed^2) > 0$. The asymptotic expansion of I_{ξ_2} in the small parameter $d^2 = J_2$ is

$$I_{\xi_2} = \sqrt{|K|} \left(1 + \frac{1}{2} \frac{E}{K} d^2 - \frac{3}{16} \frac{E^2}{K^2} d^4 + \mathcal{O}(d^6) \right).$$

On the other hand,

$$I_{\xi_1} := \frac{1}{2\pi} \oint p_{\xi_1} d\xi_1 = \frac{1}{2\pi} \oint \sqrt{K + 2d\alpha \cosh \xi_1 + 2Ed^2 \cosh^2 \xi_1}\, d\xi_1$$

is an elliptic integral of the third kind, and and its asymptotic expansion is

$$I_{\xi_1} = -\sqrt{|K|} + \frac{\alpha}{\sqrt{2|E|}} + d^2 \frac{\sqrt{|K|}}{2} \left(\frac{E}{K} - \frac{\alpha^2}{2K^2} \right) + \mathcal{O}(d^4). \tag{61}$$

Now we shall state the physically right prescription for averaging over the invariant two-tori defined by the constants E and K. We have to average $\hat{H}ds/dt$ w.r.t. the physical time-parameter t. Let $S := s(T)$, then

$$\left\langle \frac{ds}{dt}\hat{H}\right\rangle := \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{ds}{dt}\hat{H}dt = \lim_{S\to+\infty}\frac{S}{T(S)}\frac{1}{S}\int_0^S \hat{H}ds$$

$$= \frac{\lim\limits_{S\to+\infty}\frac{1}{S}\int_0^S \hat{H}ds}{\lim\limits_{S\to+\infty}\frac{1}{S}\int_0^S \frac{dt}{ds}ds} = \frac{\int \hat{H}d\varphi_1 d\varphi_2}{\int \frac{dt}{ds}d\varphi_1 d\varphi_2},$$

by the adiabatic assumption. Here φ_1 and φ_2 denote the angle variables conjugate to the actions I_{ξ_1} and I_{ξ_2}.

We want to evaluate this expression without writing $\hat{H}$ and dt/ds in action-angle coordinates but to compute $\langle \hat{H}ds/dt\rangle$ as a function of E and K. Denoting the generating function for the transformation to action-angle coordinates by $\tilde{S}(I_1, I_2, \xi_1, \xi_2)$, we have (using summation convention)

$$\frac{\partial\varphi_i}{\partial\xi_k} = \frac{\partial}{\partial I_i}\frac{\partial\tilde{S}}{\partial\xi_k} = \frac{\partial}{\partial I_i}p_{\xi_k} = \frac{\partial f_j}{\partial I_i}\frac{\partial p_{\xi_k}}{\partial f_j}$$

with $f_1 := H_0$, $f_2 := H_1$, and thus

$$d\varphi_1 d\varphi_2 = \det\left(\frac{\partial f_j}{\partial I_i}\right)\det\left(\frac{\partial p_{\xi_k}}{\partial f_j}\right)d\xi_1 d\xi_2 = -\frac{1}{4J}\frac{dt}{ds}\frac{1}{p_{\xi_1}p_{\xi_2}}d\xi_1 d\xi_2$$

with $J := \det\left(\frac{\partial I_i}{\partial f_j}\right) = -\frac{1}{16\pi^2}\oint \frac{1}{p_{\xi_1}p_{\xi_2}}\frac{dt}{ds}d\xi_1 d\xi_2$. Therefore

$$\left\langle \frac{ds}{dt}\hat{H}\right\rangle = \frac{\oint \hat{H}\frac{1}{p_{\xi_1}p_{\xi_2}}\frac{dt}{ds}d\xi_1 d\xi_2}{\oint \frac{1}{p_{\xi_1}p_{\xi_2}}\left(\frac{dt}{ds}\right)^2 d\xi_1 d\xi_2}. \tag{62}$$

We find

$$\left\langle \frac{ds}{dt}\hat{H}\right\rangle = -I_{\xi_2} - \frac{\alpha}{32LI_{\xi_2}^3(I_{\xi_1}+I_{\xi_2})}(5I_{\xi_1}^3 + 15I_{\xi_1}^2 I_{\xi_2} + 20I_{\xi_1}I_{\xi_2}^2 - 6I_{\xi_2}^3)d^4 + \mathcal{O}(d^6). \tag{63}$$

To calculate the Hannay angles $\Delta\psi_{\xi_1}, \Delta\psi_{\xi_2}$ using the adiabatic approximation we have to look at the partial derivatives (w.r.t. the action variables I_{ξ_1} and I_{ξ_2}) of the averaged term $\omega\langle \hat{H}ds/dt\rangle$ in Eq. (60). This gives the expression for the rotational frequencies. Then integrating over the time $T := 2\pi/\omega$ needed for one revolution of the planet,

$$\Delta\psi_{\xi_1} := 2\pi\frac{\partial\langle \hat{H}ds/dt\rangle}{\partial I_{\xi_1}}, \qquad \Delta\psi_{\xi_2} := 2\pi\frac{\partial\langle \hat{H}ds/dt\rangle}{\partial I_{\xi_2}}.$$

From Eq. (63) we see that

$$\Delta\psi_{\xi_1}=2\pi\frac{5\alpha^4(-5I_{\xi_1}^5-25I_{\xi_1}^4I_{\xi_2}-56I_{\xi_1}^3I_{\xi_2}^2-40I_{\xi_1}^2I_{\xi_2}^3+16I_{\xi_1}I_{\xi_2}^4+8I_{\xi_2}^5)}{8I_{\xi_2}^3(I_{\xi_1}+I_{\xi_2})^6(5I_{\xi_1}^2+10I_{\xi_1}I_{\xi_2}+2I_{\xi_2}^2)^2}d^4+\mathcal{O}(d^6),$$

$$\begin{aligned}\Delta\psi_{\xi_2}&=-2\pi+2\pi\\&\times\frac{\alpha^4(-75I_{\xi_1}^6-550I_{\xi_1}^5I_{\xi_2}-1575I_{\xi_1}^4I_{\xi_2}^2-2420I_{\xi_1}^3I_{\xi_2}^3-1580I_{\xi_1}^2I_{\xi_2}^4+64I_{\xi_1}I_{\xi_2}^5+84I_{\xi_2}^6)}{32I_{\xi_2}^4(I_{\xi_1}+I_{\xi_2})^6(5I_{\xi_1}^2+10I_{\xi_1}I_{\xi_2}+2I_{\xi_2}^2)^2}\\&\times d^4+\mathcal{O}(d^6).\end{aligned}$$

The angle -2π in the expression for $\Delta\psi_{\xi_2}$ comes from the fact that we changed to a co-rotating coordinate system and is therefore trivial. On the other hand, both $\Delta\psi_{\xi_1}$ and $\Delta\psi_{\xi_2}$ are nontrivial, the effect being of order $d^4=J_2^2$, the square of the oblateness parameter.

Note that if $\sqrt{|K|}\approx\alpha/\sqrt{2|E|}$ then I_{ξ_1} vanishes [up to order d^2, cf. Eq. (61)] and this corresponds, for $\omega=0$, to closed orbits of the satellite. In this limiting case the d^4-contributions in the Hannay angles are both positive; i.e. the adiabatic rotation of the planet leads to a drag on the satellite (in the same direction). Moreover, in the limiting case of energy $E\nearrow 0$ (where the apocentric distance of the satellite goes to infinity) both d^4 contributions disappear. These results are, of course, in agreement with physical expectation.

For the earth ($d\approx 200$ km) $J_2\approx -1.1\cdot 10^{-3}R_0^2$, R_0 denoting the equatorial radius (see Sanders and Verhulst [30, Chap. 8.7]). Thus for a near-earth satellite the effect would be of the order $10^{-6}\approx$ one arc second per day, *if* the symmetry axis of the earth were perpendicular to the axis of rotation. Since the earth is mainly a fluid, its surface is in the first approximation an equipotential surface of the gravitational and rotational field, irregularities due to the solid crust being of smaller order of magnitude. Since the gravitational acceleration on the surface of a planet (of fixed density) is inversely proportional to its radius, and since a mountain sinks into the planet if its pressure suffices to liquefy the stone, the upper bound for the height of mountains is inversely proportional to the radius of the planet. V.I. Weisskopf estimates an upper bound of 30 km (see [34]). On the other hand most local surface irregularities cancel such that the net effect is only of the order of magnitude of 100 m for the geoid (cf. Heiskanen and Moritz [20]).

From the above argument one should expect that the Hannay effect is observable for a satellite orbiting around a smaller planet such as Mars. For Mars, elevation differences of as much as 10 km are known to exist over scales of thousands of kilometers [16].

In the regions $\mathscr{P}_2$ and $\mathscr{P}_3$ of the bifurcation diagramme (Fig. 1) the Hannay angles are exactly equal to zero. The reason for this is the fact that the transformation $(P,Q)\to(-P,Q)$ is an automorphism of the invariant tori in these regions (whereas it interchanges the two tori with the same values of (E,K) in region $\mathscr{P}_1$). The perturbative Hamiltonian $\hat{H}ds/dt$ is odd under this transformation. Thus $\langle\hat{H}ds/dt\rangle=0$.

For the same reason the Hannay angles are zero in the case of a particle in a potential of the form $-1/|q|+E(t)\cdot q$ with $E(t)$ a slowly rotating electric field (Stark effect). Observe that the Stark effect can be considered as a limiting case of the two-center problem in region $\mathscr{P}_3$.

For more examples of the connection between symmetries and vanishing Hannay angles cf. Golin and Marmi [19].

A. Appendix: Proof of Lemma 4

Consider the function $\theta_\chi \in \mathscr{C}^1(\mathbb{R})$ defined by

$$\theta_\chi(y) := \begin{cases} \dfrac{1}{y} & \text{if } \ |y| \geqq \chi \\ -\dfrac{y^3}{\chi^4} + \dfrac{2y}{\chi^2} & \text{otherwise}. \end{cases} \tag{64}$$

This function is to be thought of as a smoothed version of $1/y$. By direct inspection one verifies that

$$\max_{y \in \mathbb{R}} |\theta_\chi(y)| = \sqrt{\frac{2}{3}} \frac{4}{3\chi}, \tag{65}$$

$$\int_{-\infty}^{\infty} |\theta'_\chi(y)| \, dy = \sqrt{\frac{2}{3}} \frac{16}{3\chi}. \tag{66}$$

In terms of the Fourier coefficients $f_\nu(I)$ of $f(I, \varphi)$ the function $\tilde{f}(I, \varphi)$ appearing in the homological equation has the form $\tilde{f}(I, \varphi) := \sum_{\nu \neq 0} f_\nu(I) e^{i\nu\varphi}$. Now we define

$$w(I, \varphi) := -i \sum_{\nu \neq 0} \theta_\chi(\nu \cdot \omega(I)) f_\nu(I) e^{i\nu\varphi}. \tag{67}$$

This function is to be viewed as an approximate solution of the homological equation with the resonances smoothed out by means of θ_χ. Clearly this "regularization" implies that w cannot be a pointwise solution of the homological equation but only in an averaged sense, i.e. in L^1 norm.

To prove Lemma 4 we need to estimate the Fourier coefficients $f_\nu(I)$. The smoothness condition (A2) allows for the inequality

$$\max_{I \in G} |f_\nu(I)| \leqq \|\tilde{f}\|^{\varphi}_{n+2} \frac{1}{|\nu|_\infty^{n+2}}, \tag{68}$$

where $|\nu|_\infty := \max_{i \in \{1, \ldots, n\}} |\nu_i|$.

Using Eqs. (65) and (68) we obtain inequality (i):

$$\|w\|_0 \leqq \sqrt{\frac{32}{27}} \frac{1}{\chi} \|\tilde{f}\|^{\varphi}_{n+2} \sum_{\nu \neq 0} \frac{1}{|\nu|_\infty^{n+2}} = \sqrt{\frac{32}{27}} \frac{1}{\chi} \|\tilde{f}\|^{\varphi}_{n+2} \sum_{k=1}^{\infty} \frac{1}{k^3} u_k,$$

where $u_k := k[(2 + 1/k)^n - (2 - 1/k)^n]$. We use the inequality $u_k < 3^n$ for $k \geqq 1$ so that

$$\|w\|_0 \leqq \sqrt{\frac{32}{27}} \frac{1}{\chi} \|\tilde{f}\|^{\varphi}_{n+2} 3^n \sum_{k=1}^{\infty} \frac{1}{k^3} \leqq \frac{3^{n+1}}{2\chi} \|\tilde{f}\|^{\varphi}_{n+2}. \tag{69}$$

As for inequality (ii), let $j = 1, \ldots, n$.

$$\left\| \frac{\partial w}{\partial \varphi_j} \right\|_0 \leqq \sqrt{\frac{32}{27}} \frac{1}{\chi} \|\tilde{f}\|^{\varphi}_{n+2} \sum_{\nu \neq 0} \frac{|\nu_j|}{|\nu|_\infty^{n+2}} \leqq \frac{3^{n+1}}{2\chi} \|\tilde{f}\|^{\varphi}_{n+2},$$

since

$$\sum_{\nu \neq 0} \frac{|\nu_j|}{|\nu|_\infty^{n+2}} < \sum_{|\nu|_\infty = 1} |\nu_j| + 3^n \sum_{k=2}^{\infty} \frac{1}{k^2} = 2 \cdot 3^{n-1} + 3^n \left(\frac{\pi^2}{6} - 1 \right).$$

To prove (iii) choose any $j = 1, \dots, m$. Then

$$\sum_{\nu \neq 0} \int_{G \times \mathbb{T}^n} \left| \frac{\partial f_\nu(I)}{\partial I_j} \right| |\theta_\chi(\nu \cdot \omega(I))| dI d\varphi \leqq M \frac{3^{n+1}}{2\chi} \left\| \frac{\partial \tilde{f}}{\partial I_j} \right\|_{n+2}^{\varphi},$$

and

$$\sum_{\nu \neq 0} \int_{G \times \mathbb{T}^n} |f_\nu(I)| \left| \frac{\partial}{\partial I_j} \theta_\chi(\nu \cdot \omega(I)) \right| dI d\varphi$$

$$\leqq \|\omega\|_1^I \, \|\tilde{f}\|_{n+2}^{\varphi} \sum_{\nu \neq 0} \frac{|\nu|}{|\nu|_\infty^{n+2}} \int_{G \times \mathbb{T}^n} |\theta'_\chi(\nu \cdot \omega(I))| dI d\varphi .$$

However,

$$|\nu| \int_G |\theta'_\chi(\nu \cdot \omega(I))| dI \leqq |\nu| \sum_{\alpha=1}^{l} \sup_{\Gamma_\alpha} \left| \frac{\partial(I_1^{(\alpha)}, \dots, I_n^{(\alpha)})}{\partial(\omega_1, \dots, \omega_n)} \right| \int_{\Gamma_\alpha} |\theta'_\chi(\nu \cdot \omega)| d\omega dI_{n+1}^{(\alpha)} \dots dI_m^{(\alpha)} .$$

Note that $\nu \neq 0$. By means of a rotation $\tilde{\omega} := R_\nu \omega$ we can take $\tilde{\omega}_1 = \nu \cdot \omega / |\nu|$ and carry out the substitution $x_1 := \nu \cdot \omega = |\nu| \tilde{\omega}_1$, $x_k = \tilde{\omega}_k$ $(k = 2, \dots, n)$ so that

$$|\nu| \int_{\Gamma_\alpha} |\theta'_\chi(\nu \cdot \omega)| d\omega dI_{n+1}^{(\alpha)} \dots dI_m^{(\alpha)} \leqq d_\omega^{n-1} d_I^{m-n} \int_{-\infty}^{+\infty} |\theta'_\chi(y)| dy = \sqrt{\frac{2}{3}} \frac{16}{3\chi} d_\omega^{n-1} d_I^{m-n},$$

and

$$\sum_{\nu \neq 0} \int_{G \times \mathbb{T}^n} |f_\nu(I)| \left| \frac{\partial}{\partial I_j} \theta_\chi(\nu \cdot \omega(I)) \right| dI d\varphi \leqq \frac{4M3^{n+1}}{2d_\omega \chi} \|\omega\|_1^I \, \|\tilde{f}\|_{n+2}^{\varphi} .$$

Altogether this gives

$$\int_{G \times \mathbb{T}^n} \left| \frac{\partial w}{\partial I_j} \right| dI d\varphi < \frac{M3^{n+1}}{2\chi} \left[\left\| \frac{\partial \tilde{f}}{\partial I_j} \right\|_{n+2}^{\varphi} + \frac{4}{d_\omega} \|\omega\|_1^I \, \|\tilde{f}\|_{n+2}^{\varphi} \right].$$

The last inequality, i.e. (iv), is based on the observation that for each fixed $\nu \in \mathbb{Z}^n \setminus \{0\}$, $\operatorname{meas}\{(I, \varphi) \in G \times \mathbb{T}^n \,|\, |\nu \cdot \omega(I)| < \chi\} \leqq 2M\chi / d_\omega$. Thus

$$\int_{G \times \mathbb{T}^n} \left| \tilde{f} - \frac{\partial w}{\partial \varphi} \cdot \omega \right| dI d\varphi \leqq \sum_{\nu \neq 0} \int_{G \times \mathbb{T}^n} |1 - \nu \cdot \omega(I) \theta_\chi(\nu \cdot \omega(I))| \, |f_\nu(I)| dI d\varphi$$

$$\leqq \frac{2M\chi}{d_\omega} \sum_{\nu \neq 0} \max_{I \in G} |f_\nu(I)| \leqq \frac{M\chi 3^{n+1}}{d_\omega} \|\tilde{f}\|_{n+2}^{\varphi} . \quad \square$$

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