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Erratum

The Quantum Theory of Second Class Constraints: Kinematics

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In the proof of Lemma 3.2, [1], we proved that if for all $U\in\langle\mathscr{U}\rangle$ of the form $U=\sum_{\ell}\lambda_\ell U_\ell$, $U_\ell\in\langle\mathscr{U}\rangle$ we have $\sum\lambda_\ell=1$, then $\mathbb{1}\notin C^*(\mathscr{U}-\mathbb{1})$. This proof is wrong because not all positive elements $A\in C^*(\mathscr{U})_+$ can be written as $A=B^*B$ with $B=\sum\lambda_\ell U_\ell$, though the set of such elements is of course dense in $C^*(\mathscr{U})_+$. Hence the proven inequality $\omega(B^*B)\geqq 0$ is not sufficient to ensure that ω is positive, so it does not follow that ω extends from the $*$-algebra generated by $\mathscr{U}$ to $C^*(\mathscr{U})$. In fact we know that in general this part of Lemma 3.2 is wrong:

Assertion. *There is no general condition on the $*$-algebra generated by a set of constraints $\mathscr{U}$ which is equivalent to $\mathbb{1}\in C^*(\mathscr{U}-\mathbb{1})$.*

Proof. We exhibit a $*$-algebra $\mathscr{K}$ containing a group of unitaries $\mathscr{U}$, and complete it in two different C^*-norms. In one of the resulting C^*-algebras we will have $\mathbb{1}\in C^*(\mathscr{U}-\mathbb{1})$, and in the other $\mathbb{1}\notin C^*(\mathscr{U}-\mathbb{1})$.

Let G be a discrete group acting on a unital C^*-algebra $\mathscr{F}$ with the action $\alpha: G\mapsto \mathrm{Aut}\mathscr{F}$. Construct $\tilde{\mathscr{F}}:=M(G_\alpha\times\mathscr{F})$ which contains $\mathscr{F}$ and a faithful unitary representation $U: G\mapsto\tilde{\mathscr{F}}_u$ of G which implements α, i.e. $\alpha_g=\mathrm{Ad}\,U_g$. So U_G is a group.

Lemma 1. *$U_G\subset\tilde{\mathscr{F}}$ is a linearly independent set.*

Proof. $(U_g f)(r):=\alpha_g(f(g^{-1}r))\ \forall g,r\in G$, $\forall f\in\ell^1(G,\mathscr{F})$, where $\ell^1(G,\mathscr{F}):=\left\{f:G\mapsto\mathscr{F}\,\middle|\,\sum_{g\in G}\|f(g)\|<\infty\right\}$. Assume U_G is linearly dependent, i.e. $\exists\beta_k\in\mathbb{C}\backslash 0$, $g_k\in G$ all different and $N<\infty$ such that $\sum_{k=1}^{N}\beta_k U_{g_k}=0$. Hence $\forall f\in\ell^1(G,\mathscr{F})$ we have $\sum_{k=1}^{N}\beta_k\alpha_{g_k}(f(g_k^{-1}r))=0$. Choose $f(r):=\mathbb{1}\delta(r,e)$. Then $\sum_{k=1}^{N}\beta_k\mathbb{1}\delta(g_k^{-1}r,e)=0\ \forall r\in G$, so for $r=g_k$ this implies $\beta_k=0$, which contradicts our assumption. $\square$

Take $\mathscr{U}=U_G$ for the chosen constraint set, let the $*$-algebra $\mathscr{K}$ be generated by U_G, hence it is the linear space generated by U_G. Let the C^*-algebra $\mathscr{A}$ be the C^*-

closure of $\mathscr{K}$ in $\tilde{\mathscr{F}}$, then we give below a choice of $\mathscr{F}$, G, and α for which $\mathbb{1} \in C^*(U_G - \mathbb{1})$ in $\mathscr{A}$. First note that if $\mathbb{1} \notin C^*(U_G - \mathbb{1})$, then $\exists \omega \in \mathfrak{S}(\tilde{\mathscr{F}})$ ($\equiv$ the set of states on $\tilde{\mathscr{F}}$), for which $\omega(U_G) = 1$, and since $\mathbb{1} \in \mathscr{F}$, $\omega \upharpoonright \mathscr{F}$ is a nontrivial G-invariant state on $\mathscr{F}$. Choose $\mathscr{F} = C(\mathbb{S}^1)$, and G the discrete group generated in Aut$\mathscr{F}$ by an irrational rotation of $\mathbb{S}^1$, and by a non-Lebesgue measure preserving homeomorphism of $\mathbb{S}^1$, which always exists.

Lemma 2. *There are no G-invariant states on $\mathscr{F}$, and hence $\mathbb{1} \in C^*(U_G - \mathbb{1})$.*

Proof. By the Riesz representation theorem, to each $\omega \in \mathfrak{S}(\mathscr{F})$ corresponds a unique Borel measure on $\mathbb{S}^1$. By [2, Theorem 5, p. 82], for an irrational rotation the Lebesgue measure is the only invariant Borel measure. Since the other generating element of G does not preserve the Lebesgue measure, there are no invariant measures for G, and hence no G-invariant states on $C(\mathbb{S}^1)$. □

We now construct another C^*-closure $\mathscr{B}$ of $\mathscr{K}$ such that $\mathbb{1} \notin C^*(U_G - \mathbb{1})$ in $\mathscr{B}$. Since by Lemma 1, U_G is linearly independent, each $A \in \mathscr{K}$ has a unique expression $A = \sum_{k=1}^{N} \lambda_k U_{g_k}$, $\lambda_k \in \mathbb{C} \backslash 0$, $g_k \neq g_j$ if $k \neq j$, $N < \infty$. So define a *-norm $\|A\|_1 := \sum_{k=1}^{N} |\lambda_k|$ on $\mathscr{K}$ and complete it to a Banach *-algebra $\mathscr{K}_1$. Then let $\mathscr{K}_2$ be the enveloping C^*-algebra of $\mathscr{K}_1$, with C^*-norm $\|\cdot\|_2$, and $\mathscr{B}$ is the $\|\cdot\|_2$-closure of $\mathscr{K}$ in $\mathscr{K}_2$. Since U_G is linearly independent and generates $\mathscr{K}$, a linear functional on $\mathscr{K}$ is uniquely specified by its values on U_G. Specify ω by $\omega(U_G) = 1$. Consider the set $\mathscr{P} := \{B \in \mathscr{K} | B = A^*A,\ A \in \mathscr{K}\}$, then for $B \in \mathscr{P}$, $B = A^*A$ with $A = \sum_k^N \lambda_k U_{g_k}$, we find

$$\omega(B) = \omega\left(\sum_{k,j}^{N} \bar{\lambda}_k \lambda_j U_{g_k^{-1} g_j}\right) = \sum_{k,j} \bar{\lambda}_k \lambda_j = \left|\sum_k \lambda_k\right|^2 \geqq 0\,,$$

so ω is positive on $\mathscr{P}$, and

$$\left|\omega\left(\sum_k^N \lambda_k U_{g_k}\right)\right| \leqq \sum_k |\lambda_k| \cdot |\omega(U_{g_k})| = \sum_k |\lambda_k| = \left\|\sum_k^N \lambda_k U_{g_k}\right\|_1,$$

hence ω is continuous with relation to $\|\cdot\|_1$, hence can be extended as a continuous linear functional to $\mathscr{K}_1$. Now $\mathscr{P}$ is dense in

$$(\mathscr{K}_1)_+ := \{B \in \mathscr{K}_1 | B = A^*A,\ A \in \mathscr{K}_1\}\,,$$

because if $A \in \mathscr{K}_1$ is the limit of $\{A_n\} \subset \mathscr{K}$, then A^*A is the limit of $\{A_n^* A_n\} \subset \mathscr{P}$ in the norm $\|\cdot\|_1$, by simple manipulations. Hence since ω is positive on $\mathscr{P}$, its extension is positive on $(\mathscr{K}_1)_+$, so ω is a state on the Banach *-algebra $\mathscr{K}_1$, hence by Dixmier [3] 2.7.5 has an extension as a state to the enveloping algebra $\mathscr{K}_2$. But $\omega(U_G) = 1$, hence $\mathbb{1} \notin C^*(U_G - \mathbb{1})$ in $\mathscr{B}$. □

Note. Hence the question of whether a constraint set is first or second class in general depends on the C^*-norm of the field algebra. Clearly there are some algebraic conditions which are sufficient for $\mathbb{1} \in C^*(\mathscr{U} - \mathbb{1})$, e.g. the first part of Lemma 3.2 [1].

The subsequent material in [1] is unaffected by the error in Lemma 3.2 pointed out in this erratum.

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