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# Continuous Measures in One Dimension*

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**Abstract.** Families of unimodal maps satisfying
(1) $T_\lambda:[-1,1]\mapsto[-1,1]$ with $T(\pm 1)=-1$ and $|T_\lambda'(1)|>1$,
(2) $T_\lambda(x)$ is $C^2$ in $x^2$ and $\lambda$, and symmetric in $x$,
(3) $T_0(0)=0$, $T_1(0)=1$ with $\frac{d}{d\lambda}T_\lambda(0)>0$
are considered. The results of Guckenheimer (1982) are extended to show that there is a positive measure of $\lambda$ for which $T_\lambda$ has a finite absolutely continuous invariant measure.

The appendix contains general theorems for the existence of such measures for some markov maps of the interval.

## (A) Introduction

Jakobson published a theorem in 1978 that states that for the family of unimodal maps $T_\lambda:[0,1]\mapsto[0,1]$ defined by $T_\lambda(x)=\lambda x(1-x)$, there exists a set of parameters $\lambda$ of positive lebesque measure for which $T_\lambda$ possesses a finite absolutely continuous invariant measure.

Since that time there have been several attempts to present a more comprehendable proof. Rychlik (1986) has produced a cleaner proof than Jakobson using similar techniques. Benedicks and Carleson (1983) use statistical methods and a widey different approach to achieve a similar result. Rees (1985) has presented a proof in the complex case.

In 1982 Guckenheimer proved that for reasonable families of unimodal maps, there is a positive measure set of parameter values for which the corresponding map has sensitivity to initial conditions (a map $T$ has sensitivity to initial conditions if $\exists\varepsilon>0\,\forall x$, $\delta>0\,\exists y$, $|x-y|<\delta$, $\exists n$, $|T^n(x)-T^n(y)|>\varepsilon$). Specifically, he

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considered the class of one parameter families of maps $\mathcal{G}=\{f_\lambda\}$ which satisfy
(1) $T_\lambda:[-1,1]\mapsto[-1,1]$ with $T(\pm 1)=-1$ and $|T_\lambda'(1)|>1$,
(2) $T_\lambda(x)$ is $C^2$ in $x^2$ and $\lambda$, and symmetric in $x$,
(3) $T_0(0)=0$, $T_1(0)=1$ with $\frac{d}{d\lambda}T_\lambda(0)>0$.

The idea of the proof is a simplification of Jakobson's approach, and relies on a recursive geometric construction. Since detailed estimates need only be carried out for one typical recursive step, such a construction provides a geometric skeleton for the calculations involved. This presents a conceptually simpler proof.

In this paper I demonstrate that for the same parameter values for which $T_\lambda$ was shown to have sensitivity to initial conditions, there is actually an absolutely continuous invariant measure for $T_\lambda$.

I am indebted to the guidence of John Guckenheimer and to many constructive discussions with Don Ornstein and Yitzak Katznelson.

This proof differs from other proofs of Jakobson's theorem in two ways. Firstly, relying on the geometric construction to organize and simplify allows the reader to have a more clear picture of what is involved in the proof before (or without) going into the details. Secondly, the appendix contains several theorems for the existence of a finite absolutely continuous invariant measure for certain markov maps of the interval. These theorems are tailored to be more easily reached through the technique for inducing, thus the inducing steps can be simpler without losing the conclusion of the existence of an absolutely continuous invariant measure.

A brief description of Guckenheimer's method is as follows. First, the original unimodal map $T_\lambda$ for $\lambda$ near 1 is induced on an interval $J_0$ around 0, producing a symmetric map $T_{1,\lambda}$ with monotone branches and a folded piece over an interval $J_1$ (see Fig. A.1).

This is best understood by letting $h=T_\lambda|_{I-J_0}$ and considering the preimages $h^{-n}(J_0)$. If $T_\lambda(0)\in\Gamma\in h^{-n}(J_0)$, then as $\lambda$ is varied so that $T_\lambda(0)$ moves from one end of $\Gamma$ to the other, the tip $T_{1,\lambda}(0)$ of the induced map will move from the bottom to the top of the induced picture, and the number of monotone branches will remain the same although they will go through a continuous deformation. If $\lambda$ is chosen

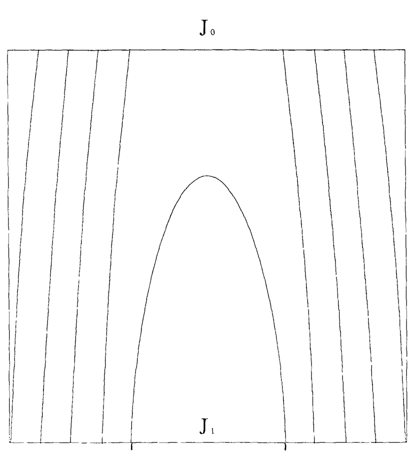


Fig. A.1

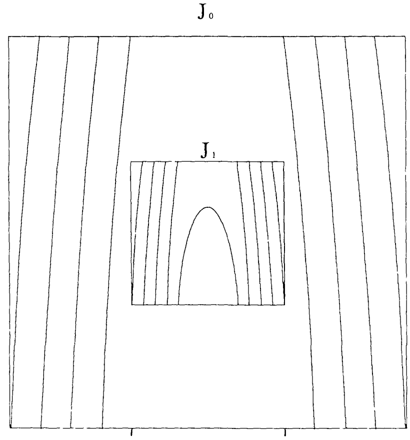


**Fig. A.2**

sufficiently close to 1, $T_{1,\lambda}$ will have a finite number of monotone branches with a uniform bound on their linearity and $|T'_{1,\lambda}| > L > 1$, and the central folded branch can be made arbitrarily sharp.

The variables can be renormalized so that the interval $J_0$ can be considered as $[-1, 1]$ and the tip $T_{1,\lambda}(0)$ moves from bottom to top as $\lambda$ moves from 0 to 1. Excluding from consideration those $\lambda$'s for which $T_{1,\lambda}(0) \in J_1$, this map is then induced on the interval $J_1$ producing a map $T_{2,\lambda}(x)$ with countably infinite monotone branches and a folded piece over a central interval $J_2$ (see Fig. A.2).

Again, this map can be understood by letting $h_{1,\lambda} = T_{1,\lambda}|_{J_0 - J_1}$ and considering the preimages $h_{1,\lambda}^{-n}(J_1)$. As $\lambda$ varies so that the tip $T_{1,\lambda}(0)$ moves from one end of a component $\Gamma \in h_{1,\lambda}^{-n_1}(J_1)$ to the other, the tip $T_{2,\lambda}(0)$ of the induced map will move from the bottom of its graph to the top, and the entire graph will go through a continuous deformation. The central folded piece of $T_{2,\lambda}$ is a result of the central folded piece of $T_{1,\lambda}$ being composed with the monotone branches of $T_{1,\lambda}$ $n_1$ times. Thus if $n_1$ is very large the fold becomes quite sharp, since $T''_{2,\lambda}(0)$ increases exponentially in $n_1$. In this way the relative length of $J_2$ is controlled by the duration of return time. The slope and distortion of the linear branches of $T_{2,\lambda}$ can be controlled by insisting that if $T_{1,\lambda}(0) \notin \Gamma \in h_{1,\lambda}^{-n}(J_1)$ then $T_{1,\lambda}(0)$ lies sufficiently far away from $\Gamma$ so as not to distort the linearity of the corresponding branch.

Thus by making some careful parameter exclusions, the map $T_{2,\lambda}$ will have sufficiently linear and steep monotone branches, and a smoothly folded and sufficiently sharp central folded branch.

This map is renormalized, parameter values for which $T_{2,\lambda}(0) \in J_2$ are excluded, and the map is induced on $J_2$ producing a map $T_{3,\lambda}$ and the recursion continues in this manner.

In order to arrive at a good final picture for this recursion process without excluding too many parameter values certain estimates must be carried throughout the process. This is done by establishing bounds which control the linearity and slope of the monotone pieces, the distortion and sharpness of the quadratic piece, the speed at which the tip moves, and the way the picture deforms with respect to changes in the parameter. Given a map which satisfies those bounds, new bounds are calculated for the next induced map. By excluding certain parameter values at each step Guckenheimer shows that the bounds remain finite or go to infinity at a controlled rate. By showing that proportionately fewer parameters can be excluded at each step, he concludes that the entire recursion can be done for a positive measure of parameter values.

One thing that must be shown in order to exclude proportionately fewer parameters at each step is that the central intervals are proportionately smaller, that is $\frac{l(J_{k+1})}{l(J_k)} \ll \frac{l(J_k)}{l(K_{k-1})}$. Letting $S_k = T''_{k,\lambda}(0)$, there is a constant $C$ depending on the above mentioned bounds for which $l(J_k) < CS_k^{-1/2}$. Since $S_k$ behaves exponentially with respect to the duration of return time, parameters can be excluded so as to make return times sufficiently long to conclude that $S_{k+1} > S_k^4$. Thus the $S_k$ grow at a fast double exponential rate and central intervals are proportionately smaller.

The final picture is a sequence of nested intervals $J_0 \subset J_1 \subset \ldots$ with the map $T_{\infty,\lambda}$ on $J_k - J_{k+1}$ consisting of branches mapping over $J_k$ with slopes $|T'_{\infty,\lambda}| > L > 1$ and distortion $\left|\frac{T''_{\infty,\lambda}}{(T_{\infty,\lambda'})^2}\right|_{J_k - J_{k+1}} < 2^k D$.

From this picture it can be concluded that the original map has sensitivity to initial conditions.

For a more detailed exposition of these ideas, see Guckenheimer (1969, 1982).

## (B) Assumptions, Definitions and Bounds

*Definition.* For a transformation $T \in C^2$ the distortion of $T$ at a point $x$ is defined as

$$\operatorname{dis}(T)(x) = \left|\frac{T''(x)}{(T'(x))^2}\right|.$$

Then dis has the composition formulae:

$$\operatorname{dis}(T_1 \circ T_2)(x) \leqq \operatorname{dis}(T_1)(T_2(x)) + \frac{1}{|T_1'(T_2(x))|} \operatorname{dis}(T_2)(x),$$

$$\operatorname{dis}(T^n)(x) \leqq \sum_{\alpha=0}^{n-1} \frac{1}{(T^\alpha)'(T^{n-\alpha}(x))} \operatorname{dis}(T)(T^{n-\alpha-1}(x)).$$

What is demonstrated by this last formula is that under repeated application of a map to a point, the distortion that is caused under any one application can be reduced through subsequent stretching. In the appendix of this paper it is shown that certain expanding markov maps with bounded distortion and arbitrarily short branches mapping onto neighborhoods of a central point $c$ may possess an absolutely continuous invariant measure by virture of having a good ratio of longer branches near $c$.

It will be assumed that a parameter value $\lambda$ is given for which Guckenheimer's construction can be carried out indefinitely. It is assumed that at the $i^{\text{th}}$ step of the recursion there is a map $T_i: J_i \mapsto J_i$ consisting of monotone branches mapping onto $J_i$, and a central folded piece over a central interval $J_{i+1}$ (see Fig. B.1).

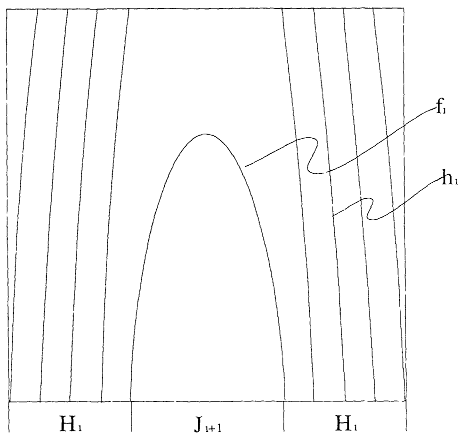


**Fig. B.1**

*Definitions.* Let $H_i = J_i - J_{i+1}$, and define $h_i = T_i|_{H_i}$ and $f_i = T_i|_{J_{i+1}}$ so that $h_i$ is the monotone branch part of $T_i$ and $f_i$ is the central folded piece (see Fig. B.1). Let $s_{i+1} = f_i''(0)$ (where 0 is the critical point of $f_i$). To exploit the quadratic nature of $f_i$ let $q(x) = x^2$ and define $p_i$ such that $f_i(x) = p_i \circ q(x)$ for $x \in J_{i+1}$.

It will be assumed that there exist positive bounds $D_0$, $L > 2$, $K_1$, $K_2$, $K_3$, $K_4$, and $K_5$ such that for all $i$:

*Bound 1.* $s_{i+1} > s_i^4$,

*Bound 2.* $\operatorname{dis}(h_i) < 2^i D_0$,

*Bound 3.* $|h_i'| > L$,

*Bound 4.* $\operatorname{dis}(p_i) < K_1$,

*Bound 5.* $\dfrac{1}{K_2} < \left|\dfrac{2p_i'}{s_{i+1}}\right| < K_2$,

*Bound 6.* $l(J_{i+1}) < K_3 s_{i+1}^{-1/2}$,

*Bound 7.* $l(J_{i+1}) > K_4 s_{i+1}^{-1}$,

*Bound 8.* $\dfrac{l(J_{i+1})}{l(J_i)} < K_5 s_{i+1}^{-1/4}$.

Bounds 1–4 are directly from Guckenheimer (1982), bounds 5 and 6 follow from bound 4, bound 7 is a consequence of a parameter exclusion in Guckenheimer (1982), and bound 8 follows from bounds 6 and 7.

## (C) An Outline of the Proof

Guckenheimer's recursive step can be viewed in terms of a stopping time $n_i(x)$. The preimages of $J_{i+1}$ in $J_i$ under $H_i$ form a set of intervals $N \subset J_i$ such that $h_i^n : N \xrightarrow{1-1\text{ onto}} J_{i+1}$ for some $n = 1, 2, 3, \ldots$. These intervals fill up all of $J_i$, so it possible to define an integer valued function $n_i(x)$ almost everywhere on $J_{i+1}$ by $n_i(x)=n$ iff $f_i(x)\in h_i^{-n}(J_{i+1})$.

This makes it possible to write $T_{i+1}$ explicitly in terms of $T_i$ as $T_{i+1}=h_i^{n_i(x)}f_i(x)$ for $x\in J_{i+1}$. If $\Gamma$ is the preimage of $J_{i+1}$ under $h_i$ that contains $f_i(0)$ and $\pi_i$ is defined as $\pi_i=n_i(0)$, then $f_{i+1}(x)=h_i^{\pi_i}f_i(x)$ for $x\in f_i^{-1}(\Gamma)$, hence $f_i^{-1}(\Gamma)$ will be the new central interval $J_{i+2}$ of $T_i+1$. Also, $h_{i+1}(x)=h_i^{n_i(x)}f_i(x)$ for $x\in J_{i+1}-f_i^{-1}(\Gamma)$.

Since $\mathrm{dis}(f_i)(x)$ behaves like $\dfrac{1}{s_{i+1}x^2}$ for $x$ near 0, the parameter exclusions in Guckenheimer (1982) must be chosen so that $\pi_i$ is sufficiently large as to ensure bounds 1 and 4, and $n_i(x)$ is sufficiently large at all $x$ to ensure bounds 2 and 3.

The idea in this paper is to leave $\pi_i$ and the definition of $f_{i+1}$ as is, but to redefine $n_i(x)$ slightly smaller so as to allow sufficient longer branches to satisfy the hypothesis of Theorem 4 in the appendix. The conclusion will be that a finite absolutely continuous invariant measure will exist for $T_0$. By restricting $\lambda$ sufficiently close to 1, $T_0$ will have a finite number of branches, thus producing a finite absolutely continuous invariant measure for $F_\lambda$. The proof is outlined as follows.

Consider $T_0$ on $J_0$, where $J_0$ is assumed to be a symmetric interval around 0 of length less than 1. Let $\mathscr{R}$ be the minimal partition of $H_0$ with respect to which $h_0$ is continuous and monotone. Let $\mathscr{P}_1$ be the partition of $J_0$ consisting of $\mathscr{R}$ and $J_1$. Let $\mathscr{P}_2=h_0^{-1}(\mathscr{P}_1)\vee J_1$, so that for any $M\in\mathscr{P}_2$ either $M=J_1$, $h_0(M)=J_1$, or $h_0(M)\in\mathscr{R}$. In general, let $\mathscr{P}_n=h_0^{-1}(\mathscr{P}_{n-1})\vee J_1$ so that for any $M\in\mathscr{P}_n$ either $h_0^k(M)=J_1$ for some $k=0,1,\ldots,n-1$ or $h_0^{n-1}(M)\in\mathscr{R}$. The partitions $\mathscr{P}_n$ form a refining sequence, and since $\mathscr{P}_n$ contains the sets $h_0^{-k}(J_0)$ for $k=0,1,\ldots,n-1$, these partitions are increasingly dominated by preimages of $J_1$. It is these partitions that will be used to define a new stopping time rather than the partition consisting entirely of preimages of $J_1$.

Define $z_1$ sucht that $(-z_1,z_1)=J_1$ and consider $f_0$ on $J_1$. The distortion $\mathrm{dis}(f_0)$ behaves like $\dfrac{1}{s_1x^2}$ for $x$ near 0. For $n=0,1,2,\ldots$ define $x_n$ such that $\mathrm{dis}(f_0)(x)<L^{n-1}D_0$ for $x\in(-z_1,\ -x_n)\cup(x_n,z_1)$. It will be shown that $x_n\to 0$ like $(s_1L^{n-1}D_0)^{-1/2}$ (If $f_0$ were quadratic with $f_0''=s_1$ then $x_n$ would equal $(s_1L^{n-1}D_0)^{-1/2}$.) Let $B_n=(-x_{n-1},\ -x_n)\cup(x_n,x_{n-1})$. These sets $B_n$ will give a lower bound on how small the stopping time may be taken and still preserve a reasonable distortion.

As before, let $\Gamma$ be the preimage of $J_1$ which contains $f_0(0)$, and let $J_2=f_0^{-1}(\Gamma)$ with $z_2$ taken to be such that $(-z_2,z_2)=J_2$. Since $\pi_0$ is such that $h_0^{\pi_0}(\Gamma)=J_1$, then $f_1$ can be defined as $f_1(x)=h_0^{\pi_0}f_0(x)$ for $x\in J_2$.

Since $f_0$ will have some amount of distortion at $z_1$, let $b_0$ be the minimal $n$ such that $x_n<z_1$; that is, such that $\mathrm{dis}(f_0)(z_1)<L^{n-1}D_0$. Let $e_0$ be the minimal $n$ such that $x_n\geqq z_2$; that is, such that $\mathrm{dis}(f_0)(z_2)<L^{n-1}D_0$. Redefine $B_{b_0}=(-z_1,x_{b_0})\cup(x_{b_0},z_1)$ and $B_{e_0}=(-x_{e_0-1},\ -z_2)\cup(z_2,x_{e_0-1})$. Now $H_1=J_1-J_2$ is partitioned into disjoint sets $B_{b_0},\ldots,B_{e_0}$, such that $\mathrm{dis}(f_0)|_{B_n}<L^{n-1}D_0$.

On $H_1$ a stopping time $\eta(x)$ will be defined creating a new function $g_1(x)=h_0^{\eta(x)}f_0(x)$ for $x\in H_1$. This will be done in such a way as to satisfy three conditions:

**Markov Condition.** *The map $g_1$ must consist of monotone branches mapping onto $J_0$ or $J_1$ with a high proportion of them mapping onto $J_0$.*

**Linearity Condition.** *The stopping time $\eta$ must be large enough to ensure that $g_1$ has reasonable distortion and good slope.*

**Return Condition.** *The stopping time $\eta$ must be small enough to allow a finite measure to be pulled back through it via a Rohklin argument to another finite measure.*

In order to satisfy the markov condition, the sets $B_n$ will be moved slightly so that $f_0(B_n)$ fits into the partition $\mathscr{P}_n$; that is, new sets $\tilde{B}_n$ will be defined such that the endpoints of the intervals $f_0(\tilde{B}_n)$ will be division points in the partition $\mathscr{P}_n$ for $n=b_0, \ldots, e_0$. This can be done in such a way that $\tilde{B}_n$ will still provide a lower bound for how large $\eta$ must be in order to satisfy the linearity condition, as well as preserving the property that $l\left(\bigcup_{k=n+1}^{e_0} \tilde{B}_k \cup J_2\right)$ is of the order $(s_1 L^{n-1} D_0)^{-1/2}$ which allows $\eta$ to be taken as small as is reasonably possible to satisfy the return condition.

The stopping time $\eta$ will be defined as follows: For $x \in \tilde{B}_n$, consider the position of $x$ in $\mathscr{P}_n$. Either $f_0(x) \in h_0^{-k}(J_1)$ for $k=0,1,\ldots,n-1$ or $f_0(x) \in h_0^{n-1}(M)$, where $M \in \mathscr{R}$ is some interval for which $h_0 : M \xrightarrow{1-1 \text{ onto}} J_0$. If $f_0(x) \in h_0^{-k}(J_1)$ for $k=0,1,\ldots,n-1$ let $\eta(x)=k$. Otherwise let $\eta(x)=n$.

Then $g_1(x)=h_0^{\eta(x)} f_0(x)$ is defined on $H_1$ and consists of monotone branches mapping onto $J_0$ or $J_1$ (see Fig. C.1) and compare with Fig. A.2). A branch of $g_1$ that maps onto $J_1$ will be identical with some branch of $h_1$. Indeed for any $x$ such that $g_1(x) \in J_1$ it will be the case that $g_1 = h_1$ for some neighborhood of $x$ since both functions represent the first return of $f_0(x)$ to $J_1$ under $h_0$. Consequently the distortion on the branches of $g_0$ that map onto $J_1$ is bounded by $2D_0$ and their slope is greater than $L$ by bounds 2 and 3 on $h_0$.

On the other hand, if $x$ lies under a branch of $g_0$ that maps onto $J_0$, let $n$ be such that $x \in B_n$, then $\operatorname{dis}(f_0)(x) < L^{n-1} D_0$ and by the way $\tilde{B}_n$ will be defined it will follow that $\eta(x) \geqq n$. Hence $\operatorname{dis}(h_0^{n-1} f_0)(x)$ will be of the order $D_0$ by the composition rule for distortion and the fact that $|h_0'| > L$. Furthermore, if $x \in B_n$ and $\eta(x) \geqq n$, then $x$

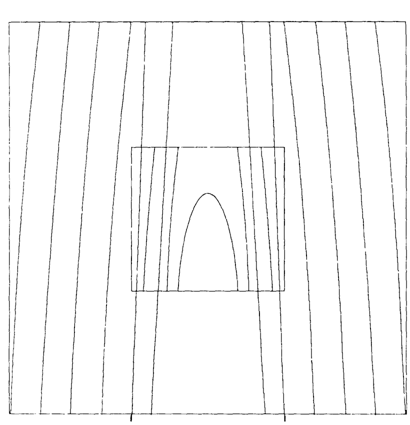

**Fig. C.1**

will be of the order $(s_1 L^{n-1} D_0)^{-1/2}$ and $g_i'(x) = (h_0^{\eta(x)} f_0(x))'(x)$ will be of the order $L^n \cdot 2s_1 \cdot (s_1 L^{n-1} D_0)^{-1/2} > s_1^{1/4}$ assuming $s_1$ is large compared with $L$ and $D_0$.

For convenience, let $g_0(x) = h_0(x)$ for $x \in H_0 = J_0 - J_1$. Then the estimates for $g_1$ on $J_0 - J_2$ are compatible with the hypothesis of Theorem 4 in the appendix.

At the $i^{\text{th}}$ step of this process it will be assumed that there is a map $g_i$ defined on $J_0 - J_{i+1}$ consisting of monotone branches each mapping onto one of the intervals $J_0, J_1, \ldots, J_i$. Let $\mathcal{R}^i$ be the minimal partition with respect to which $g_i$ is continuous and monotone and for $j = 0, 1, \ldots, i$ let $\mathcal{R}^i_j$ be the collection of intervals in $\mathcal{R}^i$ over which $g_i$ has a branch mapping onto $J_j$. It will be assumed that $\mathcal{R}^i_j \subset J_j$, and that every branch of $g_i$ in $\mathcal{R}_j \cap H_j$ is identical with some branch of $h_j$. The following bounds will be assumed for $g_i$:

*Bound a.* For $k < j \leqq i$ $\operatorname{dis}(g_i)|_{\mathcal{R}^i_k \cap J_j} < 2^k D_j$ and $\operatorname{dis}(g_i)|_{\mathcal{R}^i_j \cap J_j} < 2^j D_0$,

*Bound b.* For $k < j \leqq i$ $|g_i'||_{\mathcal{R}^i_k \cap J_j} > s_j^{1/4}$ and $|g_j'||_{\mathcal{R}^i_j \cap J_j} > L$.

A claim will be made that the $D_i$ remain bounded in the recursion.

A new map $g_{i+1}$ will be defined for $x \in H_{i+1}$ by using a stopping rule $\eta_i(x)$ and defining $g_{i+1} = g_i^{\eta_i(x)} f_i(x)$. This will be done in such a way that, firstly, $g_{i+1}$ consists of monotone branches each mapping onto one of the intervals $J_0, \ldots, J_{i+1}$ with a large proportion of them mapping onto one of $J_0, \ldots, J_i$, secondly, $\eta_i$ must be large enough to ensure that $g_{i+1}$ satisfies bounds $a$ and $b$; and finally $\eta_i$ must be small enough to allow a finite Rohklin pullback of a finite measure.

This process will utilize the following partitions; let $\mathcal{P}^i_1 = \bigvee_{k=0}^{i} \mathcal{R}^i_k \vee \{J_{i+1}\}$ and define $\mathcal{P}^i_n = g_i^{-1}(\mathcal{P}^i_{n-1}) \vee \{J_{i+1}\}$ for $n = 2, 3, \ldots$. Then for any $M \in \mathcal{P}^i_n$ either $g_i^k(M) = J_{i+1}$ for some $k = 0, 1, \ldots, n-1$ or $g_i^{n-1}(M) \in \mathcal{R}^i_j$ for some $j = 0, 1, \ldots, i$. These partitions form a refining sequence and since $\mathcal{P}^i_n$ contains the sets $g_i^{-k}(J_{i+1})$ for $k = 0, 1, \ldots, n-1$, they are increasingly dominated by preimages of $J_{i+1}$.

The set $J_{i+1} - J_{i+2}$ is then divided up into pairs of intervals $B^i_n$ for $n = b_i, \ldots, e_i$ such that $\operatorname{dis}(g_i)|_{B^i_n} < L^{n-1} 2^i D_0$ and $l\left(\bigcup_{k=n+1}^{e_i} B^i_n \cup J_{i+2}\right)$ is of the order $(s_{i+1} L^{n-1} 2^i D_0)^{-1/2}$. These sets will give a lower bound for $\eta_i$.

The sets $B^i_n$ will be moved slightly into sets $\tilde{B}^i_n$ such that $f_i(\tilde{B}^i_n)$ fits into the partition $\mathcal{P}^n_i$. This will be done in such a way that $\tilde{B}^i_n$ will still provide a lower bound for $\eta_i$ and $l\left(\bigcup_{k=n+1}^{e_i} \tilde{B}^i_k \cup J_{i+2}\right)$ is still of the order $(s_{i+1} L^{n-1} 2^i D_0)^{-1/2}$.

Then $\eta_i$ will be defined as follows: For $x \in \tilde{B}^i_n$ consider the position of $f_i(x)$ in $\mathcal{P}^i_n$. Either $f_i(x) \in g_i^{-k}(J_{i+1})$ for some $k = 0, 1, \ldots, n-1$ or $f_i(x) \in g_i^{-(n-1)}(M)$, where $M$ is a component of $\mathcal{R}^i_j$ for some $j = 0, 1, \ldots, i$. If $f_i(x) \in g_i^{-k}(J_{i+1})$ for $k = 0, 1, \ldots, n-1$ let $\eta_i(x) = k$, otherwise let $\eta_i(x) = n$.

Then $g_{i+1}(x) = g_i^{\eta_i(x)} f_i(x)$ is defined on $H_{i+1}$ and will consist of monotone branches mapping onto the intervals $J_0, J_1, \ldots, J_{i+1}$. Any branch of $g_{i+1}$ that maps onto $J_{i+1}$ will be identical with some branch of $h_{i+1}$, hence will have distortion bounded by $2^{i+1} D_0$ and slope greater than $L$ by bounds 2 and 3.

On the other hand, if $x$ lies under a branch of $g_{i+1}$ that maps onto $J_k$ for some $k < i+1$ (that is, $x \in \mathcal{R}^{i+1}_n$), let $n$ be such that $x \in \tilde{B}^i_n$. Then $\eta_i(x)$ will be greater than $n$ and $x$ will be of the order $(s_{i+1} L^{n+1} 2^i D_0)^{-1/2}$, hence $g_{i+1}'(x) = (g_i^{\eta_i(x)} f_i)'(x)$ will be of the order $L^{\eta_i} \cdot 2s_{i+1} \cdot (s_{i+1}L^{n-1}2^iD_0)^{-1/2} > s_{i+1}^{1/4}$ using bound 1 and assuming that $s_0$ is large compared with $L$ and $D_0$. The distortion will be shown to be bounded by something of the order $2^kD_0$. This is done by analyzing the distortion formula

$$\begin{aligned}\operatorname{dis}(g_{i+1})(x) \leqq \sum_{\alpha=1}^{\eta(x)} \frac{1}{(g_i^{\alpha-1})'(g_i^{\eta(x)-\alpha+1}f_i(x))} \operatorname{dis}(g_i)(g_i^{\eta_i(x)-\alpha}f_i(x)) \\ + \frac{1}{(g_i^{\eta_i(x)})'(f_i(x))} \operatorname{dis}(f_i)(x).\end{aligned}$$

This formula demonstrates that under repeated application of the map $g_i$, the distortion that is caused by $f_i$ or any application of $g_i$ is reduced through subsequent stretching.

The first term of the sum, $\alpha = 1$, contains the distortion of $g_i$ at $g_i^{\eta_i(x)-1}f_i(x)$ and is already bounded by something of the order $2^kD_0$ since $g_i^{\eta_i(x)-1}f_i(x) \in \mathscr{R}_k^i$. The last term, $\frac{1}{(g_i^{\eta_i(x)})'(f_i(x))} \operatorname{dis}(f_i)(x)$, is controlled by analyzing the slope of $g_i$ along the trajectory $f_i(x), g_if_i(x), \ldots, g_i^{\eta_i(x)-1}f_i(x)$. In controlling the remaining terms, potentially large distortions of the order $2^jD_0$ with $j > k$ are balanced against the amount of stretching that must occur on the type of trajectors that $f_i(x)$ must have in order to pick up such large distortions and then land on an $\mathscr{R}_k^i$. A final bound of the order $2^kD_0$ is established. This is the manner in which it is shown that $g_{i+1}$ satisfies bounds $a$ and $b$.

As mentioned before, the partitions $\mathscr{P}_n^i$ are increasingly dominated by preimages of $J_{i+1}$ as $n$ grows large. It is precisely these preimages that give rise to branches of $g_{i+1}$ that map into $J_{i+1}$. It will be shown that $J_{i+1}$ is sufficiently small so as to make the domination occur so slowly that such branches have relatively small measure, something of the order $s_{i+1}^{\delta}$ times the measure of $H_{i+1}$, which is sufficient for the hypothesis of Theorem 4 in the appendix.

Since points $x \in \tilde{B}_n^i$ are of the order $(s_{i+1}L^{n-1}2^iD_0)^{-1/2}$, bound 7 can be used to establish that $e_i$ is bounded above by something of the order $\log s_{i+2}$. For $x \in H_{i+1}$, $\eta_i(x)$ is uniformly bounded by $e_i$. With $\pi_i$ such that $f_i(0) \in g_i^{-\pi_i}(J_{i+1})$ the details of the construction will yield that $\pi_i < e_i$. If $m_{i+1}(x)$ is defined such that $g_{i+1}(x) = T_0^{m_{i+1}(x)}(x)$ for $x \in H_{i+1}$ then $m_{i+1}(x) \leqq \prod_{k=0}^{i} (e_i + 1)$.

The limit of the recursive process will be a map $g_\infty : J_0 \mapsto J_0$ which, satisfying the hypothesis of Theorem 4 in the appendix, posesses a finite absolutely continuous invariant measure $\mu$. Using a Rohklin argument this measure can be pulled back to an absolutely continuous invariant measure for $T_0$ iff the integral $\int m_\infty(x)d\mu$ is finite (where $m_\infty(x)$ is such that $g_\infty(x) = T_0^{m_\infty(x)}(x)$). It will be shown that this integral is finite if the sum $\sum_i \prod_{k=0}^{i-1} (e_k + 1) \cdot \mu(H_k)$ is finite. This sum converges since $e_{k-1}$ is bounded by something of the order $\log s_{k+1}$, and the results of Theorem 4 imply that $\mu(H_k)$ is bounded by something of the order $s_{k+1}^{-\varepsilon}$.

## (D) Detailed Proof

### *1. The $i^{\text{th}}$ Recursive Step*

*1.a. Assumptions on $g_i$.* To begin the recursion, let $g_0 = h_0$ on $H_0$. At the $i^{\text{th}}$ step of the recursion it will be assumed that there is a function $g_i$ defined on $J_0 - J_{i+1}$ consisting of branches $\mathscr{R}_k^i$ mapping onto the intervals $J_k$ for $k = 0, 1, \ldots, i$. It will be assumed that $\mathscr{R}_k^i \in J_k$ for $k = 0, 1, \ldots, i$. The function $g_i$ will be assumed to satisfy bounds $a$ and $b$. The functions $h_i$ on $H_i$ and $f_i = p_i(q(x))$ [where $q(x) = x^2$] on $J_{i+1}$ will be assumed to satisfy bounds 1 through 8. It will be assumed that the branches $\mathscr{R}_k^i \cap J_k$ of $g_i$ are identical with some branch of $h_k$.

*1.b. Distortion of $f_i$.* Let $z_{i+1}$ be such that $J_{i+1} = (-z_{i+1}, z_{i+1})$. Define $x_n^i = \inf\{x \in J_{i+1}, x > 0; \operatorname{dis}(f_i)(x) \leqq L^{n-1} 2^i D_0\}$, and let $b_i = \min\{n; x_n^i < z_{i+1}\}$. Let $B_{b_i}^i = (-z_{i+1}, -x_{b_i}^i) \cup (x_{b_i}^i, z_{i+1})$ and for $n > b_i$, let $B_n^i = (-x_{n-1}^i, -x_n^i) \cup (x_n^i, x_{n-1}^i)$. For convenience, redefine $x_{b_i}^i = z_{i+1}$.

Thus $J_{i+1}$ is partitioned into sets $B_n^i$ such that $\operatorname{dis}(f_i)$ over $B_n^i$ is at most $L^{n-1} 2^i D_0$.

**Lemma 1.** *$\exists C_1$ independent of $i$ such that*

$$\frac{1}{C_1}(s_{i+1} L^{n-1} 2^i D_0)^{-1/2} < x_n^i < C_1 (s_{i+1} L^{n-1} 2^i D_0)^{-1/2}.$$

*Proof.*

$$\tfrac{1}{2}(x_n^i)^{-2} = \operatorname{dis}(q)(x_n^i) \geqq (\operatorname{dis}(g_i)(x_n^i) - \operatorname{dis}(p_i)(q(x_n^i))) p_i'(q(x_n^i))$$

$$\geqq |L^{n-1} 2^i D_0 - K_1| s_{i+1} \frac{1}{2K_2} > \frac{1}{C_1'}(s_{i+1} L^{n-1} 2^i D_0)$$

for some $C_1'$ independent of $i$. Similarly,

$$\tfrac{1}{2}(x_n^i)^{-2} \leqq (L^{n-1} 2^i D_0 + K_1) s_{i+1} 2K_2 \leqq C_1''(s_{i+1} L^{n-1} 2^i D_0)$$

for some $C_1''$ independent of $i$. Let $C_1 = \max\left(\left(\frac{C_1'}{2}\right)^{1/2}, (2C_1'')^{1/2}\right)$. $\square$

Let $A_n^i = f_i(B_n^i)$ and $y_n^i = f_i(x_n^i)$.

**Lemma 2.** *$\exists C_2$ independent of $i$ such that*

$$\frac{1}{C_2}(L^{n-1} 2^i D_0)^{-1} < |y_n^i - f_i(0)| < C_2 (L^{n-1} 2^i D_0)^{-1}.$$

*Proof.*

$$|f_i(x_n^i) - f_i(0)| = \left|\int_0^{x_n^i} f_i'(t)\,dt\right| = \left|\int_0^{x_n^i} p_i'(q(t)) q'(t)\,dt\right| < \frac{K_2}{2} s_{i+1} \int_0^{x_n^i} q'(t)\,dt$$

$$= \frac{K_2}{2} s_{i+1} (x_n^i)^2 < \frac{K_2}{2}(C_1)^2 (L^{n-1} 2^i D_0)^{-1}.$$

Similarly,

$$|f_i(x_i^n) - f_i^n(0)| > \frac{1}{2K_2} s_{i+1}(x_n^i)^2 > \frac{1}{2K_2(C_1)^2}(L^{n-1}2^i D_0)^{-1}. \quad \square$$

*1.c. The Partition.* Define the partition $\mathscr{P}_1^i = \bigvee_{k=0}^{i} \mathscr{R}_k^i \vee \{J_{i+1}\}$ and recursively define $\mathscr{P}_n^i = g_i^{-1}(\mathscr{P}_{n-1}^i) \vee J_{i+1}$. Then $\{\mathscr{P}_n^i\}_{n=1}^{\infty}$ is a refining sequence of partitions, and $\mathscr{P}_n^i$ contains $g_i^{-k}(J_{i+1})$ for $k=0, 1, \ldots, n-1$, hence these partitions increasingly approximate the set of preimages of $J_{i+1}$ under $g_i$.

Define $\pi_i$ to be the return time of $f_i(0)$ to $J_{i+1}$ under $g_i$; that is, $g_i^n f_i(0) \notin J_{i+1}$ for $n < \pi_i$ and $g_i^{\pi_i} f_i(0) \in J_{i+1}$. Let $N_i$ be the component of $g_i^{-\pi_i}(J_{i+1})$ containing $f_i(0)$.

In order to ensure that $g_{i+1}$ is markov, the sets $A_n^i$ will be redefined so that they fit into the partitions $\mathscr{P}_n^i$. For $n > b_i$, if $y_n^i \in \bigcup_{k=0}^{n-1} g_i^{-k}(J_{i+1})$, then let $\tilde{y}_n^i$ be such that $(y_n^i, \tilde{y}_n^i) \subset \bigcup_{k=0}^{n-1} g_i^{-k}(J_{i+1})$ and $|\tilde{y}_n^i - f_i(0)|$ is minimized. If $y_n^i \notin \bigcup_{k=0}^{n-1} g_i^{-k}(J_{i+1})$, then let $M$ be the component of $\mathscr{P}_{n+1}^i$ containing $y_n^i$ and let $\tilde{y}_n^i$ be such that $(y_n^i, \tilde{y}_n^i) \in M$ and $|y_n^i - f_i(0)|$ is maximized (see Fig. D.1). It is quite possible for many of the $\tilde{y}_n^i$ to be identical.

Let $a_1^i$ be the endpoint of $N_i$ covered by $f_i(J_{i+1})$ and let $a_2^i$ be the remaining endpoint. With $\tilde{y}_n^i$ as defined, $\exists e_i$ such that $\tilde{y}_n^i = a_2^i$ for $n \geqq e_i$ and $\tilde{y}_n^i \notin [a_1^i, a_2^i]$ for $n < e_i$. Let $\tilde{y}_{b_i-1}^i = y_{b_i-1}^i$, and redefine $\tilde{y}_{e_i}^i$ so that $\tilde{y}_{e_i}^i = a_1^i$. Only the points $\tilde{y}_{b_i-1}, \ldots, \tilde{y}_{e_i}$ will be considered in the following arguments.

For $n = b_i, \ldots, e_i$ let $\tilde{A}_n^i = (\tilde{y}_{n-1}^i, \tilde{y}_n^i)$. The endpoints of $\tilde{A}_n^i$ lie in the division points of the partition $\mathscr{P}_n^i$, and $\tilde{A}_n^i$ is divided up by that partition into intervals each of

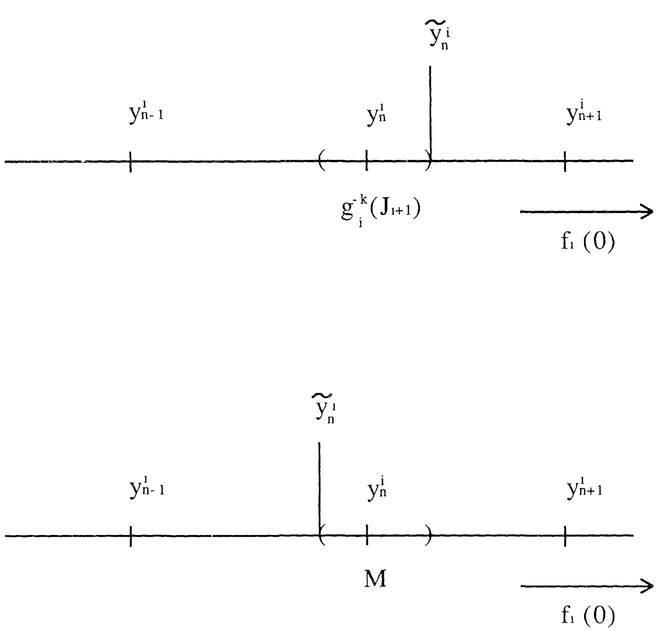


**Fig. D.1**

which is either a component of $f_i^{-k}(J_{i+1})$ for $k=0,1,\ldots,n-1$ or of $g_i^{-(n-1)}\left(\bigvee_{j=0}^{i} \mathscr{R}_j^i\right)$.

**Lemma 3.** *$\exists C_3$ independent of $i$ such that for $b_i \leqq n \leqq e_i$,*

$$|\tilde{y}_n^i - f_i(0)| < C_5 L^{-(n-b_i)} |\tilde{y}_{b_i-1}^i - f_i(0)|.$$

*Proof*

$$|y_{b_i-1} - f_i(0)| = |f_i(z_{i+1}) - f_i(0)| > C_2(L^{b_i-1} 2^i D_0)^{-1}$$

by Lemma 2. By bound 6, $l(J_{i+1}) < K_3 s_{i+1}^{-1/2}$, hence the maximum length of a component in $\mathscr{R}_i^i$ is less than $L^{-1} K_3 s_{i+1}^{-1/2}$ and the maximum length of a component in $\mathscr{R}_n^i \cap J_i$ is less than $s_i^{-1/4}$ for $n<i$. With the assumption that $L^{-1}K_3 s_{i+1}^{-1/2} < s_i^{-1/4}$, an upper bound of $s_i^{-1/4}$ can be used for the lengths of components in $\bigvee_{n=0}^{i} \mathscr{R}_n^i$. Then either $|\tilde{y}_n^i - f_i(0)| < |y_n^i - f_i(0)|$ or $|\tilde{y}_n^i - y_n^i| < L^{-(n-1)} s_i^{-1/4}$. Hence

$$\begin{aligned}|\tilde{y}_n^i - f_i(0)| &< C_2(L^{n-1} 2^i D_0)^{-1} + L^{-(n-1)} s_i^{-1/4} < L^{-(n-b_i)}(C_2(L^{b_i-1}2^i D_0)^{-1} \\ &\quad + L^{-(b_i-1)} s_i^{-1/4}) < L^{-(n-b_i)} C_3 |f_i(z_{i+1}) - f_i(0)|\end{aligned}$$

for some $C_3$ independent of $i$, assuming $s_i^{-1/4}$ is small compared to $C_2(2^i D_0)^{-1}$. □

Let $\tilde{x}_i^n = f_i^{-1}(\tilde{y}_n^i) \cap [0,1]$ and $\tilde{B}_n^i = f_i^{-1}(\tilde{A}_n^i)$.

**Lemma 4.** *$\exists C_4$ independent of $i$, such that for $n = b_i, \ldots, e_i$,*

$$|\tilde{x}_n^i| < C_4 L^{-\frac{n-b_i}{4}} |z_{i+1}|.$$

*Proof*

$$|f_i(\tilde{x}_n^i) - f_i(0)| = \left|\int_0^{\tilde{x}_n^i} f_i'(t)dt\right| = \left|\int_0^{\tilde{x}_n^i} p_1'(q(t))q'(t)dt\right| > \frac{1}{2K_2} s_{i+1}(\tilde{x}_n^i)^2.$$

Hence by Lemma 3, $\exists C_4$ independent of $i$ such that $|\tilde{x}_n^i| < C_4 L^{-\frac{n-b_i}{2}} |z_{i+1}|$. □

The following will allow the measure to be pulled back in a finite fashion at the final section of the proof.

**Lemma 5.** *$\exists C_5$ independent of $i$ such that $b_i < C_5 \log s_{i+1}$ and $e_i < C_5 \log s_{i+2}$.*

*Proof.* $|x_{e_i-1}| > |z_{i+2}|$ implies by Lemma 1 and bound 7 that

$$C_1(s_{i+1} L^{e_i-2} 2^i D_0)^{-1/2} > |x_{e_i-1}| > |z_{i+2}| > \tfrac{1}{2} K_4 s_{i+2}^{-1},$$

hence

$$L^{e_i-2} < s_{i+2}^2 \frac{2C_1}{s_{i+1} 2^i D_0 K_4} < C' s_{i+2}^2$$

for some $C'$ independent of $i$, thus $e_i < C_5' \log s_{i+1}$ for some $C_5'$ independent of $i$. Similarly, $b_i$ being the minimum such that $|x_{b_i}^i| < |z_{i+1}|$ yields

$$C_1(s_{i+1} L^{b_i-2} 2^i D_0)^{-1/2} > |z_{i+1}| > \tfrac{1}{2} K_4 s_{i+1}^{-1},$$

hence
$$L^{b_i-1} < s_{i+1}^2 \frac{2C_1}{s_{i+1}2^i D_0 K_4} = s_{i+1}\frac{2C_1}{2^i D_0 K_4} < C'' s_{i+1}$$

for some $C''$ independent of $i$, thus $b_i < C_5'' \log s_{i+1}$ for some $C_5''$ independent of $i$. Let $C_5 = \max(C_5', C_5'')$. □

*1.d. Definition of* $g_{i+1}$. $H_{i+1}$ is divided into sets $\tilde{B}_n^i$ for $n = b_i, \ldots, e_i$ and each $\tilde{B}_n^i$ is further divided by $f_i^{-1}(\mathscr{P}_n^i)$ into intervals. Take any $M \in \tilde{B}_n^i \cap f_i^{-1}(\mathscr{P}_n^i)$ for any $n = b_i, \ldots, e_i$. Then either $f_i(M)$ is a component of $g_i^{-k}(J_{i+1})$ for some $k < n$ or $f_i(M)$ is a component of $g^{-n}\left(\bigvee_{j=0}^{i} \mathscr{R}_j^i\right)$. For $x \in M$ define $\eta_i(x) = k$ if $f_i(M)$ is a component of $g_1^{-k}(J_{i+1})$ for $k < n$ and $\eta(x) = n$ otherwise.

For $x \notin J_{i+1}$ let $g_{i+1}(x) = g_i(x)$ and for $x \in H_{i+1}$ let $g_{i+1}(x) = g_i^{\eta_i(x)} f_i(x)$. For $x \in J_{i+2}$ let $f_{i+2}(x) = g_i^{\pi_i} f_i(x)$, where $\pi_i$ is the return time of $f_i(0)$ to $J_{i+1}$ under $g_i$.

For $x \in J_0 - J_{i+2}$ then, $g_{i+1}$ consists of monotone branches mapping onto $J_0$, $J_1, \ldots, J_{i+1}$. For $k = 0, 1, \ldots, i+1$, let $\mathscr{R}_k^{i+1} \supset \mathscr{R}_k^i$ be the collection of intervals that map onto $J_k$ under an entire branch of $g_{i+1}$. Note that $g_i$ had no branches that map onto $J_{i+1}$ but $g_{i+1}$ might, hence $\mathscr{R}_{i+1}^{i+1} \subset J_{i+1}$.

In the definition of $\tilde{A}_n^i$, either $\tilde{A}_n^i$ contained $y_{n-1}^i$ or else $y_{n-1}^i$ was in a component $M$ of $g_i^{-k}(J_{i+1})$ for $k < n-1$ in which case $\tilde{A}_n^i$ and $M$ are disjoint and share a common endpoint (see Fig. D.2). Thus is established the essential fact that

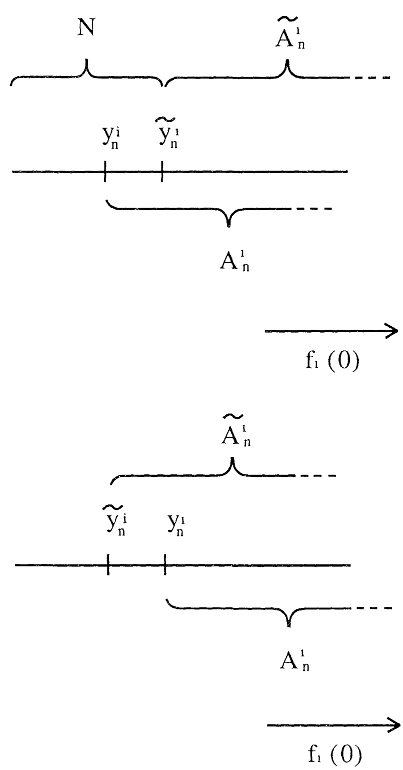


**Fig. D.2**

for $x \in B_n^i$, either $\eta_i(x) \geqq n$ or $\eta_i(x)$ is the first return of $f_i(x)$ to $J_{i+1}$ under $g_i$. If $\eta_i(x)$ is the first return, then $g_{i+1}(x) = h_{i+1}(x)$, and hence the distortion and slope is controlled by bounds 2 and 3. Otherwise the distortion and slope is the subject of the following calculations.

*1.e. Verification of Recursive Bounds.* Recall the recursive bounds $a$ and $b$ stated earlier

*Bound a.* For $k < j \leqq i$ $\operatorname{dis}(g_i)|_{\mathscr{R}_k^i \cap J_i} < 2^k D_j$ and $\operatorname{dis}(g_i)|_{\mathscr{R}_j^i \cap J_j} < 2^j D_0$.

*Bound b.* For $k < j \leqq i$ $|g_i'||_{\mathscr{R}_k^i \cap J_j} > s_j^{1/4}$ and $|g_j'||_{\mathscr{R}_j^i \cap J_j} > L$.

**Proposition.** *Under the recursive definition of $g_{i+1}$, bound a and bound b are satisfied in such a way that the $D_i$ remain bounded.*

*Proof.* Bound $b$ is satisfied by

$$\begin{aligned} |g_{i+1}'(x)| &= |(g_i^\eta f_i)'(x)| = |g_i^{\eta\prime}(p_i q(x)) p_i'(q(x)) q'(x)| \\ &> (L^\eta)\left(\frac{1}{2K_2} s_{i+1}\right)\left(\frac{2}{C_1}(s_{i+1} L_{n-2} 2^i D_0)^{-1/2}\right) > s_{i+1}^{1/4}, \end{aligned}$$

assuming that $s_{s+i}^{1/4} > C_1 K_2 2^{\frac{i}{2}} D_0^{1/2}$.

Bound $a$ is more involved. Recall the distortion formula:

$$\begin{aligned} \operatorname{dis}(g_{i+1})(x) \leqq & \sum_{\alpha=1}^{\eta(x)} \frac{1}{(g_i^{\alpha-1})'(g_i^{\eta(x)-\alpha+1} f_i(x))} \operatorname{dis}(g_i)(g_i^{\eta_i(x)-\alpha} f_i(x)) \\ & + \frac{1}{(g_i^{\eta_i(x)})'(f_i(x))} \operatorname{dis}(f_i)(x). \end{aligned}$$

First it is possible that $g_i^\alpha f_i(x)$ all lie in $\mathscr{R}_i^i \cap H_i$ for $\alpha = 0, 1, \ldots, \eta - 1$. Then

$$\operatorname{dis}(g_{i+1})(x) < \sum_{\alpha=1}^{\eta} \frac{1}{L^{\alpha-1}} 2^i D_0 + \frac{1}{L^\eta} L^{n-1} 2^i D_0 < \frac{L+1}{L-1} 2^i D_0.$$

In particular this starts the recursion for $i = 0$.

Otherwise one of $g_i^\alpha f_i(x)$ for $\alpha = 0, 1, \ldots, \eta - 1$ lies in $\mathscr{R}_k^i \cap H_i$ for some $k < i$, making $g_i^{\eta\prime} f_i(x) > s_i^{1/4}$, and so for the last term in the distortion formula,

$$\frac{1}{g_i^{\eta\prime} f_i(x)} \operatorname{dis}(f_i)(x) < s_i^{-1/4} L^{-\eta-1} \cdot L^{n-1} 2^i D_0 v < s_i^{-1/4} 2^i D_0.$$

With $j$ taken to be such that $g_i^{\eta-1} \in \mathscr{R}_j^i$ (that is, $x \in \mathscr{R}_j^{i+1}$), consideration of the sum from $\alpha = 1$ to $\alpha = \eta$ divides into two cases; either $g_i^{\eta-1} f_i(x) \in \mathscr{R}_j^i \cap H_j$ or $g_i^{\eta-1} f_i(x) \in \mathscr{R}_j^i \cap J_{j+1}$ (note that if $j = i$ then the latter possibility is vacuous).

Suppose that $g_i^{\eta-1} f_i(x) \in \mathscr{R}_j^i \cap H_j$. Then for $\alpha = 1$, $\operatorname{dis}(g_i)(g_i^{\eta-1} f_i(x)) < 2^j D_0$ and $g_i'(g_i^{\eta-1} f_i(x)) > L$. For any $\alpha = 2, 3, \ldots, \eta$ consider the term

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1} f_i(x))} \operatorname{dis}(g_i)(g_i^{\eta-\alpha} f_i(x)),$$

and let $k$ be such that $g_i^{\eta-\alpha} f_i(x) \in \mathscr{R}_k^i$.

If $k \leqq j$ then

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x))}\operatorname{dis}(g_i)(g_i^{\eta-\alpha}f_i(x)) < \frac{1}{L^{\alpha-1}}2^kD_i < \frac{1}{L^{\alpha-1}}2^jD_i.$$

If $k>j$ then one of $g_i^{\eta-\alpha+1}f_i(x), \ldots, g_i^{\eta-2}f_i(x)$ must land in $\mathscr{R}_m^i \cap J_k$ for some $m<k$, because $g_i^{\eta-\alpha} \in \mathscr{R}_k^i$ implies $g_i^{\eta-\alpha+1}f_i(x) \in J_k$ and since $g_i^{\eta-1}f_i(x) \in H_j$ with $k>j$ the trajectory of $g_i^{\eta-\alpha+1}f_i(x)$ must bounce out of $J_k$ sometime. Then $(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x)) > L^{\alpha-2}s_k^{1/4}$ making

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x))}\operatorname{dis}(g_i)(g_i^{\eta-\alpha}f_i(x)) < \frac{1}{L^{\alpha-2}s_k^{-1/4}}2^kD_i < L^{-(\alpha-1)}2^jD_i,$$

assuming that $2^k s_k^{-1/4} < 2^j$.
Under these circumstances then,

$$\operatorname{dis}(g_{i+1})(x) < 2^jD_0 + \sum_{\alpha=2}^{\eta-1}\frac{1}{L^{\alpha-1}}2^jD_i + s_i^{-1/4}2^iD_0 < 2^jD_0 + \frac{1}{L-1}2^jD_i + s_i^{-1/4}2^iD_0.$$

Suppose now that $g_i^{\eta-1}f_i(x) \in \mathscr{R}_j^i \cap J_{i+1}$ and let $w>j$ be such that $g_i^{\eta-1}f_i(x) \in \mathscr{R}_j^i \cap H_w$. Then for $\alpha=1$, $\operatorname{dis}(g_i)\,(g_i^{\eta-1}f_i(x)) < 2^iD_w$ and $g_i'(g_i^{\eta-1}f_i(x)) > s_w^{1/4}$. For any $\alpha = 2, 3, \ldots, \eta-1$, consider the term

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x))}\operatorname{dis}(g_i)(g_i^{\eta-\alpha}f_i(x)),$$

and let $k$ be such that $g_i^{\eta-\alpha}f_i(x) \in \mathscr{R}_k^i$.

If $k \leqq j$ then

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x))}\operatorname{dis}(g_i)(g_i^{\eta-\alpha}f_i(x)) < \frac{1}{L^{\alpha-2}s_w^{1/4}}2^kD_i < L^{-(\alpha-2)}s_w^{-1/4}2^jD_i.$$

If $k>w$ then one of $g_i^{\eta-\alpha+1}f_i(x), \ldots, g_i^{\eta-2}f_i(x)$ must land in $\mathscr{R}_m^i \cap J_k$ for some $m>k$ making

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x))}\operatorname{dis}(g_i)(g_i^{\eta-\alpha}f_i(x))$$
$$< \frac{1}{L^{\alpha-3}}s_w^{-1/2}s_k^{-1/4}2^kD_i < L^{-(\alpha-2)}s_w^{-1/4}2^jD_i,$$

assuming that $Ls_k^{-1/4}2^k < 2^j$.

If $j<k<w$ then

$$\frac{1}{(g_i^{\alpha-1})'(g_i^{\eta-\alpha+1}f_i(x))}\operatorname{dis}(g_i)(g_i^{\eta-\alpha}f_i(x))$$
$$< \frac{1}{L^{\alpha-2}s_w^{1/4}}2^kD_i < L^{-(\alpha-2)}s_w^{-1/4}2^wD_i < L^{-(\alpha-2)}s_w^{-1/8}2^jD_i,$$

assuming that $s_w^{-1/8}2^w < 2^j$.

Under these circumstances then,

$$\begin{aligned}\operatorname{dis}(g_{i+1})(x) &< 2^j D_w + \sum_{\alpha=2}^{\eta} L^{-(\alpha-2)} s_w^{-1/8} 2^j D_i + s_i^{-1/4} 2^i D_0 \\ &< 2^j D_w + \frac{L}{L-1} s_w^{-1/8} 2^j D_i + s_i^{-1/4} 2^i D_0 .\end{aligned}$$

Thus for $x \in B_n^i$ and $g_i^{\eta-1} f_i(x) \in \mathscr{R}_j^i$ it is shown that:

$$\operatorname{dis}(g_{i+1})(x) < \max \left\{ \begin{array}{l} 2^j D_0 \dfrac{L+1}{L-1} \\ 2^j D_0 + \dfrac{1}{l-1} 2^j D_i + s_i^{-1/4} 2^i D_0 \\ \max\limits_{w<i} 2^j D_w + \dfrac{L}{L-1} s_w^{-1/8} 2^j D_i + s_i^{-1/2} 2^i D_0 \end{array} \right\}.$$

Thus for recursive bound $a$, define

$$D_{i+1} = \max \left\{ \begin{array}{l} D_0 \dfrac{L+1}{L-1} \\ D_0 + \dfrac{1}{L-1} D_i + s_i^{-1/4} D_0 \\ \max\limits_{w<i} D_w + \dfrac{L}{L-1} s_w^{-1/8} D_i + s_i^{-1/4} D_0 \end{array} \right\}.$$

This is sufficient to claim that the $D_i$'s are bounded by the following lemma.

**Lemma 6.** *For $P_0 > 0$ and $0 < \gamma < 1$, define $\{P_i\}_{i=0}^{\infty}$ by the recursion*

$$P_{i+1} = \max_{w \leqq i} \{P_w + \gamma^{w+1} P_i\}.$$

*Then*

$$P_{i+1} = P_0 \prod_{n=0}^{i} (1 + \gamma^{n+1}).$$

*Proof.* Assume that

$$P_m = P_0 \prod_{n=0}^{m-1} (1 + \gamma^{n+1})$$

for $m \leqq i$, and check that

$$P_m + \gamma^{m+1} P_i \leqq P_0 \prod_{n=0}^{i} (1 + \gamma^{n+1})$$

for $m \leqq i$. By assumption,

$$\begin{aligned} P_m + \gamma^{m+1} P_i &= P_0 \prod_{n=0}^{m-1} (1 + \gamma^{n+1}) + \gamma^{m+1} P_0 \prod_{n=0}^{i-1} (1 + \gamma^{n+1}) \\ &= P_0 \prod_{n=0}^{m-1} (1 + \gamma^{n+1}) \left( 1 + \gamma^{m+1} \prod_{n=m}^{i-1} (1 + \gamma^{n+1}) \right). \end{aligned}$$

Hence it suffices to show that

$$\left(1+\gamma^{m+1}\prod_{n=m}^{i-1}(1+\gamma^{n+1})\right) \leqq \prod_{n=m}^{i}(1+\gamma^{n+1}),$$

or

$$1 \leqq (1-\gamma^{m+1}+\gamma^{i+1})\prod_{n=m}^{i-1}(1+\gamma^{n+1}).$$

Let $v=\left[\log_2 \frac{i}{m+1}\right]$, and $r=2^{v+1}(m+1)$ so that $\frac{1}{2}\tau \leqq j \leqq \tau$. Let

$$\Gamma=\prod_{n=m}^{i-1}(g_{i+1}) \quad \text{and} \quad \tilde{\Gamma}=\frac{\Gamma}{\prod_{r=0}^{v}(1+\gamma^{2^r(m+1)})}.$$

Note that $\tilde{\Gamma} \geqq 1$. Then

$$\begin{aligned}
&(1-\gamma^{m+1}+\gamma^{i+1})\prod_{n=m}^{i-1}(g_{i+1}) \\
&\quad =(1-\gamma^{m+1})\Gamma+\gamma^{i+1}\Gamma=(1-\gamma^{m+1})(1+\gamma^{m+1})(1+\gamma^{2(m+1)}) \\
&\qquad \times(1+\gamma^{4(m+1)})\ldots(1+\gamma^{2^v(m+1)})\tilde{\Gamma}+\gamma^{i+1}\Gamma \\
&\quad =(1+\gamma^{\tau})\tilde{\Gamma}+\gamma^{i+1}\Gamma=\tilde{\Gamma}-\gamma^{\tau}\tilde{\Gamma}+\gamma^{i+1}\Gamma \geqq \tilde{\Gamma} \geqq 1 . \quad \square
\end{aligned}$$

This concludes the proof of the proposition.

*1.f. Density of Long Branches.* In the definition of $g_{i+1}$, points $x \in \tilde{B}_n^i$ were assigned a stopping time of $\eta_i(x)=n$ unless $g_i^k f_i(x) \in J_{i+1}$ for some $k=0,1,\ldots,n-1$. Thus the subset of $\tilde{B}_n^i$ where $\eta_i(x)=n$ is given by $\tilde{B}_n^i-\bigcup_{k=0}^{n-1} f_i^{-1} g_i^{-k}(J_{i+1})$. Since points $x \in \tilde{B}_n^i$, where $\eta_i(x)=n$ are exactly the points of $\mathscr{R}_{i+1}^{i+1}$, this will be used to estimate the proportion of $\mathscr{R}_k^{i+1}$ branches in $J_{i+1}$ with $k<i+1$.

**Lemma 7.** *$\exists C_7, \delta>0$ independent of i such that*

$$\frac{l(H_{i+1}-\mathscr{R}_{i+1}^{i+1})}{l(H_{i+1}^{i+1})}>1-C_7 s_{i+1}^{-\delta}.$$

*Proof.* Let $r$ be maximal such that $l\left(\bigcup_{n=r+1}^{\infty} \tilde{B}_n^i\right)>2 s_{i+1}^{-1/8} l(J_{i+1})$. Then $|\tilde{x}_r^i|>s_{i+1}^{-1/8}|z_{i+1}|$ (see Fig. D.3). Lemmas 4 and 5 then imply

$$\begin{aligned}
r &< \frac{2}{\log L}\left(\tfrac{1}{8}\log s_{i+1}+\log C_4\right)+b_i \\
&< \frac{2}{\log L}\left(\tfrac{1}{8}\log s_{i+1}+\log C_4\right)+C_5 \log s_{i+1}<C \log s_{i+1}
\end{aligned}$$

for some $C$ independent of $i$.

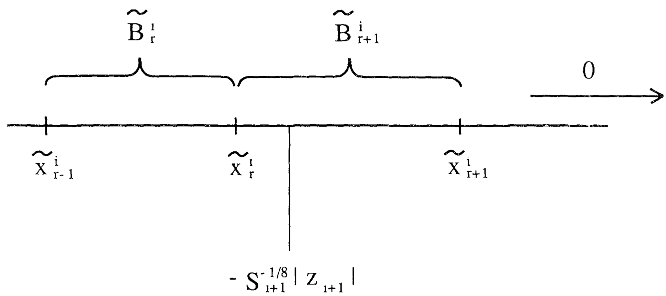


**Fig. D.3**

Let $M=\bigcup_{n=r+1}^{\infty}\tilde{B}^i_n$ and $N=f_i(J_{i+1}-M)=\bigcup_{n=b_i}^{r}\tilde{A}^i_n$. Note that $\dfrac{l(N-J_{i+1})}{l(N)}>1-2K_3s_{i+1}^{-1/2}$. Let $U=\bigcup_{k=0}^{r-1}g_i^{-k}(J_{i+1})$. Then $H_{i+1}-\mathscr{R}^{i+1}_{i+1}\supset f_i^{-1}(N-U)$.

Using the composition rule for distortion it follows that $\operatorname{dis}(g_i^k)(y)<2^iD_i\dfrac{L}{L-1}$ whenever $g_i^k$ is defined at $y$. Thus if $g_i^n(x)$ and $g_i^n(y)$ lie under the same branches of $g_i$ for $n=0,1,\dots,k-1$ then $\left|\dfrac{g_i^{k\prime}(x)}{g_u^{k\prime}(y)}\right|<s_{i+1}^{1/8}$ by assuming that

$$s_{i+1}^{1/8}>e^{2^iD_i\frac{L}{L-1}}.$$

To estimate $l(N-U)$, first remove $J_{i+1}$ from $N$, then remove $\bigcup_{k=1}^{r-1}g_i^{-k}(J_{i+1})$ from whats left. This yields

$$\begin{aligned}l(N-U)&<l(N)(1-2K_3s_{i+1}^{-1/2})\left(1-s_{i+1}^{1/8}\frac{l(J_{i+1})}{l(J_i)}\right)^{r-1}\\&<l(N)(1-2K_3s_{i+1}^{-1/2})(1-s_{i+1}^{1/8}K_5s_{i+1}^{-1/4})^{C\log s_{i+1}}\\&<l(N)(1-C's_{i+1}^{-1/4})^{C\log s_{i+1}+1}\end{aligned}$$

for some $C'$ independent of $i$. This implies that $\dfrac{l(U)}{l(N)}<C''(\log s_{i+1})s_{i+1}^{-1/4}$ for some $C''$ independent of $i$.

For $x\in J_{i+1}-M$ it follows that

$$|f_i'(x)|=|p_i'(q(x))q'(x)|>\frac{1}{2K_2}s_{i+1}\cdot 2(s_{i+1}^{-1/8}|z_{i+1}|)=\frac{1}{K_2}s_{i+1}^{1-1/8}|z_{i+1}|,$$

and similarly

$$|f_i'(x)|<\frac{K_2}{2}s_{i+1}2|z_{i+1}|=K_2s_{i+1}|z_{i+1}|,$$

hence

$$\max_{x,y\in J_{i+1}-M}\left|\frac{f_i'(x)}{f_i'(y)}\right|<(K_2)^2s_{i+1}^{1/8}$$

or equivalently,

$$\max_{u,v\in N}\left|\frac{f_i^{-1\prime}(u)}{f_i^{-1\prime}(v)}\right| < (K_2)^2 s_{i+1}^{1/8} .$$

The sets $H_{i+1}$ and $J_{i+1}-M$ are reasonably comparable:

$$\frac{l(J_{i+1}-M)}{l(H_{i+1})} > \frac{l(J_{i+1})-2s_{i+1}^{-1/8}l(J_{i+1})}{l(J_{i+1})} = 1-2s_{i+1}^{-1/8} .$$

Therefore,

$$\begin{aligned}\frac{l(H_{i+1}-\mathscr{R}_{i+1}^{i+1})}{l(H_{i+1})} &= \frac{l(J_{i+1}-M)}{l(H_{i+1})}\,\frac{l(H_{i+1}-\mathscr{R}_{i+1}^{i+1})}{l(f_i^{-1}(N))} > (1-2s_{i+1}^{-1/8})\,\frac{l(f_i^{-1}(N-U))}{l(f_i^{-1}(N))}\\ &= (1-2s_{i+1}^{-1/8})\left(1-\frac{l(f_i^{-1}(U))}{l(f_i^{-1}(N))}\right) > (1-2s_{i+1}^{-1/8})\left(1-(K_2)^2 s_{i+1}^{1/8}\,\frac{l(U)}{l(N)}\right)\\ &> (1-2s_{i+1}^{-1/8})(1-(K_2)^2 s_{i+1}^{1/8} C''(\log s_{i+1}) s_{i+1}^{-1/4}) > (1-C_7 s_{i+1}^{-\delta})\end{aligned}$$

for some $C_7$, $\delta>0$ independent of $i$. □

*(2) The Invariant Measure*

*2.a. The Measure for $g_\infty$.* The limit of this recursive process is a map $g_\infty: J_0 \mapsto J_0$ consisting of monotone branches $\mathscr{R}_j^\infty$ mapping onto intervals $J_j$. This map satisfies the bounds:

*Bound $a_\infty$.* $\operatorname{dis}(g_\infty)|_{\mathscr{R}_k^\infty} < 2^k D_\infty$.

*Bound $b_\infty$.* For $k<j$, $|g'_\infty|\,|_{\mathscr{R}_k^\infty \cap J_j} > S_j^{1/4}$ and $|g'_\infty|\,|_{\mathscr{R}_j^\infty \cap J_j} > L$.

This, along with Lemma 7, bounds 1 and 8, and the markov properties of $g_\infty$, is sufficient to conclude the existence of a finite absolutely continuous invariant measure $\mu$ for $g_\infty$ by Theorem 4 in the appendix. This also yields the property that $\mu(J_j) < G_1 s_j^{-\varepsilon}$ for some $G_1$, $\varepsilon>0$.

*2.b. Pulling $\mu$ Back to $T_0$.* The argument here is the typical Rohklin type reasoning with special care being taken to account for the map being many to one and for the use of a stopping rule rather than a return time.

Let $\mathscr{Q}$ be the minimal partition with respect to which $g_\infty$ is continuous and monotone, and for a component $M\in\mathscr{Q}$ define $\eta_\infty(M)$ so that $g_\infty(x) = T_0^{\eta_\infty(M)}(x)$ for $x\in M$. Thus $\eta_\infty$ is an integer valued function on $\mathscr{Q}$.

With $\mu$ as the absolutely continuous invariant measure for $g_\infty$, then $\tilde{\mu}$ is defined for any measurable set $E$ by

$$\tilde{\mu}(E) = \sum_{M\in\mathscr{Q}} \sum_{n=0}^{\eta_\infty(M)-1} \mu(T_0^{-n}(E\cap T_0^n(M))\cap M)$$

is an absolutely continuous invariant measure for $T_0$ as the following calculation indicates:

$$\begin{aligned}\tilde{\mu}(T_0^{-1}(E)) &= \sum_{M\in\mathcal{Q}} \sum_{n=0}^{\eta_\infty(M)-1} \mu(T_0^{-n}(T_0^{-1}(E)\cap T_0^n(M))\cap M)\\ &= \sum_{M\in\mathcal{Q}} \sum_{n=0}^{\eta_\infty(M)-1} \mu(T_0^{-(n+1)}(E\cap T_0^{n+1}(M))\cap M)\\ &= \sum_{M\in\mathcal{Q}} \sum_{n=0}^{\eta_\infty(M)-1} \mu(T_0^n(E\cap T_0^n(M))\cap M)\\ &\quad + \sum_{M\in\mathcal{Q}} [\mu(T_0^{-\zeta(M)}(E\cap T_0^{\zeta(M)}(M))\cap M)-\mu(E\cap M)]\\ &= \tilde{\mu}(E)+\mu(g_\infty^{-1}(E\cap J_0))-\mu(E)=\tilde{\mu}(E).\end{aligned}$$

**Proposition.** *$\tilde{\mu}$ is finite.*

*Proof.* By definition,

$$\tilde{\mu}(J_0)=\sum_{M\in\mathcal{Q}} \sum_{n=0}^{\eta_\infty(M)-1} \mu(M)=\sum_{M\in\mathcal{Q}} \eta_\infty(M)\mu(M).$$

Consider the sets $\{\tilde{B}_n^i\}_{k=b_i}^{e_i}$ and the stopping time $\eta_i$ defined on the $i^{\text{th}}$ step of the recursion. It follows from the construction that $\max_{x\in H_{i+1}} \eta_i(x)=e_i$. Thus if $M\in\tilde{B}_n^i$, then

$$\eta_\infty(M)\leqq \prod_{j=0}^{i-1}(e_j+1)\cdot(n+1).$$

Hence,

$$\begin{aligned}\tilde{\mu}(J_0) &= \sum_{M\in\mathcal{Q}} \eta_\infty(M)\cdot\mu(M)<\mu(H_0)\\ &\quad+\sum_{i=0}^{\infty}\left[\prod_{j=0}^{i-1}(e_j+1)\right]\cdot\sum_{n=b_i}^{e_i}(n+1)\mu(\tilde{B}_n^i).\end{aligned}$$

From Theorem 5 in the appendix and the definition of $\mu$ it follows that if $U\in H_{i+1}$, then

$$\frac{\mu(U)}{l(U)}<G_2\frac{\mu(J_{i+1})}{l(J_{i+1})} \quad\text{or}\quad \mu(U)<G_2\frac{\mu(J_{i+1})}{l(J_{i+1})}l(U).$$

Hence,

$$\begin{aligned}\sum_{n=b_i}^{e_i}(n+1)\mu(\tilde{B}_n^i) &= b_i\mu(H_{i+1})+\sum_{n=0}^{e_i-b_i} n\mu(\tilde{B}_{b_i+n}^i)\\ &= b_i\mu(H_{i+1})+\sum_{n=0}^{e_i-b_i}\mu\left(\bigcup_{j=b_i+n}^{e_i}\tilde{B}_j^i\right)\\ &< b_i\mu(H_{i+1})+\sum_{n=0}^{e_i-b_i} G_2\frac{\mu(J_{i+1})}{l(J_{i+1})}\,l\left(\bigcup_{j=b_i+n}^{e_i}\tilde{B}_j^i\right)\\ &< b_i\mu(H_{i+1})+G_2\frac{\mu(J_{i+1})}{l(J_{i+1})}\sum_{n=1}^{e-i-b_i} C_4L^{-\frac{n}{2}}l(J_{i+1})\\ &< b_i\mu(H_{i+1})+G_2C_4\mu(J_{i+1})\frac{L^{1/2}}{L^{1/2-1}}\\ &< Cb_i\mu(J_{i+1})\end{aligned}$$

for some constant $C$. Therefore,

$$\begin{aligned}\tilde{\mu}(J_0) &< \mu(H_0) + \sum_{i=1}^{\infty}\left[\prod_{j=0}^{i-1}(e_j+1)\right]\cdot\left[\sum_{n=b_i}^{e_i+1} n\mu(\tilde{B}_n^i)\right]\\ &< \mu(H_0) + \sum_{i=1}^{\infty}\left[\prod_{j=0}^{i-1}(e_j+1)\right]\cdot[Cb_i\mu(J_{i+1})]\\ &< \mu(H_0) + \sum_{i=1}^{\infty}\left[\prod_{j=0}^{i-1}(e_j+1)\right]\cdot[Cb_iG_4s_{i+1}^{-\varepsilon}].\end{aligned}$$

This sum converges since by bound 1, $s_{i+1}^{-\varepsilon} < s_{i+1}^{-\frac{3\varepsilon}{4}} \cdot s_i^{-\frac{3\varepsilon}{4}} \cdot s_{i-1}^{-\frac{3\varepsilon}{4}} \dots s_1^{-\frac{3\varepsilon}{4}}$, hence

$$\begin{aligned} b_i s_{i+1}^{-\varepsilon}\prod_{j=0}^{i-1}(e_j+1) &< (b_i s_{i+1}^{-\frac{3\varepsilon}{8}})(s_{i+1}^{-\frac{3\varepsilon}{8}}(e_{i-1}+1))\prod_{j=0}^{i-2}(s_{j+2}^{-\frac{3\varepsilon}{4}}(e_j+1))\\ &< (C_5\log s_{i+1}\cdot s_{i+1}^{-\frac{3\varepsilon}{8}})(s_{i+1}^{-\frac{3\varepsilon}{8}}(C_5\log s_{i+1}+1))\prod_{j=0}^{i-2} s_{j+2}^{-\frac{3\varepsilon}{4}}(C_5\log s_{j+2}+1)),\end{aligned}$$

and this product approaches 0 faster than an exponential rate. □

## Appendix

In this appendix I give some background material on sufficient conditions for the existence of finite absolutely continuous invariant measures for some Markov maps on an interval.

*Continuous Invariant Measures and the Folklore Theorem.* The simplest non-trivial example of a continuous invariant measure is for a linear map consisting of branches that map onto an interval. Let the unit interval $I$ be divided into a countable union of disjoint intervals, $I = \bigcup H_i$, and $T: I \mapsto I$ maps each $H_i$ linearly onto $I$. Then for any measurable set $E \subset I$,

$$l(T^{-1}(E)) = \sum_i l(T^{-1}(E)\cap H_i) = \sum_i \frac{l(E)}{l(I)}\, l(H_i) = l(E),$$

hence lebesgue measure is invariant.

If such a transformation is distorted in such a way that $\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right|$ is uniformly bounded on branches of $T^n$, then a continuous invariant measure is given by any Banach limit $\mu = LIM\, l \circ T^{-n}$: By the properties of the Banach limit, $\mu$ is finite, positive, finitely additive, and invariant under $T$. It remains to be shown that $\mu$ is countably additive and continuous. Let $\mathcal{Q}$ be the partition formed by $\{H_i\}$ and $\mathcal{Q}^n = T^{-n}(\mathcal{Q})$. Suppose that $\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right| < B$ for $\forall n$ and $\forall x, y \in J \in \mathcal{Q}^n$. If $F$ is a measurable set in any $J \in \mathcal{Q}^n$ and $E = T^n(F)$, then $T^n: F \xrightarrow{1-1 \text{ onto}} E$, and so

$$l(E) = \int_F |T^{n\prime}|\, dl.$$

But $T^n : J \xrightarrow{1-1 \text{ onto}} I$ and $\left|\dfrac{T^{n\prime}(x)}{T^{n\prime}(y)}\right| < B$ imply $\dfrac{1}{B}\dfrac{l(I)}{l(J)} < |T^{n\prime}(x)| < B\dfrac{l(I)}{l(J)}$ for $x \in J$. Hence

$$\frac{1}{B}\frac{l(I)}{l(J)}\int_F dl < l(E) < B\frac{l(I)}{l(J)}\int_F dl$$

or

$$\frac{1}{B}\frac{l(E)}{l(I)} < \frac{l(F)}{l(J)} < B\frac{l(E)}{l(I)}.$$

Thus for any measurable set $E$ and any $J \in \mathcal{Q}^n$,

$$\frac{1}{B}\frac{l(E)}{l(I)} < \frac{l(T^{-n}(E)\cap J)}{l(J)} < B\frac{l(E)}{l(I)}.$$

Summing over all $J \in \mathcal{Q}^n$ yields

$$l(T^{-n}(E)) = \sum_{J\in\mathcal{Q}^n} l(T^{-n}(E)\cap J) \leqq \sum_{J\in\mathcal{Q}^n} B\frac{l(E)}{l(I)}\, l(J) = Bl(E).$$

Combining this with a similar argument yields

$$Bl(E) > l(T^{-n}(E)) > \frac{1}{B}\, l(E).$$

Thus $B \cdot l(E) \geqq \mu(E) \geqq \dfrac{1}{B}\, l(E)$ implying that $\mu$ is countably additive and continuous with respect to lebesgue measure.

Weiss, Flatto and Adler (see Adler's afterword to Bowen's paper [1979] for a discussion of the somewhat nebulous origins of these ideas) used the notion of the logarithmic derivative $\dfrac{d}{dx}\log|T'| = \left|\dfrac{T''}{T'}\right|$ to establish a bound on the quantity

$$\log\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right| = \sum_{k=0}^{n-1}\log|T'(T^k(x))| - \log|T'(T^k(y))| = \sum_{k=0}^{n-1}\left|\int_{T^k(x)}^{T^k(y)}\frac{T''(u)}{T'(u)}\,du\right|$$

for $x, y \in J \in \mathcal{Q}^n$. Thus if $\left|\dfrac{T''}{T'}\right|$ is bounded by $B$ and $T$ is expanding, $T' > L > 1$, it can be concluded that

$$\begin{aligned}\log\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right| &= \sum_{k=0}^{n-1}\left|\int_{T^k(x)}^{T^k(y)}\frac{T''(u)}{T'(u)}\,du\right| \\ &\leqq \sum_{k=0}^{n-1} B\left|\int_{T^k(x)}^{T^k(y)} du\right| \leqq \sum_{k=0}^{n-1} B\left(\frac{1}{L}\right)^{n-k} \leqq B\frac{1}{L-1}.\end{aligned}$$

This approach does not involve checking conditions for arbitrarily large iterates of $T$. A similar conclusion can be reached if the quantity $\left|\dfrac{T''}{(T')^2}\right|$ is bounded. This is a weaker bound to establish and is independent of scale changes in the domain, hence is a more desirable notion of distortion to work with. If $\left|\frac{T''}{(T')^2}\right| < B$, then a change of variables $w = T(u)$ in the integration yields

$$\log\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right| = \sum_{k=0}^{n-1}\left|\int_{T^k(x)}^{T^k(y)} \frac{T''(u)}{T'(u)}\,du\right|$$
$$= \sum_{k=0}^{n-1}\left|\int_{T^{k+1}(x)}^{T^{k+1}(y)} \frac{T''(T^{-1}(w))}{T'(T^{-1}(w))}\,\frac{1}{T'(T^{-1}(w))}\,dw\right| \leqq B\frac{L}{L-1}$$

[where if $w \in (T^{k+1}(u), T^{k+1}(u))$ then $T^{-1}(w)$ is taken to be in $(T^k(x), T^k(y))$.]

These results plus a result on bernoullicity was finally stated by Adler [1975], and due to its nebulous origins was called the Folklore theorem:

**Folklore Theorem.** *Suppose $T: I \mapsto I$ where $I$ is a union of disjoint intervals $I = \bigcup H_i$ such that $T$ is $C^2$ over each interval $H_i$ and $T: H_i \xrightarrow{1-1\text{ onto}} I$. If $\exists B, L > 1$ such that $|T'| > L$ and $\left|\frac{T''}{(T')^2}\right| < B$, then there exists a finite absolutely continuous invariant measure with respect to which $T$ is bernoulli.*

*Generalization to Markov Maps.* These results are readily generalized to certain Markov maps. Let $T: I \mapsto I$ be a piecewise monotone $C^2$ transformation with base partition $\mathcal{Q}$ formed by the intervals over which $T$ is monotone, and image partition $\mathcal{R}$ formed by $T(\mathcal{Q})$. Then $T$ is Markov if $\mathcal{Q}$ refines $\mathcal{R}$, and is said to have a finite image partition if $\mathcal{R}$ is finite.

If $T$ is linear over each interval in $\mathcal{Q}$ then the set of measures that have constant densities over the intervals of $\mathcal{R}$ is a finite simplex mapped into itself under composition with $T^{-1}$ and thus contains a fixed point, i.e. an invariant measure with constant densities over each element in $\mathcal{Q}$.

Let $\mathcal{Q}^n = T^{-n}(\mathcal{Q})$, then $T$ is said to be of bounded distortion if $\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right| < B$ for $\forall x, y \in J \in \mathcal{Q}^n$. The map $T$ is said to be uniformly expanding if $|T'| > L > 1$. As before, $T$ will be of bounded distortion if $T$ is uniformly expanding and $\left|\frac{T''}{(T')^2}\right|$ is bounded. The following theorem is a generalization of the folklore theorem and the proof is left for the reader.

**Theorem 1.** *If $T$ is a uniformly expanding $C^2$ Markov map with finite image partition and bounded distortion then $T$ has a finite absolutely continuous invariant measure.*

*Proof.* Let $\mu = LIM\, l \circ T^{-n}$, then $\mu$ is positive, shift invariant, and finitely additive.

Let $E$ be any measurable set. With $\{R_i\} = \mathcal{R}$ let $E_i = E \cap R_i$.

Fix $i$ and let $J$ be any component of $\mathcal{Q}^n$ such that $T^n(J) = R_i$. As in the proof of the Folklore Theorem,

$$\frac{1}{B}\frac{l(E_i)}{l(R_i)} < \frac{l(T^{-n}(E_i) \cap J)}{l(J)} < B\frac{l(E_i)}{l(R_i)}.$$

Summing over such $J$'s yields

$$\frac{1}{B}\frac{l(E_i)}{l(R_i)} < \frac{l(T^{-n}(E_i))}{l(T^{-n}(R_i))} < B\frac{l(E_i)}{l(R_i)}.$$

Hence

$$\frac{1}{B}\,l(E_i)\frac{l(T^{-n}(R_i))}{l(R_i)} < l(T^{-n}(E_i)) < Bl(E_i)\frac{l(T^{-n}(R_i))}{l(R_i)}.$$

There are only a finite number of $R_i$'s and the Banach limit is finitely additive. Hence $l(T^{-n}(I)) = l(I)$ implies that there is some nonempty collection $\mathscr{I}$ of $i$'s such that $LIM\, l(T^{-n}(R_i)) > 0$ for $i \in \mathscr{I}$. Let $K_1 = \min_{i \in \mathscr{I}} LIM \frac{l(T^{-n}(R_i))}{l(R_i)}$ and $K_2 = \max_{i \in \mathscr{I}} LIM \frac{l(T^{-n}(R_i))}{l(R_i)}$. Then for $i \in \mathscr{I}$,

$$K_1 \frac{1}{B}\, l(E_i) < \mu(E_i) < K_2 B l(E_i),$$

hence $\mu$ is a finite absolutely continuous invariant measure. □

*Markov Maps with Infinite Image Partition.* If $\mathscr{R}$ is allowed to be infinite the difficulty arises that points may become absorbed into arbitrarily small portions of $I$. For example, take $0 < \alpha < 1$ and let $T$ be the transformation on $I = [0, 1]$ that consists of linear branches mapping each interval $[1-\alpha^n, 1-\alpha^{n+1})$ linearly onto $[1-\alpha^{n-1}, 1)$ for $n = 1, 2, \ldots$ and mapping $[0, 1-\alpha)$ linearly onto $[0, 1)$ (see Fig. A1). This transformation has a finite absolutely continuous invariant measure iff $\alpha < 1/2$. If $\alpha > 1/2$ then high iterates of the map become increasingly dominated by arbitrarily short branches.

It is possible to take $\alpha < 1/2$ and very close to $1/2$ and perturb $T$ in such a way that it is expanding, has bounded distortion, and yet will not have a finite continuous invariant measure. [The proof of Theorem 1 breaks down in this case in that it is possible for $LIM\, l(T^{-n}(H_i)) = 0$ for all $i$.] Thus expanding maps of this type with bounded distortion can behave quite differently from their linear counterparts and the need for additional criteria is clear.

The remainder of this appendix is concerned with establishing sufficient conditions for the existence of a finite absolutely continuous invariant measure for some expanding markov maps of bounded distortion for which $\mathscr{R}$ is infinite. These criteria will be used in the main paper and are based on establishing an asymptotic bound on the ratio of longer to shorter branches.

Let $T$ be a uniformly expanding Markov map with base partition $\mathscr{Q}$ and image partition $\mathscr{R}$. Let $\mathscr{S}$ be the collection of intervals that are images of intervals in $\mathscr{Q}$. It will be assumed that $\mathscr{S} = \{S_i\}$ is countable, with $l\left(\bigcup_{i=k}^{\infty} S_i\right) \to 0$ as $k \to \infty$. Let $\mathscr{Q}^n = T^{-n}(\mathscr{Q})$ and let $\mathscr{Q}^n_k$ be the collection of intervals in $\mathscr{Q}^n$ which map onto $S_k$ under a single branch of $T^n$. If $\exists \varepsilon > 0$ such that $l(\mathscr{Q}^n_k) > \varepsilon$ for arbitrarily large $n$ and $k$ and $T$

is transitive, then $T$ will not have an absolutely continuous invariant measure. It is thus necessary that $l\left(\bigcup_{i=k}^{\infty} \mathcal{Q}_i^n\right) \to 0$ uniformly in $n$ as $k \to \infty$ for the existence of an absolutely continuous invariant measure. This turns out to be sufficient.

**Theorem 2.** *If $T$ is a uniformly expanding $C^2$ Markov map with bounded distortion and $l\left(\bigcup_{i=k}^{\infty} \mathcal{Q}_i^n\right) \to 0$ uniformly in $n$ as $k \to \infty$, then $T$ has a finite absolutely continuous invariant measure.*

*Proof.* Let $\mu = LIM\, l \circ T^{-n}$. Then $\mu$ is finite, shift invariant, and finitely additive. $\sum_{k=0}^{\infty} l(\mathcal{Q}_k^n)$ converges uniformly in $n$, so for measurable sets $E$,

$$l(T^{-n}(E)) < \sum_{k=0}^{\infty} Bl(\mathcal{Q}_k^n) \frac{l(E \cap S_k)}{l(S_k)} \to 0 \quad \text{as} \quad l(E) \to 0$$

uniformly in $n$. Hence $\mu$ is countably additive and absolutely continuous. $\square$

The property that $l\left(\bigcup_{i=k}^{\infty} \mathcal{Q}_i^n\right) \to 0$ uniformly in $n$ as $k \to \infty$ involves arbitrarily high iterates of $T$. If there are sufficiently good estimates on the ratios of longer to shorter branches for $T$, this property can be established by analyzing how short branches become longer and long branches become shorter under iterates of $T$.

Pick an arbitrary integer $k$ and call branches that map onto $S_0, S_1, \ldots, S_{k-1}$ long branches and branches that map onto $S_k, S_{k+1}, \ldots$ short branches. Then

$$\frac{l\left(\bigcup_{j=0}^{k-1} \mathcal{Q}_j^1 \cap S_i\right)}{l(S_i)}$$

is the ratio of points in $S_i$ that lie under long branches of $T$. Since $T^m(\mathcal{Q}_i^m) = S_i$, and $T$ is of bounded distortion, this is an estimate of the ratio of points in $\mathcal{Q}_i^m$ that lie under long branches of $T^{m+1}$.

It will be hypothesised that there is a good ratio of points in $S_i$ that lie under branches of $T$ mapping onto $S_0, S_1, \ldots, S_{k-1}$. The ratio of long branches for $T^{m+1}$ will be estimated in terms of the ratio for $T^m$ and an asymptotic bound will be established.

**Theorem 3.** *Suppose $T$ is a uniformly expanding $C^2$ Markov map with bounded distortion and $\exists \tau_k$ with $\sum_{k=1}^{\infty} \tau_k < \infty$ such that*

$$\frac{l\left(\bigcup_{j=k+1}^{\infty} \mathcal{Q}_j^1 \cap S_i\right)}{l(S_i)} < \tau_k$$

*for $i = 0, 1, \ldots, k+1$. Then there exists a finite absolutely continuous invariant measure.*

*Proof.*

$$l\left(\bigcup_{i=0}^{k} \mathcal{Q}_i^{m+1}\right) > \sum_{i=0}^{\infty} l(\mathcal{Q}_i^m)\left(1 - B\frac{l\left(\bigcup_{j=k+1}^{\infty} \mathcal{Q}_j^1 \cap S_i\right)}{l(S_i)}\right)$$
$$> \sum_{i=0}^{k+1} l(\mathcal{Q}_i^m)(1 - B\tau_k)$$
$$> l\left(\bigcup_{i=0}^{k+1} \mathcal{Q}_i^m\right)(1 - B\tau_k).$$

Hence by repeated application,

$$l\left(\bigcup_{i=0}^{k} \mathcal{Q}_i^{m+1}\right) > l\left(\bigcup_{i=0}^{k+m} \mathcal{Q}_i^1\right)\prod_{j=k}^{k+m}(1 - B\tau_j),$$

and this quantity approaches 1 uniformly in $m$ as $k \to \infty$, hence a finite absolutely continuous invariant measure exists for $T$ by Theorem 2. □

The case where the $S_i$ are nested, $S_0 \supset S_1 \supset \ldots$, and $\mathcal{Q}_i^1 \subset S_i$ so that the branches in $S_i - S_{i+1}$ map onto $S_i$ or larger arises naturally in repeated induction of certain unimodal maps. The following result holds.

**Theorem 4.** *Suppose $T$ is a uniformly expanding $C^2$ Markov map with bounded distortion and image partition $\mathcal{S} = \{S_i\}$, where $S_0 \supset S_1 \supset \ldots$ and $\mathcal{Q}_i^1 \subset S_i$. If $\exists \tau < 1$ such that*

$$\frac{l(S_{k+1})}{l(S_k)} < \tau^k \quad \textit{and} \quad \frac{l\left(\bigcup_{i=k+1}^{\infty} \mathcal{Q}_i^1 \cap S_{k+1}\right)}{l(S_{k+1})} < \tau^k,$$

*then there exists a finite absolutely continuous invariant measure $\mu$ for $T$ with $\mu(S_{k+1}) < G_1\tau^k$ for some $G_1$ independent of $k$.*

*Proof.* The measure exists by Theorem 3, and is a Banach limit of $l \circ T^{-n}$,

$$l(T^{-m}(S_{k+1})) < \sum_{i=0}^{k} B\frac{l(S_{k+1})}{l(S_i)} l(\mathcal{Q}_i^m) + l\left(\bigcup_{i=k+1}^{\infty} \mathcal{Q}_i^m\right)$$
$$< B\frac{l(S_{k+1})}{l(S_k)} l\left(\bigcup_{i=0}^{k} \mathcal{Q}_i^m\right) + l\left(\bigcup_{i=k+1}^{\infty} \mathcal{Q}_i^m\right)$$
$$< B\tau^k + l(S_0) - l\left(\bigcup_{i=0}^{k} \mathcal{Q}_i^m\right)$$
$$< B\tau^k + l(S_0) - l\left(\bigcup_{i=0}^{k+m} \mathcal{Q}_i^m\right)\prod_{j=k}^{\infty}(1 - B\tau^j),$$

hence

$$\limsup_{m\to\infty} l(T^{-m}(S_{k+1})) < B\tau^k + l(S_0) - l(S_0)\prod_{j=k}^{\infty}(1 - B\tau^j)$$
$$< B\tau^k + l(S_0)\left(1 - \prod_{j=k}^{\infty}(1 - B\tau^j)\right)$$
$$< G_1\tau^k$$

for some $G_1$ independent of $k$, since $\left(1-\prod_{j=k}^{\infty}(1-B\tau^j)\right)$ is dominated by $\sum_{j=k}^{\infty}\tau^j$ which in turn is dominated by $\tau^k$. $\square$

If the sum of $l(S_m)$ is finite, then it is possible to allow $\left|\frac{T''}{(T')^2}\right|$ on any branch of $T$ to vary according to the length of the branch and still preserve bounded distortion.

**Theorem 5.** *If* $\exists\{\sigma_m\}$, $L>1$ *such that* $|T'|>L$, $\left|\frac{T''}{(T')^2}\right|_{\mathcal{Q}_m^1}<\sigma_m$, *and* $\sum_{m=0}^{\infty}\sigma_m l(S_m)<\infty$, *then* $\exists G_2$ *such that*

$$\sup_{x,y\in M\in\mathcal{Q}^n}\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right|<G_2.$$

*Proof.* Fix any $n$ and $x,y\in M\in\mathcal{Q}^n$. Then for $k=0,1,\ldots,n-1$, $T$ is continuous and monotone on $[T^k(x), T^k(y)]$. Let $\Psi_m=\{k<n: T^k(x)\in\mathcal{Q}_m^1\}$ and $k_m=\max\{k\in\Psi_m\}$. Then

$$\begin{aligned}
&\log\left|\frac{T^{n\prime}(x)}{T^{n\prime}(y)}\right|\\
&=\sum_{k=1}^{n-1}\log|T'(T^k(x))|-\log|T'(T^k(y))|=\sum_{m=0}^{\infty}\sum_{k\in\Psi_m}\left|\int_{T^k(x)}^{T^k(y)}\frac{T''(t)}{T'(t)}\,dt\right|\\
&=\sum_{m=0}^{\infty}\sum_{k\in\Psi_m}\left|\int_{T^{k+1}(x)}^{T^{k+1}(y)}\frac{T''(T^{-1}(u))}{(T'(T^{-1}(u)))^2}\,du\right|<\sum_{m=0}^{\infty}\sum_{k\in\Psi_m}\sigma_m l([T^{k+1}(x), T^{k+1}(y)])\\
&<\sum_{m=0}^{\infty}\sum_{k\in\Psi_m}\sigma_m\left(Ts\frac{1}{L}\right)^{k_m-k} l(S_m)<\sum_{m=0}^{\infty}\sigma_m Ts\frac{L}{L-1}\,l(S_m)<\infty. \quad\square
\end{aligned}$$

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