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Large Field Renormalization. II. Localization, Exponentiation, and Bounds for the R Operation

Tadeusz Balaban*

Department of Mathematics, Boston University, Boston, MA 02215, USA

Abstract. In this paper we conclude the discussion of the basic properties and bounds for the **R** operation. This allows us to complete the proof of the ultraviolet stability of four-dimensional pure gauge field theories, as formulated in Theorem 1.

Introduction

In this paper we perform all the remaining operations defining **R**, like cancellations of the large terms in numerators and denominators in (1.100) [IV], localizations in disjoint large field regions, and exponentiations. Then we prove bounds for the obtained expressions. The main bounds are combinatorial bounds proving convergence of the whole expansion defined by the **R** operation. This completes the proof of the inductive representation of the effective densities, the representation described in detail in Sect. 2 [III]. Thus we prove the following theorem:

Theorem 1. *If the sequence of the effective coupling constants is contained in an interval* $]0,\gamma]$ *with a sufficiently small positive* γ, *then the effective densities* ρ_k *have the form, and satisfy all the conditions and bounds, described in Sect. 2 [III].*

This is the main result of the whole sequence of papers of the present author on non-Abelian gauge field theories. Theorem 2 of [I] allows us to remove the assumption on the effective coupling constants in the above theorem, because this assumption follows from the basic inequality (0.31) [II], which is the result of Theorem 2. The proof of Theorem 2, which is based on second order perturbative calculations, is very awkward and long in the context of the renormalization group approach to lattice gauge field theories, and has not been published yet, so we have Theorem 1 with the assumption. As an immediate consequence of this theorem, we get the ultraviolet stability bounds of the same type as for superrenormalizable models in [16]:

$$\chi_k \exp\left[-\frac{1}{g_k^2} A(U_k) - E_- |T_\eta| \right] \leqq \rho_k \leqq \exp E_+ |T_\eta|, \tag{0.1}$$

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with the constants E_-, E_+ independent of k, T_η, U_k. Theorem 1 is the basis for many other applications, for example, for an analysis of expectation values of physical observables, like loop variables, averaged loop variables, etc. These problems are very important for construction of the four-dimensional gauge field theories, and they deserve detailed analysis and further publication. Another important direction for future development is to include interactions with matter fields, e.g. Fermi fields in quantum chromodynamics.

1. The Conclusion of the Operation R—Localization, Bounds, and Exponentiation

In this section we describe in detail the remaining operations necessary to conclude the **R**-operation, and mentioned briefly at the end of [IV]. The definition of the operation **R**′ was written in the form (1.100) [IV] in order to make clear the fundamental issues, like construction of a polymer expansion, and the exponentiation, but now we come back again to the simpler notation used throughout the last section, e.g., in the formula (1.99) [IV]. Consider one term on the right-hand side of (1.99) [IV]. In this term the operations $\mathbf{T}_k''$, $\mathbf{T}_h$, and the integration with respect to the variables V_h, are left unchanged by operations and considerations of this section, therefore we omit them in formulas below. The result of the **R**′-operation on the obtained expression can be written in the form

$$\chi_k \chi_{k,\Lambda} \delta_{G_0}(V_k') \chi \exp\left[-\frac{1}{g_k^2} A(\zeta_0, U_{k,Z}(V_k' V_\Lambda)) \right] \cdot \frac{\chi_{h,1/2} \int dV' \Gamma_{B_0} \delta_{T_0}(V') \chi' \exp A_k''}{\int dV' \Gamma_\Lambda \delta_{G_0}(V') \chi \exp\left[-\frac{1}{g_k^2} A(\zeta_0, U_{k,Z}(V' V_\Lambda)) \right]}, \tag{1.1}$$

where we have omitted also the averagings over the choices of G_0, T_0. Most of this section we will work with the above expression.

The basic problem for this expression is connected with the Wilson terms in the exponentials in the integrals. A part of this term from the nominator has to be combined with the term in the first exponential, to give the Wilson term of the new action. The remaining part has to be cancelled, or approximately cancelled, with the term from the denominator. This problem has to be analyzed carefully, because the terms are multiplied by g_k^{-2}, so they are potentially dangerous and they may spoil bounds. More precisely, the term from the denominator is dangerous, and we start the analysis with the integral in the denominator. Writing $V' = \exp i g_k B'$, identifying Λ with one of its components, and using (2.8) [I], we get

$$\begin{aligned}
&\int dV' \Gamma_\Lambda \delta_{G_0}(V') \chi \exp\left[-\frac{1}{g_k^2} A(\zeta_0, U_{k,Z}(V' V_\Lambda)) \right] \\
&= \exp\left[-\frac{1}{g_k^2} A(\zeta_0, U_0) + \left(-\tfrac{1}{2} d(g) \log g_k^{-2} + \log \sigma_0 \right) |\Lambda^{(k)} \backslash G_0| \right] \\
&\quad \cdot \int dB' \Gamma_\Lambda \sigma(g_k B') \delta_{G_0}(B') \chi(\{ |B'| < M_0 g_k^{-1} \varepsilon_k \})
\end{aligned}$$

$$\cdot\exp\left[-\tfrac{1}{2}\langle H_{1,k}B', \Delta_1(\zeta_0)H_{1,k}B'\rangle - \frac{1}{g_k}\langle DH_{1,k}B', \zeta_0\eta^{-2}\operatorname{Im}\partial U_0\rangle\right.$$
$$\left. - \frac{1}{g_k^2}V(\zeta_0, g_k H_{1,k}B')\right]. \tag{1.2}$$

The last term in the exponential is small, because the function V is at least of third order in the argument. It can be estimated by $O(g_k^{1-\beta})|\Lambda|$. To consider the second term in the exponential we make the following remark about the configuration U_0. It is a minimum of the functional

$$U \to A(U), \quad \text{for} \quad U: U \text{ defined and regular on } Z,$$
$$M_{\mathbf{B}_k}(U) = M_{\mathbf{B}_k(Z)}(Q_k^{s*}V_k) \quad \text{on} \quad Z\backslash\Lambda. \tag{1.3}$$

By the results of Sect. G [15] we have the condition

$$\langle \delta A, J_0\rangle = 0 \quad \text{for all} \quad \delta A: Q_{\mathbf{B}_k(Z)}\delta A = 0 \quad \text{on} \quad Z\backslash\Lambda, \tag{1.4}$$

where J_0 is the current defined by U_0, and $Q_{\mathbf{B}_k(Z)}$ is the linear averaging operation defined by U_0 and $\mathbf{B}_k(Z)$. In particular, the above condition is satisfied for all δA with $\operatorname{supp}\delta A \subset \Lambda$, therefore $J_0 = 0$ on Λ. The functions $\delta A = H_{1,k}B'$ satisfy also the condition in (5.4), therefore $\langle H_{1,k}B', J_0\rangle = \langle DH_{1,k}B', \operatorname{Im}\partial U_0\rangle = 0$, and we have

$$-\frac{1}{g_k}\langle DH_{1,k}B', \zeta\eta^{-2}\operatorname{Im}\partial U_0\rangle = \frac{1}{g_k}\langle DH_{1,k}B', (1-\zeta_0)\eta^{-2}\operatorname{Im}\partial U_0\rangle. \tag{1.5}$$

By the exponential decay of the minimizer, and the localizations of $1-\zeta_0$ and B', the expression on the right-hand side above can be bounded by

$$\frac{1}{g_k}B_3 M_0 A_0 p_0(g_k)\exp(-\delta\operatorname{dist}(\Omega_k, \Lambda))3\varepsilon_k|\Lambda|$$
$$< 3B_3 M_0 A_0^2 p_0^2(g_k)\exp(-R_k)(100M)^4, \tag{1.6}$$

where we have used the fact that Λ is contained in a cube of the size $100M$. The bound on the right-hand side above can be made arbitrarily small for sufficiently small g_k, but it is enough to have an absolute bound, e.g., the number 1.

Consider now the quadratic form in the exponential in (1.2). This quadratic form is positive definite, and it is simple to estimate the lower bound. At first, we apply the construction of Sect. F [15] to the configuration U_0, and doing a proper gauge transformation we represent it on the domain Z as $\exp i\xi A_0$, with A_0 satisfying the bound $|A_0|, |\nabla^\eta A_0| < O(1)M^6 R_k\varepsilon_k$. We also have to do the compensating adjoint gauge transformation on the variables B'. Then we expand the expressions defining the quadratic form with respect to A_0, up to the first order in A_0. This was discussed in Sect. B [13]. The leading term in the expansion is the quadratic form with the background field identically equal to 1. Replacing the minimizer in this form by the k^{th} minimizer defined on the whole lattice, and the function ζ_0 by the function identically equal to 1, we change the form by a quadratic form bounded by $O(\exp(-R_k))$. Now the leading quadratic form is equal to

$\langle B', \Delta_k B' \rangle$ defined by (1.65), (1.66) [10]. Using the bound (1.67) [10] for this form, we obtain

$$\langle H_{1,k}B', \Delta_1(\zeta_0)H_{1,k}B' \rangle \geqq \gamma_0 \| \partial B' \|^2 - O(1)(M^6 R_k \varepsilon_k + \exp(-R_k)) \| B' \|^2. \quad (1.7)$$

Consider the quadratic form $\| \partial B' \|^2$ on the fields B' defined on Λ, and equal to 0 on bonds of the graph G_0. Using the fact that Λ is a rectangular parallelepiped contained in a cube of the size $100M$, and that G_0 determines the axial gauge in Λ, we obtain the inequality

$$\sum_{b \in \Lambda} |B'(b)|^2 \leqq d(100M)^{d+1} \sum_{p \in \Lambda} |(\partial B')(p)|^2. \quad (1.8)$$

It follows by the same simple argument as in the proof of Lemma 2.4 in [11], it is even simpler in this case, because we do not have the averaging operations. The inequality is also much more general, holding for general domains Λ, and graphs G_0 fixing a gauge in Λ, but with different constants. For example, if in this domain Λ we fix an axial gauge in the direction of x_1-axis, i.e., we put $B'(\langle x - e_1, x \rangle) = 0$ for $x \in \Lambda$, then the inequality (1.8) holds with the constant $(100M)^3$. The inequalities (1.7), (1.8) imply finally

$$\langle H_{1,k}B', \Delta_1(\zeta_0)H_{1,k}B' \rangle \geqq \frac{\gamma_0}{2d(100M)^5} \| B' \|^2, \quad (1.9)$$

for g_k sufficiently small. The above inequality holds for U_0 in an arbitrary gauge. It allows us to prove Proposition 1 [IV], but at first we estimate the integral on the right-hand side of (1.2). This integral is in the denominator of (1.1), hence we are interested only in a lower bound, which follows from the boundedness of the quadratic form. The integral is bounded from below by $\exp(-O(1)|\Lambda|) = \exp(-O(M^4))$. In fact, we need such a bound for the integral extended analytically on G^c-valued configurations $\mathbf{U}$. We construct such an extension replacing the variables V_k in the function U_0 by the averages $M^k(\mathbf{U})$ for $\mathbf{U} \in U_k^c(Z, \alpha_{0,k}, \alpha_{1,k})$. The expressions in the integral are analytic functions of $\mathbf{U}$, and all the bounds above hold with ε_k replaced by $O(\alpha_{0,k} + \alpha_{1,k})$. This is an unessential change; it may result only in stronger restrictions on g_k, or γ. Thus we have

$$\left(\int dV' \Gamma_\Lambda \delta_{G_0}(V') \chi \exp\left[-\frac{1}{g_k^2} A(\zeta_0, U_{k,Z}(V'V_\Lambda)) \right] \right)^{-1}$$
$$= \exp\left[+\frac{1}{g_k^2} A(\zeta_0, U_0) + I(U_0) - E_k(\Lambda) \right],$$
$$E_k(\Lambda) = (-\tfrac{1}{2} d(\mathrm{g}) \log g_k^{-2} + \log \sigma_0) |\Lambda^{(k)} \backslash G_0|, \quad (1.10)$$

and the analytically extended $I(U_0)$ satisfies the bound

$$|I(U_0)| < O(1) \log M |\Lambda| < O(1) M^5. \quad (1.11)$$

From the equality (1.10) it is clear that we have to cancel the first term in the exponential on the right-hand side, otherwise we will not get good bounds for this exponential.

Now we prove Proposition 1 [IV]. For simplicity of the argument let us use the fact that V_k belongs to the domain determined by the characteristic functions $\chi_k \chi_{k,\Lambda}$, i.e., it has an extension from $Z \cap \Lambda^c$ onto Z, such that $U_{k,Z}(V_k)$ satisfies the inequality in (1.75) [IV]. We fix such an extension, and we consider the variational problem for the function $V' \upharpoonright_\Lambda \rightarrow A(U_{k,Z}(V'V_k))$, where V' satisfies mild regularity conditions. Fixing the gauge G_0 for V' we get a small configuration, and we can write $V' = \exp iB'$. We expand the function with respect to B', the expansion has the same form as in the exponentials in (5.2), but without the powers of g_k, and with $\zeta_0 = 1$. Now the condition for a critical configuration is the equation

$$\langle \delta B', H_{1,k}^* J_{k,Z} \rangle + \langle \delta B', H_{1,k}^* \Delta_1 H_{1,k} B' \rangle + \left\langle \delta B', H_{1,k}^* \left(\frac{\delta}{\delta A} V \right) (H_{1,k} B') \right\rangle = 0. \tag{1.12}$$

Denote by P_0 the projection onto the subspace of B' satisfying the gauge condition $B' \upharpoonright_{G_0} = 0$. By the inequality (1.9) the operator $P_0 H_{1,k}^* \Delta_1 H_{1,k} P_0$ is positive, hence invertible on this subspace, and the inverse is bounded by $\gamma_0^{-1} 2d(100M)^5$. Equation (1.12) can be written as

$$\begin{aligned} B' &+ (P_0 H_{1,k}^* \Delta_1 H_{1,k} P_0)^{-1} P_0 H_{1,k}^* \left(\frac{\delta}{\delta A} V \right) (H_{1,k} B') \\ &= -(P_0 H_{1,k}^* \Delta_1 H_{1,k} P_0)^{-1} P_0 H_{1,k}^* J_{k,Z}. \end{aligned} \tag{1.13}$$

Using Proposition 4 [15] and the fixed point theorem for contractive mappings, we can easily prove that the above equation has exactly one solution, which has a bound equal to twice a bound of the right-hand side of the equation, i.e., it can be bounded by $2\gamma_0^{-1} 2d(100M)^5 B_3^2 4\varepsilon_k$. This proves the existence and the uniqueness statements of Proposition 1 [IV], and the bound (1.78) [IV]. The above equations, bounds and statements are valid for $\mathbf{g}^c$-valued fields, hence the existence of the analytic extension follows immediately, and Proposition 1 [IV] is proved. It is proved for ε_k instead of a general ε, but the generalization is obvious.

Now we analyze the integral in the nominator of (1.1). The crucial term in the action A_k'' is the Wilson term $-A(1/(g_k''(\cdot))^2, U_k'')$. We have to extract from it the Wilson term $-(1/g_k^2)A(\zeta_0, U_0)$, in order to cancel the corresponding term in the exponential on the right-hand side of (1.10), and to preserve enough of it to get the small factors from large fields. For the remaining terms in the action we have to discuss the localization only. To analyze the Wilson term we write at first

$$A\left(\frac{1}{(g_k''(\cdot))^2}, U_k'' \right) = \frac{1}{g_k^2} A(U_k'') + A\left(\frac{1}{(g_k''(\cdot))^2} - \frac{1}{g_k^2}, U_k'' \right). \tag{1.14}$$

The second term on the right-hand side is localized in Λ as a function of the field U_k''. We leave it in this form. In the future we will apply to it the localization operation in the same way as for other terms in the action, and we will use it to get the bounds for large fields. The first term we divide into two parts:

$$A(U_k'') = A(\zeta_0, U_k'') + A(1 - \zeta_0, U_k''). \tag{1.15}$$

We localize the configuration U_k'' in the first term on the right-hand side in the domain Z, i.e., we apply the representation

$$U_k'' = U_{k,Z}''(M^{\cdot}(U_k''))$$
$$= \left(\exp i\eta \mathbf{H}_{k,Z}'' \left(\frac{1}{i} \log [M^{\cdot}(U_k'')(M^{\cdot}(Q_k^{s*} V_k))^{-1}] \upharpoonright_{Z \setminus Z^{\sim -1}} \right) U_{k,Z}'' \right)^{u''-1}. \tag{1.16}$$

The argument of the function $\mathbf{H}_{k,Z}''$ has a support in a layer of width $2M_1$ at the boundary ∂Z, and it can be bounded by $44d^2\varepsilon_k$, as it follows from the restrictions given by the characteristic functions χ_k. We expand the first term on the right-hand side of (1.15) with respect to the function $\mathbf{H}_{k,Z}''$:

$$A(\zeta_0, U_k'') = A(\zeta_0, \exp i\eta \mathbf{H}_{k,Z}'' U_{k,Z}'') = A(\zeta_0, U_{k,Z}'') + \langle D\mathbf{H}_{k,Z}'', \zeta_0 \eta^{-2} \operatorname{Im} \partial U_{k,Z}'' \rangle$$
$$+ \tfrac{1}{2} \langle \mathbf{H}_{k,Z}'', \Delta(\zeta_0) \mathbf{H}_{k,Z}'' \rangle + V_0(\zeta_0, \mathbf{H}_{k,Z}''). \tag{1.17}$$

The last three terms on the right-hand side are small, even after dividing by g_k^2. This follows from the above stated localization of the argument of the function $H_{k,Z}''$, and from the exponential decay in the scaled distance of this function (the inequality (190) [15]). From this exponential decay, and the localization of ζ_0, we get an additional exponential factor $\exp(-(1/2)\delta M R_k)$. It suppresses all powers of $\log g_k^{-2}$, and all powers of M. A bound of this type will be discussed later in a more complicated situation, e.g., see the estimate (1.47), so we omit details now. This implies also that the above terms are small if we take proper analytic extensions. These extensions will be discussed later. Now let us only notice that $\mathbf{H}_{k,Z}''$ is an analytic function of the configuration U_k'' restricted to the above mentioned boundary layer. The second term on the right-hand side of (1.15) is represented in a similar way. We apply (1.16) with Z replaced by $\Omega_k^{\sim}$, and $Z \setminus Z^{\sim -1}$ replaced by $\Omega_k^{\sim} \setminus \Omega_k$, and then the formula (1.17) with ζ_0 replaced by $1-\zeta_0$, and Z by $\Omega_k^{\sim}$. The last three terms in this expansion are also small for the same reason, more precisely we introduce a decomposition of unity connected with the partition π_k, and terms of the obtained sum are small, and exponentially small in the distance to Ω_k^c.

Consider the main term in the expansion (1.17). We take a function $\zeta_1 \in C_0^\infty(Z_{h+1}'')$, which changes from 0 to 1 on a neighborhood of the boundary $\partial Z_{h+1}''$ (on a layer of the width 2 at the boundary, in L^{-h}-scale). We divide this term again:

$$A(\zeta_0, U_{k,Z}'') = A(\zeta_0(1-\zeta_1), U_{k,Z}'') + A(\zeta_1, U_{k,Z}''). \tag{1.18}$$

Denote $\zeta_0(1-\zeta_1) = \zeta$. The configuration $U_{k,Z}''$ depends on V'', which is decomposed into $V'V_0$ on $\mathbf{B}_0$, according to (1.81) [IV]. The field $V' = \exp ig_k B$ is small, more precisely $|B| < g_k^{-1}\delta_k'$ by the restrictions (1.82) [IV] in the characteristic function χ', hence we have

$$U_{k,Z}'' = U_{k,Z}''(\exp ig_k B M^{\cdot}(U_0) \upharpoonright_{\Omega_{h+1}''^{\sim 2}}, V'' \upharpoonright_{(\Omega_{h+1}''^{\sim})^c})$$
$$= (\exp i\eta \mathbf{H}_{k,Z}''(g_k B) U_{k,Z}''(M^{\cdot}(U_0) \upharpoonright_{\Omega_{h+1}''^{\sim 2}}, V'' \upharpoonright_{(\Omega_{h+1}''^{\sim 2})^c})^{u_{k,Z}^{-1}}. \tag{1.19}$$

Notice that the field V'' on the domain $(\Omega_{h+1}''^{\sim 2})^c$ is equal to the old field V. We denote the new background field by $U_{k,Z}^0$. Expanding the first term on the right-hand side of (1.18) as in the exponential in (1.2), i.e., using (2.8) [I], we get

$$A(\zeta, U''_{k,Z}) = A(\zeta, \exp i\eta \mathbf{H}''_{k,Z}(g_k B) U^0_{k,Z}) = A(\zeta, U^0_{k,Z}) + g_k \langle DH''_{1,k,Z}B, \zeta\eta^{-2} \operatorname{Im} \partial U^0_{k,Z} \rangle$$
$$+ \tfrac{1}{2} g_k^2 \langle H''_{1,k,Z}B, \Delta_1(\zeta) H''_{1,k,Z}B \rangle + V(\zeta, g_k H''_{1,k,Z}B). \tag{1.20}$$

The last term above is at least of third order in $g_k B$, therefore it is small after dividing by g_k^2. Notice that this time bounds are a bit more difficult and larger, because of the many scales in the definitions of the operators and the field B. Thus the last term is bounded by $O(1)|\mathbf{B}_0|\delta_k'^3 \leqq O(1)(100MR_k)^d N^2 \delta_k'^3 \leqq g_k^3 O(1) M^d R_k^{d+2} A_1^3 p_1^3(g_k)$, and dividing the number on the right-hand side by g_k^2 we get still a small number, if g_k is small enough. We have assumed here that $N \leqq R_k$. Consider the quadratic form in (1.20). According to the formulas (79) [15], (3.10) [13], it is equal to a sum of several terms, which are small except the one term determined by the operator $D^*\zeta D$. To see that the other terms are small we have to establish bounds for the background field $U^0_{k,Z}$. Here we are interested only in simplest global bounds on the support of ζ. Using the estimate (1.80) [IV] and Theorem 1 [15] we see that $U^0_{k,Z}$ satisfies the usual regularity conditions (2) [15], for the sequence $\{\Omega''_j\}$ restricted to $\Omega_k^c \cap \Omega''^{\sim}_{h+1}$, with the constant $O(1)B_3^2 B_5 M^5 \varepsilon_k$ instead of ε_0. From these conditions we obtain that the sum of the above mentioned terms can be bounded by $O(1)B_3^5 B_5 M^5 A_1^2 p_1^2(g_k)\varepsilon_k |\mathbf{B}_0| \leqq O(1) B_3^5 B_5 M^{5+d} A_1^2 p_1^2(g_k) R_k^{d+2} \varepsilon_k$, and this bound is small for g_k small enough. Notice that we obtain a similar bound, with a larger power of $\log g_k^{-2}$ only, if we extend the configuration U_0 to G^c-valued fields in the way described in the paragraph before (1.10). Thus the quadratic form in (1.20) can be written as a sum of the nonnegative form

$$\sum_{p \in T_\eta} \eta^d \zeta(x(p)) |(DH''_{1,k,Z}B)(p)|^2, \tag{1.21}$$

and the small form, with the above written bound. In the effective action we have the form (1.21) with the minus sign, therefore it can be bounded by 0. Other bounds for this form will be discussed later.

Now consider the first two terms on the right-hand side of (1.20). The first, the Wilson action term, is a desirable term, which we will transform further, but the second term, linear in B, should be small. This cannot be seen just by a simple estimate of this term, and we have to use the fact that on the support of ζ the field $U^0_{k,Z}$ differs only a little bit from the field U_0, which is a critical configuration, and for which the linear term is equal to 0. To use it, we represent the configuration $U^0_{k,Z}$ again in the familiar way. We introduce the determining set $\mathbf{B}_1 = \mathbf{B}''_k(Z) \cup \mathbf{B}_h(\Omega''^{\sim 2}_{h+1})$, where the operation of joining of two determining sets is given by (2.14) [III], and we take the corresponding function $U_{\mathbf{B}_1}$. It is supported in the domain $Z \cap \Omega''^{\sim 2}_{h+1}$, and on this domain we have

$$U^0_{k,Z} = U_{\mathbf{B}_1}([M^\cdot(U^0_{k,Z})(M^\cdot(U_0^{(AL)}))^{-1}] M^\cdot(U_0^{(AL)}))$$
$$= \left(\exp i\eta \mathbf{H}_{\mathbf{B}_1}\left(\frac{1}{i} \log [M^\cdot(U^0_{k,Z})(M^\cdot(U_0^{(AL)}))^{-1}]\right) U_{\mathbf{B}_1}(M^\cdot(U_0^{(AL)}))\right)^{u_1^{-1}} \tag{1.22}$$

The argument of the function $\mathbf{H}_{\mathbf{B}_1}$ has a support in the boundary layer of the width $2M_1$ (in the L^{-h}-scale), at the boundary $\partial\Omega''^{\sim 2}_{h+1}$. The configuration $U^0_{k,Z}$ satisfies the regularity condition (1.96) [IV] on $(\Omega''^{\sim}_{h+1})^c \cap \Omega_h$. In fact by the same argument as the one leading to (1.96) [IV], we have to forget only about the field B' in the bounds. By the usual reasoning the field in the argument of the function $\mathbf{H}_{\mathbf{B}_1}$ can be bounded by $11d^2\varepsilon_h \leqq 11d^2(1+\beta_0)^2 N^{\beta_0}\varepsilon_k \leqq 11d^2(1+\beta_0)^2 R_k^{\beta_0}\varepsilon_k$, hence

$$L^j\eta|\mathbf{H}_{\mathbf{B}_1}|, (L^j\eta)^2|\nabla^\eta_{U^0(\mathbf{B}_1)}\mathbf{H}_{\mathbf{B}_1}| < B_3 \exp(-\delta M R_{h+1}) 11 d^2(1+\beta_0)^2 R_k^{\beta_0}\varepsilon_k \tag{1.23}$$

on $\Omega''_j \cap \Omega^c_k$, for $j = h+1, \ldots, k$. Consider the Wilson action term in (1.20). Using (1.22) we expand it with respect to $\mathbf{H}_{\mathbf{B}_1}$, as in (1.17):

$$\begin{aligned} A(\zeta U^0_{k,Z}) = A(\zeta, \exp i\eta \mathbf{H}_{\mathbf{B}_1} U^0_{\mathbf{B}_1}) = A(\zeta, U^0_{\mathbf{B}_1}) + \langle D\mathbf{H}_{\mathbf{B}_1}, \zeta\eta^{-2}\operatorname{Im}\partial U^0_{\mathbf{B}_1}\rangle \\ + \tfrac{1}{2}\langle \mathbf{H}_{\mathbf{B}_1}, \Delta(\zeta)\mathbf{H}_{\mathbf{B}_1}\rangle + V_0(\zeta, \mathbf{H}_{\mathbf{B}_1}). \end{aligned} \tag{1.24}$$

For the new background field $U^0_{\mathbf{B}_1}$ we have, by (4.87):

$$\begin{aligned} U^0_{\mathbf{B}_1} &= U_{\mathbf{B}_1}(M^\cdot(U_0^{(AL)})) = U_{\mathbf{B}_1}(M^\cdot(U_0^{u_0})) \\ &= (U_{\mathbf{B}_1}(M^\cdot(U_0)))^{\bar u_0} = U_0^{\bar u_0} = (U_0^{(AL)})^{\bar u_0 u_0^{-1}}, \end{aligned} \tag{1.25}$$

where $\bar u_0$ is a gauge transformation constant on blocks of the determining set $\mathbf{B}_1$, and equal to u_0 at centers of the blocks, i.e., on $\mathbf{B}_1$. This equality, the bounds for the configuration U_0, and the bounds (1.23) imply that the last three terms on the right-hand side of the equality (1.24) are small, after dividing by g_k^2. For example, the second term can be bounded by

$$\begin{aligned} &O(1)B_3B_5M^5\varepsilon_k B_3\exp(-\delta M R_{h+1})11d^2(1+\beta_0)^2R_k^{\beta_0}\varepsilon_k \\ &\quad\cdot(|\Omega''_k \cap \Omega^c_k| + L^{d-2}|\Omega''_{k-1}\backslash\Omega''_k| + \cdots + L^{(N-1)(d-2)}|\Omega''_{h+1}\backslash\Omega''_{h+2}|) \\ &\leqq g_k^2 O(1)A_0^2B_3^2B_5M^{d+5}R_k^{d+1}p_0^2(g_k)\exp(-R_k), \end{aligned}$$

and the number multiplying g_k^2 can be made arbitrarily small for g_k small enough. Similar, or better, bounds hold for the remaining two terms. Thus, the only important large term is the Wilson action term on the right-hand side of (1.24). We will consider it later, now we analyze the second term on the right-hand side of (1.20). At first we localize the minimizer $H''_{1,k,Z}$. We construct the generalized random walk expansion for it, including $Z \cap \Omega''^{\sim}_{h+1}$ as one of the localization domains X, see Sect. C [13] for details. Other localization domains are the same as in that paper; they are based on the partitions into M_1-cubes in corresponding scales, and they intersect the domain $(\Omega''^{\sim}_{h+1})^c$. The expansion has the form (3.107) [13], where $X_0 = Z \cap \Omega''^{\sim}_{h+1}$, and the terms of the expansion satisfy (3.108) [13]. We consider them on the domain $\Omega^c_k \cap \Omega''_{h+1}$. More precisely the first argument of their kernels is restricted to the support of ζ, hence the first term of the expansion corresponds to the walk $(0, X_0)$, and is given by the function H''_{1,k,X_0}. The remaining terms correspond to walks intersecting $(\Omega''^{\sim}_{h+1})^c$, hence the bound (3.108) [13] provides the additional exponential factor $\exp(-(1/4)\delta_0 M R_{h+1})$. We represent $H''_{1,k,Z}$ as a sum of the first term and the sum of the remaining terms:

$$H''_{1,k,Z} = H''_{1,k,X_0} + H''_{X_0,(\Omega''^{\sim}_{h+1})^c}. \tag{1.26}$$

The decomposition yields the corresponding decomposition of the linear term in (1.20). The bound

$$|DH_{X_0(\Omega''^{\sim}_{h+1})^c}B| < O(1)(L^j\eta)^{-2}\exp(-\tfrac{1}{4}\delta_0 M R_{h+1})A_1 p_1(g_k) \tag{1.27}$$

allows us to estimate the second term of this decomposition in the same way as the second term of the expansion (1.24) above, hence it is a small term. The first term is equal to

$$g_k\langle DH''_{1,k,X_0}B,\zeta\eta^{-2}\operatorname{Im}\partial U^0_{k,Z}\rangle. \tag{1.28}$$

This expression depends on the background field $U^0_{k,Z}$ restricted to the domain $Z\cap\Omega''^{\sim}_{h+1}$ (or rather to the neighborhood of this domain obtained by adding the boundary layer of the width $5M_1L^{-N+1}$ at the boundary $\partial\Omega''^{\sim}_{h+1}$), and the dependence is analytic. We apply the representation (1.22), and we expand (1.28) up to the first order in $\mathbf{H}_{\mathbf{B}_1}$. A simple estimate of the expression (1.28) gives the bound $g_k^2O(1)A_0^2B_3^2B_5M^{d+5}R_k^dp_0^2(g_k)$, hence the first order term in this expansion can be bounded by this number multiplied by the right-hand side of the inequality (1.23), and by some absolute constant. Thus the first order term of this expansion is small, and we have to consider the expression (1.28) with $U^0_{k,Z}$ replaced by $U^0_{\mathbf{B}_1}$. Using (1.25), we obtain

$$\begin{aligned}
&g_k\langle DH''_{1,k,X_0}B,\zeta\eta^{-2}\operatorname{Im}\partial U^0_{\mathbf{B}_1}\rangle\\
&=g_k\langle H''_{1,k,X_0}R(\bar u_0^{-1})B,D^*\zeta\eta^{-2}\operatorname{Im}\partial U_0\rangle\\
&=g_k\langle\zeta H''_{1,k,X_0}R(\bar u_0^{-1})B,J_0\rangle+g_k\sum_{x,\mu}\eta^{d-3}\operatorname{tr}(H''_{1,k,X_0}R(\bar u_0^{-1})B)_\mu(x)\\
&\quad\cdot\sum_{\nu\neq\mu}(\partial_\nu^{1*}\zeta)(x)R(U_0(\langle x,x-\eta e_\nu\rangle))\operatorname{Im}U_0(\partial p_{\nu\mu}(x-\eta e_\nu)).
\end{aligned} \tag{1.29}$$

The sum on the right-hand side above is localized in two domains on which $\partial^{1*}\zeta\neq 0$: the first is the boundary layer at $\partial\Omega_k$, and the second is the boundary layer at $\partial\Omega''_{h+1}$. On the first domain we estimate the second sum in (1.29) by $(d-1)3\varepsilon_k\eta^3$, hence the whole expression can be bounded by

$$g_k(100MR_k)^{d-1}4dB_3A_0p_0(g_k)\exp(-\delta 8MR_k)3(d-1)\varepsilon_k,$$

which is a small number, even after dividing by g_k^2. On the second domain the second sum is estimated by $(d-1)L^{-h}O(1)B_3B_5M^5\varepsilon_k\eta^2$, and the expression is bounded by

$$\begin{aligned}
&g_k(L^h\eta)^{d-1}(100MR_{h+1})^{d-1}4dB_3A_0p_0(g_k)(d-1)O(1)B_3B_5M^5\varepsilon_k\\
&\leqq g_k^2L^{-(d-1)N}N^{(d-1)\beta_0}O(1)A_0^2B_3^2B_5M^{d+5}R_k^{d-1}p_0^2(g_k).
\end{aligned}$$

The number multiplying g_k^2 is small if L^{-N} is small enough, for example if $L^{-N}\leqq(\log g_k^{-2})^{-\nu}$ for sufficiently large ν, or $N\geqq(\nu/\log L)\log\log g_k^{-2}$. For the smallness of the above bound we need $\nu\geqq 2p_0+dr_0$. Thus, with the above assumptions, the second term on the right-hand side of (1.29) is small, after dividing it by g_k^2. Consider now the first term. We write it as a sum of two terms corresponding to the decomposition $\zeta=(1-\zeta_1)-(1-\zeta_0)(1-\zeta_1)$. For the second term the support of the function $(1-\zeta_0)(1-\zeta_1)=1-\zeta_0$ is sufficiently far from the support of the field B. The distance is larger than $8MR_k$, hence a bound for this term has the additional exponential factor $\exp(-\delta 8MR_k)$, and the term is small. The first term is equal to $g_k\langle(1-\zeta_1)H''_{1,k,X_0}R(\bar u_0^{-1})B,J_0\rangle$, and the function multiplying J_0 satisfies the condition

$$Q_{\mathbf{B}_k(Z)}(1-\zeta_1)H''_{1,k,X_0}R(\bar u_0^{-1})B=0 \quad \text{on} \quad Z\backslash\Lambda.$$

It follows from the definition of the function H''_{1,k,X_0}, and from the fact that the support of the field B is contained in Λ. This condition is exactly the same as the condition for δA in (1.4), therefore the above term is equal to 0. Thus we have proved that in the combined expansions (1.20), (1.24) all terms are small, except the Wilson action term on the right-hand side of (1.24). This term is equal to

$$A(\zeta, U_0) = A(\zeta_0, U_0) - A(\zeta_1, U_0). \tag{1.30}$$

The second term on the right-hand side can be estimated as follows:

$$0 \leqq A(\zeta_1, U_0) < g_k^2 L^{-4N} N^{4\beta_0} O(1) A_0^2 B_3^2 B_5^2 M^{14} R_k^4 p_0^2(g_k). \tag{1.31}$$

For N satisfying the conditions discussed above the bound on the right-hand side above is small, after dividing by g_k^2. The only term which remains as possibly large is the first term on the right-hand side of (1.30).

Let us summarize the result of the above analysis of the Wilson action term in the effective action A''_k. We have

$$\begin{aligned} -A\left(\frac{1}{(g''_k(\cdot))^2}, U''_k\right) &= -A\left(\frac{1}{(g''_k(\cdot))^2} - \frac{1}{g_k^2}, U''_k\right) - \frac{1}{g_k^2} A(1-\zeta_0, U''_{k,Z^{\sim-3}}) \\ &\quad - \frac{1}{g_k^2} A(\zeta_1, U''_{k,Z}) - \frac{1}{2} \sum_{p \in T_\eta} \eta^4 \zeta(x(p)) |(DH''_{1,k,Z} B)(p)|^2 \\ &\quad - \frac{1}{g_k^2} A(\zeta_0, U_0) + O(1), \end{aligned} \tag{1.32}$$

where the term $O(1)$ denotes the sum of all the small terms obtained by the above transformations and expansions. We will transform further the first two terms on the right-hand side of the above equality. The first term can be written as a sum of terms localized in components of the large field region. We divide them into two parts, one is localized in Z, another in $Z_k \backslash Z$. The first is transformed as in (1.17), i.e., it is written as the sum of the Wilson term

$$-A\left(\left(\frac{1}{(g''_k(\cdot))^2} - \frac{1}{g_k^2}\right) Z, U''_{k,Z}\right)$$

and the remaining terms, corresponding to the terms in (1.17). The remaining terms are small by the same reason as in the case of the expansion (1.17). We have to notice only that $(1/(g''_k(\cdot))^2) - 1/g_k^2$ is bounded by $O(1)(k-j)$ on $\Omega''_j \backslash \Omega''_{j+1}$, and the exponential decay of $H''_{k,Z}$ suppresses this bound. The second part, localized in $Z_k \backslash Z$, is transformed in the same way as $A(1-\zeta_0, U''_k)$, namely it is represented as

$$-A\left(\left(\frac{1}{(g''_k(\cdot))^2} - \frac{1}{g_k^2}\right)(Z_k \backslash Z), U''_{k,Z^{\sim-3}}\right)$$

plus the sum of the corresponding small terms, as in (1.17) again. The above Wilson action term is combined together with the second term on the right-hand side of (1.32), and the sum of these two terms is equal to

$$-A\left(\frac{1}{(g''_k(\cdot))^2}(1-\zeta_0), U''_{k,Z^{\sim-3}}\right).$$

Now we introduce a new determining set $\mathbf{B}_k$, and a new background field U_k. We define $\mathbf{B}_k$ as equal to $\mathbf{B}_k''$ on Z^c, and to $\{Z^{(k)}\}$ on Z. More precisely

$$\mathbf{B}_k=\{\Gamma_j\}, \quad \text{where} \quad \Gamma_j=\Gamma_j''\cap Z^c \quad \text{for} \quad j<k, \quad \Gamma_k=\Gamma_k''\cup Z^{(k)}. \tag{1.33}$$

On this determining set we define new gauge field variables V:

$$V\lceil_{Z^c}=V''\lceil_{Z^c}, \quad V\lceil_Z=V_k'V_\Lambda\lceil_Z. \tag{1.34}$$

Let us recall that $V_k'=1$ on Λ^c, and V_Λ is defined on the whole set $Z^{(k)}$, and equal to V_k outside Λ. The new background field U_k is defined by

$$U_k=U_{\mathbf{B}_k}(V). \tag{1.35}$$

We define also the new function $1/(g_k(\cdot))^2=1/(g_k''(\cdot))^2Z^c+1/(g_k^2)Z$. We use here and above the same notation for a set, and for its characteristic function. Consider the last Wilson action term above. We expand the configuration $U''_{k,Z^{\sim -3}}$ on the support of $1-\zeta_0$ around the new background field U_k. We use a representation inverse to (1.16), i.e., we write

$$\begin{aligned} U''_{k,Z^{\sim-3}} &= U(\mathbf{B}_k''\cup\mathbf{B}_k(Z^{\sim-3}),[M^{\cdot}(Q_k^{s*}V_k)(M^{\cdot}(U_k))^{-1}]M^{\cdot}(U_k)) \\ &= \left(\exp i\eta\mathbf{H}''_{k,Z^{\sim-3}}\left(-\frac{1}{i}\log[M^{\cdot}(U_k)(M^{\cdot}(Q_k^{s*}V_k))^{-1}]\right)U_k\right)^{u_k^{-1}}. \end{aligned} \tag{1.36}$$

The argument of the $\mathbf{H}$-function above has a support in the boundary layer of the width $2M_1$ at the boundary $\partial Z^{\sim-3}$, and it is bounded by $44d^2\varepsilon_k$. Expanding the Wilson action with respect to this function, we get again an expansion of the type (1.17), with all terms small except the first one, which in this case is equal to

$$-A\left(\frac{1}{(g_k(\cdot))^2}(1-\zeta_0),U_k\right).$$

This expression is now combined together with the Wilson action in the first exponential in (1.1), only transformed in the same way as above, i.e., with the configuration $U_{k,Z}$ replaced by U_k, and with some new small terms. The sum of the two actions is equal to

$$-A\left(\frac{1}{(g_k(\cdot))^2},U_k\right),$$

which is the new Wilson action term for a final effective action, the rest of which will be constructed later.

All the above transformations of the Wilson action terms from all the expressions in (1.1) yield

$$\begin{aligned} &-\frac{1}{g_k^2}A(\zeta_0,U_{k,Z}(V_k'V_\Lambda))-A\left(\frac{1}{(g_k''(\cdot))^2},U_k''\right)+\frac{1}{g_k^2}A(\zeta_0,U_0) \\ &= -A\left(\frac{1}{(g_k(\cdot))^2},U_k\right)-A\left(\left(\frac{1}{(g_k''(\cdot))^2}-\frac{1}{g_k^2}\right)\zeta,U''_{k,Z}\right)-A\left(\frac{1}{(g_k''(\cdot))^2}\zeta_1,U''_{k,Z}\right) \\ &\quad -\frac{1}{2}\sum_{p\in T_\eta}\eta^4\zeta(x(p))|(DH''_{1,k,Z}B)(p)|^2+O(1). \end{aligned} \tag{1.37}$$

The first term on the right-hand side is the basic Wilson term of a new effective action, and we write it in the first exponential in (1.1), instead of the previous local one. The second term is in fact a small term, because

$$\frac{1}{(g_k''(\cdot))^2}-\frac{1}{g_k^2}=O(N)\leqq O(R_k)$$

on the support of ζ, so it can be bounded by

$$O(1)N^{2+4\beta_0}(100MR_k)^4(O(1)B_3B_5M^5\varepsilon_k+O(1)B_3\delta_k')^2\leqq O(1)B_3^2B_5^2M^9R_k^7\varepsilon_k^2,$$

and the bound is small for g_k small enough. We include this term into the sum of small terms denoted by $O(1)$ in (1.37). Let us repeat the remark about the analyticity properties of these terms, formulated already several times, e.g., before the formula (1.10). Each term depends analytically on one of the background fields which appeared in the above constructions, and the bounds are preserved if we replace this field by the corresponding G^c-valued field. This replacement will be described more precisely later, when we will discuss localization. Consider now the third term on the right-hand side of (1.37). It is a very important term, because the exponential of this term gives the small factors connected with large plaquette variables in the domain Ω_h^c. We have separated this term from the old action exactly for this purpose. There is one problem connected with this term. We have to extend the function $U_{k,Z}''$, or rather the function $U_0^{(AL)}$ in its argument, to the G^c-function, and then the obvious positivity properties of the Wilson action are lost. We will show that this extension can be written as a small perturbation of a positive Wilson action, from which we can get all the small factors. The extension is constructed by replacing in

$$U_0^{(AL)}=U_{k,Z}^{(L)}(V_k\lceil_{Z\cap\Lambda^c},V_\Lambda(V_k\lceil_{Z\cap\Lambda^c}))$$

the variables V_k by $M^k(\mathbf{U})$, where $\mathbf{U}\in U_k^c(Z,\alpha_{0,k},\alpha_{1,k})$. By the definition we have $\mathbf{U}=U'U$, where U is a G-valued configuration satisfying the regularity condition $|\partial U-1|<\alpha_{0,k}\eta^2$, and $U'=\exp i\eta A'$, where A' is a $\mathbf{g}^c$-valued function satisfying the bounds $|A'|$, $|\nabla_U^\eta A'|<\alpha_{1,k}$. We write $M^k(\mathbf{U})=\tilde{M}^k(U')M^k(U)$, and $\tilde{M}^k(U')=\exp i\tilde{Q}_k(\eta A')=\exp i\tilde{B}'$, $|\tilde{B}'|<O(1)\alpha_{1,k}$. In the proof of Proposition 1 [IV] we have noticed that the function V_Λ is an analytic function of the field V_k, but now we need a more quantitative statement. We would like to establish an identity of the form

$$V_\Lambda(M^k(\mathbf{U}))=V_\Lambda'(\tilde{M}^k(U'))V_\Lambda(M^k(U)),\tag{1.38}$$

holding in the axial gauge, and to prove analyticity properties and bounds for the function V_Λ'. Of course the identity is a definition of this function, so the bounds are important. We may replace also the k^{th} averages by fields $\tilde{V}'$, V, satisfying proper smallness, or regularity conditions. The function V_Λ' is determined as a critical point of the function

$$V'\lceil_\Lambda\rightarrow A(U_{k,Z}(\tilde{V}'V,V'V_\Lambda(V))),\tag{1.39}$$

where $V'=\exp iB'$, and B' is sufficiently small. We expand the above function with respect to $(\tilde{B}',B')$, $\tilde{B}'=(1/i)\log\tilde{V}'$. The expansion has the same form as the one in the exponentials in (1.2), only with $\zeta_0 = 1$ and $g_k = 1$. Differentiating it with respect to B', using the criticality condition for $U_0 = U_{k,Z}(V, V_\Lambda(V))$, and inverting the linear operator as in (1.13), we obtain the equation

$$B' + (P_0 H_{1,k}^* \Delta_1 H_{1,k} P_0)^{-1} P_0 H_{1,k}^* \left(\frac{\delta}{\delta A} V \right) (H_{1,k} B' + H_{1,k} \tilde{B}')$$
$$= -(P_0 H_{1,k}^* \Delta_1 H_{1,k} P_0)^{-1} P_0 H_{1,k}^* \Delta_1 H_{1,k} \tilde{B}'. \tag{1.40}$$

The notation above is a bit simplified, in fact we should write $H_{1,k,Z}$ instead of $H_{1,k}$ and $Q_{\mathbf{B}_k(Z)} Q_k^{s*} \tilde{B}'$ instead of $\tilde{B}'$, but it does not matter, because we restrict the operators to Λ, and the boundary effects at the boundary ∂Z are negligible. We can prove again that Eq. (1.40) has exactly one solution $B'_\Lambda(\tilde{B}')$, which is an analytic function of the $\mathbf{g}^c$-valued small field $\tilde{B}'$, satisfying the bound

$$|B'_\Lambda(\tilde{B}')| \leqq 4d(100M)^5 \gamma_0^{-1} B_3^2 |\tilde{B}'|. \tag{1.41}$$

This yields the formula

$$V'_\Lambda(\tilde{M}^k(U')) = \exp i B'_\Lambda \left(\frac{1}{i} \log \tilde{M}^k(U') \right) \tag{1.42}$$

for the function in the identity (1.38), and a quantitative control of the analytically extended function V_Λ. Consider now the extended function

$$U_0(\mathbf{U}) = U_{k,Z}(M^k(\mathbf{U}), V_\Lambda(M^k(\mathbf{U}))) = U_{k,Z}(\tilde{M}^k(U') M^k(U), V'_\Lambda(\tilde{M}^k(U')) V_\Lambda(M^k(U))) \tag{1.43}$$

in the axial gauge. Existence of the analytic extension follows from the results of Sect. G [15], and from the results and bounds formulated above. From these results it follows also that a j^{th} average of the above configuration can be represented in the form

$$M^j(U_0(\mathbf{U})) = \tilde{M}^j(U'_{k,Z}(\tilde{M}^k(U'), V'_\Lambda(\tilde{M}^k(U')))) M^j(U_0(U))$$
$$= \tilde{V}_{j,Z}(\tilde{M}^k(U')) M^j(U_0(U)), \tag{1.44}$$

where $\tilde{V}_{j,Z}(V')$ is an analytic function of $(1/i)\log V'$, satisfying the bound

$$|\tilde{V}_{j,Z}(V') - 1| \leqq O(1) B_3 B_5 M^5 |\log V'|. \tag{1.45}$$

The representation and the bound hold on the j^{th} domain of the determining set $\mathbf{B}_k(Z)$, for $j = 0, 1, \ldots, k$. They all hold on the domain Λ, and the field $M^j(U_0(U))$ is also small on Λ; it satisfies the bound (1.84) [IV], with $\alpha_{0,k}$ instead of ε_k. From the results of Sect. G [15] it follows again that the function $\mathbf{H}_{k,\Lambda}$ in the representation (1.86) [IV] has the analytic extension $\mathbf{A}_0 = \mathbf{H}_{k,\Lambda}((1/i)\log M^{\cdot}(U_0(\mathbf{U})))$ satisfying (1.87) [IV], with $\alpha_{0,k} + \alpha_{1,k}$ instead of ε_k. This is the information we need to discuss the third term on the right-hand side of (1.37). It will be used also later, in the discussion of localization. Consider now the expression

$$A\left(\frac{1}{(g_k''(\cdot))^2} \zeta_1, U''_{k,Z} \right).$$

To simplify the discussion we localize it again in the domain Z''_{h+2} instead of Z.

We will not repeat again all the considerations connected with this operation. They should be obvious; let us state only that all terms of the expansion corresponding to (1.17) are small by the exponential decay of the **H**-function, except the term $A((1/(g_k''(\cdot))^2)\zeta_1, U''_{k,Z''_{h+2}})$. The configuration $U''_{k,Z''_{h+2}}$ depends on the field $V'V_0 = \exp ig_k B \exp i\tilde{Q}.(\eta \mathbf{A}_0)$ on $\Omega''^{\sim 2}_{h+1} \cap Z''_{h+2}$, and on the field V'' on $(\Omega''^{\sim 2}_{h+1})^c$. We consider here the function $\mathbf{A}_0$ extended analytically to the fields **U**. Denote for simplicity the above configuration by U''_{h+2}. It is an analytic function of $\mathbf{A}_0$, hence of **U**, and we have the expansion

$$\begin{aligned} &U''_{h+2}([R(\exp ig_k B)\exp i\tilde{Q}.(\eta \mathbf{A}_0)]\exp ig_k B, V'') \\ &\quad = (\exp i\eta \mathbf{H}''_{h+2}(R(\exp ig_k B)\tilde{Q}.(\eta \mathbf{A}_0))U''_{h+2}(\exp ig_k B, V''))^{u''^{-1}_{h+2}}. \end{aligned} \tag{1.46}$$

The argument of the function $\mathbf{H}''_{h+2}$ has a support in the domain $\Omega''^{\sim 2}_{h+1} \cap Z''_{h+2}$. This is very important, because the exponential decay of this function suppresses strongly large values of $1/(g_k''(\cdot))^2$. The above expansion yields the usual expansion of the Wilson action, i.e., the expansion of the form (1.17) with proper changes. Again, all the terms in this expansion are small, except the term $A((1/(g_k''(\cdot))^2)\zeta_1, U''_{h+2})$. For example, consider the most dangerous term, the term linear in $\mathbf{H}''_{h+2}$. It can be bounded in the following way:

$$\begin{aligned} &\left|\left\langle D\mathbf{H}''_{h+2}, \frac{1}{(g_k''(\cdot))^2}\zeta_1 \eta^{-2}\operatorname{Im}\partial U''_{h+2}\right\rangle\right| \\ &< \sum_{j=1}^{h}\sum_{x\in\Gamma_j''} B_3 \exp(-\delta d(x,\Omega''^{\sim 2}_{h+1}))O(1)L^{-N}B_3^2 B_5 M^6(\alpha_{0,k}+\alpha_{1,k}) \\ &\quad \cdot\left(\frac{1}{g_k^2}+O(k-j)\right)\varepsilon_j \leqq O(1)\frac{1}{g_k^2}(\alpha_{0,k}+\alpha_{1,k})\varepsilon_k L^{-N}N^{1+\beta_0} \\ &\quad \cdot B_3^3 B_5 M^6 \sum_{j=1}^{h}\sum_{x\in\Gamma_j''}\exp(-\tfrac{1}{2}\delta d(x,\Omega''^{\sim 2}_{h+1}))\exp(-\tfrac{1}{2}\delta M(h-j))(1+h-j)^{1+\beta_0} \\ &\leqq O(1)A_0 C_1 B_3^3 B_5 M^6 p_0(g_k)q_1(g_k)R_k^2 L^{-N}|\Gamma_h''\cap\Omega''^{\sim 2}_{h+1}| \\ &\leqq O(1)A_0 C_1 B_3^3 B_5 M^{10} p_0(g_k)q_1(g_k)R_k^7 L^{-N}. \end{aligned} \tag{1.47}$$

The last bound above is small under the usual conditions on N. Bounds for the other terms in the expansion are similar. Thus we have

$$A\left(\frac{1}{(g_k''(\cdot))^2}\zeta_1, U''_{h+2}(\mathbf{U})\right) = A\left(\frac{1}{(g_k''(\cdot))^2}\zeta_1, U''_{h+2}(\exp ig_k B, V'')\right) + O(1), \tag{1.48}$$

where the terms in $O(1)$ are small, and depend analytically on the field **U** restricted to the domain Z. The gauge field in the Wilson action on the right-hand side is G-valued, hence the action is positive, and it provides the necessary bounds for large plaquette variables.

We have finished the operations connected with the Wilson action terms in (1.1). It is an important part of the procedure, and also the most difficult one, because it involves the problem of cancellation of the terms in the numerator and the denominator in (1.1). This problem dictates some basic aspects of the **R**-operation, for example it determines the number N of the usual renormalization steps we have to perform for a given large field region, before we can do this operation. In the course of the above transformations we have created many new terms in the effective action. They have been obtained by expanding some Wilson actions, so they have a similar structure; in particular they are easily localizable, but they depend on several different background fields. They have, however, two important common features: they are all small, in fact their sum is also small, although we need boundedness only, and they depend on background fields restricted to a neighborhood of the region Z, therefore they are boundary terms, according to our classification of terms in effective actions. We denote the sum of all these terms, including the term $I(U_0)$ from the denominator, by $\mathbf{C}_k$. As the result of all the transformations above we obtain the following equality:

$$
\begin{aligned}
(1.1) = {} & \chi_k \chi_{k,A} \delta_{G_0}(V'_k) \chi \exp\left[-A\left(\frac{1}{(g_k(\cdot))^2}, U_k \right) \right] \\
& \cdot \chi_{h,1/2} \int dB \sigma(g_k B) \delta_{T_0}(B) \chi' \exp\left[-\tfrac{1}{2} \langle DH''_{1,k,Z} B, \zeta DH''_{1,k,Z} B \rangle \right. \\
& - A\left(\frac{1}{(g''_k(\cdot))^2} \zeta_1, U''_{h+2}(\exp i g_k B, V'') \right) + \mathrm{C}_k \\
& + \mathbf{E}_k(U''_k) + \mathbf{R}_k(U''_k) + \mathbf{B}_k(U''_k, A) + \mathbf{B}''_k(U''_k, A) \\
& \left. - E_k + (-\tfrac{1}{2} d(\mathbf{g}) \log g_k^{-2} + \log \sigma_0) |\mathbf{B}_0 \setminus T_0| - E_k(A) \right]. \qquad (1.49)
\end{aligned}
$$

Let us recall that the term $\mathbf{R}_k$ is not complete yet at this stage. It is equal to $\mathbf{R}_{k-1} + \mathbf{R}''^{(k)}$, where $\mathbf{R}''^{(k)}$ is the contribution from $\mathbf{R}_{k-1}$ connected with the last $\mathbf{T}$-operation. The essential contribution to $\mathbf{R}^{(k)}$ will be constructed by this $\mathbf{R}$-operation. Let us recall also that the term $\mathbf{B}''_k$ is the sum of the new boundary terms created by the N preliminary integrations considered in the previous section.

Our problem now is to reconstruct the effective action corresponding to the new determining set $\mathbf{B}_k$ and the background field U_k, possible with some new boundary terms. We have done it for the main term in the action, and we have obtained the Wilson action in the first exponential. Now we will extract other terms from the effective action in (1.49), more precisely these terms which contribute to the new action. We divide the terms in the second exponential in (1.49) into two parts: the first part includes the first three terms, and these terms of the rest of the effective action, for which their localization domains intersect the region Z, the second part includes the remaining terms, i.e., the terms with localization domains disjoint with Z. The second part contributes to the new action, more precisely if we replace in it the field U''_k by the new background field U_k, then we obtain all the terms of the new action with localization domains disjoint with Z. To obtain such terms, we expand the terms of the second part around their values at the configuration U_k. These expansions are obtained from an expansion of the configuration U''_k on the domain Z^c, around the configuration U_k, which we describe first.

Let us write explicitly the configuration U''_k, and its dependence on the variable fields:

$$U_k'' = U_{\mathbf{B}_k''}(V''\lceil_{\Lambda^c}, \exp ig_k BM^\cdot(U_0^{(AL)})\lceil_{\Lambda\cap\Omega_{h+1}''^{\sim 2}}, V''\lceil_{(\Omega_{h+1}''^{\sim 2})^c}). \tag{1.50}$$

The determining set $\mathbf{B}_k''$ restricted to the domain Λ^c, and the field $V''\lceil_{\Lambda^c}$, coincide with the corresponding determining set, and the field, for the new configuration U_k. Therefore the expansions of U_k'' are connected with changes made inside the domain Λ. The first expansion is with respect to B, and we have as in (1.19)

$$U_k'' = (\exp i\eta \mathbf{H}_k''(g_k B)U_k^0)^{u_k''^{-1}}, \tag{1.51}$$

where U_k^0 is the configuration in (1.50) with $B=0$. This configuration considered on Z^c depends weakly on the field $V''\lceil_{(\Omega_{h+1}''^{\sim 2})^c}$, and we try to cut off this dependence completely. The argument of U_k^0 is equal to $M^\cdot(U_0^{(AL)}) = \exp i\tilde{Q}_\cdot(\eta\mathbf{A}_0)$ on $Z_k''\cap\Omega_{h+1}''^{\sim 2}$. We expand with respect to it, but around some special configuration. We define it in the following way. Take the determining sets

$$\mathbf{B}_1 = \mathbf{B}_k''\cup\mathbf{B}_h(\Omega_{h+1}''^{\sim}), \quad \mathbf{B}_2 = \mathbf{B}_k''\cup\mathbf{B}_h((\Omega_{h+1}''^{\sim})^c), \tag{1.52}$$

where the operation of joining two determining sets is given by (2.14) [III], and take the corresponding functions $U_{\mathbf{B}_1}$, $U_{\mathbf{B}_2}$. These functions are supported in the domains $\Omega_{h+1}''^{\sim}, (\Omega_{h+1}''^{\sim})^c$ correspondingly, and they are introduced in order to break the configuration U_k^0 into two independent parts by the boundary conditions at the boundary $\partial\Omega_{h+1}''^{\sim}$. Define the configurations

$$U_1 = U_{\mathbf{B}_1}(V''\lceil_{\Lambda^c}, M^k(U_0^{(AL)})\lceil_{\Lambda\cap\Omega_k''}, 1\lceil_{Z_k''\cap\Omega_{h+1}''^{\sim}}),$$

$$U_2 = U_{\mathbf{B}_2}(1\lceil_{(\Omega_{h+1}''^{\sim})^c\cap\Omega_{h+1}''^{\sim 2}}, V''\lceil_{(\Omega_{h+2}''^{\sim 2})^c}),$$

$$U_{1,2} = \begin{cases} U_1 & \text{on} \quad \Omega_{h+1}''^{\sim}, \\ U_2 & \text{on} \quad (\Omega_{h+1}''^{\sim})^c. \end{cases} \tag{1.53}$$

In the configuration $U_{1,2}$ the arguments are separated completely across the boundary $\partial\Omega_{h+1}''^{\sim}$. It seems to be singular on a neighborhood of this boundary, but in fact it is regular, and satisfies the regularity conditions typical for an h-order configuration on the domain $(\Omega_{h+1}'')^c\cap\Omega_{h+1}''^{\sim 2}$. It is so, because the arguments of the corresponding functions are equal to 1 on this domain, and the exponential decay properties enforce the additional regularity. We construct the expansion of U_k^0 around the configuration $U_{1,2}$. From the results of Sects. D, E [15] we obtain

$$U_k^0 = (\exp i\eta H_k''(\mathbf{G}_k''J_{1,2}, H_{1,k}''\tilde{Q}_\cdot(\eta\mathbf{A}_0)\lceil_{Z_k''\cap\Omega_{h+1}''^{\sim 2}})U_{1,2})^{(u_k^0)^{-1}}. \tag{1.54}$$

We have written explicitly the form of the operator dependence of the function $\mathbf{H}_k''$ on the fields $J_{1,2}$, $\tilde{Q}_h(\eta\mathbf{A}_0)$. The second field is obviously localized in the domain $Z_k''\cap\Omega_{h+1}''^{\sim 2}$, and we will use this localization and the exponential decay of the function $\mathbf{H}_k''$ to get exponential factors in the localization procedure. The field $J_{1,2}$ is not localized, but in the composition $\mathbf{G}_k''J_{1,2}$ we may localize it, because of the properties of the kernel of the operator $\mathbf{G}_k''$. The property important here is the equality $\mathbf{G}_k''Q_{\mathbf{B}_k''}^* = 0$, see the definition (3.148) [13] and the identities (3.149)–(3.153) [13]. The field $J_{1,2}$ restricted to $\Omega_{h+1}''^{\sim}$ is equal to J_1, and it satisfies the criticality condition $\langle\delta A, J_1\rangle = 0$ for δA such, that $Q_{\mathbf{B}_1}\delta A = 0$. This condition for δA, and the condition for $\mathbf{G}_k''$ coincide on the domain $\Omega_{h+1}''^{\sim}$, except the boundary layer of the width $2M_1$. We use this fact to localize the considered expression. We introduce a function $\theta \in C_0^\infty((\Omega_{h+1}'')^c)$, such that $\theta = 1$ on the union of $(\Omega_{h+1}''^{\sim})^c$ and the above boundary layer, and changes from 1 to 0 near the boundary of this layer. We write $\mathbf{G}_k'' J_{1,2} = \mathbf{G}_k''(1-\theta)J_1 + \mathbf{G}_k''\theta J_{1,2}$, and the second term of the sum is localized properly. For the first term the kernel of $\mathbf{G}_k''(1-\theta)$ does not satisfy the condition for averages on the domain where the function $1-\theta$ is neither equal to 1, nor to 0. We improve this writing

$$\mathbf{G}_k''(1-\theta) = \mathbf{G}_k''(1-\theta)(1 - Q_{\mathbf{B}_1}^* H_{\mathbf{B}_1}^*) + \mathbf{G}_k''(1-\theta) Q_{\mathbf{B}_1}^* H_{\mathbf{B}_1}^*.$$

Of course, we have the equality

$$\mathbf{G}_k''(1-\theta)(1 - Q_{\mathbf{B}_1}^* H_{\mathbf{B}_1}^*) Q_{\mathbf{B}_1}^* = 0,$$

hence the above condition for J_1 implies

$$\mathbf{G}_k''(1-\theta)(1 - Q_{\mathbf{B}_1}^* H_{\mathbf{B}_1}^*) J_1 = 0.$$

Thus we have

$$\mathbf{G}_k'' J_{1,2} = \mathbf{G}_k''(1-\theta) Q_{\mathbf{B}_1}^* \theta_1 H_{\mathbf{B}_1}^* J_1 + \mathbf{G}_k'' \theta J_{1,2}, \tag{1.55}$$

where the function θ_1 is equal to 1 on the domain where $\theta \neq 0, 1$, and changes from 1 to 0 on a neighborhood of this domain. The expression on the right-hand side above has a good localization property, which yields the required exponential factors. The two formulas (1.51), (1.54) give the expansion of U_k'' around the configuration $U_{1,2}$. This configuration restricted to Z^c is equal to U_1. Next, we expand U_1 around the configuration

$$U_{k,\Lambda} = U_k(V''\lceil_{\Lambda^c}, V_\Lambda^{(A)}). \tag{1.56}$$

The field $V_\Lambda^{(A)}$ is in the axial gauge in Λ, hence it is small inside Λ, and averages of $U_{k,\Lambda}$ are also small there. More precisely the condition (1.84) [IV] is satisfied for it. This allows us to write

$$\begin{aligned} U_1 &= U_{\mathbf{B}_1}(V''\lceil_{\Lambda^c}, M^k(U_{k,\Lambda})\lceil_{\Lambda \cap \Omega_k''}, (M^\cdot(U_{k,\Lambda}))^{-1} M^\cdot(U_{k,\Lambda})\lceil_{Z_k''}) \\ &= \left(\exp i\eta \mathbf{H}_{\mathbf{B}_1}\left(-\frac{1}{i} \log M^\cdot(U_{k,\Lambda})\lceil_{Z_k''} \right) U_{k,\Lambda} \right)^{u_1^{-1}}. \end{aligned} \tag{1.57}$$

The field in the argument of the function $\mathbf{H}_{\mathbf{B}_1}$ has again a right localization, and the function is decaying exponentially off the domain Z_k''. Finally, we obtain an expansion around U_k introducing the field V_k' into the argument of the function $U_{k,\Lambda}$. We have

$$U_{k,\Lambda} = U_k(V''\lceil_{\Lambda^c}, V_k'^{-1} V_k' V_\Lambda^{(A)}) = \left(\exp i\eta \mathbf{H}_k \left(-\frac{1}{i} \log V_k' \lceil_\Lambda \right) U_k \right)^{u_{k,\Lambda}^{-1}}. \tag{1.58}$$

The sequence of the four formulas (1.51), (1.54), (1.57), and (1.58) yields the required expansion of the configuration U_k'' around U_k.

Now we apply the above expansions to terms of the second part of the effective action. We repeat all the considerations and constructions of Sect. 3, [III], and

Sects. 3–6 [I] so we will be very brief here, and we will mention only basic issues. Consider a term $\mathbf{E}^{(j)}(X, U''_k, z)$ with $X \subset Z^c$, and let $z \in \Omega_n \backslash \Omega_{n+1}$ for $n \geqq j$. As in Sect. 3 we take the cube $\square \in \pi_n$ containing the point z, and we consider the case $X \subset \square^{\sim 2}$. Using a somewhat simplified version of the transformation (3.3), (3.18), (3.19) [I] we write

$$\begin{aligned}\mathbf{E}^{(j)}(X, U''_k, z) = \mathbf{E}^{(j)}(X, U^0_k, z) + [\mathbf{E}^{(j)}(X, U_j(\mathbf{B}_j(\square_0), \exp iQ.(\eta \mathbf{H}''_k(g_k B)) M^{\cdot}(U^0_k)), z) \\ - \mathbf{E}^{(j)}(X, U_j(\mathbf{B}_j(\square_0), M^{\cdot}(U^0_k)), z)],\end{aligned} \tag{1.59}$$

where $\square_0 = \square^{\sim 5}$ (the operation $\sim$ is defined here in terms of M-cubes of the L^{-n}-lattice). To the expression in the square bracket above we apply all the transformations and estimates of Sects. 3–5 [I] (with η replaced by L^{-n}). We obtain a sum of expressions with good bounds, depending analytically on $\mathbf{H}''_k(g_k B)$ restricted to the cube $\square_0$. The field B is localized in Λ, and we try to decouple it from the cube $\square_0$, and also we try to decouple components of Z. More precisely, we apply the expansion (6·9) [I] to the above expressions, the expansion with σ_0 defined as the set of cubes Δ disjoint with $\square_0$ and $\Lambda^{\sim}$. We write this expansion in the form (6.10) [I], where the summation is over domains $Y_0 \in \mathbf{D}_k$ containing $\square_0$, and at least one of the components of $\Lambda^{\sim}$. The other terms in the expansion vanish, because of the localization of the field B. The nonvanishing terms in the expansion can be estimated as in (6.24)–(6.29) [I], in fact the bounds are even simpler and better in this case, because on the domains with nonzero parameters s the field B is equal to 0. Terms $\mathbf{R}^{(j)}(X, U''_k)$ are considered in the same way. For the terms with localization domains X not contained in $\square^{\sim 2}$, and for the boundary terms in the second group, we write the following simpler identity

$$\mathbf{E}^{(j)}(X, U''_k, z) = \mathbf{E}^{(j)}(X, U^0_k, z) + [\mathbf{E}^{(j)}(X, \exp i\eta \mathbf{H}''_k(g_k B) U^0_k, z) - \mathbf{E}^{(j)}(X, U^0_k, z)], \tag{1.60}$$

and we repeat all the above described operations for the expression in the square bracket. As a result of the final localization expansion we get a sum of terms with the above described localization domains Y_0. For a fixed domain Y_0 we resum all the expressions constructed above, and having Y_0 as the final localization domain. This resummation is almost exactly the same as the one discussed in Sect. 6 [I], it is controlled by the renormalization procedure for $\mathbf{E}$-terms, the renormalization and the bounds (2.31) [III] for $\mathbf{R}$-terms, and the bounds (2.42) [III] for $\mathbf{B}$-terms. In effect, for a given Y_0 we obtain a term which is small, and satisfies the exponential bound (2.42) [III] in $d_k(Y_0)$. We consider all these terms as boundary terms connected with corresponding components of Z, and we will consider them later, together with other terms of the first part of the effective action in (1.49). Let us notice only that the above expressions are analytic functions of U^0_k on Y_0, and the analyticity domain is restricted on the cube $\square_0$, or on the domain X, by the analyticity properties of the original term. Outside the cube $\square_0$, or the domain X, this analyticity domain is quite large, restricted only by the analyticity properties of the function $\mathbf{H}''_k$. The bounds mentioned above are satisfied for these analytically extended expressions.

Next, we consider the first terms on the right-hand side of the equalities (1.59), (1.60). We apply again the same equalities to them, but using the representation

(1.54). Then we repeat all the other operations described above, in particular the localization expansions, which give the second set of expressions localized in the domains Y_0, but depending analytically on the background field $U_{1,2}$. The first terms on the right-hand sides of the equalities depend on the field U_1, and we repeat again the procedure using the representation (1.57), and then once more using the representation (1.58). After the last operation we obtain again the original terms on the right-hand sides of the last set of equalities, but taken at the background field U_k. These are the desired terms we need to reconstruct the new action, in fact we get almost the whole new action, except the terms with localization domains intersecting the domain Z, and the new terms of $\mathbf{R}^{(k)}$, which will be obtained by the **R**-operation.

Now we consider terms of the first part of the effective action in (1.49), except the first two terms, and all the above constructed new terms with the localization domains Y_0. The first operation is exactly the same as in the previous case, i.e., we apply either the equality (1.59), or (1.60), and the localization operation is determined by the set σ_0 of cubes Δ disjoint with $\square_0$, or X, and with $\Lambda^{\sim}$. We obtain some new terms, and all the terms depend on the background field U_k^0. Let us take one term in the above sum. It has a localization domain X intersecting Z. We assume that U_k^0 is the background field for this term, and we denote it by $\mathbf{E}(X, U_k^0)$, although it may be any term in the sum. Take the smallest localization domain $Y_1 \in \mathbf{D}_k$ containing X, and containing these components of Z, which intersect X. For the domain Y_1 we take the determining set $\mathbf{B}_k''(Y_1) = \mathbf{B}(Y_1) \cup \mathbf{B}_k''$, and the representation

$$U_k^0 = U(\mathbf{B}_k''(Y_1), M^{\cdot}(U_k^0)) = U(\mathbf{B}_k''(Y_1), ((u_k^0)_-^{-1} \exp i\tilde{Q}.(\eta \mathbf{H}_k'') R((u_k^0)_+) M^{\cdot}(U_1) \upharpoonright_{(Y_1^0)^c \cap Y_1},$$
$$\cdot V'' \upharpoonright_{\Lambda^c \cap Y_1^0}, M^{\cdot}(U_0^{(AL)}) \upharpoonright_{\Lambda \cap \Omega_{h+1}''^{\sim 2} \cap Y_1}, V'' \upharpoonright_{(\Omega_{h+1}''^{\sim 2})^c \cap Y_1})). \tag{1.161}$$

Here Y_1^0 is the domain on which the determining sets $\mathbf{B}_k''(Y_1)$ and $\mathbf{B}_k''$ coincide, hence $Y_1 \setminus Y_1^0$ is a boundary layer at the boundary δY_1, of the width $2M_1$ at most, for the corresponding scale. Notice that we have to keep the gauge transformation u_k^0 in the above configuration. We cannot remove it, because we use the two different representations for fields at the boundary of the boundary layer. We use the fact that $u_k^0(x)$ is an analytic function of $\mathbf{H}_k''$ restricted to this block of the set $\mathbf{B}_k''$, to which the point x belongs. This function is given by the explicit formula (106) [12]. We substitute the representation (1.61) into the function $\mathbf{E}(X, U_k^0)$. We apply the localization operation to the function $\mathbf{H}_k''$, with the set $\tilde{\sigma}_0$ of cubes Δ disjoint with Λ, and we represent this expansion in the form of the expansion (6.10) [I], where the summation is over domains Y_2' such that each component of Y_2' belongs to $\mathbf{D}_k$, contains at least one component of Λ, and intersects the boundary layer $Y_1 \setminus Y_1^0$. A term in this expansion depends on the background field $U_{1,2}$ restricted to $Y_2 = Y_1 \cup Y_2'$. Consider the configuration U_1, given by (1.53). For the domain Y_2 we take the determining set $\mathbf{B}_1(Y_2) = \mathbf{B}(Y_2) \cup \mathbf{B}_1$, and the representation

$$U_1 = U(\mathbf{B}_1(Y_2), M^{\cdot}(U_1)) = U(\mathbf{B}_1(Y_2), (u_{1,-}^{-1} \exp i\tilde{Q}.(\eta \mathbf{H}_{\mathbf{B}_1}) R(u_{1,+}) M^{\cdot}(U_{k,\Lambda}) \upharpoonright_{(Y_2^0)^c \cap Y_2},$$
$$\cdot V'' \upharpoonright_{\Lambda^c \cap Y_2}, M^{\cdot}(U_0^{(AL)}) \upharpoonright_{\Lambda \cap \Omega_k''}, 1 \upharpoonright_{Z_k'' \cap \Omega_{h+1}''^{\sim}}), \tag{1.62}$$

where we have used the expansion (1.57). The same remarks apply to this representation, as to (1.61), in particular Y_2^0 is the domain on which the determining sets $\mathbf{B}_1(Y_2)$ and $\mathbf{B}_1$ coincide. The function

$$\mathbf{H}_{\mathbf{B}_1} = \mathbf{H}_{\mathbf{B}_1}\left(-\frac{1}{i}\log M^{\cdot}(U_{k,\Lambda})\restriction_{Z_k''}\right)$$

is small on the boundary layer, and it depends analytically on the field $U_{k,\Lambda}$. We substitute the function on the right-hand side of (1.62) into the considered term of the expansion. For the obtained expression we construct again the localization expansion for the function $\mathbf{H}_{\mathbf{B}_1}$ with the set $\tilde{\sigma}_0$ of cubes Δ disjoint with Λ, and we resum it and write it in the form of the expansion (6.10) [I]. The summation is over domains Y_3' such that each component belongs to $\mathbf{D}_k$, contains a component of Λ, and intersects $Y_2 \setminus Y_2^0$. A term of this expansion depends on the field $U_{k,\Lambda}$ restricted to $Y_3 = Y_2 \cup Y_3'$, which depends still nonlocally on the gauge field variables V''. We repeat once more the above construction for the field $U_{k,\Lambda}$. It is given by (1.56), and we take the determining set $\mathbf{B}_k(Y_3) = \mathbf{B}(Y_3) \cup \mathbf{B}_k$, and the representation

$$\begin{aligned} U_{k,\Lambda} &= U(\mathbf{B}_k(Y_3), M^{\cdot}(U_{k,\Lambda})) \\ &= U(\mathbf{B}_k(Y_3), ((u_{k,\Lambda})_-^{-1} \exp i\tilde{Q}.(\eta\mathbf{H}_k)R((u_{k,\Lambda})_+)M^{\cdot}(U_k)\restriction_{(Y_3^0)^c \cap Y_3}, V''\restriction_{\Lambda^c \cap Y_3}, V_\Lambda^{(A)}), \end{aligned} \tag{1.63}$$

where we have used the expansion (1.58). Here the function $\mathbf{H}_k$ is given by

$$\mathbf{H}_k = \mathbf{H}_k\left(-\frac{1}{i}\log V_k'\restriction_\Lambda\right) = \mathbf{H}_k\left(-\frac{1}{i}\log V_k(V_\Lambda^{(A)})^{-1}\restriction_\Lambda\right). \tag{1.64}$$

We substitute the representation (1.63) into the term of the last expansion, corresponding to the domain Y_3, and we construct the localization expansion for the function $\mathbf{H}_k$, determined by the set $\tilde{\sigma}_0$ of cubes Δ disjoint with Λ. We resum it and we get the expansion (6.10) [I], with the summation over domains Y_4' such that each component belongs to $\mathbf{D}_k$, contains a component of Λ, and intersects the domain $Y_3 \setminus Y_3^0$. A term of this expansion depends on the background field U_k restricted to $Y_4 = Y_3 \cup Y_4'$. This is the final localization, because all other functions in the constructed expressions depend on the new gauge field variables, or on the background field U_k, restricted to this domain. Combining all the expansions we represent the term $\mathbf{E}(X, U_k^0)$ as a sum of terms, the summation over domains Y_2', Y_3', Y_4' satisfying all the conditions described above; the term corresponding to the given domains depends on the background field U_k restricted to the domain $Y_4 = Y_1 \cup Y_2' \cup Y_3' \cup Y_4'$. The dependence is analytic. Now we should resum all terms having the same localization domain, but at first we have to discuss more carefully the analyticity properties, and bounds for terms of the above expansion.

Let us start with the analyticity properties. We express a term in the above expansion as a function of the variables $(\mathbf{U}, \mathbf{J})$. We introduce these variables in the usual way into the localized function $\mathbf{H}_k$ in (1.63), hence into $u_{k,\Lambda}$. We replace $M^{\cdot}(U_{k,})$, V'' by the corresponding $M^{\cdot}(\mathbf{U})$, and $V_\Lambda^{(A)}$ by the function $V_\Lambda^{(A)}(M^k(\mathbf{U}))$. Such an extended function on the right-hand side of (1.63) is an analytic function of $(\mathbf{U}, \mathbf{J})$ with a large domain of analyticity. The function $\mathbf{H}_k(s(Y_4'))$ on $(Y_3^0)^c \cap Y_3$

is also very small, because of the localization of the argument, and the exponential decay, hence the regularity properties of the configuration (1.63) are almost the same, as the regularity properties of $U(\mathbf{B}_k(Y_3),(M^{\cdot}(\mathbf{U})\lceil_{\Lambda^c}, V_{\Lambda}^{(A)}(M^k(\mathbf{U}))))$, which, considered on the domain $(\Lambda^{\sim})^c$, are almost the same as these properties for the configuration $U(\mathbf{B}_k(Y_3), M^{\cdot}(\mathbf{U}))$. We substitute the extended function (1.63) into the function of the right-hand side of (1.62), in the place of the field $U_{k,\Lambda}$, and into $\mathbf{H}_{\mathbf{B}_1}$ as the background field and in the argument of this function. The remaining fields in (1.62) are replaced by functions of $\mathbf{U}$ in the usual way. The function $\mathbf{H}_{\mathbf{B}_1}$ is very small on $Y_2 \setminus Y_2^0$, again by the exponential decay and the localization of its argument. Hence the regularity properties of (1.62) are almost the same as these properties of $U(\mathbf{B}_1(Y_2),(M^{\cdot}(U_{k,\Lambda})\lceil_{(Y_2^0)^c \cap Y_2}, M^{\cdot}(\mathbf{U})\lceil_{\Lambda^c \cap Y_2}, M^{\cdot}(U_0^{(AL)}(\mathbf{U}))\lceil_{\Lambda \cap \Omega_k''}, 1\lceil_{Z_k'' \cap \Omega_{h+1}''^{\sim}}))$, which, on $(\Lambda^{\sim})^c$, are almost the same as of the configuration $U(\mathbf{B}_1(Y_2), M^{\cdot}(\mathbf{U}))$. We substitute the extended function (1.62) into (1.61), and we introduce the variables $(\mathbf{U},\mathbf{J})$ into other expressions in the usual way. We obtain the final function of $(\mathbf{U},\mathbf{J})$, for which the regularity properties on the domain $(\Lambda^{\sim})^c$ are almost the same as for the configuration $U(\mathbf{B}_k''(Y_1), M^{\cdot}(\mathbf{U}))$. On the domain $\Lambda^{\sim}$ the regularity properties are similar to the properties of the configuration U_k''. A precise description of the regularity properties is given by the following statement:

the extended function U_k^0 described above is an analytic function of the variables $(\mathbf{U},\mathbf{J})$ in the space $\tilde{U}_k^c(Y_4,\tilde{\alpha}_0,\tilde{\alpha}_1)$, with values in the space $\tilde{U}_k''^c(Y_1,\tilde{\alpha}_0,\tilde{\alpha}_1)$, i.e.,

$$(\mathbf{U},\mathbf{J}) \in \tilde{U}_k^c(Y_4,\tilde{\alpha}_0,\tilde{\alpha}_1) \to U_k^0(\mathbf{U},\mathbf{J}) \in \tilde{U}_k''^c(Y_1,\tilde{\alpha}_0,\tilde{\alpha}_1). \tag{1.65}$$

Let us recall that the last space above, the space of values for the function U_k^0, is the space defined by (1.64)–(1.69) [IV] for $n = N$. The above statement has been proved already, in a slightly less general form, in the proof of the equality (1.89) [IV]. The arguments needed for the proof of (1.65) are almost exactly the same as there; in many places it is enough to replace ε_j by $\alpha_{0,j}$, or to make similar simple changes, therefore we do not repeat them here. Let us explain only that the statement (1.65), similar to the equality (1.89) [IV], is one of the basic points of our construction, and the preliminary integrations of the previous section were done in order to create a smaller space of values of the function (1.65). In particular, the space $\tilde{U}_k''^c(Y_1,\tilde{\alpha}_0,\tilde{\alpha}_1)$ is contained in the analyticity domain of the term $\mathbf{E}(X)$ we consider, hence in the analyticity domains of all terms in the obtained effective action. It is clear for terms of $\mathbf{B}_k''$, by the inductive assumptions formulated in the previous section (see the inequality (1.70) [IV] and the assumptions connected with it). It is also clear for all new terms created by the previous localization operations. Finally, it is simple to see for all the old terms, because they are connected with the old determining set, for which the regularity conditions on the crucial domain $\Omega_{k_0+1}'' \setminus \Omega_{k_0+1}$ are weaker than the regularity conditions for the new determining set. This implies the basic fact that the function $\mathbf{E}(X, U_k^0)$ is an analytic function of the variables $(\mathbf{U},\mathbf{J})$ on the space $\tilde{U}_k^c(Y_4,\tilde{\alpha}_0,\tilde{\alpha}_1)$.

Let us consider now bounds for the terms of the obtained expansion. We discuss here only exponential factors, the other factors, like the powers of $L^j\eta$ or $\alpha_{0,k}$ coming from the renormalization procedure, were discussed thoroughly before. The exponential factors are important here, because they control the resummations we have to do, and they give an exponential factor for the final resummed terms. The inductive assumption yields the factor

$$\begin{aligned}\exp(-\kappa d_j(X)) &\leqq \exp(-\beta\kappa d_j(X))\exp(-(1-\beta)\kappa(L^j\eta)^{-1}d_k(Y_0))\\ &\leqq \exp(-\beta\kappa d_j(X))\exp(-(1+3\beta)\kappa d_k(Y_0)) \quad \text{for} \quad j<k,\end{aligned}$$

where Y_0 is the smallest domain from $\mathbf{D}_k$ containing X, and $\exp(-(1+4\beta)\kappa d_k(X))$ for $j=k$, from the improved bounds after the k^{th} renormalization transformation T. Every localization operation above yields, after the resummations connected with a fixed domain Y_i', $i=2,3,4$, the factor $\exp(-(\kappa_1-1)M^{-d}|Y_i'|)$. If $Y_2'\cup Y_3'\cup Y_4'\neq\emptyset$, then the exponential decay of the functions $\mathbf{H}_k''$, $\mathbf{H}_{\mathbf{B}_1}$, $\mathbf{H}_k$ and $\mathbf{H}(\mathbf{B}_k''(Y_1),\cdot)$ yields the additional factor $\exp(-\delta d((Y_1^0)^c\cap Y_1\cap Y_i', X))$, where i is the first index with a nonempty domain Y_i', and $d(\cdot,\cdot)$ is the scaled distance, see the definition (2.36) [11]. Thus we obtain the follwing exponential factor for the term in the expansion, corresponding to domains Y_2', Y_3', Y_4' with a nonempty union:

$$\exp\left(-\beta\kappa d_j(X)-(1+3\beta)d_k(Y_0)-\delta d((Y_1^0)^c\cap Y_1\cap Y_{i_0}', X)-\sum_{i=2}^{4}(\kappa_1-1)M^{-d}|Y_i'|\right), \tag{1.66}$$

where i_0 is the index of the first nonempty domain Y_i'. We have the equality $Y_4\setminus Z=(Y_0\setminus Z)\cup\bigcup_{i=2}^{4}(Y_i'\setminus Z)$; therefore the above exponential provides a complete exponential factor for the domain $Y_4\setminus Z$. The domains Y_i' connect $Y_i'\setminus Z$ with Λ, and the third term in the exponential yields a connection of X with Y_{i_0}'. Therefore we also have exponential factors connecting different components of $Y_4\setminus Z$. Introduce the following definition of the relative linear size $d_{k,Z}(Y)$ of a domain $Y\in\mathbf{D}_k$:

$$\begin{aligned}d_{k,Z}(Y)=M^{-1}\ &\text{(the length of a shortest tree graph contained in } Y\\ &\text{and intersecting all } M\text{-cubes of the components of } Y\setminus Z).\end{aligned} \tag{1.67}$$

The above considerations imply that we can extract the factor $\exp(-(1+3\beta)\kappa d_{k,Z}(Y_4))$ from the exponential (1.66), and the following exponential remains:

$$\exp\left(-\beta\kappa d_j(X)-\tfrac{1}{2}\delta d((Y_1^0)^c\cap Y_1\cap Y_{i_0}', X)-\sum_{i=2}^{4}\tfrac{1}{2}(\kappa_1-1)M^{-d}|Y_i'|\right).$$

This we use now to control the resummations. We fix the domain $Y=Y_4$, and we resum at first over *all* the admissible domains Y_i'. Then, for a particular type of terms, we sum over all admissible localization domains X. For **E**-terms and **R**-terms this includes renormalization in the form described in the proof of Theorem 2 in Sect. 2 [III]. We sum also over j. For the boundary terms we have just the summations over X and j. Finally, we sum over different types of terms. All the above summations were discussed in Sects. 6, 7 [I], and in Sects. 2, 3 [III]. Consider now the exceptional terms, for which all the domains Y_i' are empty. Then there are only the first two terms in the exponential (1.66), $Y_1=Y_4=Y$ and $Y_0\setminus Z=Y_1\setminus Z$. If the last domain is nonempty, then $d_k(Y_0)\geqq d_{k,Z}(Y_1)=d_{k,Z}(Y)$, and we can still resum over all terms with domains X such that $Y_0\setminus Z$ is fixed and equal to $Y\setminus Z$. The sum is bounded by a constant times the same exponential factor as before.

The sum of the remaining exceptional terms, and the sum of terms with the new localization domains equal to a component of Z, satisfies a much worse bound, similar to the bound (2.48) [III] for the sum of the boundary terms. This sum is localized in a component of Z. We make the last resummation, we sum all the terms with localization domains contained in a component of $Z^{\sim}$. For the remaining terms we use a part of the exponential factor to get a small constant in the bound. Thus, after all these resummations we obtain a sum of expressions, the summation is over domains $Y \in \mathbf{D}_k$ satisfying the property that the intersection $Y \cap Z^{\sim}$ is a union of components of $Z^{\sim}$, containing at least one component. The expression corresponding to such a domain Y depends on the new background field U_k restricted to Y, and also on the fields A, B, V'' localized in Y. We denote this expression by $V(Y, U_k)$. It has an analytic extension $V(Y,(\mathbf{U},\mathbf{J}))$ defined on the space $\tilde{U}_k^c(Y, \tilde{\alpha}_0, \tilde{\alpha}_1)$, and satisfying the bounds:

$$|V(Y,(\mathbf{U},\mathbf{J}))| \leqq \alpha \exp(-(1+2\beta)\kappa d_{k,Z}(Y)) \tag{1.68}$$

for $Y \backslash Z^{\sim} \neq \emptyset$, with a constant α, which can be chosen arbitrarily small for γ small enough, and

$$|V(Y,(\mathbf{U},\mathbf{J}))| \leqq O(1) \sum_{j=1}^{k} |\Gamma_j^0 \cap Y|, \tag{1.69}$$

if Y is a component of $Z^{\sim}$. Here $\{\Gamma_j^0\}$ denotes the old determining set, obtained after the last renormalization transformation, with which we have started the analysis in Sect. [IV].

We have reconstructed almost the whole new action connected with the new determining set $\mathbf{B}_k$, except the terms with localization domains intersecting Z, and, of course, except new $\mathbf{R}^{(k)}$-terms. Now we add all the missing terms with localization domains intersecting the domain Z, more exactly we add these terms to complete the effective action, and we subtract them from the remaining expression. They may be resummed in domains Y described above, and the obtained expressions satisfy the bounds (1.68), (1.69). We subtract them from the corresponding terms $V(Y)$, and we obtain new terms satisfying these bounds. For simplicity we denote them by $V(Y)$ also. After the addition of the above described terms we obtain a new action, which contains all the necessary $\mathbf{E}$-terms, and all the $\mathbf{R}$-terms and boundary terms connected with the first $k-1$ operations $\mathbf{R}T$. It contains also some $\mathbf{R}$-terms and boundary terms connected with the k^{th} renormalization transformation T, but these will be completed yet. Denote this action by A_k'. The result of all the preceding operations can be written as the equality:

$$\begin{aligned}(1.1) = {} & \chi_k \chi_{k,A} \delta_{G_0}(V_k') \chi \exp A_k'\left(\frac{1}{(g_k(\cdot))^2}, U_k\right) \\ & \cdot \chi_{h,1/2} \int dB \sigma(g_k B) \delta_{T_0}(B) \chi' \exp\Bigg[-\tfrac{1}{2}\langle DH_{1,k,Z}'' B, \zeta DH_{1,k,Z}'' B\rangle \\ & - A\left(\frac{1}{(g_k''(\cdot))^2}\zeta_1, U_{h+2}''(\exp ig_k B, V'')\right) + \sum_Y V(Y, U_k, B) \\ & + E_k(Z) + (-\tfrac{1}{2}d(\mathbf{g})\log g_k^{-2} + \log \sigma_0)|\mathbf{B}_0 \backslash T_0| - E_k(A)\Bigg],\end{aligned} \tag{1.70}$$

where $E_k(Z)$ denotes the sum of normalization terms and vacuum energy terms with localizations intersecting the large field domain Z.

Now we try to remove the characteristic functions $\chi_{k,\Lambda}\chi$. We introduce the decompositions of unity

$$1 = \chi_k((\Omega_k^{\sim 4})^c) + \chi_k^c((\Omega_k^{\sim 4})^c)$$

in components of Z. The components with the second, large field function are included again into the large field domain of the new action, except that we will transform further the integral expression in (1.70), and we will write it in a more convenient from. For the components with the first, small field function we can remove the characteristic functions $\chi_{k,\Lambda}\chi$, because they are equal to 1. This is clear for $\chi_{k,\Lambda}$. The restrictions introduced by χ_k imply that $|\partial V_k - 1| < 2\varepsilon_k$ on the corresponding components of Z, and the bound (1.78) [IV] for the function V_Λ. These bounds and the axial gauge conditions for $V'_k = V_k(V_\Lambda)^{-1}$ imply the bound (1.83) [IV] for V'_k, i.e., $|V'_k - 1| < O(1)B_5 M^6 \varepsilon_k$, by the same reasoning as in the proof of Lemma 1 [14] (even a simpler one). The number M_0 in the definition (1.101) [IV] is not fixed yet; we choose it as equal to the above bound, i.e., $M_0 = O(1)B_5 M^6$. This implies that the characteristic function χ is also equal to 1. Thus, the components of Z are divided into two classes. For components of the first class, denoted by $\{X_1, \ldots, X_n\}$, there are only the characteristic functions $\chi_k(X_i^{\sim -6})$, which are combined with the function $\chi_k(\Omega_k^{\sim 4})$, here Ω_k is the old k^{th} domain. These functions are gauge invariant, and the effective action is gauge invariant; therefore we can remove the gauge fixing δ-functions $\delta_{G_0 \cap X_i}(V'_k)$ by a reversed Faddeev–Popov procedure. This is possible only if we integrate the density with respect to the gauge field variables V_k, at least on the components X_i. Thus, the density without the δ-functions connected with these components is equivalent, but not equal, to the density with these functions, the equivalence understood in the same sense as for the formula (1.99) [IV]. For the components of the second class, denoted by $\{Y_1, \ldots, Y_m\}$, we have the large field characteristic functions $\chi_k^c(Y_i^{\sim -6})$, and the previous functions $\chi_{k,\Lambda_i}\chi_i$ together with the δ-functions $\delta_{G_i}(V'_k)$, where $\Lambda_i = \Lambda \cap Y_i$, $G_i = G_0 \cap Y_i$. The above operations, i.e., the introduction of the decompositions of unity, and the removal of the δ-functions leading to the equivalence relation, are the last operations defining the complete **R**-operation. To write its final form, we introduce a new **T**-operation. It is defined for a component of Z by a composition of the integration in (1.70) localized in this component, and the integration together with the $\mathbf{T}_h$-operation in (1.100) [IV], also localized in this component. We use the fact that these integrations factorize in components of Z. We denote the component by X, it may be any component of the first, or of the second class, and we define

$$\mathbf{T}'_k(X) = \frac{1}{2^d d!} \sum_{\{\Omega_j^c \cap X, Z_j \cap X\}, r} \int dB \Gamma_X \sigma(g_k B) \delta_{rT}(B) \chi'(X)$$

$$\cdot \int dV_h \Gamma_{(\Omega''^{\sim 2}_{h+1})^c \cap X} \chi_{h,1/2}((\Omega''^{\sim}_{h+1})^c \cap \Omega_h \cap X) \mathbf{T}_h(Z_h \cap X)$$

$$\cdot \exp\Big[-\tfrac{1}{2}\langle DH''_{1,k,X} B, \zeta DH''_{1,k,X} B \rangle$$

$$-A\left(\frac{1}{(g_k''(\cdot))^2}\zeta_1\Gamma_X, U_{h+2}''(\exp ig_k B, V'')\right)+V(X^{\sim}, U_k)$$

$$+E_k(X)+(-\tfrac{1}{2}d(\mathbf{g})\log g_k^{-2}+\log\sigma_0)|(\mathbf{B}_0\backslash T_0)\cap X|-E_k(\Lambda\cap X)\Bigg]. \quad (1.71)$$

Here $T=T_0\cap X$, and rT denotes the usual axial gauge tree graph, described in the previous section, but constructed for the coordinate system obtained by applying the Euclidean transformation r to the original coordinate system. As it was mentioned in connection with the definition (1.100) [IV], we take only the 2^d reflections for subsets of coordinates, and the $d!$ permutations of the coordinates. All the expressions in the above integral operation are localized in the domain X. With the help of these operations we define the **R**-operation:

$$(\mathbf{R}\rho_k)(V_k)=\sum_{Z_k}\sum_{\{Y_1,\dots,Y_m\}}\chi_k((\Omega_k\cap(Y_1^c)^{\sim 2}\cap\cdots\cap(Y_m^c)^{\sim 2})^{\sim 4})$$
$$\cdot\left(\sum_{\{\Omega_j^c,Z_j\}}\mathbf{T}_k''(Z_k)\right)\exp A_k'\left(\frac{1}{(g_k(\cdot))^2},U_k\right)$$
$$\cdot\prod_{i=1}^{m}\chi_k^c(Y_i^{\sim -6})\chi_{k,A_i}\chi_i\delta_{G_i}(V_k')\mathbf{T}_k'(Y_i)$$
$$\cdot\left\{\sum_{n\geqq 0}\sum_{\{X_1,\dots,X_n\}}\prod_{j=1}^{n}\mathbf{T}_k'(X_j)\exp\sum_Y V(Y,U_k)\right\}, \quad (1.72)$$

where Ω_k is the new k^{th} domain defined by the determining set of the configuration U_k. Notice that the terms $V(Y,U_k)$ depend on the sequence $\{\Omega_j^c,Z_j\}$ restricted to the components of Z contained in Y, and that the sum in the definition (1.71) acts also on the last exponential in (1.72). The sum in the last exponential does not include terms with localization domains equal to one of the components of $Z^{\sim}$; these terms are included into the operations $\mathbf{T}_k'$.

In order to proceed with further operations, we have to find bounds for the operation $\mathbf{T}_k'(X)$. We consider the analytically extended expressions. In the definition (1.71) only the quadratic form in B and the term $V(X^{\sim})$ depend on $(\mathbf{U},\mathbf{J})$. We expand them up to the first order in $A',\mathbf{J}$, the first order terms are already small. This is clear for the expansion of the quadratic form, for which the first order term can be bounded by $O(1)B_3^2A_1^2p_1^2(g_k)|\mathbf{B}_0|(\alpha_{0,k}+\alpha_{1,k})<O(1)B_3^2A_1^2C_0M^dR_k^{d+2}p_1^2(g_k)q_0(g_k)g_k$, and this bound is small. The expression $V(X^{\sim})$ is a sum of a big number of terms, mainly boundary terms from all the renormalization steps, depending on the configuration (1.61)–(1.63) localized in $X^{\sim}$. For all these terms the expansion in $A',\mathbf{J}$ creates a connection between their localization domains and the set $\mathbf{B}_0\cup(\Gamma_k''\cap X^{\sim})$, a connection with a proper exponential decay. This allows us to resum all these terms, in the way discussed in connection with the other terms $V(Y)$. The resummed expression has the same bound as above, with a different constant $O(1)$. Thus the operation $\mathbf{T}_k'(X)$ can be represented as the operation for a regular G-valued configuration U, with the additional small, and possible complex valued, term in the exponential. All the expressions in the definition (1.71), for the configuration U, are real, and the

exponential density in the integral is positive. This implies the inequalities

$$\begin{aligned}|\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))F| &= |\mathbf{T}'_k(X,(U,0))e^{\sigma}F| \\ &\leqq \mathbf{T}'_k(X,(U,0))|e^{\sigma}F| \leqq (\mathbf{T}'_k(X,(U,0))1)\sup e^{|\sigma|}|F|, \qquad (1.73)\\ |\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))1| &= |\mathbf{T}'_k(X,(U,0))e^{\sigma}| \\ &\geqq \mathbf{T}'_k(X,(U,0))e^{-2|\sigma|} \geqq (\mathbf{T}'_k(X,(U,0))1)e^{-2\sup|\sigma|}, \qquad (1.74)\end{aligned}$$

where σ is the small term of the first order in $A', \mathbf{J}$, and F is a function of the integration variables in the integral (1.71). The expression $\mathbf{T}'_k(X,(U,0))1$ is obviously positive, although it may be very small, hence $\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))1 \neq 0$, and from the above inequalities we obtain

$$|(\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))1)^{-1}\mathbf{T}'_k(X,(\mathbf{U},\mathbf{J}))F| \leqq e^{3\sup|\sigma|}\sup|F|. \qquad (1.75)$$

From the inequality (1.73) it is clear that we have to find a bound of $\mathbf{T}'_k(X,(U,0))1$, which we denote for simplicity by $\mathbf{T}'_k(X)1$. At first we bound all the expressions in the exponential, except the Wilson action and the quadratic forms. The expression $V(X^{\sim})$ is bounded in (1.69). Similar bounds hold for vacuum energy counterterms and normalization constants, except that for the last we have the corresponding constant $O(\log g_j^{-2})$ instead of $O(1)$. In effect we get a bound, to which every large field domain Z_j contributes the constant $O(1)\log g_j^{-2}|Z_j| \leqq O(1)\log g_j^{-2}(MR_j)^d d'_j(Z_j) \leqq O(1)M^d R_j^{d+1} d'_j(Z_j)$, where d'_j is the linear size defined in terms of MR_j-cubes instead of M-cubes. Next, we bound the Wilson action and the quadratic forms, or rather the corresponding Gaussian integrals. We have to obtain all small factors connected with large fields. Here the situation is almost exactly the same as in [16], so we summarize the results, and we discuss only some new issues. Consider large field characteristic functions introduced in the j^{th} step. There is the function $\chi_j^c(P_j)$, which yields the factor

$$\exp\left(-\gamma_0\frac{1}{2g_j^2}(B_3^{-1}\varepsilon_j)^2(LM_2R_j)^{-d}|P_j|\right)$$

in the usual way, using Theorem 1 [15] and (71) [16]. Here the volume is for the L^{-j}-scale, and γ_0 is an absolute positive constant (in this factor we may take $\gamma_0 = 1/2$). The function $\chi_j'^c(Q_j)$ yields the factor

$$\exp\left(-\tfrac{1}{2}\gamma_0\frac{1}{g_j^2}(2\delta_j)^2(LM_2R_j)^{-2}|Q_j|\right),$$

by expanding the Wilson action locally in the approximate fluctuation field, and using the positivity bound for the resulting quadratic form. The characteristic function $\chi^{(j-1)c}(R_j)$ yields the factor

$$\exp\left(-\gamma_0\frac{1}{g_j^2}\delta_j^2(LM_2R_j)^{-d}|R_j|\right),$$

by the same positivity bound for the quadratic form. Take the cover of the large field region $P_j \cup Q_j \cup R_j$ by MR_j-cubes, i.e., the smallest domain containing this region, which is a union of MR_j-cubes. This domain is a union of components denoted by $Z_j^{(i)}$, hence it is equal to $\bigcup_i Z_j^{(i)}$. For each component the above large field factors yield an exponential factor, which can be estimated by

$$\begin{aligned}&\exp(-\gamma_0 \min\{\tfrac{1}{2}B_3^{-2}A_0^2, 2A_1^2, A_1^2\}p_0^2(g_j)(d_j'(Z_j^{(i)})+1))\\&\leqq \exp(-\tfrac{1}{2}\gamma_0 A_1^2 p_0^2(g_j)(d_j'(Z_j^{(i)})+1)-2p_0(g_j)).\end{aligned}$$

These are the fundamental large field factors, which control convergence of the expansion of the effective densities. The above large field domain has also the following important property:

$$Z_j \subset Z_{j-1}'^{\sim 10} \cup \bigcup_i (Z_j^{(i)})^{\sim 2}. \tag{1.76}$$

Besides the above large field factors there are also the factors arising in the preparatory steps to the **R**-operation, and described in Sect. 1 [IV]. These factors are connected with components of a large field region Z_j, which satisfies the conditions (i), (ii) at the beginning of Sect. 1 [IV]. Thus, if one of the functions $1-\chi_j^{(n)}$ is introduced in such a component, we get a factor

$$\exp\left(-\frac{1}{4}\frac{1}{g_m^2}B_3^{-2}\frac{1}{2}\varepsilon_m^2\right)$$

for some m satisfying $j-N \leqq m \leqq j$. It can be bounded by

$$\exp(-\tfrac{1}{8}B_3^{-2}(1+\beta_0)^{-2}A_0^2 p_0^2(g_j)) \leqq \exp(-A_1^2 p_0^2(g_j)).$$

The function $1-\chi_m'$, where χ_m' is given by (1.27) [IV], yields the factor

$$\exp\left(-\gamma_0\frac{1}{g_m^2}(2\delta_m')^2\right) \leqq \exp(-\gamma_0 4(1+\beta_0)^{-2}A_1^2 p_1^2(g_j)) \leqq \exp(-\gamma_0 A_1^2 p_1^2(g_j)).$$

The second function in the decomposition (1.28) [IV] yields the same final factor. The function $1-\chi_{j,\Lambda}$ yields the factor

$$\exp\left(-\frac{1}{4}\frac{1}{g_j^2}(O(1)B_3 M^2)^{-1}\varepsilon_j^2\right) \leqq \exp(-A_1^2 p_0^2(g_j)),$$

as it was proved in Sect. 1 [IV] after the definition (1.75) [IV]. Finally, consider the function $1-\chi'$, where χ' is given by (1.82) [IV]. Now the situation is much more complicated, and to get a small factor we have to consider two cases. The restrictions introduced by the function $1-\chi'$ mean that at least one bond variable $B'(b)$, $b \in \mathbf{B}_0$, satisfies the inequality $|B'(b)| \geqq \delta_j'$, where j is the index of the considered large field region satisfying (i), (ii). In the first case we assume that $b \subset \Omega_{h+1}''^{\sim 2} \setminus \Omega_{h+1}''$, so it is a bond belonging to the last domain $P_1 \setminus P_2$, in which the axial gauge was fixed, the gauge defined in Sect. 1 [IV] before the definition (1.82) [IV]. In this case we will get a small factor using the Wilson action. Assume that the plaquette variables of V'' on this domain are small, i.e., $|V''(\partial p')-1| < \varepsilon$ for $p' \subset \Omega_{h+1}''^{\sim 2} \setminus \Omega_{h+1}''$, and transform the field V'' into a field satisfying the axial gauge conditions. The gauge transformation v is determined by V_0 only, because V' satisfies the conditions, and is given by $v(x) = V_0(\Gamma_{y,x})$, where the contour $\Gamma_{y,x}$ was defined in Sect. 1 [IV]. We obtain the field $V_1 = V''^v = v_- V' V_0 v_+^{-1}$, which satisfies the regularity condition above, and the axial gauge conditions. By elementary reasoning, the same as in the proof of Lemma 1 [14], we obtain the estimate

$$|V_1(b)-1|<6(100MR_{j-N+1})^2\varepsilon\leqq 6(100M(L+1)N^{\beta_0}R_j)^2\varepsilon,$$

where we have taken into account that Ω''^{c}_{h+1} satisfies the condition (i). The estimate holds for $b\subset\Omega''^{\sim 2}_{h+1}\backslash\Omega''_{h+1}$, and on this domain the field V_0 is given by $V_0=\exp i\tilde{Q}_{j-N}(L^{-j}\mathbf{A}_0)$, where $\mathbf{A}_0$ satisfies (1.87) [IV]. This implies the estimate

$$|V_0(b)-1|\leqq|\tilde{Q}_{j-N}(L^{-j}\mathbf{A}_0)|<O(1)B_3^2B_5M^6L^{-N}\varepsilon_j,$$

hence

$$\begin{aligned}|v(x)-1|=|V_0(\Gamma_{y,x})-1|&<(d+1)100MR_{j-N+1}O(1)B_3^2B_5M^6L^{-N}\varepsilon_j\\&<O(1)B_3^2B_5M^7R_jN^{\beta_0}L^{-N}\varepsilon_j.\end{aligned}$$

From the above estimates we obtain

$$|V'(b)-1|<6(100M(L+1)R_j^{1+\beta_0})^2\varepsilon+O(1)\frac{A_0}{A_1}B_3^2B_5M^7R_j^{1+\beta_0}\frac{p_0(g_j)}{p_1(g_j)}L^{-N}\delta'_j.$$

By our assumptions on N the number multiplying δ'_j is small, e.g., it is smaller than 1/4, hence we have $|B'(b)|<12R_j^4\varepsilon+1/2\delta'_j$. If we take $\varepsilon=(24R_j^4)^{-1}\delta'_j$, then $|B'(b)|<\delta'_j$. This implies that at least one plaquette variable has to satisfy the opposite inequality, i.e., $|V''(\partial p')-1|\geqq(24R_j^4)^{-1}\delta'_j$ for some $p'\subset\Omega''^{\sim 2}_{h+1}\backslash\Omega''_{h+1}$. Then the Wilson action yields the factor

$$\exp\left(-\frac{1}{4}\frac{1}{g_j^2}(24R_j^4)^{-2}\delta_j'^2\right)=\exp(-A_1^2 48^{-2}R_j^{-8}p_1^2(g_j))\leqq\exp(-R_j^{-8}p_1^2(g_j)).$$

Consider now the second case, i.e., we assume that $|B'|<\delta'_j$ on $\Omega''^{\sim 2}_{h+1}\backslash\Omega''_{h+1}$, but $|B'(b)|\geqq\delta'_j$ for a bond $b\in\Omega''_{h+1}\cap\mathbf{B}_0$. To get a small factor in this case, we use the quadratic form in the expansion (1.20). We make all the previous transformations of the Wilson action localized by the function ζ, with the only difference that we take now the function ζ_1 changing from 1 to 0 in the neighborhood of $\partial\Omega''^{\sim}_{h+1}$, instead of the neighborhood of $\partial\Omega''_{h+1}$. We estimate the terms in the expansions in the same way as before. Although we get a worse bound this time, because of the worse bound for the field B', the bound is still a small constant. The remaining quadratic form (1.21) is treated in a similar way as the form in (1.7). For the field U_0 inside $U^0_{j,Z}$ we change the gauge again, we fix the axial gauge for $M^k(U_0)$ on a neighborhood of Z, and then the Landau gauge for U_0 on such a neighborhood. We obtain (1.83)–(1.87) [IV] with k replaced by j, and with an additional factor R_j on the right-hand sides. Next, we fix the Landau gauge for $U^0_{j,Z}$ on $Z\cap\Omega''^{\sim 2}_{h+1}$, and we get a representation corresponding to (1.87) [IV]. We expand the quadratic form up to the first order with respect to the corresponding field $\mathbf{A}_0$. The first order term can be estimated by

$$\|B\|^2O(1)B_3^4B_5M^6R_j\varepsilon_j<A_1^2p_1^2(g_j)N(100MN^{\beta_0}R_j)^dO(1)B_3^4B_5M^6R_jA_0p_0(g_j)g_j.$$

The leading term in the expansion is the quadratic form (1.21) with the minimizer calculated at the external gauge field equal to 1. Using the local version of the bound (1.7) we can bound this quadratic form from below by

$$\gamma_0 \sum_{p' \in \mathbf{B}_0 \cap \Omega''^{\sim}_{h+1}} |(\partial B)(p')|^2.$$

Thus we obtain the inequality

$$\langle DH''_{1,j,Z}B, \zeta DH''_{1,j,Z}B \rangle > \gamma_0 \sum_{p' \in \mathbf{B}_0 \cap \Omega''^{\sim}_{h+1}} |(\partial B)(p')|^2$$
$$- O(1) A_0 A_1^2 B_3^4 B_5 M^{d+6} R_j^{d+3} p_0(g_j) p_1^2(g_j) g_j. \quad (1.77)$$

We assume that g_j is sufficiently small, so that the constant on the right-hand side above is small, or $O(1)$. For the quadratic form on the right-hand side we have an inequality similar to (1.8), but now we need a simpler inequality. We have to bound one bond variable $|B(b)|^2$ by the quadratic form. By the same remark as in the case of the inequality (1.8), we get

$$|B(b)|^2 \leqq 6(d+3)(100M(L+1)N^{\beta_0}R_j)^{d+2} \sum_{p' \in \mathbf{B}_0 \cap \Omega''^{\sim}_{h+1}} |(\partial B)(p')|^2, \quad (1.78)$$

for a bond $b \in \mathbf{B}_0 \cap \Omega''^{\sim}_{h+1}$. We take the bond b, for which $|B(b)| \geqq g_j^{-1} \delta'_j = A_1 p_1(g_j)$, and the inequalities (1.77), (1.78) yield the following large field factor:

$$\exp(-\tfrac{1}{2}\gamma_0 [6(d+3)(100M(L+1)N^{\beta_0}R_j)^{d+2}]^{-1} A_1^2 p_1(g_j)) < \exp(-R_j^{-d-5} p_1^2(g_j)).$$

This is the largest factor among all the small factors we have obtained from the large field characteristic functions in the preparatory steps. We assume that $2p_1 - (d+5)r_0 > p_0$, and we estimate the factors by $\exp(-p_0(g_j))$.

All the integrals with respect to the group valued variables in the definition (1.71) are estimated by 1. Similar to the integral with respect to B, we have to include only the constant with the logarithms in the last line into the integration measure, and we estimate the quadratic form in B from below by 0. The integrals with respect to the fields A_j in (2.21) [III], or (1.25) [IV], are estimated using the positivity properties of the quadratic forms, and we get the factors $\exp O(1)|Z_j \cap \Omega_j|$, where the volume is for the corresponding scale. Finally, the summations over the admissible sequences can be replaced by the factors $\exp O(1)(MR_j)^{-d}|Z_j|$.

Let us summarize now the results of the above estimates. We have obtained the following inequality:

$$\mathbf{T}'_k(X)1 \leqq \sup \exp\left\{ \sum_{j=1}^{k} O(1) M^d R_j^{d+1} d'_j(Z_j) \right\}$$
$$\cdot \prod_{j=1}^{k} \prod_i \exp(-\tfrac{1}{2}\gamma_0 A_1^2 p_0^2(g_j)(d'_j(Z_j^{(i)})+1) - 2p_0(g_j)) \prod{}' \exp(-p_0(g_j)), \quad (1.79)$$

where the last product is over components of Z_j satisfying the conditions (i), (ii), for which some large fields are created during the preparatory steps. The supremum is taken over all admissible domains, satisfying all the conditions described in the previous sections, in particular the conditions (1.76). Notice that the above inequality holds for all large field regions, not only for the regions satisfying the conditions (i), (ii). For the last regions we will prove that the expression on the right-hand side can be estimated by $\exp(-2p_0(g_k))$. For general regions we will prove the inductive statement below. The expression on the right-hand side of (1.79) is a product of factors coming from the successive renormalization steps. The product of factors coming from the first j steps is connected with the region Z_j, and can be factorized in components of Z_j. If Z is a component of Z_j, and $j(Z)$ is the index of a first large field region contained in Z, then we write the factor connected with Z in the form $\exp(-\kappa_j(Z)-2p_0(g_{j(Z)}))$. The basic inductive statement concerns these factors, or the numbers $\kappa_j(Z)$. To formulate it, we introduce an operation S, naturally connected with our procedure. If Z is a union of MR_j-cubes of the lattice T_ξ, $\xi = L^{-j}$, then we take the cover Z' of Z by a smallest union of LMR_{j+1}-cubes, and we add ten layers of such cubes. We denote the obtained domain by $S(Z)$, i.e., $S(Z)=Z'^{\sim 10}$. Such a domain arises as a new large field region in our procedure, if no large fields are created in a neighborhood of Z, more precisely in $S(Z)\backslash Z$. The operation S may be iterated. Next, notice that the expression in the first exponential in (1.79) has a universal character, it can be written for an arbitrary sequence of domains, in particular for the sequence arising by applying the operation S many times to a given domain Z. We use this remark formulating the following statement:

> The factor $\exp(-\kappa_j(Z))$ controls K renormalization steps, under the assumption that no large fields are created in these steps, where the number K is the smallest positive integer having the property that the domain $S^K(Z)$, considered as a domain in the lattice of the scale $L^{-(j+K)}$, satisfies the conditions (i), (ii), with $N=R_j$. More precisely this means that
>
> $$\kappa_j(Z) \geqq \sum_{n=j+1}^{j+K} O(1) M^d R_n^{d+1} d_n'(S^{n-j}(Z)). \tag{1.80}$$

We prove this statement by an induction with respect to j. At first we estimate the sum on the right-hand side above. Consider the domain $S^{n-j}(Z)$. The domain Z is a union of MR_j-cubes $\square$, $Z=\bigcup_{\square\subset Z}\square$, and by the definition of the operation S we have $S^{n-j}(Z)=\bigcup_{\square\subset Z} S^{n-j}(\square)$. It is easy to see that $S^{n-j}(\square)$ is a cube, which is a union of $L^d_{n-j}MR_n$-cubes, where

$$L_{n-j} \leqq 42\frac{1-L^{-(n-j)}}{1-L^{-1}} < 63.$$

The MR_n-cube in the center of $S^{n-j}(\square)$ contains $\square$. Denote by $Z^{(n-j)}$ the cover of Z by MR_n-cubes in the corresponding scale (i.e., by $L^{n-j}MR_n$-cubes in L^{-j}-scale). Thus $S^{n-j}(Z) \subset \bigcup_{\square_0\subset Z^{(n-j)}} \square_0^{\sim 31}$, and this implies

$$d_n'(S^{n-j}(Z)) \leqq (63)^d (MR_n)^{-d} |Z^{(n-j)}| \leqq (63)^d 3\cdot 2^{d-1} d_n'(Z^{(n-j)})$$

if the linear size on the right-hand side is different from 0, or $d_n'(S^{n-j}(Z)) \leqq (64)^d$ if it is equal to 0. From the scaling property (6.31) [I] we obtain

$$d_n'(Z^{(n-j)}) \leqq L^{-1/2(n-j)} d_j'(Z) \leqq 2^{-(n-j)} d_j'(Z)$$

for $n-j>1$. The square root appears here, because for some steps we do not gain the scaling factor L^{-1} (then $R_{m+1}=LR_m$). Consider now the sum in (1.80). Let n_0 be the last index n such that $d'_n(Z^{(n-j)})>0$. Then $S^{n_0+1-j}(Z)$ is contained in a cube of the size $64MR_{n_0+1}$, hence it satisfies the condition (i), and doing at most R_j further steps we obtain a domain satisfying both conditions (i), (ii). Thus $K \leqq n_0-j+R_j$, and we have

$$\begin{aligned}
&\sum_{n=j+1}^{j+K} O(1)M^d R_n^{d+1} d'_n(S^{n-j}(Z)) \\
&\quad \leqq \sum_{n=j+1}^{n_0} O(1)M^d R_n^{d+1} 3(126)^d 2^{-(n-j)} d'_j(Z) \\
&\quad + \sum_{n=n_0+1}^{n_0+R_j} O(1)M^d d R_n^{d+1}(64)^d \leqq O(1)3(126)^d M^d L^{d+1} R_j^{d+1} d'_j(Z) \\
&\quad + O(1)(64)^d M^d L^{d+1} R_j^{d+2} \leqq O(1)(64)^d M^d L^{d+1} R_j^{d+2}(d'_j(Z)+1). \qquad (1.81)
\end{aligned}$$

Now we prove the statement for $j=1$. Take a component Z of the region Z_1. It is determined by some of the large field regions $Z_1^{(i)}$, in the sense that the property (1.76) is satisfied, i.e., $Z \subset \bigcup_i (Z_1^{(i)})^{\sim 2}$. Then $d'_1(Z) \leqq \sum_i d'_1((Z_1^{(i)})^{\sim 3})$, and

$$d'_1((Z_1^{(i)})^{\sim 3}) \leqq 7^d(3\cdot 2^{d-1} d'_1(Z_1^{(i)})+2^d) \leqq 2(14)^d(d'_1(Z_1^{(i)})+\tfrac{1}{2}).$$

This implies

$$d'_1(Z)+1 \leqq 2(14)^d \sum_i (d'_1(Z_1^{(i)})+1),$$

and

$$\prod_i \exp(-\tfrac{1}{2}\gamma_0 A_1^2 p_0^2(g_1)(d'_1(Z_1^{(i)})+1)) \leqq \exp(-\tfrac{1}{4}\gamma_0(14)^{-d} A_1^2 p_0^2(g_1)(d'_1(Z)+1)).$$

From the definition of $\kappa_1(Z)$ we get the following bound:

$$\kappa_1(Z) \geqq \tfrac{1}{4}\gamma_0(14)^{-d} A_1^2 p_0^2(g_1)(d'_1(Z)+1) - O(1)M^d R_1^{d+1} d'_1(Z). \qquad (1.82)$$

Thus, by the estimate (1.81), the statement holds for $j=1$, i.e., the inequality (1.80) holds for $j=1$, if

$$\tfrac{1}{4}\gamma_0(14)^{-d} A_1^2 p_0^2(g_1) \geqq O(1)2(64)^d M^d L^{d+1} R_1^{d+2}.$$

This condition is satisfied for p_0 large, and g_1 sufficiently small. Assume that the statement is true for some j, and take a component Z of Z_{j+1}. We consider two cases. In the first case no large fields were introduced in the last step, hence $Z=S(Z_0)$, where Z_0 is a component of Z_j, and from the definition of $\kappa_{j+1}(Z)$ we have

$$\kappa_{j+1}(Z) = \kappa_j(Z_0) - O(1)M^d R_{j+1}^{d+1} d'_{j+1}(Z). \qquad (1.83)$$

The equality $Z=S(Z_0)$, and the inequality (1.80) holding for j and Z_0, imply that this inequality holds for $j+1$ and Z. In this case we should consider also the domains Z such that $Z=S(Z_0)$, Z_0 satisfies the conditions (i), (ii), and a new large field was introduced in the preparatory operations. Then $\kappa_j(Z_0) \geqq 0$, and we do not have a better bound for it, but we have the new factor $\exp(-p_0(g_j))$. We define $\kappa_{j+1}(Z)=p_0(g_j)-O(1)M^dR_{j+1}^{d+1}d'_{j+1}(Z)$. It satisfies (1.80), because Z is a small domain, it is contained in a cube of the size $100MR_{j+1}$, hence $K=R_{j+1}$ for Z. In the second case Z is obtained from some number of components of Z_j, and some number of new large field regions, joined together into the one component of Z_{j+1} by the operations of the last step, in particular by adding layers of MR_{j+1}-cubes. Denote the components of Z_j by $Z_j^{(n)}$, and the new large field regions by $Z_{j+1}^{(i)}$. The property (1.76) implies

$$Z\subset\bigcup_n(Z_j^{(n)})^{\sim 10}\cup\bigcup_i(Z_{j+1}^{(i)})^{\sim 2}. \tag{1.84}$$

For the domains $Z_j^{(n)}$ there are defined the numbers $\kappa_j(Z_j^{(n)})$, and for $Z_{j+1}^{(i)}$ we have the factors from (1.79). By the definition we have

$$\begin{aligned}\kappa_{j+1}(Z)=&\sum_n(\kappa_j(Z_j^{(n)})+2p_0(g_{j(Z_j^{(n)})}))+\sum_i(\gamma_0A_1^2p_0^2(g_{j+1})(d'_{j+1}(Z_{j+1}^{(i)})+1)+2p_0(g_{j+1}))\\&-O(1)M^dR_{j+1}^{d+1}d'_{j+1}(Z)-2p_0(g_{j(Z)}).\end{aligned} \tag{1.85}$$

We show that this number satisfies (1.80) by an induction with respect to the number of domains in $\{Z_j^{(n)},Z_{j+1}^{(i)}\}$. We have proved (1.80), if this number is equal to 1, because then we have either the situation covered by (1.83), or by the first induction step. Assume that (1.80) holds for a number of domains smaller than the number for Z. Define the graph G in the following way: the set of vertices of G is $\{Z_j^{(n)},Z_{j+1}^{(i)}\}$, and a pair of domains is a line in G if the union of corresponding domains in (1.84) is a connected domain, i.e., if the corresponding domains intersect, or touch each other. By (1.84) the graph G is connected. Take a maximal tree graph contained in the graph G. This tree graph has some nonzero number of endpoints, i.e., vertices which are connected by one line with the rest of the graph. Take one of them, and denote the corresponding domain in (1.84) by X. Remove the vertex, and the connecting line, from the graph. The obtained graph is still a connected tree graph, hence the union of the domains in (1.84) corresponding to the vertices of the obtained graph is a connected domain. Denote this domain by Y. It is equal to the corresponding union (1.84), and the number of domains in the union is smaller than the number for Z, hence the assumption of the induction hypothesis is satisfied, and the corresponding number $\kappa_{j+1}(Y)$, defined as in (1.85), satisfies (1.80). The statement (1.80) holds also for X and $\kappa_{j+1}(X)$, because there is the exactly one domain in X. By the definitions of X and Y, and by (1.84), we have $Z\subset X\cup Y$. By the definition (1.85) we have

$$\begin{aligned}\kappa_{j+1}(Z)=&\kappa_{j+1}(X)+\kappa_{j+1}(Y)+2p_0(g_{j(X)})+2p_0(g_{j(Y)})-2p_0(g_{j(Z)})\\&+O(1)M^dR_{j+1}^{d+1}d'_{j+1}(X)+O(1)M^dR_{j+1}^{d+1}d'_{j+1}(Y)-O(1)M^dR_{j+1}^{d+1}d'_{j+1}(Z).\end{aligned} \tag{1.86}$$

The index $j(Z)$ is equal to one of the indices $j(X)$, $j(Y)$, hence

$$2p_0(g_{j(X)})+2p_0(g_{j(Y)})-2p_0(g_{j(Z)})\geqq 2(1+\beta_0)^{-1}p_0(g_{j+1}).$$

The domain Z is connected, and $Z\subset X\cup Y$, hence

$$d'_{j+1}(X)+d'_{j+1}(Y)+2d\geqq d'_{j+1}(Z),$$

and

$$\kappa_{j+1}(Z) \geqq \kappa_{j+1}(X) + \kappa_{j+1}(Y) + 2(1+\beta_0)^{-1} p_0(g_{j+1}) - O(1) 2dM^d R_{j+1}^{d+1}. \quad (1.87)$$

Consider the sum on the right-hand side of (1.80), with $j+1$ instead of j. The domains X, Y intersect, or at least touch each other, hence the intersection of $S^{n-j-1}(X)$, $S^{n-j-1}(Y)$, for $n > j+1$, contains at least a cube of the size $20MR_n$. This implies that

$$d_n'(S^{n-j-1}(Z)) \leqq d_n'(S^{n-j-1}(X)) + d_n'(S^{n-j-1}(Y)).$$

The domains X, Y determine the corresponding indices K_1, K_2. Now the situation is symmetric in X, Y, so we may assume that, for example $K_1 \leqq K_2$. The domain $S^{K_1}(X)$ satisfies the conditions (i), (ii), in particular it is contained in a cube of the size $100MR_{j+1+K_1}$, and it intersects the domains $S^{K_1}(Y)$. It is clear that applying n_1 times the operation S to the last domain, where n_1 is a rather small number, e.g., $n_1 < 10$, we obtain the domain $S^{K_1+n_1}(Y)$ containing $S^{K_1+n_1}(X)$. This implies that $S^{n-j-1}(Z) = S^{n-j-1}(Y)$ for $n \geqq j+1+K_1+n_1$, and that $K \leqq K_2 + n_1 + R_{j+1}$. To prove the statement for $\kappa_{j+1}(Z)$ we estimate the sum in (1.80):

$$\begin{aligned}
&\sum_{n=j+2}^{j+1+K} O(1) M^d R_{j+1}^{d+1} d_n'(S^{n-j-1}(Z)) \\
&\leqq \sum_{n=j+2}^{j+1+K_1+n_1} O(1) M^d R_{j+1}^{d+1} d_n'(S^{n-j-1}(X)) \\
&\quad + \sum_{n=j+2}^{j+1+K_2+n_1+R_{j+1}} O(1) M^d R_{j+1}^{d+1} d_n'(S^{n-j-1}(Y)) \\
&\leqq \kappa_{j+1}(X) + \kappa_{j+1}(Y) + O(1) 2(100M)^d R_{j+1}^{d+2} \\
&\leqq \kappa_{j+1}(Z) - 2(1+\beta_0)^{-1} p_0(g_{j+1}) + O(1) 3(100M)^d R_{j+1}^{d+2} \leqq \kappa_{j+1}(Z),
\end{aligned} \quad (1.88)$$

for p_0 large and γ small enough. This completes the inductive proof of the statement.

Let us draw some conclusions from the statement. At first, the domain X in the definition (1.71) satisfies the assumption of the statement with $K=0$, therefore $\kappa_k(X) \geqq 0$, and we have the fundamental inequality

$$\mathbf{T}_k'(X) 1 \leqq \exp(-2(1+\beta_0)^{-1} p_0(g_k)). \quad (1.89)$$

Next, we have noticed already that the inequality (1.79) holds for the $\mathbf{T}$-operation connected with an arbitrary large field region. The inequality (1.80) holds quite generally for such regions, hence also an improved bound (1.89), with the additional term $-\kappa_1 d_k(X)$ in the exponential. This implies the inequality (2.50) [III], hence Corollary 3.

The next step is a construction of an exponentiated cluster expansion for the expression in the curly bracket in (1.72). The first operation is the Mayer expansion of the exponential, the same as in (7.1) [I]. We obtain a sum of terms over $\{X_1, \ldots, X_n\}$ and subfamilies $\mathbf{D} \subset \mathbf{D}_k$. Each term determines the localization domain $X_0' = \bigcup_{Y \in \mathbf{D}} Y \cup \bigcup_{j=1}^{n} X_j$. This domain is decomposed into components, but now we define them in a different way. At first we take components of $X_0' \setminus \bigcup_{i=1}^{m} Y_i$, and we consider two such components as connected, if they are contained in one localization domain Y of a term in the product. This means that each component has a correct tree graph decay factor, controlling summations over components. Thus we write $X'_0 = \bigcup_{p=1}^{r} X'_p$, and the expression corresponding to X'_0 factorizes in the components. For the fixed decomposition we resum all the expressions determining the same components. We obtain the following polymer expansion

$$\{\cdots\} = 1 + \sum_{r \geqq 1} \sum_{\{X'_1,\ldots,X'_r\}} \prod_{p=1}^{r} F(X'_p), \tag{1.90}$$

where the activities $F(X')$ are defined by

$$F(X') = \sum_{q} \sum_{\{X_{j_1},\ldots,X_{j_q}\}}' \sum_{\mathbf{D}}' \prod_{Y \in \mathbf{D}} \int_0^1 dt(Y) \prod_{h=1}^{q} \mathbf{T}'_k(X_{j_h}) \prod_{Y \in \mathbf{D}} V(Y) \exp \sum_{Y \in \mathbf{D}} t(Y) V(Y). \tag{1.91}$$

Here the summation is over $\{X_{j_1},\ldots,X_{j_q}\}$ and $\mathbf{D}$ such that the connected localization domain they determine is equal to X'. To get a convergent exponentiated expansion of the right-hand side of (1.90), we have to obtain the bounds for the activities. Using (1.68), (1.73), (1.89), we obtain

$$\begin{aligned} |F(X')| \leqq \sum_{q} \sum_{\{X_{j_1},\ldots,X_{j_q}\}}' \sum_{D}' \prod_{h=1}^{q} \exp(-2(1+\beta_0)^{-1} p_0(g_k) + 1) \\ \cdot \left(\prod_{Y \in \mathbf{D}} \alpha \exp(-(1+2\beta)\kappa d_{k,Z'}(Y)) \right) \exp \sum_{Y \in \mathbf{D}} \alpha \exp(-(1+2\beta)\kappa d_{k,Z'}(Y)), \end{aligned} \tag{1.92}$$

where $Z' = \bigcup_{i=1}^{m} Y_i \cup \bigcup_{h=1}^{q} X_{j_h}$, and we have estimated the expression $|\sigma|$ in (1.73) by 1. We estimate the second product above in the following way:

$$\begin{aligned} &\prod_{Y \in \mathbf{D}} \alpha \exp(-(1+2\beta)\kappa d_{k,Z'}(Y)) \\ &\quad \leqq \exp(-(1+\beta)\kappa d_{k,\cup Y_i}(X')) \exp\left(-\frac{1}{3 \cdot 2^d} \beta \kappa M^{-d} |X' \setminus \cup Y_i| \right) \\ &\quad \cdot \prod_{h=1}^{q} \exp\left(\left(1 + \beta + \frac{1}{3 \cdot 2^d} \beta \right) \kappa M^{-d} |X_{j_h}| \right) \prod_{Y \in \mathbf{D}} \alpha^{2/3} \exp(-\tfrac{1}{2} \beta \kappa d_{k,Z'}(Y)), \end{aligned} \tag{1.93}$$

where we have used the fact that α is sufficiently small, e.g., $\alpha^{1/3} \leqq \exp(-(1+\beta)\kappa 2d)$, to produce the exponential factors connecting graphs in domains Y, in the cases they intersect outside Z'. The sum of the last products above is estimated in the usual way:

$$\begin{aligned} &\sum_{\mathbf{D}} \prod_{Y \in \mathbf{D}} \alpha^{2/3} \exp(-\tfrac{1}{2} \beta \kappa d_{k,Z'}(Y)) \\ &\quad = \sum_{n} \frac{1}{n!} \sum_{(Y_1,\ldots,Y_n)}' \prod_{j=1}^{n} \alpha^{2/3} \exp(-\tfrac{1}{2} \beta k d_{k,Z'}(Y)) \leqq c_0 \exp \sum_{Y \subset X'} \alpha^{1/3} \exp(-\tfrac{1}{2} \beta \kappa d_{k,Z'}(Y)) \end{aligned}$$

$$\leqq c_0 \exp\left(\sum_{i: Y_i \subset X'} O(1)\alpha^{1/3} M^{-d+1} |\partial Y_i \cap \partial(X' \setminus Y_i)| \right.$$

$$\left. + \sum_{h=1}^{q} O(1)\alpha^{1/3} M^{-d+1} |\partial X_{j_h} \cap \partial(X' \setminus X_{j_h})| \right) \leqq c_0 \exp O(1)\alpha^{1/3} M^{-d} |X' \setminus Z'|, \tag{1.94}$$

where $c_0 = 1$, if the empty subfamily is admissible in the sum over $\mathbf{D}$, and $c_0 = \alpha^{1/3}$ in the remaining cases. The first case is possible, if X' is one of the domains $\mathbf{X}_j^{\sim}$, and then we have the small factor from the bound (1.89) of the operation $\mathbf{T}'_k(X_j)$. The last exponential on the right-hand side of (1.92) can be also estimated by the above bound. We have $|X' \setminus Z'| \leqq |X' \setminus \cup Y_i|$, and we use the fact that $O(1)\alpha^{1/3} \leqq 1 \leqq 1/(6 \cdot 2^d)\beta\kappa$ for κ large enough, hence the above bound is cancelled by the corresponding part of the second factor on the right-hand side of (1.93). The product over h there is estimated by $\prod_{h=1}^{q} \exp 2\kappa(100 R_k)^d$, and this product is combined with the first product on the right-hand side of (1.92). We assume that g_k is so small that

$$-2(1+\beta_0)^{-1} p_0(g_k) + 1 + 2\kappa(100 R_k)^d \leqq -\tfrac{3}{2} p_0(g_k)$$

(for example, take $\beta_0 = 1/7$, then this means that $-(1/4)p_0(g_k) + 1 + 2\kappa(100R_k)^d \leqq 0$, and before we had stronger restrictions on $p_0(g_k)$). Combining together all the estimates we obtain

$$|F(X')| \leqq \sum_q \sum_{\{X_{j_1},\ldots,X_{j_q}\}}' \exp(-q\tfrac{3}{2} p_0(g_k)) c_0$$

$$\cdot \exp(-(1+\beta)\kappa d_{k,\cup Y_i}(X')) \exp\left(-\frac{1}{6 \cdot 2^d} \beta\kappa M^{-d} |X' \setminus \cup Y_i| \right). \tag{1.95}$$

Now there are two cases to consider. Either the domain X' contains one of the domains Y_i, then the sum over q above begins with $q = 0$, but then $c_0 = \alpha^{1/3}$, or the domain X' is disjoint with $\bigcup_{i=1}^{m} Y_i$, and then the sum begins with $q = 1$. In the last case we extract the factor $\exp(-p_0(g_k))$ before the sum, and in both cases we estimate the obtained sum by

$$\sum_{q \geqq 0} \sum_{\{X_{j_1},\ldots,X_{j_q}\}}' \exp(-q\tfrac{1}{2} p_0(g_k)) \leqq \sum_{q \geqq 0} \frac{1}{q!} \sum_{(X_{j_1},\ldots,X_{j_q})}' \exp(-q\tfrac{1}{2} p_0(g_k))$$

$$\leqq \exp(\exp(-\tfrac{1}{2} p_0(g_k)) 100^d (M R_k)^{-d} |X' \setminus \cup Y_i|). \tag{1.96}$$

Finally, we have

$$\exp(-\tfrac{1}{2} p_0(g_k)) 100^d R_k^{-d} < \exp(-\tfrac{1}{2} p_0(g_k)) \leqq \frac{1}{6 \cdot 2^d} \beta\kappa$$

for g_k small, hence the above bound is cancelled by the last exponential in (1.95), and we obtain

$$|F(X')| \leqq c_1 \exp(-(1+\beta)\kappa d_{k,\cup Y_i}(X')), \tag{1.97}$$

where $c_1 = \exp(-p_0(g_k))$ if X' does not intersect $\bigcup_{i=1}^{m} Y_i$, and $c_1 = \alpha^{1/3}$ in the remaining cases.

The estimate (1.97) is sufficient for convergence of the exponentiated cluster expansion, and we have

$$\{\cdots\} = \exp \mathbf{R}'^{(k)} = \exp \sum \mathbf{R}'^{(k)}(X). \tag{1.98}$$

The summation here is over domains $X \in \mathbf{D}_k$, which have nonempty intersections with Z_k^c. In fact, admissible domains, i.e., domains with nonzero expressions $\mathbf{R}'^{(k)}(X)$, have the intersections sufficiently large, because they contain at least one large field region connected with one of the $\mathbf{T}_k'$-operations. The expression $\mathbf{R}'^{(k)}(X)$ depends on the background field U_k restricted to X, and it can be extended as an analytic function of the variables $(\mathbf{U}, \mathbf{J})$, defined on the space $\tilde{U}_k^c(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$. It is given by the convergent series (7.13) [I] (with proper notational changes), and it satisfies the inequality

$$|\mathbf{R}'^{(k)}(X, (\mathbf{U}, \mathbf{J}))| \leqq O(1) c_1 \exp(-(1 + \tfrac{1}{2}\beta)\kappa d_{k, \cup Y_i}(X)). \tag{1.99}$$

The exponentiation (1.98) completes the $\mathbf{R}$-operation. Now we divide the terms $\mathbf{R}'^{(k)}(X)$ into two groups. To the first group we assign all the terms with the localization domains X intersecting the large field region $Z_k^{\sim} \cup \bigcup_{i=1}^{m} Y_i^{\sim}$, to the second group the terms with the domains disjoint with this region. The terms of the first group are new boundary terms, and they are denoted by $\mathbf{B}'^{(k)}(X)$. The terms of the second group give the basic contribution to the $\mathbf{R}$-terms in the k^{th} renormalization step. By their definition the linear size of the domain X in (1.99) can be replaced by $d_k(X)$, and c_1 is equal to $\exp(-p_0(g_k))$, hence we have the following more precise bound for them:

$$|\mathbf{R}'^{(k)}(X, (\mathbf{U}, \mathbf{J}))| \leqq \exp(-p_0(g_k)) \exp(-\kappa d_k(X)). \tag{1.100}$$

By their construction it is also clear that they are Euclidean covariant, i.e., they have the property (2.32) [III].

With the above definitions we have completed the construction and the description of the new action after the k^{th} transformation $\mathbf{R}T$. We have

$$A_k\left(\frac{1}{(g_k(\cdot))^2}, U_k\right) = A_k'\left(\frac{1}{(g_k(\cdot))^2}, U_k\right) + \sum_X \mathbf{R}'^{(k)}(X, U_k) + \sum_X \mathbf{B}'^{(k)}(X, U_k). \tag{1.101}$$

This new action satisfies the induction hypothesis, as it follows from the construction and the properties of the new terms. The new large field region is equal to $Z_k \cup \bigcup_{i=1}^{m} Y_i$, and the operation $\mathbf{T}_k$ for a component Y of the set $\bigcup_{i=1}^{m} Y_i$ has the form

$$\mathbf{T}_k(Y) = \chi_k^c(Y^{\sim -6}) \chi_{k,\Lambda} \delta_G(V_k V_\Lambda^{-1}) \mathbf{T}_k'(Y). \tag{1.102}$$

From (1.72), (1.98), and the above definitions, it follows that the result $\mathbf{R}\rho_k$ of the $\mathbf{R}$-operation can be written in the form (2.18) [III], with all the expressions satisfying the induction hypothesis described in Sect. 2 [III]. This completes the proof of Theorem 1 and Corollary 3.

Let us make the final remark about another possible representation of the k^{th} density. We divide the terms of the sum in (1.98) in the same way as before, and we consider the part of the expression (1.72) with the product of the operations $\mathbf{T}'_k(Y_i)$ acting on the exponential in (1.98) with the boundary terms only, in fact with the terms with localization domains intersecting $\cup Y_i$. We write this expression in the following form

$$\prod_{i=1}^{m}(\mathbf{T}'_k(Y_i)1)\left[\prod_{i=1}^{m}(\mathbf{T}'_k(Y_i)1)^{-1}\mathbf{T}'_k(Y_i)\exp\sum\mathbf{B}'^{(k)}(X)\right], \tag{1.103}$$

and we consider the expression in the square bracket. We apply successively the same steps as for the expression in the curly bracket in (1.72), so we describe now changes and differences only. We apply the Mayer expansion, and we write the polymer expansion (1.90), using the fact that the $\mathbf{T}'_k$-operations are normalized, i.e., if such an operation is applied to a function which does not depend on the integration variables connected with the operation, then it is equal to 1. The activities in the polymer expansion are given by (1.91), but without the first two sums, and the product of the $\mathbf{T}'_k$-operations is restricted to the operations with regions Y_i contained in the localization domain of the activity. Also, we use the ordinary notion of connectedness to define components. The activities are estimated as in (1.92), with the corresponding changes, i.e., without the first two sums, the first product over h is replaced by the product of e^3 over indices i such that $Y_i \subset X'$, the number β is replaced by $(1/4)\beta$, and α by $O(1)c_1$. All these changes have to be done in the next formulas, and we do not mention them any more. We have the inequality (1.93), but with the linear size $d_k(X'\setminus \cup Y_i)$ in the first exponential on the right-hand side, and without the product over h. The sum over $\mathbf{D}$ is estimated as in (1.94), and we get the bound (1.95) without the first two sums and the first exponential. This bound is enough for the convergence of the exponentiated cluster expansion. This requires a comment, because in this bound there are no tree decay exponential factors for the domains Y_i. In fact, looking carefully at a standard proof of the convergence of the expansion (e.g., in [26]), we see that in the present situation such factors are not needed, because there are no summations over these domains, they are fixed. Thus, we obtain (1.98) for the expression in the square bracket in (1.103). We denote the terms in the sum again by $\mathbf{B}'^{(k)}(X)$, because the domains X intersect $\bigcup_{i=1}^{m} Y_i$. We substitute the right-hand side of (1.98) into the expression for the effective density, and we obtain

$$\mathbf{R}\rho_k = \sum_{Z_k}\sum_{\{Y_1,\dots,Y_m\}}\chi_k\left(\sum_{\{\Omega_j^c,Z_j\}}\mathbf{T}''_k(Z_k)\right)\prod_{i=1}^{m}\chi_k^c(Y_i^{\sim -6})\chi_{k,\Lambda_i}\chi_i\delta_{G_i}(V_k V_{\Lambda_i}^{-1})(\mathbf{T}'_k(Y_i)1)$$
$$\cdot\exp\left[A'_k+\sum_X\mathbf{R}'^{(k)}(X)+\sum_X\mathbf{B}'^{(k)}(X)\right]. \tag{1.104}$$

For this representation the domains Y_i are still large field domains, but there are no integral operations connected with them, there are only the characteristic functions, the δ-functions and the functions $\mathbf{T}'_k(Y_i)1$ multiplying the action density, hence $\mathbf{T}_k(Y_i)$ is the multiplication operation. These functions depend only on the new field variables V_k restricted to Y_i, therefore they do not participate in operations connected with next renormalization transformations. The action is also simple; it is the usual small field action outside Z_k, with the additional boundary terms with localization domains containing one of the domains Y_i. The contributions from the previous large field regions is isolated in these boundary terms, and in the functions $\mathbf{T}'_k(Y_i)1$. The representation (1.104) can be used alternatively in the inductive description of the effective actions, and the above considerations leading to it give the proof of Theorem 1.

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References

[I] Balaban, T.: Renormalization group approach to lattice gauge field theories. I. Commun. Math. Phys. **109**, 249–301 (1987)
[II] Balaban, T.: Renormalization group approach to lattice gauge field theories. II. Commun. Math. Phys. **116**, 1–22 (1988)
[III] Balaban, T.: Convergent renormalization expansions for lattice gauge theories. Commun. Math. Phys. **119**, 243–285 (1988)
[IV] Balaban, T.: Large field renormalization. I. The basic step of the R operation. Commun. Math. Phys. **122**, 175–202

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