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Uniform and L^2 Convergence in One Dimensional Stochastic Ising Models*

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Abstract. We study the rate of convergence to equilibrium of one dimensional stochastic Ising models with finite range interactions. We do *not* assume that the interactions are ferromagnetic or that the flip rates are attractive. The infinitesimal generators of these processes all have gaps between zero and the rest of their spectra. We prove that if one of these processes is observed by means of local observables, then the convergence is seen to be exponentially fast with an exponent that is any number less than the spectral gap. Moreover this exponential convergence is uniform in the initial configuration.

0. Introduction

The stochastic Ising model (often called the kinetic Ising model) was introduced by R. J. Glauber [RG] in 1963. The model that Glauber introduced is one dimensional and was carefully chosen so that one could explicitly compute the rate at which local observables relax to their equilibrium values. As a consequence of these explicit calculations one can see for Glauber's model that the rate at which convergence takes place when measured in the uniform norm is exactly the same as the rate when measured in the L^2 norm. It is the purpose of this paper to prove that, in one dimension, the same equality holds for all translation invariant, finite range interactions and all choices of flip rates that are translation invariant, have finite range, and satisfy the appropriate detailed balance condition.

By an *interaction* we mean any collection $\{J_R:R\subseteq \mathbf{Z}\}\subseteq \mathbb{R}$. We say that the interaction $\{J_R:R\subseteq \mathbf{Z}\}$ is *translation invariant* if, for every $R\subseteq Z$, $J_R=J_{R+k}$ for any $k\in\mathbf{Z}$; and we say that it has *finite range* if there is a finite number L (the range) such that $J_R=0$ whenever $\operatorname{diam}(R)>L$. We assume throughout that our interactions are translation invariant and have finite range.

Next, set $E=\{-1,1\}^{\mathbf{Z}}$ and think of the elements σ of E as configurations on $\mathbf{Z}$ of ± 1 valued spins. Thus, $\sigma_k\in\{-1,1\}$ is the spin at site $k\in\mathbf{Z}$ of the configuration

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$\sigma \in E$. Also, given a finite, non-empty set $R \subseteq \mathbf{Z}$, we will use the notation

$$\sigma_R \equiv \prod_{i \in R} \sigma_i.$$

The Gibbs state determined by the interaction $\{J_R : R \subseteq \mathbf{Z}\}$ is the unique (we are in one dimension) probability measure, μ, on E whose finite dimensional conditional distributions are given by

$$\mu(\sigma_i = \omega_i, i \in \Lambda \mid \sigma_j = \eta_j, j \in \Lambda^c) = \frac{\exp\left[\sum_{R \subset \Lambda} J_R \omega_R + \sum_{R \cap \Lambda^c \neq \emptyset, R \cap \Lambda \neq \emptyset} J_R \omega_{R \cap \Lambda} \eta_{R \cap \Lambda^c}\right]}{Z(\Lambda, \eta)}$$

for any finite set Λ. Here $Z(\Lambda, \eta)$ is the normalizing constant needed to make the expression on the right-hand side a probability measure.

The *flip rates* of a stochastic *Ising* model are a family $\{c_k : k \in \mathbf{Z}\}$ of functions $c_k : E \to (0, \infty)$ which satisfy the *detailed balance condition*:

$$c_k(\sigma) \exp\left[\sum_{R \ni k} J_R \sigma_R\right] = c_k(\sigma^k) \exp\left[-\sum_{R \ni k} J_R \sigma_R\right],$$

where σ^k is the configuration of spins that agrees with σ expect at k, at which site the spin is $-\sigma_k$. We will be assuming throughout that, in addition, the c_k's are *translation invariant* and have *finite range* $L \in \mathbf{Z}^+$. That is, $c_k(\sigma) = c_j(\tau)$ if $\sigma_i = \tau_{j-k+i}$ for all $i \in Z$, and $c_k(\sigma) = c_k(\omega)$ if $\sigma_l = \omega_l$ for all $|k - l| \leqq L$. Clearly, there are many ways to choose rates so that they satisfy all of these conditions. Glauber took the rates to be

$$c_k(\sigma) = \frac{1}{2}\left(1 - \sigma_k \tanh\left(\sum_{R \ni k} J_R \sigma_{R \setminus \{k\}}\right)\right)$$

and considered the case when $J_R = 0$ unless $R = \{k, k+1\}$ for some k. In this case the flip rates become $c_k(\sigma) = \frac{1}{2}(1 - (\gamma/2)\sigma_k(\sigma_{k-1} + \sigma_{k+1}))$, where $\gamma = \tanh(2J_{\{0,1\}})$. One of the main points of this paper is that results obtained here do *not* depend on the particular choice of flip rates.

Given flip rates $\{c_k : k \in \mathbf{Z}\}$, we define the operator $\mathscr{L}$ on the space $\mathscr{F}$ of cylinder functions by

$$\mathscr{L} f(\sigma) = \sum_{k \in Z} c_k(\sigma)(f(\sigma^k) - f(\sigma)), \quad \sigma \in E, \quad \text{for} \quad f \in \mathscr{F}.$$

The corresponding stochastic Ising model is the (unique) Markov process whose infinitesimal generator $\hat{\mathscr{L}}$ extends $\mathscr{L}$; and we denote by $\{T_t : t \geqq 0\}$ the associated semigroup. The following facts are quite well known:

1. $\mathscr{F}$ is a core for $\hat{\mathscr{L}}$,
2. $\{T_t : t \geqq 0\}$ is Feller continuous (i.e. it takes $C(E; \mathbb{R})$ into itself) and has the Gibbs state μ as its one and only stationary measure,
3. $\mathscr{L}$ is essentially self-adjoint on the space $L^2(\mu)$.

In what follows, we will use $\bar{\mathscr{L}}$ to denote the unique $L^2(\mu)$-self-adjoint extension of $\mathscr{L}$ (i.e. $\bar{\mathscr{L}}$ is the closure of $\mathscr{L}$ in $L^2(\mu)$) and $\{\bar{T}_t : t > 0\}$ to denote the

$L^2(\mu)$-semigroup generated by $\bar{\mathscr{L}}$. Thus, for each $t>0$, $\bar{T}_t$ is the closure in $L^2(\mu)$ of T_t on $C(E;\mathbb{R})$. Since μ is stationary for $\{T_t: t \geqq 0\}$ it follows that 0 is an eigenvalue for $\bar{\mathscr{L}}$; and, since μ is the only stationary measure, one knows that the corresponding eigenspace is the set of constant functions (see [L], [H&S 1], and [H&S 2]). It is also known (cf. [H 1]) that there is a positive gap between 0 and the rest of the spectrum of $\bar{\mathscr{L}}$ as an operator on $L^2(\mu)$. We denote this gap by gap_2. Thus for all $f \in L^2(\mu)$ we have

$$\| \bar{T}_t f - \langle f \rangle \|_2 \leqq e^{-\text{gap}_2 t} \| f - \langle f \rangle \|_2, \quad t \in [0,\infty), \tag{0.1}$$

where we have introduced the notation $\langle f \rangle$ to stand for $\int_E f\, d\mu$.

Remark (0.2). In conjunction with the fact that, for each $f \in \mathscr{F}, t \in [0,\infty) \mapsto T_t f(\sigma)$ has a bounded derivative for each $\sigma \in R$, the estimate (0.1) leads (via an easy Borel-Cantelli argument) to the conclusion that $T_t f(\sigma) \to \langle f \rangle$ as $t \nearrow \infty$ for μ-almost every $\sigma \in E$. Thus, since $\mathscr{F}$ is dense in $C(E;\mathbb{R})$ and the T_t's are contractions with respect to the uniform norm $\|\cdot\|_u$, it is obvious that the same statement holds for all $f \in C(E;\mathbb{R})$. (In fact, by using a theorem of Stein (see ES, Maximal Theorem, page 73]) one can even get the same conclusion for all $f \in L^2(\mu)$.)

Because the preceding statements are all modulo sets of μ-measure 0, none of them tells us anything about any specific σ. In particular, a much more useful statement would be one which says that there is a $\gamma \in (0,\infty)$ and a map $f \in C(E;\mathbb{R}) \mapsto A_f \in (0,\infty)$ for which

$$\| T_t f(\cdot) - \langle f \rangle \|_u \leqq A_f e^{-\gamma t}, \quad t \in (0,\infty), \tag{0.3}$$

for every $f \in C(E;\mathbb{R})$. Unfortunately no such statement can hold as can be seen from the fact that there are $\sigma \in E$ such that for all $t \geqq 0$ the measure determined by $f \mapsto T_t f(\sigma)$ is singular with respect to μ. For such a σ one can construct an $f \in C(E;R)$ such that for all $\gamma > 0, \overline{\lim}_{t\to\infty} |T_t f(\sigma) - \langle f \rangle| e^{\gamma t} = \infty$. On the other hand, as we will show in Sect. 1, if one is less ambitious and replaces $C(E;\mathbb{R})$ by $\mathscr{F}$, then not only one can show how to choose $f \in \mathscr{F} \mapsto A_f \in (0,\infty)$ so that (0.3) holds for *some* $\gamma > 0$ but even that such a choice is possible for *every* $\gamma \in (0, \text{gap}_2)$. To be more precise, in Sect. 1 we prove the following theorem.

Theorem (0.4). *For any one-dimensional stochastic Ising model with translation invariant, finite range flip rates, there is uniformly exponentially fast convergence to equilibrium in the sense that*

$$\inf_{f \in \mathscr{F}} \underline{\lim}_{t \to \infty} -\frac{1}{t} \log [\| T_t f(\cdot) - \langle f \rangle \|_u] = \text{gap}_2. \tag{0.5}$$

Remark (0.6). There are several results in the literature which are closely related to Theorem (0.3). Dobrushin [D] and Sullivan [WS] have proved general theorems (in particular, they apply both to higher dimensional lattices and to spin-flip processes which are not necessarily stochastic Ising models), which, when applied to stochastic Ising models with sufficiently weak interactions (i.e., high temperature), have conclusions similar to be one in (0.3) for all $f \in \mathscr{F}$. Moreover, in [H 1], a theorem which is nearly as strong as Theorem (0.4) is provided in the case when the interaction is ferromagnetic and the flip rates are attractive. However, besides having more restrictive hypotheses, these earlier results fail to determine the optimal range of γ's for which (0.3) can hold.

Remark (0.7). In [H&S 3] and in [H 2] there are theorems with the same hypotheses as Theorem (0.4), but the conclusion is only that there is a $\gamma \in (0, \infty)$ such that for all $f \in \mathscr{F}$,

$$\| T_t f(\cdot) - \langle f \rangle \|_u \leqq A_f \exp\left[-\frac{\gamma t}{\log(t+1)} \right], \quad t \in [0, \infty).$$

Remark (0.8). As will be apparent in Sect. 1, our proof relies heavily on our assumption that we are dealing with stochastic Ising models. We have no idea whether one can get away without making this assumption. In fact, an interesting open problem is to determine whether for an arbitrary one dimensional spin-flip process (i.e. one which is not necessarily a stochastic Ising model) with translation invariant, finite range, strictly positive flip rates there is a $\gamma > 0$ for which (0.3) holds whenever $f \in \mathscr{F}$. In fact it is not even known whether or not the invariant measure is unique! At the moment, it is not at all clear what one should expect. The only thing that is clear is that the techniques which we use here shed no light on the more general situation.

1. Proof of Theorem (0.4)

The proof of Theorem (0.4) is accomplished by approximating the infinite system whose semigroup is $\{T_t : t \geqq 0\}$ by finite systems. In the finite systems only the spins inside of a finite interval are permitted to change, and all other spins are held fixed. To be more precise, let $\Lambda \subseteq \mathbf{Z}$ be a finite, non-empty interval and define $\Phi_\Lambda \equiv E_\Lambda \{-1, 1\}^\Lambda \to E$ by

$$\Phi_\Lambda(\sigma)_k = \begin{cases} \sigma_k & \text{if } k \in \Lambda \\ 1 & \text{if } k \in \mathbf{Z} \backslash \Lambda. \end{cases}$$

Next, define the operator $\mathscr{L}^\Lambda$ on $C(E_\Lambda; \mathbb{R})$ by

$$\mathscr{L}^\Lambda f(\sigma) = \sum_{k \in \Lambda} c_k \circ \Phi_\Lambda(\sigma)(f(\sigma^k) - f(\sigma)), \quad \sigma \in E_\Lambda.$$

It is then an easy matter to check (see [H&S 3]) that $\mathscr{L}^\Lambda$ is self-adjoint on $L^2(\mu_\Lambda)$, where

$$\mu_\Lambda(\{\omega\}) = \mu(\sigma_k = \omega_k, k \in \Lambda \mid \sigma_j = 1, j \notin \Lambda), \quad \omega \in E_\Lambda.$$

Finally, let $\{T_t^\Lambda : t > 0\}$ be the Markov semigroup of $L^2(\mu_\Lambda)$-self-adjoint contractions generated by $\mathscr{L}^\Lambda$. It will be convenient to allow T_t^Λ to act on functions $f : E \to \mathbb{R}$ by setting

$$T_t^\Lambda f(\sigma) = [T_t^\Lambda (f \circ \Phi_\Lambda)](\sigma), \quad \sigma \in E_\Lambda.$$

Our proof of Theorem (0.4) rests on several facts about the semigroups $\{T_t^\Lambda : t > 0\}$ and the degree to which they approximate $\{T_t : t > 0\}$ as $\Lambda \nearrow \mathbf{Z}$. We will state these facts in a sequence of lemmata and will simply cite the place in the literature where their proofs can be found.

Lemma (1.1). [H&S 4] *There is a finite constant C, independent of* Λ, *and a map* $f \in \mathscr{F} \mapsto A_f \in [0, \infty)$ *such that*

$$\| T_t f - T_t^{\Lambda} f \|_u \leqq A_f \left(e^{Ct} - \sum_{j=0}^{N} \frac{(Ct)^j}{j!} \right), \quad f \in \mathscr{F}, \tag{1.2}$$

where

$$N \equiv \frac{\text{the distance between } \Lambda^c \text{ and the sites on which } f \text{ depends}}{L}$$

and L is the range of the interaction.

Before stating the next result, it will be helpful to recall the notion of a Dirichlet form. Namely, given a Polish space M and a weakly continuous transition probability function $Q(t, x, \cdot)$ which is symmetric with respect to the probability measure m (in the sense that

$$Q(t, x, dy)m(dx) = Q(t, y, \mathrm{d}x)m(dy) \text{ on } M \times M$$

for $t \in (0, \infty)$), one can use the Spectral Theorem to check that, for each $\phi \in L^2(m)$,

$$t \in (0, \infty) \mapsto \frac{1}{2t} \int_M \left(\int_M (\phi(y) - \phi(x))^2 Q(t, x, dy) \right) m(dx) \in [0, \infty]$$

is non-increasing. In fact, if $S_t \phi(x) = \int_M \phi(y) Q(t, x, dy), (t, x) \in (0, \infty) \times M$ for $\phi \in B(M; \mathbb{R})$, then each S_t admits a unique extension as a self-adjoint contraction $\bar{S}_t$ on $L^2(m)$, $\{\bar{S}_t : t > 0\}$ is a strongly continuous semigroup, and

$$\frac{1}{2t} \int_M \left(\int_M (\phi(y) - \phi(x))^2 Q(t, x, dy) \right) m(dx) \nearrow \int_{[0, \infty)} \lambda d(E_\lambda \phi, \phi)_{L^2(m)}, \quad \phi \in L^2(m),$$

as $t \searrow 0$, where $\{E_\lambda : \lambda \in [0, \infty)\}$ is the spectral resolution of the identity for $-\bar{L}$ ($\bar{L}$ is the generator of $\bar{S}_t$). In particular, when $\phi \in \mathrm{Dom}(\bar{L})$,

$$\lim_{t \searrow 0} \frac{1}{2t} \int_M \left(\int_M (\phi(y) - \phi(x))^2 Q(t, x, dy) \right) m(dx) = -(\phi, \bar{L}\phi)_{L^2(m)}.$$

The quadratic mapping

$$\phi \in L^2(m) \mapsto \mathscr{E}(\phi, \phi) \equiv \lim_{t \searrow 0} \frac{1}{2t} \int_M \left(\int_M (\phi(y) - \phi(x))^2 Q(t, x, dy) \right) m(dx) \in [0, \infty]$$

is called the *Dirichlet form* determined by the symmetric Markov semigroup $\{S_t : t > 0\}$ on $L^2(m)$. The original statement of the following result was proved by L. Gross [LG]. For a proof which covers the present setting, see [DS, Theorem (9.10)].

Lemma (1.3). *Referring to the preceding discussion, suppose that there exists an* $\alpha \in (0, \infty)$ *such that*

$$\int \phi^2(x) \log(\phi^2(x)) m(dx) \leqq \alpha \mathscr{E}(\phi, \phi) + \| \phi \|_{L^2(m)}^2 \log[\| \phi \|_{L^2(m)}^2], \quad \phi \in L^2(m),$$

and set $q(t) = 1 + \exp[4t/\alpha], t \in [0, \infty)$. *Then for every* $\phi \in L^2(m)$,

$$\| S_t \phi \|_{L^{q(t)}(m)} \leqq \| \phi \|_{L^2(m)}, \quad t \in [0, \infty). \tag{1.4}$$

An estimate of the form in (1.4) is called a *logarithmic Sobolev inequality*, and the smallest α for which (1.4) holds is called the *logarithmic Sobolev constant.*

In what follows, we will use $\mathscr{E}^\Lambda$ to denote the Dirichlet form on $L^2(\mu_\Lambda)$ determined by $\{T_t^\Lambda : t > 0\}$; and, in keeping with our definition of the action of T_t^Λ on $B(E; \mathbb{R})$, we define $\mathscr{E}^\Lambda(f, f) = \mathscr{E}^\Lambda(f \circ \Phi_\Lambda, f \circ \Phi_\Lambda)$ for $f : E \to \mathbb{R}$.

Because, for each finite Λ, the state space is finite and the measure μ_Λ charges every point, it is not very difficult to check that there will always be an $\alpha_\Lambda \in (0, \infty)$ for which (1.4) holds for μ_Λ and $\mathscr{E}^\Lambda$. The key to our proof of Theorem (0.4) will be the following estimate on the rate at which α_Λ grows as $\Lambda \nearrow \mathbf{Z}$.

Lemma (1.5). [H&S 3] *There is a constant* $\gamma \in (0, \infty)$ *such that, for every finite interval* Λ *with* $|\Lambda| \geqq 2$ *and all* $f \in L^2(\mu_\Lambda)$,

$$\int_{E_\Lambda} f^2 \log f^2 \, d\mu_\Lambda \leqq \gamma \log(|\Lambda|) \mathscr{E}^\Lambda(f, f) + \| f \|^2_{L^2(\mu_\Lambda)} \log[\| f \|^2_{L^2(\mu_\Lambda)}]. \tag{1.6}$$

The proof of Theorem (0.4) is now just a matter of applying these lemmata carefully. We begin by taking $a > 0$ and setting $\Lambda_t = [-at, at]$.

Lemma (1.7). *For every* $\delta > 0$ *there is an a such that for every* $f \in \mathscr{F}$ *and all sufficiently large t,*

$$\| T_s f - T_s^{\Lambda_t} f \|_u \leqq e^{-\delta t} \quad \textit{for all} \quad 0 \leqq s \leqq t. \tag{1.8}$$

Proof. By Lemma (1.1),

$$\| T_s f - T_s^{\Lambda_t} f \|_u \leqq A_f \left(\sum_{k = N_t + 1}^{\infty} \frac{(Cs)^k}{k!} \right) \leqq A_f \left(\sum_{k = N_t + 1}^{\infty} \frac{(Ct)^k}{k!} e^{(k - N_t)} \right) \leqq A_f e^{Cet - N_t}, \tag{1.9}$$

where

$$N_t = \frac{\text{the distance between } \Lambda_t^c \text{ and the sites on which } f \text{ depends}}{L}.$$

From the definition of Λ_t, we see that if a is sufficiently large ($a > 2L(\delta + Ce + 1)$ will do) then no matter which (finitely many) sites f depends on, $N_t \geqq (\delta + Ce + 1)t$ when t is sufficiently large. Therefore for all sufficiently large t and all $0 \leqq s \leqq t$,

$$\| T_s f - T_s^{\Lambda_t} f \|_u \leqq A_f e^{-(N_t - Cet)} \leqq e^{-\delta t}. \quad \blacksquare$$

We now take a_0 so large that (1.8) holds when $\delta = \text{gap}_2$. For the rest of this paper $\Lambda_t = [-a_0 t, a_0 t]$.

In view of Lemma (1.7), proving Theorem (0.4) comes down to checking that, for each $f \in \mathscr{F}$ with $\langle f \rangle = 0$,

$$\varliminf_{t \to \infty} -\frac{1}{t} \log[\| T_t^{\Lambda_t} f \|_u] \geqq \text{gap}_2. \tag{1.10}$$

The next lemma follows immediately from Lemmas (1.3) and (1.5).

Lemma (1.11). *Let*

$$q_{s,t}=1+\exp\left[\frac{4s}{\gamma\log(2a_0t+1)}\right],\quad 0\leqq s<t,$$

where γ is as in Lemma (1.5) and a_0 is as in the preceding definition of Λ_t. Then for all $f:E_{\Lambda_t}\to\mathbb{R}$ and all $s\in(0,\infty)$:

$$\|T_s^{\Lambda_t}f\|_{L^{q_{s,t}}(\mu_{\Lambda_t})}\leqq\|f\|_{L^2(\mu_{\Lambda_t})}. \tag{1.12}$$

The following lemma is an essentially obvious consequence of the fact that there exists a $\Gamma\in(0,\infty)$ such that, for every finite $\Lambda\subset\mathbf{Z}$, $\mu_\Lambda(\sigma)\geqq\exp[-\Gamma|\Lambda|]$ for all $\sigma\in E_\Lambda$.

Lemma (1.13). *There is a $\Gamma<\infty$ such that for all finite intervals Λ and all functions $f:E_\Lambda\to\mathbb{R}$,*

$$\|f\|_{L^\infty(\mu_\Lambda)}\leqq e^{\Gamma|\Lambda|/q}\|f\|_{L^q(\mu_\Lambda)}\quad \textit{for all}\quad q\geqq 1. \tag{1.14}$$

Note that since for every finite interval Λ we have $\mu_\Lambda(\sigma)>0$ for every $\sigma\in\{-1,1\}^\Lambda$, it follows that $\|f\|_{L^\infty(\mu_\Lambda)}=\|f\|_u$ for every function $f:E_\Lambda\to\mathbb{R}$.

Lemma (1.15). *Let $s\geqq 0$ be given and set $t=s^2+s$. Then for all $f:E_{\Lambda_t}\to\mathbb{R}$,*

$$\|T_t^{\Lambda_t}f\|_u\leqq\exp\left[\frac{\Gamma|\Lambda_t|}{q_{s,t}}\right]\|T_{s^2}^{\Lambda_t}f\|_{L^2(\mu_{\Lambda_t})}. \tag{1.16}$$

Proof. Note that $T_t^{\Lambda_t}f=T_s^{\Lambda_t}T_{s^2}^{\Lambda_t}f$ and apply Lemmas (1.11) and (1.13) to $T_s^{\Lambda_t}$ operating on $T_{s^2}^{\Lambda_t}f$. ■

Lemma (1.17). *There is a finite constant D such that, for each $f\in\mathscr{F}$ with $\langle f\rangle=0$,*

$$\|T_{s^2}^{\Lambda_t}f\|_{L^2(\mu_{\Lambda_t})}\leqq D(\|f\|_{L^2(\mu)}+1)e^{-\text{gap}_2 s^2}\quad \textit{with}\quad t=s^2+s \tag{1.18}$$

for all sufficiently large s.

Proof. From the definition of μ_Λ it is easily seen that there is a constant $D^2<\infty$ such that if ν_Λ is the marginal distribution of μ on E_Λ then $\|d\mu_\Lambda/d\nu_\Lambda\|_u\leqq D^2$. Thus

$$\begin{aligned}\|T_{s^2}^{\Lambda_t}f\|^2_{L^2(\mu_{\Lambda_t})}&=\int_{E_{\Lambda_t}}(T_{s^2}^{\Lambda_t}f(\sigma))^2\mu_{\Lambda_t}(d\sigma)\\&\leqq D^2\int_{E_{\Lambda_t}}(T_{s^2}^{\Lambda_t}f(\sigma))^2\nu_{\Lambda_t}(d\sigma)=D^2\int_E(T_{s^2}^{\Lambda_t}f(\sigma))^2\mu(d\sigma);\end{aligned}$$

and therefore $\|T_{s^2}^{\Lambda_t}f\|_{L^2(\mu_{\Lambda_t})}\leqq D\|T_{s^2}^{\Lambda_t}f\|_{L^2(\mu)}$.

But by Lemma (1.7)

$$\|T_{s^2}^{\Lambda_t}f\|_{L^2(\mu)}\leqq\|T_{s^2}f\|_{L^2(\mu)}+e^{-\text{gap}_2 t};$$

and, since $\|T_{s^2}f\|_{L^2(\mu)}\leqq e^{-\text{gap}_2 s^2}\|f\|_{L^2(\mu)}$ while $e^{-\text{gap}_2 t}\leqq e^{-\text{gap}_2 s^2}$, we get the desired conclusion from this. ■

After combining (1.16) with (1.18), we have, for every $f\in\mathscr{F}$, that for all sufficiently large $s \in (0, \infty)$:

$$\| T_t^{\Lambda_q} f \|_u \leqq \exp\left[\frac{\Gamma|\Lambda_t|}{q_{s,t}}\right] D(\| f \|_{L^2(\mu)} + 1) \exp[-\text{gap}_2 s^2], \tag{1.19}$$

where $t = s^2 + s$.

Finally note that, since $|\Lambda_t| \leqq 2a_0(s + s^2) + 1$ and

$$q_{s,t} = 1 + \exp\left[\frac{4s}{\gamma \log(2a_0(s + s^2) + 1)}\right]$$

when $t = s^2 + s$,

$$\frac{\Gamma|\Lambda_t|}{q_{s,t}} \to 0 \quad (t = s^2 + s) \quad \text{as} \quad s \to \infty. \tag{1.20}$$

Hence, because $(s^2/s^2 + s) \to 1$, (1.10) follows from (1.19) and (1.20).

Concluding Remark

As we have said before, the key to this proof is the logarithmic Sobolev inequality in Lemma (1.5). The reason why this logarithmic Sobolev inequality is sufficient for our purposes is that the logarithmic Sobolev constant grows at most logarithmically fast with the length of the interval. Actually, it may very well be that, for these one-dimensional Ising models, the logarithmic Sobolev constant does not really diverge as $\Lambda \nearrow \mathbf{Z}$; but, at the moment, we cannot say one way or the other.

In contrast to the situation with the logarithmic Sobolev constant, one can show that although an ordinary Sobolev inequality holds for each finite interval the corresponding Sobolev constant blows up exponentially fast as $\Lambda \nearrow \mathbf{Z}$.

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