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A New Proof of the Propagation Theorem for N-Body Quantum Systems

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Abstract. A new proof of I. Sigal's and A. Soffer's propagation theorem is given. This theorem describes a large class of operators which are Kato-smooth with respect to an N-body Schrödinger operator.

1. Introduction

One can learn a lot about the Schrödinger operator H by studying the asymptotic behavior of certain observables in the Heisenberg picture as the time goes to infinity. There exists a number of various results on this subject, which say roughly that for large times many observables behave to some extent in a semiclassical way. For example, various estimates that are used in the proofs of the asymptotic completeness by the Enss method (see e.g. [E1, 2, 3, Pe]) belong to this category.

Another class of estimates that describe propagation of observables is related to the concept of the Kato-smoothness. We say that an operator B is locally H-smooth on the interval Δ if and only if the estimate

$$\int_{-\infty}^{\infty} \| Be^{iHt}\phi \|^2 dt < \infty$$

is satisfied for any vector ϕ that belongs to the range of the spectral projection of H onto Δ. (In the sequel we will just say "H-smooth" instead of "locally H-smooth".) This concept has been introduced by Kato in [Ka1, 2]. It has been used to prove various properties of Schrödinger operators such as the asymptotic completeness and the absence of the singular continuous spectrum. Let us name for instance the following references which used the H-smoothness (sometimes in a disguised form) [Pu, La1, 2, 3, 4, Ar, RS4, IoOC, Ha, HaPe, MS, Sig, SigSof1, De1].

The problem of finding H-smooth operators is especially interesting and nontrivial in the case of N-body Schrödinger operators. First of all, it can be shown that in this case, under quite mild conditions on the potentials, the operator

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$(1+|x|)^{-\frac{1}{2}-\epsilon}$ is locally H-smooth outside thresholds and bound states (see M1, 2, PSS, CFKS, Ya]). This result can be used to prove the absence of the singular continuous spectrum.

Another result on this subject is contained in [SigSof1], where I. Sigal and A. Soffer have been able to describe a very rich nontrivial family of H-smooth operators. This result, which they called the propagation theorem, was a crucial step in the proof of the asymptotic completeness of the short range N-body scattering contained in [SigSof1].

The fact that I. Sigal and A. Soffer proved says roughly that if Q is a function on the configuration space homogeneous of degree $-\frac{1}{2}, g$ is a bounded function on the momentum space and $Q(\cdot)g(\cdot)$ is supported outside of a certain subset of the phase space, then $g(D)Q(x)$ is H-smooth on a certain energy interval. The original proof of this theorem by I. Sigal and A. Soffer is based on very intuitive and beautiful ideas, although some of its technical details may seem quite complicated. In this paper we present a somewhat different proof. Our proof essentially uses ideas and techniques very similar to those employed in the original proof. Nevertheless, we think it is more transparent. We avoid for instance the so-called channel expansion, which is one of the technical steps used in [SigSof1]. We also formulate our theorem is a somewhat different way, which seems to be more convenient.

2. *H*-Smooth Operators

In this section we introduce the notion of the H-smooth operators and present its basic properties (see [Ka1, 2, RS4]).

Let $\mathscr{H}$ be a Hilbert space, H—a self adjoint operator on $\mathscr{H}$ and B—a bounded operator on $\mathscr{H}$. We say that B is H-smooth (or Kato smooth with respect to H) if and only if for every $\psi\in\mathscr{H}$,

$$\int_{-\infty}^{\infty} \|Be^{iHt}\psi\|^2 dt < \infty.$$

Let Δ be a measurable subset of $\mathbb{R}$. Let $E_\Delta(H)$ denote the spectral projection of H onto Δ. We say that B is H-smooth on Δ if and only if $BE_\Delta(H)$ is H-smooth.

Let us state the most obvious properties of the H-smooth operators.

Lemma 2.1. a) *Let B_1 and B_2 be H-smooth on Δ. Then B_1+B_2 is H-smooth on Δ.*

b) *Let B be H-smooth on Δ and C be bounded. Then CB is H-smooth on Δ.*

c) *Let B_1 be H-smooth on Δ and $B_1^*B_1 \geqq B_2^*B_2$. Then B_2 is H-smooth on Δ.*

Proof. b) and c) are obvious. To show a) it is enough to note that

$$\int_{-\infty}^{\infty} \|(B_1+B_2)e^{iHt}\psi\|^2 dt \leqq \int_{-\infty}^{\infty} \|B_1e^{iHt}\psi\|^2 dt + \int_{-\infty}^{\infty} \|B_2e^{iHt}\psi\|^2 dt$$
$$+2\left(\int_{-\infty}^{\infty} \|B_1e^{iHt}\psi\|^2 dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} \|B_2e^{iHt}\psi\|^2 dt\right)^{\frac{1}{2}}. \quad \text{QED.}$$

The following criterion for the H-smoothness of an operator is a minor modification of the so-called Putman–Kato theorem (see [Ka1, Pu, RS4, SigSof1]).

Lemma 2.2. *Let B and $E_\Delta(H)CE_\Delta(H)$ be bounded. Let B_i and B_i' be H-smooth on Δ on Δ for $i=1,\ldots,k$. Suppose also that*

$$E_\Delta(H)i[H,C]E_\Delta(H)\geqq B^*B+\sum_{i=1}^{k} B_i^*B_i'.$$

Then B is H-smooth on Δ.

3. Basic Definitions

Throughout the paper X will denote a fixed vector space isomorphic to $\mathbb{R}^n$ endowed with a scalar product. $|x|$ will denote the Euclidian norm of a vector x.

Let $x_0\in X$ and $R>0$. Then $B(x_0,R)=\{x\in X:|x-x_0|\leqq R\}$ and

$$\mathrm{Cone}(x_0,R)=\left\{x\in X:\ \left|\frac{x}{|x|}-\frac{x_0}{|x_0|}\right|<R\right\}=\left\{x\in X:\ \frac{x\cdot x_0}{|x||x_0|}>1-\frac{R^2}{2}\right\}.$$

$B(R)$ will denote $B(0,R)$ and S will denote the unit sphere in X.

An important role in our paper will be played by a certain fixed family $\{X_a: a\in\mathscr{A}\}$ of subspaces of X. To be consistent with the notation in the literature devoted to N-body Schrödinger operators (e.g. [Hag, PSS, Sig, SigSof1, RS3, A, FH1]) we will write $a_1\subset a_2$ whenever $X_{a_1}\supset X_{a_2}$ and $a_1\cup a_2=a_3$ whenever $X_{a_1}\cap X_{a_2}=X_{a_3}$. $X_{a_{\min}}$ will denote X and $X_{a_{\max}}=\{0\}$. We will assume the following properties of $\mathscr{A}$:

1. $a_{\min}, a_{\max}\in\mathscr{A}$,
2. if $a_1,a_2\in\mathscr{A}$ then $a_1\cup a_2\in\mathscr{A}$.

The orthogonal complement of X_a in X will be denoted X^a. Dual spaces to X, X_a and X^a will be denoted by K, K_a and K^a respectively. π_a will stand for the projection of K onto K_a and π^a—the projection of X onto X^a.

It will also be useful to introduce the following symbols:

$$Z_a=X_a\setminus\bigcup_{b\not\subset a}X_b,$$

and

$$Y_a^\varepsilon=X\setminus\bigcup_{b\not\subset a}\left\{x:\ \mathrm{dist}\left(\frac{x}{|x|},X_b\right)\leqq\varepsilon\right\}.$$

We fix a certain positive C^∞ function $X\ni x\mapsto\langle x\rangle\in\mathbb{R}$ such that for $|x|>1$, we have $\langle x\rangle=|x|$. (Similarly, we will use functions $\langle k\rangle$, $\langle x^a\rangle$, etc.).

The Greek latter α will denote a multiindex; $|\alpha|$ will denote the length of this multiindex.

We define

$$S^m(\mathbb{R}^n)=\{Q\in C^\infty(\mathbb{R}^n):\ |Q(x)|\leqq c\langle x\rangle^m \text{ and } |\partial^\alpha Q(x)|\leqq \varkappa_\alpha\langle x\rangle^{m-1} \text{ for } |\alpha|\geqq 1\}.$$

Typical examples of functions from $S^m(\mathbb{R}^n)$ are C^∞ functions that are homogeneous of degree m outside $B(1)$.

We will say that a subset Ω of $X \times K$ is conical if and only if for any $t > 0$ $(x, k) \in \Omega$ implies $(xt, k) \in \Omega$.

Let Ω be a conical subset of $X \times K$ and $\varepsilon, \beta > 0$. Then

$$\Omega^{\varepsilon,\beta} = \bigcup_{\substack{x,k \in \Omega \\ x \neq 0}} \mathrm{Cone}(x, \varepsilon) \times B(k, \beta).$$

D and D_a will denote the operators $(1/i)\nabla$ and $(1/i)\nabla_a$. Δ, Δ_a and Δ^a will stand for the Laplacians on X, X_a and X^a respectively.

An important role will be played by the generator of dilations $A = \frac{1}{2}(D \cdot x + x \cdot D)$, and the operator $\gamma = \frac{1}{2}(D \cdot (x/\langle x \rangle) + (x/\langle x \rangle) \cdot D)$, whose importance was first realized in [SigSof 1]. It is easy to show that both γ and A are essentially self adjoint on $\mathcal{S}(X)$ (the space of Schwartz test functions on X).

If B is an operator then $B + hc$ will mean $B + B^*$. The spectrum of B is denoted by $\sigma(B)$. If B is an unbounded operator then $\mathcal{D}(B)$ denotes its domain.

4. Hamiltonian

N-body Schrödinger operators arise naturally in the many particle nonrelativistic quantum mechanics. The reader will find their basic properties e.g. in [RS3, 4]. In our paper we consider operators which are slightly more general than the regular N-body Schrödinger operators, similar to those considered in [A, FH1, 2].

We begin this section with stating the assumptions on the N-body Schrödinger operators that we will use in our paper. Those assumptions are essentially the same as in [SigSof 1].

We assume that μ is a real number such that $0 < \mu < 1$ and for every $a \in \mathcal{A}$ we are given a real function v_a on X^a such that

$$\text{a) } v_a(x^a)(-\Delta^a + 1)^{-1} \quad \text{is compact on} \quad L^2(X^a), \tag{4.1}$$

$$\text{b) } \langle x^a \rangle^{\mu} v_a(x^a)(-\Delta^a + 1)^{-1} \quad \text{is bounded,} \tag{4.2}$$

$$\text{c) } \langle x^a \rangle^{1+\mu} \nabla v_a(x^a)(-\Delta^a + 1)^{-1} \quad \text{is bounded,} \tag{4.3}$$

and

$$\text{d) } (x^a \cdot \nabla)^2 v_a(x^a)(-\Delta^a + 1)^{-1} \quad \text{is bounded.} \tag{4.4}$$

Note that $v_a(\pi^a x)$'s are relatively bounded perturbations of $-\Delta$ with an infinitesimal bound.

Set $V = \sum_{a \in \mathcal{A}} v_a(\pi^a x)$. We define H to be the self adjoint operator on $L^2(X)$ such that $\mathcal{D}(H) = \mathcal{D}(-\Delta)$ and $H = -\Delta + V$. Define also $V_a(x) = \sum_{b \subset a} v_b(\pi^b x)$ and $I_a = V - V_a$. The so-called "cluster Hamiltonians" H_a are defined as the self-adjoint operators such that $\mathcal{D}(H_a) = \mathcal{D}(-\Delta)$ and $H_a = -\Delta + V_a$. Note that $H_{a_{\min}} = -\Delta$ and $H_{a_{\max}} = H$. $-\Delta$ will be also denoted by H_0. We will often write E_Δ instead of $E_\Delta(H)$. Note also that H_a's and H are bounded from below.

In the sequel λ will denote a fixed real number that lies below the spectrum of H. (Consequently, by the HVZ-theorem λ lies below the spectrum of H_a for all $a \in \mathcal{A}$—see [RS4] and references therein.)

If we identify $L^2(X)$ with $L^2(X_a)\otimes L^2(X^a)$, then the cluster Hamiltonians can be decomposed as

$$H_a = -\Delta_a \otimes 1_{X^a} + 1_{X_a} \otimes H^a,$$

where $H^a = -\Delta^a + V_a$ is a self adjoint operator on $L^2(X^a)$.

Let $\tilde{\mathscr{T}}_a$ denote the set of eigenvalues of H^a if $a \neq a_{\min}$ and $\tilde{\mathscr{T}}_{a_{\min}} = \{0\}$. We define also $\mathscr{T}_a = \bigcup_{b\subset a} \tilde{\mathscr{T}}_b$. If $E\in\mathbb{R}$, then we set

$$\tilde{\Sigma}_a(E) = \{\pm\sqrt{E-\tau}:\ E-\tau \geqq 0,\ \tau\in\tilde{\mathscr{T}}_a\},$$

and

$$\Sigma_a(E) = \{\pm\sqrt{E-\tau}:\ E-\tau \geqq 0,\ \tau\in\mathscr{T}_a\}.$$

We will often drop $a_{\max}$ from $\tilde{\mathscr{T}}_{a_{\max}}$, $\mathscr{T}_{a_{\max}}$, $\tilde{\Sigma}_{a_{\max}}(E)$ and $\Sigma_{a_{\max}}(E)$. Note that the elements of $\bigcup_{a\neq a_{\max}} \tilde{\mathscr{T}}_a$ are usually called the thresholds of H. Thus $\mathscr{T}$ is the set of all thresholds and bound states of H. Clearly the sets $\mathscr{T}_a$ are bounded from below and consequently the sets $\Sigma_a(E)$ are bounded.

Now we are going to introduce a definition of a certain family of subsets of the phase space $X\times K$ which will play the central role in our paper. This notion is a variation of the concept of the propagation set which was invented and studied in [SigSof1].

Let $E\in\mathbb{R}$ and let Ω be a conical subset of $X\times K$. We say that there is no propagation in Ω at the energy E, and we write $\Omega\in\mathscr{N}\mathscr{P}_E$ if and only if there exists an interval Δ containing E such that if $Q\in S^{-\frac{1}{2}}(X)$, $g\in L^\infty(K)$ and $\operatorname{supp} Q(\cdot)g(\cdot)\subset\Omega$, then $g(D)Q(X)$ is H-smooth on Δ.

Let us now state the main result of our paper.

Theorem 4.1. *Let $E\notin\mathscr{T}$. Define*

$$PS_E = \bigcup_{a\in\mathscr{A}} \bigcup_{\substack{x\in X_a\\ x\neq 0}} \bigcup_{M\in\Sigma_a(E)} \left\{(x,k):\ k\in K,\ \pi_a k = M\frac{x}{|x|}\right\}.$$

Then for any $\varepsilon,\beta>0$,

$$X\times K\backslash (PS_E)^{\varepsilon,\beta}\in\mathscr{N}\mathscr{P}_E.$$

The above theorem, in a somewhat different form, was stated and proved in [SigSof1]. We are going to present a proof that, we think, is simpler and more transparent.

Next let us present a number of facts about N-body Schrödinger operators that we will use in this paper without proofs. The proofs of these facts can be found in the literature.

We start with two variants of the so-called Mourre inequality. For any $a\in\mathscr{A}$ define $\Gamma_a(E)=\varnothing$ if $E\notin\tilde{\mathscr{T}}_a$ and $\Gamma_a(E)=\{0\}$ if $E\in\tilde{\mathscr{T}}_a$. Let θ_a and ρ_a be the real functions on $\mathbb{R}$ such that

$$\theta_a(E) = 2\inf\left[\Gamma_a(E)\cup\bigcup_{b\subset a,\, b\neq a}\Sigma_b(E)\right]^2,$$

and

$$\rho_a(E) = 2\sup\left[\Gamma_a(E) \cup \bigcup_{b\subset a, b\neq a} \Sigma_b(E)\right]^2.$$

(We set $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$, moreover we assume that $0\cdot\infty = 0$).

Proposition 4.2. *Suppose that instead of the conditions (4.1)–(4.4) the potentials v_a satisfy the following hypotheses:*

$$\text{a) } v_a(x^a)(-\Delta^a+1)^{-1} \quad \textit{is compact on} \quad L^2(X^a), \tag{4.5}$$

$$\text{b) } (-\Delta^a+1)^{-1}x^a\cdot\nabla v_a(x^a)(-\Delta^a+1)^{-} \quad \textit{is compact on} \quad L^2(X^a). \tag{4.6}$$

Then for any $a\in\mathscr{A}$, $E\in\mathbb{R}$ and $\delta>0$ there exists an open interval Δ containing E such that

$$(\rho_a(E)+\delta)E_\Delta(H^a) \geqq E_\Delta(H^a)i[H^a, A]E_\Delta(H^a) \geqq (\theta_a(E)-\delta)E_\Delta(H^a). \tag{4.7}$$

Moreover, the sets $\mathscr{T}_a$ are countable and closed.

The original Mourre inequality is the part of (4.7) which estimates the commutator from below. It was proved by E. Mourre in [M1] in the 3-body case and by P. Perry, I. Sigal and B. Simon in [PSS] in the N-body case. Another proof of the Mourre inequality was given by R. Froese and I. Herbst in [FH1]. The part of (4.7) which estimates the commutator from above is due to I. Sigal and A. Soffer (see [SigSof1]). See also [CFKS] and [De2].

We would like to rephrase Proposition 4.2 in a form that contains H_a instead of H^a. To this end, if $\kappa>0$ and $k_a\in K_a$, we define

$$\theta_a^\kappa(E, k_a) = 2|k_a|^2 + \inf\{\theta_a(E'-k_a^2): |E-E'| \leqq \kappa\},$$

and

$$\rho_a^\kappa(E, k_a) = 2|k_a|^2 + \sup\{\rho_a(E'-k_a^2): |E-E'| \leqq \kappa\}.$$

The following corollary is an easy consequence of Proposition 4.2 (see [FH1] and [De1] for similar statements).

Corollary 4.3. *Suppose that the same conditions hold as in Proposition 4.2. Then for any $E\in\mathbb{R}$ and $\delta,\kappa>0$ there exists an open interval Δ containing E such that*

$$(\rho_a^\kappa(E, D_a)+\delta)E_\Delta(H_a) \geqq E_\Delta(H_a)i[H_a, A]E_\Delta(H_a) \geqq (\theta_a^\kappa(E, D_a)-\delta)E_\Delta(H_a).$$

Another important property of N-body Schrödinger operators that will find an extensive application in our paper is the local H-smoothness of $\langle x\rangle^{-\frac{1}{2}-\varepsilon}$ outside the thresholds and bound states. This fact is a consequence of the Mourre estimate; its proof is due to E. Mourre [M1], see also [PSS, CFKS]. There also exists another interesting proof due to I. Sigal and A. Soffer ([SigSof2]).

A precise formulation of this result is the following:

Proposition 4.4. *Assume that in addition to (4.5) and (4.6) the following conditions are true:*

$$\text{a) } (-\Delta^a+1)^{-\frac{1}{2}}x^a\cdot\nabla v_a(x^a)(-\Delta^a+1)^{-1} \qquad \textit{is bounded,}$$

b) $(-\Delta^a+1)^{-1}(x^a\cdot\nabla)^2 v_a(x^a)(-\Delta^a+1)^{-1}$ *is bounded.*

Suppose that Δ *is a compact subset of* $\mathbb{R}$ *such that* $\mathscr{T}\cap\Delta=\emptyset$. *Then for any* $\varepsilon>0$ *the operator* $\langle x\rangle^{-\frac{1}{2}-\varepsilon}$ *is* H*-smooth on* Δ.

It is easy to see that the hypotheses (4.1)–(4.4) imply the conditions on the potentials that are imposed in Proposition 4.2, Corollary 4.3 and Proposition 4.4. From now on we will always assume that the hypotheses (4.1)–(4.4) hold.

5. Approximately Commuting Operators

In this section we study the properties of certain operators that we will extensively use in our paper. One may look at these operators as noncommutative versions of functions on phase space. It turns out that in many situations these operators commute modulo higher order terms (in a sense which will be clear in a moment). The proof of this fact will be the main subject of this section. The results and techniques applied here are based on those of [SigSof1].

First let us define what we mean by "the order" of an operator. Let B be a densely defined operator on $L^2(X)$ and let $m\in\mathbb{R}$. We will write $B=0(\langle x\rangle^m)$ if and only if for any $k\in\mathbb{R}$ the set $\mathscr{D}(B)\cap\mathscr{D}(\langle x\rangle^k)$ is dense in $L^2(X)$ and $\langle x\rangle^{-m+k}B\langle x\rangle^{-k}$ extends to a bounded operator. We will also write

$$B_1=B_2+0(\langle x\rangle^m)$$

if and only if $B_1-B_2=0(\langle x\rangle^m)$ and

$$B_1\geqq B_2+0(\langle x\rangle^m)$$

if there exists $C=0(\langle x\rangle^m)$ such that $B_1\geqq B_2+C$.

Next let us introduce a certain number of classes of functions that we will use in this section:

$$BC^\infty(\mathbb{R}^n)=\{g\in C^\infty(\mathbb{R}^n)\colon |\partial_x^\alpha g|\leqq c_\alpha \text{ for any } \alpha\},$$
$$B_1C^\infty(\mathbb{R}^n)=\{g\in BC^\infty(\mathbb{R}^n)\colon \hat{g}(k)k\langle k\rangle^m \text{ is a finite measure on } \mathbb{R}^n \text{ for any } m\geqq 0\},$$
$$B_2C^\infty(\mathbb{R}^n)=\{g\in BC^\infty(\mathbb{R}^n)\colon k\cdot\nabla\hat{g}(k)\in BC^\infty(\mathbb{R}^n)\},$$
$$B_3C^\infty(\mathbb{R}^n)=\{h\in BC^\infty(\mathbb{R})\colon h(t)=\sum_{i=1}^{N}(\lambda_i-t)^{-n_i}+h_1(t) \text{ such that } \lambda_i\notin\sigma(H),$$
$$n_i\in\mathbb{N} \text{ and } h_1\in\mathscr{S}(\mathbb{R})\}.$$

Let us state the proposition that describes the commutation properties of various operators of interest to us.

Proposition 5.1.

I. The following operators are $0(\langle x\rangle^0)$:

a) $g(D)$ *for* $g\in BC^\infty(K)$,
b) $(H-\lambda)^{-1}(H_0+1)$,
c) $(H-\lambda)^{-1}\gamma$, $\gamma(H-\lambda)^{-1}\gamma$ *and* $(i+\gamma)^{-1}$,
d) $(H-\lambda)^{-1}[H,A]$,
e) $h(H)$ *for* $h\in B_3C^\infty(\mathbb{R})$,
f) $f(\gamma)$ *for* $f\in B_1C^\infty(\mathbb{R})$.

II. Let $Q\in S^m(X)$. The following operators are $0(\langle x\rangle^{m-1})$:

a) $\langle D\rangle^{-m_1+1}[Q(x),g(D)]$ *for* $g\in S^{m_1}(K)$,
b) $[Q(x),(H-\lambda)^{-1}(H_0+1)]$,
c) $[Q(x),(H-\lambda)^{-1}\gamma]$,
d) $[Q(x),(H-\lambda)^{-1}[H,A]]$,
e) $[Q(x),h(H)]$ *for* $h\in B_3C^\infty(\mathbb{R})$,
f) $[Q(x),f(\gamma)]$ *for* $f\in B_1C^\infty(\mathbb{R})$.

III. Let $f\in B_1C^\infty(\mathbb{R})$. The following operators are $0(\langle x\rangle^{-1})$:

a) $[f(\gamma),g(D)]$ *for* $g\in S^0(K)$,
a′) $(i+\gamma)^{-1}[f(\gamma),g(D)]$ *for* $g\in B_2C^\infty(K)$,
b) $[f(\gamma),(H-\lambda)^{-1}(H_0+1)]$,
c) $[f(\gamma),(H-\lambda)^{-1}\gamma]$,
d) $[f(\gamma),(H-\lambda)^{-1}[H,A]]$,
e) $[f(\gamma),h(H)]$ *for* $h\in B_3C^\infty(\mathbb{R})$.

The proof of the above proposition is rather standard and lengthy. Therefore we indicate only its main steps and omit some of the details, which are easy to fill in.

The starting point for the proof is the following easy lemma.

Lemma 5.2. *Suppose that $B\in B(L^2(X))$ and for $k=1,\ldots,\ j=1,\ldots,k$, and $i_j=1,\ldots,n$ we have*

$$[x_{i_1}[\ldots[x_{i_k},B]\ldots]]\in B(L^2(X)).$$

Then $B=0(\langle x\rangle^0)$. Moreover, if $|k'|\leqq k$, then

$$\|\langle x\rangle^{k'}B\langle x\rangle^{-k'}\|\leqq c_k\left(\|B\|+\cdots+\sum_{i_1,\ldots,i_k=1}^{n}\|[x_{i_1},[\ldots[x_{i_k},B]\ldots]]\|\right).$$

We omit an easy proof of the above lemma and proceed directly to the proof of Proposition 5.1.

Proof of Proposition 5.1. I a) follows easily from Lemma 5.2. II a) follows from the calculus of the pseudodifferential operators (see [Hö, Ta and De2]). The proofs of I b), c), d) and II b), c), d) are straightforward applications of Lemma 5.2 and the identity

$$[B,(H-\lambda)^{-1}]=(H-\lambda)^{-1}[H,B](H-\lambda)^{-1}.$$

To deal with the remaining statements of the proposition we have to develop a certain technique which was extensively used in [SigSof1]. First we present this technique in an abstract form. We begin with the following identity.

Lemma 5.3. *Let B and C be self adjoint operators on a Hilbert space $\mathscr{H}$.*
a) Suppose that $\mathscr{D}_1\subset\mathscr{D}(B)\cap\mathscr{D}(C)$, $\mathscr{D}_1$ is a core for B, B maps $\mathscr{D}_1$ into $\mathscr{D}_1$ and $[B,C]$ extends from a quadratic form on $\mathscr{D}(B)\cap\mathscr{D}(C)$ to a bounded operator. Then $[B,e^{iC}]$ extends from a quadratic form on $\mathscr{D}(B)$ to a bounded operator, and the following identity is true:

$$[B,e^{iC}]=\int_0^1 d\tau e^{i(1-\tau)C}i[B,C]e^{i\tau C}. \tag{5.1}$$

b) *If* $\mathcal{D}_2 \subset \mathcal{D}(B) \cap \mathcal{D}(C)$ *and* $e^{i\tau C}$ *maps* $\mathcal{D}_2$ *into itself for all* τ, *then* (5.1) *is true in the sense of quadratic forms on* $\mathcal{D}_2$.

Proof. The proof of b) is straightforward. We shall show only a).

Let $\operatorname{Im} z \neq 0$. Let us prove first that $[C, (B+z)^{-1}]$ defined as a quadratic form on $\mathcal{D}(C)$ extends to a bounded operator equal

$$(B+z)^{-1}[B, C](B+z)^{-1}. \tag{5.2}$$

To this end note first that both $(B+z)\mathcal{D}_1$ and $(B+\bar{z})\mathcal{D}_1$ are dense in $\mathcal{H}$ because $\mathcal{D}_1$ is a core for B. Take $\phi \in (B+\bar{z})\mathcal{D}_1$ and $\psi \in (B+z)\mathcal{D}_1$. Clearly $\phi, \psi \in \mathcal{D}(C)$. Thus by definition

$$(\phi, [C, (B+z)^{-1}]\psi) = (C\phi, (B+z)^{-1}\psi) - ((B+\bar{z})^{-1}\phi, C\psi). \tag{5.3}$$

But $(B+\bar{z})^{-1}\phi, (B+z)^{-1}\psi \in \mathcal{D}_1 \subset \mathcal{D}(B) \cap \mathcal{D}(C)$. So (5.3) equals

$$\begin{aligned}&((B+\bar{z})(B+\bar{z})^{-1}\phi, C(B+z)^{-1}\psi) - (C(B+\bar{z})^{-1}\phi, (B+z)(B+z)^{-1}\psi)\\ &\quad = ((B+\bar{z})^{-1}\phi, [B, C](B+z)^{-1}\psi).\end{aligned}$$

This is clearly equal to the matrix element of (5.2).

Now let $\phi, \psi \in \mathcal{D}(B) \cap \mathcal{D}(C)$. Then clearly

$$(\phi, [B, e^{iC}]\psi) = \lim_{\varepsilon \to 0}\left(\phi, \left[\frac{B}{1+i\varepsilon B}, e^{iC}\right]\psi\right).$$

By the Stone Theorem we can write

$$\begin{aligned}\left(\phi, \left[\frac{B}{1+i\varepsilon B}, e^{iC}\right]\psi\right) &= \int_0^1 \frac{d}{d\tau}\left(\phi, e^{i(1-\tau)C}\frac{B}{1+i\varepsilon B}e^{i\tau C}\psi\right)d\tau\\ &= \int_0^1\left(\phi, e^{i(1-\tau)C} i\left[\frac{B}{1+i\varepsilon B}, C\right]e^{i\tau C}\psi\right)d\tau\\ &= \int_0^1\left(\phi, e^{i(1-\tau)C}\frac{1}{1+i\varepsilon B} i[B, C]\frac{1}{1+i\varepsilon B}e^{i\tau C}\psi\right)dt.\end{aligned}$$

This tends to the right-hand side of the matrix element of (5.1) as $\varepsilon \to 0$. QED.

It will be convenient to introduce the following definition. We will write $C = \tilde{o}(\langle x \rangle^0)$ if and only if C is a self adjoint operator on $L^2(X)$, $\mathcal{S}(X) \subset \mathcal{D}(C)$ and for $k = 1, 2, \ldots; j = 1, 2, \ldots, k$; and $i_j = 1, \ldots, n$,

$$[x_{i_1}, \ldots, [x_{i_k}, C] \ldots] \in B(L^2(X)).$$

(Note that C itself need not be bounded).

The following lemma follows from Lemma 5.2 by a repeated application of Lemma 5.3 a) with $\mathcal{D}_1 = \mathcal{S}(X)$.

Lemma 5.4. *Let* $C = \tilde{o}(\langle x \rangle^0)$, *then for any* $k \in \mathbb{Z}$,

$$\| \langle x \rangle^k e^{iCt} \langle x \rangle^{-k} \| \leqq c_k (1 + |t|)^{|k|}.$$

The following lemma is the basis for the proof of most of the statements of Proposition 5.1.

Lemma 5.5. a) *Let* $C = \tilde{o}(\langle x\rangle^0$ *and* $g \in B_1 C^\infty(\mathbb{R})$. *Then* $g(C) = 0(\langle x\rangle^0)$.

b) *Assume in addition that* $\mathscr{S}(X) \subset \mathscr{D}(B) \cap \mathscr{D}(C)$ *and* $[C, B] = 0(\langle x\rangle^m)$. *Then* $[g(C), B] = 0(\langle x\rangle^m)$.

c) *Assume moreover that* $B = \tilde{o}(\langle x\rangle^0)$ *and* $f \in B_1 C^\infty(\mathbb{R})$. *Then* $[g(C), f(B)] = 0(\langle x\rangle^m)$.

Proof. To show a) we use the representation

$$g(C) = (2\pi)^{-1} \int dt \hat{g}(t) e^{iCt}, \tag{5.4}$$

Lemma 5.2 and the fact that

$$\| [x_{i_1}, \ldots, [x_{i_k}, e^{iCt}] \ldots] \| \leqq C_k(|t| + |t|^k)$$

(the above estimate is a by-product of the proof of Lemma 5.4).

To prove b) we also use the representation (5.4). We apply Lemma 5.3. b) to it with $\mathscr{D}_2 = \mathscr{S}(X)$. Thus we can write

$$[g(C), B] = (2\pi)^{-1} \int dt \hat{g}(t) \int_0^t d\tau e^{i(t-\tau)C} i[C, B] e^{i\tau C}$$

in the sense of quadratic forms on $\mathscr{S}(X)$. Now b) follows easily from Lemma 5.4.

The proof of c) is similar. A double application of (5.1) yields

$$[g(C), f(B)] = (2\pi)^{-2} \int\int dt_1 dt_2 \hat{g}(t_1) \hat{f}(t_2) \int_0^{t_1} \int_0^{t_2} d\tau_1 d\tau_2 e^{i(t_1-\tau_1)C} e^{i(t_2-\tau_2)B} [C, B] e^{i\tau_2 B} e^{i\tau_1 C}$$

in the sense of quadratic forms on $\mathscr{S}(X)$. Then we use Lemma 5.4. QED.

Note that in Lemma 5.5 instead of a single operator C we can take k operators $C_1, \ldots, C_k$, assume that they are $\tilde{o}(\langle x\rangle^0)$ and that $g \in B_1 C^\infty(\mathbb{R}^k)$. The proof remains essentially the same.

Now we can return to the proof of Proposition 5.1.

Proof of Proposition 5.1 continued. Note that if $h \in B_3 C^\infty(\mathbb{R})$ then we can find $h_1 \in C_0^\infty(\mathbb{R})$ such that $h_1(t) = h(1/t + \lambda)$ for $t \in [0, (\inf \sigma(H))^{-1}]$. Clearly $h_1((H-\lambda)^{-1}) = h(H)$.

We easily see that γ and $(H-\lambda)^{-1}$ are $\tilde{o}(\langle x\rangle^0)$. Thus I e), f) follow from Lemma 5.5 a). Similarly, II e), f) and III a), b), c), d) follow from 5.5 b). To show III e) we use Lemma 5.5 c).

It remains to prove III a′). To this end we need the following fact.

Lemma 5.6. *Let* $g \in B_2 C^\infty(K)$. *Then there exist* $B_1 = 0(\langle x\rangle^{-1})$ *and* $B_2 = 0(\langle x\rangle^{-1})$ *such that* $[\gamma, g(D)] = \gamma B_1 + B_2$.

Proof. Clearly $\gamma = A(1/\langle x\rangle) + Q(x)$, where $Q \in S^{-1}(X)$. Thus

$$[\gamma, g(D)] = D \cdot \nabla g(D) \frac{1}{\langle x\rangle} + A\left[\frac{1}{\langle x\rangle}, g(D)\right] + 0(\langle x\rangle^{-2}).$$

Clearly we can write

$$A\left[\frac{1}{\langle x\rangle}, g(D)\right] = \gamma B_1 + 0(\langle x\rangle^{-2}),$$

where $B_1 = \langle x\rangle[(1/\langle x\rangle, g(D)]$. By Proposition 5.1 II a) $B_1 = 0(\langle x\rangle^{-1})$. It is obvious that $D\cdot\nabla g(D)(1/\langle x\rangle) = 0(\langle x\rangle^{-1})$. QED.

Proof of Proposition 5.1 III a'). Let us write:

$$(i+\gamma)^{-1}[f(\gamma), g(D)] = (2\pi)^{-1}\int dt\hat{f}(t)\int_0^t d\tau e^{i(t-\tau)\gamma}(i+\gamma)^{-1}i[\gamma, g(D)]e^{i\tau\gamma}. \tag{5.5}$$

By Lemma 5.6 we can write

$$(i+\gamma)^{-1}[\gamma, g(D)] = (i+\gamma)^{-1}\gamma B_1 + (i+\gamma)^{-1}B_2. \tag{5.6}$$

It is easy to see that (5.6) is $0(\langle x\rangle^{-1})$. This implies III a'). QED.

Our next proposition is an example of the so-called geometric method, which proved quite successful in the study of the N-body Schrödinger operators (see [CFKS], Chap. 4 and the references therein).

Proposition 5.7. *Let* $a\in\mathscr{A}$, $\varepsilon > 0$, $Q\in S^0(X)$ *and* $\operatorname{supp} Q \subset Y_a^\varepsilon$. *Then*

a) $Q(x)(h(H) - h(H_a)) = 0(\langle x\rangle^{-\mu})$ *for* $h\in B_3C^\infty(\mathbb{R})$,
b) $Q(x)((H-\lambda)^{-1}(H_0+1) - (H_a-\lambda)^{-1}(H_0+1)) = 0(\langle x\rangle^{-\mu})$,
c) $Q(x)([H, A](H-\lambda)^{-1} - [H_a, A](H_a-\lambda)^{-1}) = 0(\langle x\rangle^{-\mu})$,
d) $Q(x)[g(D_a), (H-\lambda)^{-1}] = 0(\langle x\rangle^{-1-\mu})$ *for* $g\in B_1C^\infty(K_a)$,
e) $Q(x)[g(D_a), (H-\lambda)^{-1}(H_0+1)] = 0(\langle x\rangle^{-1-\mu})$ *for* $g\in B_1C^\infty(K_a)$.

The proof of the above proposition is based on the following lemma.

Lemma 5.8. *Suppose that the assumptions of Proposition 5.7 hold. Then*

a) $Q(x)I_a(H_0+1)^{-1} = 0(\langle x\rangle^{-\mu})$,
b) $Q(x)\nabla I_a(H_0+1)^{-1} = 0(\langle x\rangle^{-1-\mu})$.

Proof. Let $b\not\subset a$. Then hypotheses (4.2) and (4.3) imply that $\langle\pi^b x\rangle^\mu v_b(\pi^b x)(H_0+1)^{-1}$ and $\langle\pi^b x\rangle^{1+\mu}\nabla v_b(\pi^b x)(H_0+1)^{-1}$ are $0(\langle x\rangle^0)$. Moreover it is easy to see that for any ν,

$$Q(x)\langle\pi^b x\rangle^{-\nu} = 0(\langle x\rangle^{-\nu}).$$

This implies immediately the statement of the lemma. QED.

Proof of Proposition 5.7. First let us show that

$$Q(x)((H-\lambda)^{-1} - (H_a-\lambda)^{-1}) = 0(\langle x\rangle^{-\mu}). \tag{5.7}$$

In fact, the left-hand side of (5.7) equals

$$[(H-\lambda)^{-1}, Q(x)]I_a(H_a-\lambda)^{-1} - (H-\lambda)^{-1}Q(x)I_a(H_a-\lambda)^{-1}. \tag{5.8}$$

Clearly, the first term of (5.8) is $0(\langle x\rangle^{-1})$; the second is $0(\langle x\rangle^{-\mu})$ by Lemma 5.8. a).

Next introduce the function h_1 exactly as in the proof of Proposition 5.1. We can write

$$\begin{aligned} Q(x)(h(H) - h(H_a)) &= (2\pi)^{-1}\int dt\hat{h}_1(t)Q(x)(e^{i(H-\lambda)^{-1}t} - e^{i(H_a-\lambda)^{-1}t}) \\ &= (2\pi)^{-1}\int dt\hat{h}_1(t)\Bigg[\int_0^t d\tau e^{i(H-\lambda)^{-1}(t-\tau)}Q(x)((H-\lambda)^{-1} \end{aligned}$$

$$-(H_a-\lambda)^{-1})e^{i(H_a-\lambda)^{-1}\tau}+(2\pi)^{-1}\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2 e^{i(H-\lambda)^{-1}(t-\tau_1)}$$
$$\cdot[Q(x),(H-\lambda)^{-1}]e^{i(H-\lambda)^{-1}}(\tau_1-\tau_2)((H-\lambda)^{-1}$$
$$-(H_a-\lambda)^{-1}e)^{i(H_a-\lambda)^{-1}}.$$

Now the first term of the above expression is $0(\langle x\rangle^{-\mu})$ by (5.7) and the second is $0(\langle x\rangle^{-1})$.

The proof of b) is similar to that of (5.7). c) follows easily from Lemma 5.8. b).

In the proof of d) we proceed as follows. We write

$$Q(x)[g(D_a),(H-\lambda)^{-1}]=(2\pi)^{-1}\int_{X_a}\hat{g}(y)dyQ(x)\int_0^1 d\tau e^{i(1-\tau)D_a\cdot y}[D_a\cdot y,(H-\lambda)^{-1}]e^{i\tau D_a\cdot y}. \tag{5.9}$$

We commute $Q(x)$ through $e^{i(1-\tau)D_a\cdot y}$ using (5.1). Thus we get a term containing $[D_a\cdot y, Q(x)]$, which is of the order $0(\langle x\rangle^{-1})$. This is still unsatisfactory, so we commute $[D_a\cdot y, Q(x)]$ further to the right. In effect we obtain

$$\int_0^1 d\tau Q(x)e^{i(1-\tau)D_a\cdot y}[D_a\cdot y,(H-\lambda)^{-1}]e^{i\tau D_a\cdot y}$$
$$=\int_0^1 d\tau e^{i(1-\tau)D_a\cdot y}Q(x)[D_a\cdot y,(H-\lambda)^{-1}]e^{i\tau D_a\cdot y}$$
$$+\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2 e^{i(1-\tau_2)D_a\cdot y}i[D_a\cdot y,Q(x)][D_a\cdot y,(H-\lambda)^{-1}]e^{i\tau_2 D_a. y}$$
$$-\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\int_0^{\tau_2} d\tau_3 e^{i(1-\tau_2)D_a\cdot y}[D_a\cdot y,[D_a\cdot y,Q(x)]]$$
$$\cdot e^{i(\tau_2-\tau_3)D_a\cdot y}[D_a\cdot y,(H-\lambda)^{-1}]e^{i\tau_3 D_a\cdot y}.$$

Now we easily see that the first term of the above expression is $0(\langle x\rangle^{-1-\mu})$, the second is $0(\langle x\rangle^{-2-\mu})$ and the third is $0(\langle x\rangle^{-2})$. Thus (5.9) is $0(\langle x\rangle^{-1-\mu})$. QED.

By Proposition 5.1 and 5.7, if we are given a product of various operators studied in this section we can very often change the order of factors producing an error of a smaller order. It will be convenient to systematize various possibilities of changing this order.

Proposition 5.9. *Let* $a\in\mathscr{A}$ *and* $\varepsilon>0$. *Let* $Q_i\in S^{m_i}(X)$ *for* $i=0,1,\dots,N$, $m=m_0+m_1+\dots+m_N$ *and* $\operatorname{supp} Q_0\subset Y_a^\varepsilon$.

a) *Let* B_1 *be the product of the operators* $Q_0(x), Q_1(x),\dots,(H-\lambda)^{-1}$ *and a certain number of operators belonging to the following classes*:

1) $f(\gamma)$ *where* $f\in B_1C^\infty(\mathbb{R})$,
2) $g(D_a)$ *where* $g\in B_2C^\infty(K_a)$,
3) $G(D)$ *where* $G\in S^0(K)$,
4) $h(H)$ *where* $h\in B_3C^\infty(\mathbb{R})$,
5) $(H-\lambda)^{-1}(H_0+1)$,
6) $(H-\lambda)^{-1}[H,A]$.

(The order of the factors in this product is arbitrary). Moreover, suppose that B_2 is a product of the same operators as in B_1 except that H is replaced by H_a in the classes 4), 5) *and* 6) *and the order of the factors may be permuted without changing the order of the operators of the classes* 3), 4), 5) *and* 6). *Then*

$$B_1 = 0(\langle x \rangle^m), \tag{5.10}$$

and

$$B_1 = B_2 + 0(\langle x \rangle^{m-\mu}). \tag{5.11}$$

b) *Let C_1 be a product of the operators $Q_0(x), Q_1(x), \ldots, Q_N(x), (H-\lambda)^{-1}$ and a certain number of operators of the following classes:*

1) $f(\gamma)$ *where* $f \in B_1C^\infty(\mathbb{R})$,
2) $g(D_a)$ *where* $g \in B_1C^\infty(K_a) \cap B_2C^\infty(K_a)$,
3) $G(D)$ *where* $G \in S^0(K)$,
4) $(H-\lambda)^{-1}$,
5) $(H-\lambda)^{-1}(H_0+1)$.

(The order is arbitrary.) Suppose also that C_2 is a product of the same operators as in C_1 except that the order of the factors is permuted without changing the order of the operators of the classes 3), 4) *and* 5). *Then*

$$C_1 = 0(\langle x \rangle^m), \tag{5.12}$$

and

$$C_1 = C_2 + 0(\langle x \rangle^{m-\mu}). \tag{5.13}$$

Proof. Clearly all the factors of B_1 and C_1 are $0(\langle x \rangle^0)$ except that $Q_i(x) = 0(\langle x \rangle^{m_i})$. This implies (5.10) and (5.12). Proposition 5.1.II. shows that commuting $Q_i(X)$ in B_1 and C_1 produces errors of the order $0(\langle x \rangle^{m-1})$. Let us show now how one can commute $g(D_a)$ with the operators containing H. This is done in a different way in a) and in b). Consider first a). First we move $Q_0(X)$ to a position adjacent to any of the operators of the class 4), 5) or 6) and replace H with H_a. By Proposition 5.7 a), b), c) this produces an error of the order $0(\langle x \rangle^{m-\mu})$. Now $g(D_a)$ commutes with thus modified operators of the class 4), 5) or 6). In the case b) we move $Q_0(x)$ to a position adjacent to any of the operators of the class 4) or 5). Now by Proposition 5.7 d) or e) if we commute $g(D_a)$ with them we get an $0(\langle x \rangle^{m-1-\mu})$ error. Now if we take into account Proposition 5.1.III we know how to move freely $g(D_a)$ and $f(\gamma)$ in B_1 and C_1 producing errors of a small enough order except that we do not know how to commute $g(D_a)$ with $f(\gamma)$. To do this we move those operators in such a way that one of them becomes adjacent to $(H_a-\lambda)^{-1}$ (in the case a)) or to $(H-\lambda)^{-1}$ (in the case b)). Now we can commute $g(D_a)$ with $f(\gamma)$ producing an $0(\langle x \rangle^{m-1})$ error because by Proposition 5.1.I c) and III a′) we have:

$$(H_a-\lambda)^{-1}[g(D_a), f(\gamma)] = [(H_a-\lambda)^{-1}(\gamma+i)][(\gamma+i)^{-1}[g(D_a), f(\gamma)]] = 0(\langle x \rangle^{-1}).$$

QED.

6. Proof of the Propogation Theorem

In this section we fix $E \notin \mathcal{T}$ and prove Theorem 4.1 for the energy E. The proof is broken into a number of steps. In each step we consider a different class of operators and prove that they are H-smooth on a certain vicinity of E.

The general strategy of the proof is to show that the commutator of a certain observable with H is positive around the energy E modulo some "higher order terms" and then to use Lemma 2.2. An additional technical device that turns out to be helpful is using observables that are equal to $g(D)Q(x)$ times a function of γ.

The commutator of H and of a function of γ has an especially nice from which is the subject of our first lemma in this section.

Lemma 6.1. *Suppose that* $F, f \in C^\infty(\mathbb{R})$, $f' \in C_0^\infty(\mathbb{R})$ *and* $F' = f^2$. *Then*

$$\begin{aligned}&(H-\lambda)^{-1} i[H, F(\gamma)](H-\lambda)^{-1}\\ &= \frac{1}{\sqrt{\langle x\rangle}} f(\gamma)(H-\lambda)^{-1} i[H, A](H-\lambda)^{-1} f(\gamma) \frac{1}{\sqrt{\langle x\rangle}}\\ &\quad -2\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1} f^2(\gamma)\gamma^2 (H-\lambda)^{-1}\frac{1}{\sqrt{\langle x\rangle}} + 0(\langle x\rangle^{-2}).\end{aligned} \tag{6.1}$$

The proof of this lemma is contained in Sect. 7. Now, following [SigSof1] let us study the propagation for large $|\gamma|$.

Proposition 6.2. *Let* $w > \sup \Sigma(E)$, $f \in C^\infty(\mathbb{R})$, $f' \in C_0^\infty(\mathbb{R})$, $f \geqq 0$ *and* $\operatorname{supp} f \subset (-\infty, -w] \cup [w, \infty)$. *Then there exists an open interval* Δ *containing E such that* $(1/\sqrt{\langle x\rangle}) f(\gamma)$ *is H-smooth on* Δ.

Proof. Define

$$F(t) = -\int_0^t f^2(s)ds.$$

Set $w_0 = \sup \Sigma(E)$. Choose a positive number δ such that $0 < 2w^2 - 2w_0^2 - \delta$. Clearly $\rho_{a_{\max}}(E) = 2w_0^2$. Thus by Proposition 4.2 we can find an open interval Δ_1 containing E such that

$$E_{\Delta_1} i[H, A] E_{\Delta_1} \leqq (2w_0^2 + \delta) E_{\Delta_1}. \tag{6.2}$$

Let Δ be an open interval containing E such that $\bar{\Delta} \subset \Delta_1$ and $\bar{\Delta} \cap \mathcal{T} = \emptyset$. Now Proposition 6.2 will follow easily from the boundedness of $F(\gamma)E_\Delta$, the H-smoothness of $\langle x\rangle^{-1} E_\Delta$, Lemma 2.2 and the next lemma. QED.

Lemma 6.3. *These exists* c_1 *and* $c > 0$ *such that*

$$E_\Delta i[H, F(\gamma)] E_\Delta \geqq c E_\Delta f(\gamma) \frac{1}{\langle x\rangle} f(\gamma) E_\Delta - c_1 E_\Delta \frac{1}{\langle x\rangle^2} E_\Delta. \tag{6.3}$$

Proof. Let $h \in C_0^\infty(\mathbb{R})$ such that $h = 1$ on Δ and $\operatorname{supp} h \subset \Delta_1$. By (6.1) multiplied from both sides with $h(H)(H-\lambda)$, we can write

$$\begin{aligned}&h(H) i[H, F(\gamma)] h(H)\\ &= -\frac{1}{\sqrt{\langle x\rangle}} f(\gamma) h(H) i[H, A] h(H) f(\gamma) \frac{1}{\sqrt{\langle x\rangle}}\end{aligned}$$

$$+2h(H)(H-\lambda)\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1}f^2(\gamma)\gamma^2(H-\lambda)^{-1}\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)h(H)$$
$$+0(\langle x\rangle^{-2}). \tag{6.4}$$

By (6.2) the first term of (6.4) is greater than or equal to

$$-(2w_0^2+\delta)\frac{1}{\sqrt{\langle x\rangle}}f(\gamma)h^2(H)f(\gamma)\frac{1}{\sqrt{\langle x\rangle}}.$$

Thus (6.4) is greater than or equal to

$$(2w^2-2w_0^2-\delta)h(H)f(\gamma)\frac{1}{\sqrt{\langle x\rangle}}f(\gamma)h(H)+O(\langle x\rangle^{-2}).$$

This implies immediately (6.3). QED.

The above proposition is the only place in our paper where we use the reverse Mourre inequality. Our next propositions are based on the regular Mourre inequality. More exactly, we will need the following consequence of it.

Proposition 6.4. *Let* $a\in\mathscr{A}$ *and* $\varepsilon,\kappa,v,\delta>0$. *Let* $F\in C^\infty(\mathbb{R})$, $f\in C_0^\infty(\mathbb{R})$ *such that* $F'=f^2$. *Suppose that* $g\in B_2C^\infty(K_a)$ *and* $Q\in S^0(X)$ *is a function homogeneous of degree zero outside the unit ball such that* $\operatorname{supp} Q\subset Y_a^\varepsilon$. *Define* θ^κ *to be the function on* $\mathbb{R}\times(X\backslash\{0\})\times K$ *such that if* $x\in Z_b$ *and* $k\in K$, *then* $\theta^\kappa(E,x,k)=\theta_b^\kappa(E,\pi_b k)$. *Let*

$$\Theta=\inf\left\{\theta^k(E,x,k)\colon x\in\operatorname{supp}Q,\ \pi_a k\in\operatorname{supp}g\ and\operatorname{dist}\left(\frac{x}{|x|}\cdot k,\operatorname{supp}f\right)\leqq v\right\}$$

and

$$\Psi=g(D_a)\sqrt{Q(x)}\frac{1}{\sqrt{\langle x\rangle}}f(\gamma).$$

Then there exists an open interval Δ *containing* E *and a number* c_1 *such that*

$$\begin{aligned}&\tfrac{1}{2}E_\Delta(H-\lambda)i[F(\gamma),(H-\lambda)^{-1}]Q(x)g^2(D_a)(H-\lambda)E_\Delta+hc\\&\geqq(\Theta-\delta-2\sup\{t^2\colon t\in\operatorname{supp}f\})E_\Delta\Psi^*\Psi E_\Delta-c_1E_\Delta\langle x\rangle^{-1-\mu}E_\Delta.\end{aligned}$$

The proof of the above proposition is given in Sect. 8.

To facilitate our study of the propagation of observables it will be convenient to introduce the following definition. Suppose that $M,E\in\mathbb{R}$ and Γ is a conical subset of $X\times K$. We will say that $\Gamma\in\mathscr{N}\mathscr{P}_{E,M}$ if and only if there exists an open interval Δ containing E and an open interval $\tilde{\Delta}$ containing M such that if $f\in C_0^\infty(\mathbb{R})$, $f\geqq 0$, $\operatorname{supp}f\subset\tilde{\Delta}$, $Q\in S^{-\frac{1}{2}}(X)$, $g\in L^\infty(K)$ and $\operatorname{supp}Q(\cdot)g(\cdot)\subset\Gamma$, then $g(D)Q(x)f(\gamma)$ is H-smooth on Δ.

The rest of this section is devoted to finding a sufficiently rich family of sets from $\mathscr{N}\mathscr{P}_{E,M}$ for various values of M. We start with the easiest case (equivalent to Theorem 7.1 of [SigSof1]).

Proposition 6.5. *Let* $M\notin\Sigma(E)$. *Then* $X\times K\in\mathscr{N}\mathscr{P}_{E,M}$.

Proof. Let $w_0,\zeta_0\in\Sigma(E)$ such that $(w_0,\zeta_0)\cap\Sigma(E)=\emptyset$ and $M\in(w_0,\zeta_0)$. Note that since $E \notin \mathscr{T}$, both w_0 and ζ_0 are nonzero. Choose w, ζ such that $w_0 < w < M < \zeta < \zeta_0$. Let $f \in C_0^\infty(\mathbb{R})$, $f \geqq 0$ and $\operatorname{supp} f \subset [w, \zeta]$. Set $F(t) = \int_{-\infty}^{t} f^2(s)ds$. Proposition 6.5 will be proven if we show that there exists an open interval Δ containing E such that $(1/\sqrt{\langle x \rangle})f(\gamma)$ is locally H-smooth on Δ. This will follow from the following lemma. QED.

Lemma 6.6. *There exists an open interval Δ containing E, c_1 and $c > 0$ such that*

$$E_\Delta i[H, F(\gamma)]E_\Delta \geqq cE_\Delta f(\gamma)\frac{1}{\langle x \rangle}f(\gamma)E_\Delta - c_1 E_\Delta \langle x \rangle^{-1-\mu} E_\Delta. \tag{6.5}$$

Proof. The lemma will follow from Proposition 6.4 with $Q = g = 1$ and $a = a_{\max}$.

First assume that w_0 and ζ_0 are of the same sign, e.g. $0 < w_0 < \zeta_0$. Choose positive numbers v, κ and δ such that $w_0^2 + \kappa < (w - v)^2$ and $0 < 2\zeta_0^2 - 2\kappa - \delta - 2\zeta^2$. Then

$$\begin{aligned}\Theta &\geqq \inf\left\{\theta^\kappa(E, x, k) \colon (x, k) \in X \times K,\ w - v \leqq \frac{x}{|x|}\cdot k \leqq \zeta + v\right\}\\ &\geqq \inf\{\theta_b^\kappa(E, \pi_b k) \colon b \in \mathscr{A},\ k \in K,\ (w - v)^2 \leqq |\pi_b k|^2\}\\ &\geqq \inf\{\theta_b^\kappa(E, k_b) \colon b \in \mathscr{A},\ k_b \in K_b,\ w_0^2 + \kappa \leqq |k_b|^2\}\\ &\geqq 2\zeta_0^2 - 2\kappa.\end{aligned}$$

If w_0 and ζ_0 are of a different sign, then $w_0 = -\zeta_0$ and we can assume that $w = -\zeta$. Moreover:

$$\Theta \geqq \inf\{\theta^\kappa(E, x, k) \colon (x, k) \in (X \backslash 0) \times K\} \geqq 2\zeta_0^2 - 2\kappa.$$

In any case, Proposition 6.4 implies that (6.5) holds with $c = 2\zeta_0^2 - 2\kappa - \delta - 2\zeta^2$. QED.

Next we are going to study the propagation for $\gamma \in \Sigma(E)$. [SigSog1] also contains results about this that are sufficient to prove the asymptotic completeness (see Theorem 8.1 of [SigSof1]). Our analysis is somewhat different and leads, we think, to a better understanding of the propagation for N-body Schrödinger operators.

It turns out that even if $\gamma \in \Sigma(E)$, then in some directions of the configuration space there is no propagation. This fact is described in the following proposition.

Proposition 6.7. *Let $a \in \mathscr{A}$, $M \notin \Sigma_a(E)$ and $y \in Z_a \cap S$. Then there exists $\varepsilon > 0$ such that* $\operatorname{Cone}(y, \varepsilon) \times K \in \mathscr{N}\mathscr{P}_{E,M}$.

To prove the above proposition we will analyze separately the propagation in two regions of phase space. This analysis is the subject of the following proposition, which immediately implies Proposition 6.7.

Proposition 6.8. *Let a, M and y satisfy the hypotheses of Proposition 6.7. Set* $\sigma = \sup\{(x/|x|)\cdot y \colon x \neq 0,\ x \notin Y_a^0\}$.

a) *Let $1 > \sigma^- > \sigma$ and $\lambda^- < M\sigma^-$. Then*

$$\operatorname{Cone}(y, \sqrt{2(1 - \sigma^-)}) \times \{k \in K \colon k \cdot y < \lambda^-\} \in \mathscr{N}\mathscr{P}_{E,M}.$$

b) *Let* $1 > \sigma^+ > \sigma$ *and* $\lambda^+ > M\sigma^+$. *Then*

$$\mathrm{Cone}(y, \sqrt{2(1-\sigma^+)}) \times \{k \in K: k\cdot y > \lambda^+\} \in \mathcal{N}\mathcal{P}_{E,M}.$$

Proof. Let $w_0, \zeta_0 \in \Sigma_a(E)$ such that $(w_0, \zeta_0) \cap \Sigma_a(E) = \phi$ and $M \in (w_0, \zeta_0)$. Note, that we may assume that both w_0 and ζ_0 are positive. Using the fact that $\Sigma_a(E)$ is closed and countable we can choose w_1, w_2, ζ_1, ζ_2, $\lambda_1^\pm$ and $\sigma_1^\pm$ such that:

$w_0 < w_1 < w_2 < M < \zeta_2 < \zeta_1 < \zeta_0$,
$[w_1, w_2] \cap \Sigma(E) = \emptyset$,
$[\zeta_2, \zeta_1] \cap \Sigma(E) = \emptyset$,
$1 > \sigma^\pm > \sigma_1^\pm > \sigma$,
$\lambda^- < \lambda_1^- < w_1 \sigma_1^-$,

and $\lambda^+ > \lambda_1^+ > \zeta_1 \sigma^+$.

Let f_-, f_0 and f_+ be nonnegative $C_0^\infty(\mathbb{R})$ functions such that

$\operatorname{supp} f_- \subset [w_1, w_2]$,
$\operatorname{supp} f_0 \subset [w_2, \zeta_2]$,
$\operatorname{supp} f_+ \subset [\zeta_2, \zeta_1]$,

and $\int_{-\infty}^{\infty} f_-^2(s)ds = \int_{-\infty}^{\infty} f_0^2(s)ds = \int_{-\infty}^{\infty} f_+^2(s)ds$.

Define $F_\pm(t) = \int_{-\infty}^{t} f_\pm^2(s)ds$ and $F_0(t) = \int_{-\infty}^{t} f_0^2(s)ds$.

We fix also $\tilde{g}_\pm \in C^\infty(\mathbb{R})$ such that $1 \geqq \tilde{g}_\pm \geqq 0$, $\tilde{g}_- = 1$ on $(-\infty, \lambda^-]$, $\operatorname{supp} \tilde{g}_- \subset (-\infty, \lambda_1^-]$, $\tilde{g}_+ = 1$ on $[\lambda^+, \infty)$ and $\operatorname{supp} \tilde{g}_+ \subset [\lambda_1^+, \infty)$. We set $g_\pm(k_a) = \tilde{g}_\pm(k_a \cdot y)$.

Finally we choose $\tilde{q}_\pm \in C_0^\infty(\mathbb{R})$ such that $\tilde{q}_\pm \geqq 0$ and $\operatorname{supp} \tilde{q}_\pm \subset [\sigma_1^\pm, \sigma^\pm]$. We set $\tilde{Q}_\pm(t) = \int_{-\infty}^{t} \tilde{q}_\pm^2(s)ds$, $q_\pm(x) = \tilde{q}_\pm((x/\langle x\rangle)\cdot y)$ and $Q_\pm(x) = Q_\pm((x/\langle x\rangle)\cdot y)$. Note that $f_\pm, f_0, F_\pm, F_0$, $\sqrt{\pm(F_\pm - F_0)} \in B_1 C^\infty(\mathbb{R})$, $g_\pm \in B_1 C^\infty(K_a) \cap B_2 C^\infty(K_a)$ and $q_\pm, Q_\pm$, $\sqrt{Q_\pm} \in S^0(X)$.

Define also

$$\Phi_\pm = \tfrac{1}{2} F_0(\gamma) Q_\pm(x) g_\pm^2(D_a) + \tfrac{1}{2} F_\pm(\gamma)(1 - Q_\pm(x) g_\pm^2(D_a)) + hc$$

and

$$\Psi_\pm = g_\pm(D_a)\sqrt{Q_\pm(x)}\frac{1}{\sqrt{\langle x\rangle}} f_0(\gamma).$$

Lemma 6.9 that we state and prove below implies that there exists an open interval Δ containing E such that Ψ_+ are H-smooth on Δ. Proposition 6.8 follows easily from this fact by an application of Lemma 2.1 c). QED.

Lemma 6.9. *There exist an open interval* Δ *containing* E, $c > 0$ *and operators* B_i, B_i' *that are* H*-smooth on* Δ *such that*

$$E_\Delta i[H, \Phi_\pm] E_\Delta \geqq c E_\Delta \Psi_\pm^* \Psi_\pm E_\Delta + \sum_{i=1}^{k} B_i^* B_i'.$$

Proof. Clearly

$$
\begin{aligned}
(H-\lambda)^{-1} i[H, \Phi_{\pm}](H-\lambda)^{-1} = & \tfrac{1}{2} i[F_0(\gamma), (H-\lambda)^{-1}] Q_{\pm}(x) g_{\pm}^2(D_a) + hc \\
& + \tfrac{1}{2} i[F_{\pm}(\gamma), (H-\lambda)^{-1}](1 - Q_{\pm}(x) g_{\pm}^2(D_a)) + hc \\
& + \tfrac{1}{2}(F_0(\gamma) - F_{\pm}(\gamma)) i[Q_{\pm}(x), (H-\lambda)^{-1}] g_{\pm}^2(D_a) + hc \\
& + \tfrac{1}{2}(F_0(\gamma) - F_{\pm}(\gamma)) Q_{\pm}(x) i[g_{\pm}^2(D_a), (H-\lambda)^{-1}] + hc.
\end{aligned}
\tag{6.6}
$$

Lemma 5.7 d) implies that the fourth term on the right-hand side of (6.6) is $0(\langle x \rangle^{-1-\mu})$.

By methods employed in the proof of Proposition 6.4 we easily show that the second term is greater than or equal to

$$c_1 B_1^* B_1 + c_2 B_2^* B_2 + c_3 E_\Delta \langle x \rangle^{-1-\mu} E_\Delta,$$

where

$$B_1 = (H-\lambda)^{-1} \frac{1}{\sqrt{\langle x \rangle}} f_{\pm}(\gamma),$$

$$B_2 = g_{\pm}(D_a) \sqrt{Q_{\pm}(x)} (H-\lambda)^{-1} \frac{1}{\sqrt{\langle x \rangle}} f_{\pm}(\gamma),$$

and c_1, c_2 and c_3 are some numbers. Clearly, by Proposition 6.5 the operators B_1 and B_2 are H-smooth on some neighborhood of E. To deal with the first term we use Proposition 6.4. We choose some positive numbers ν, κ and δ such that $w_0^2 + \kappa < (w_2 - \nu)^2$ and $0 < 2\zeta_0^2 - 2\kappa - \delta - 2\zeta^2$. We note also that $\operatorname{supp} Q_{\pm} \subset Y_a^{\varepsilon}$ for some $\varepsilon > 0$. Thus both in the $+$ and $-$ cases we have

$$
\begin{aligned}
\Theta & \geqq \inf \left\{ \theta^{\kappa}(E, x, k) \colon (x, k) \in Y_a^{\varepsilon} \times K,\ w_2 - \nu \leqq \frac{x}{|x|} \cdot k \leqq \zeta_2 + \nu \right\} \\
& \geqq \inf \{ \theta_b^{\kappa}(E, \pi_b k) \colon b \subset a,\ k \in K,\ (w_2 - \nu)^2 \leqq |\pi_b k|^2 \} \\
& = \inf \{ \theta_b^{\kappa}(E, k_b) \colon b \subset a,\ k_b \in K_b,\ w_0^2 + \kappa \leqq |k_b|^2 \} \geqq 2\zeta_0^2 - 2\kappa.
\end{aligned}
$$

Now Proposition 6.4 guarantees that we can find an open interval Δ containing E and a number c_1 such that

$$
\begin{aligned}
& \tfrac{1}{2} E_\Delta (H-\lambda) i[F_{\pm}(\gamma), (H-\lambda)^{-1}] Q_{\pm}(x) g_{\pm}^2(D_a)(H-\lambda) E_\Delta + hc \\
& \geqq (2\zeta_0^2 - 2\kappa - \delta - 2\zeta_2^2) E_\Delta \Psi_{\pm}^* \Psi_{\pm} E_\Delta - c_1 E_\Delta \langle x \rangle^{-1-\mu} E_\Delta.
\end{aligned}
$$

It remains to handle the third term of (6.6). This is the subject of our next lemma, which completes the proof of Lemma 6.9. QED.

Lemma 6.10.

$$\tfrac{1}{2}(F_0(\gamma) - F_{\pm}(\gamma)) i[Q_{\pm}(x), (H-\lambda)^{-1}] g_{\pm}^2(D_a) + hc \geqq 0(\langle x \rangle^{-2}). \tag{6.7}$$

Proof. We easily compute that

$$
\begin{aligned}
i[Q_{\pm}(x), (H-\lambda)^{-1}] = & (H-\lambda)^{-1} \left(\frac{1}{\langle x \rangle} q_{\pm}^2(x) D_a \cdot y - \gamma q_{\pm}^2(x) \frac{x \cdot y}{\langle x \rangle^2} \right) (H-\lambda)^{-1} \\
& + 0(\langle x \rangle^{-2}).
\end{aligned}
$$

First consider the "$-$" case. By the methods of Proposition 5.9 b)

$$\tfrac{1}{2}(F_-(\gamma)-F_0(\gamma))(H-\lambda)^{-1}\gamma q_-^2(x)\frac{x\cdot y}{\langle x\rangle^2}(H-\lambda)^{-1}g_-^2(D_a)+hc$$

equals

$$g_-(D_a)q_-(x)\sqrt{\frac{x\cdot y}{\langle x\rangle^2}}(H-\lambda)^{-1}\gamma(F_-(\gamma)-F_0(\gamma))(H-\lambda)^{-1}\sqrt{\frac{x\cdot y}{\langle x\rangle^2}}q_-(x)g_-(D_a)$$

plus $0(\langle x\rangle^{-2})$. This is greater than or equal to

$$w_1 g_-(D_a)q_-(x)\sqrt{\frac{x\cdot y}{\langle x\rangle^2}}(H-\lambda)^{-1}(F_-(\gamma)-F_0(\gamma))(H-\lambda)^{-1}\sqrt{\frac{x\cdot y}{\langle x\rangle^2}}q_-(x)g_-(D_a).$$

Now the above term equals

$$w_1 g_-(D_a)(H-\lambda)^{-1}\sqrt{F_-(\gamma)-F_0(\gamma)}q_-^2(x)\frac{x\cdot y}{\langle x\rangle^2}\sqrt{F_-(\gamma)-F_0(\gamma)}(H-\lambda)^{-1}g_-(D_a)$$

plus $0(\langle x\rangle^{-2})$. This is greater than or equal to

$$w_1\sigma_1^- g_-(D_a)(H-\lambda)^{-1}\sqrt{F_-(\gamma)-F_0(\gamma)}q_-^2(x)\frac{1}{\langle x\rangle}\sqrt{F_-(\gamma)-F_0(\gamma)}(H-\lambda)^{-1}g_-(D_a). \tag{6.8}$$

Next consider the expression

$$-\tfrac{1}{2}(F_-(\gamma)-F_0(\gamma))(H-\lambda)^{-1}\frac{1}{\langle x\rangle}q_-^2(x)D_a\cdot y(H-\lambda)^{-1}g_-^2(D_a)+hc. \tag{6.9}$$

Instead of $D_a\cdot y(H-\lambda)^{-1}$ we can write

$$[D_a\cdot y(H_0+1)^{-1}][(H_0+1)(H-\lambda)^{-1}]$$

and apply Proposition 5.9 b). Thus (6.9) equals

$$-\sqrt{F_-(\gamma)-F_0(\gamma)}\frac{q_-(x)}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1}D_a\cdot yg_-^2(D_a)(H-\lambda)^{-1}\frac{q_-(x)}{\sqrt{\langle x\rangle}}\sqrt{F_-(\gamma)-F_0(\gamma)}$$

plus $0(\langle x\rangle^{-2})$. This is greater than or equal to

$$-\lambda_1^-\sqrt{F_-(\gamma)-F_0(\gamma)}\frac{q_-(x)}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1}g_-^2(D_a)(H-\lambda)^{-1}\frac{q_-(x)}{\sqrt{\langle x\rangle}}\sqrt{F_-(\gamma)-F_0(\gamma)}.$$

By another application of Proposition 5.9 b) the above expression equals

$$-\lambda_1^- g_-(D_a)(H-\lambda)^{-1}\sqrt{F_-(\gamma)-F_0(\gamma)}q_-^2(x)\frac{1}{\langle x\rangle}\sqrt{F_-(\gamma)-F_0(\gamma)}(H-\lambda)^{-1}g_-(D_a) \tag{6.10}$$

plus $0(\langle x\rangle^{-2})$.

Thus, the left-hand side of (6.7) in the "$-$" case is greater than or equal to the sum of (6.8) and (6.10) plus $0(\langle x\rangle^{-2})$. But $w_1\sigma_1^- - \lambda_1^- > 0$. Consequently, the left-hand side of (6.7) in the "−" case is greater or equal than $0(\langle x\rangle^{-2})$.

To prove the "+" case we proceed in a very similar way. We show that in this case the left hand side of (6.7) is greater than or equal to

$$(\lambda_1^+ - \zeta_1\sigma^+)g_+(D_a)(H-\lambda)^{-1}\sqrt{F_0(\gamma)-F_+(\gamma)}q_+^2(x)$$
$$\cdot\frac{1}{\langle x\rangle}\sqrt{F_0(\gamma)-F_+(\gamma)}(H-\lambda)^{-1}g_+(D_a)$$

plus $0(\langle x\rangle^{-2})$, which immediately implies the desired statement. QED.

Classical intuition suggests that a particle moving with velocity k after a long enough time will travel within an arbitrarily small cone around the direction of k. This intuition motivates our next proposition which describes the propagation for threshold values of γ.

Proposition 6.11. *Let $M\in\mathbb{R}$, $a\in\mathscr{A}$, $y\in Z_a\cap S$ and $\beta>0$. Then there exists $\varepsilon>0$ such that* $\mathrm{Cone}(y,\varepsilon)\times\{k\in K\colon\ |\pi_a k - yM|\geqq\beta\}\in\mathcal{NP}_{E,M}$.

The following elementary lemma is an important step of the proof of the above proposition

Lemma 6.12. *Suppose that the assumptions of Proposition 6.11 hold. Let $\beta_0>0$. Then there exist $v_0>0$ and $\varepsilon_0>0$ such that if $b\subset\mathscr{A}$, $x\in\mathrm{Cone}(y,\varepsilon_0)\cap X_b$, $k\in K$, $(x/|x|)\cdot k>M-v_0$,* and *$|\pi_a k - yM|\geqq\beta_0$, then $b\subset a$ and $|\pi_b k|\geqq M+v_0$.*

Proof. To simplify the notation we drop the subscript from β_0, v_0 and ε_0. Clearly if we assume that ε is small enough then $x\in\mathrm{Cone}(y,\varepsilon)\cap X_b$ implies $b\subset a$. Then we can write

$$\left|\pi_b k - M\frac{x}{|x|}\right|^2 = |\pi_b k|^2 - 2\frac{x}{|x|}\cdot kM + M^2 \leqq |\pi_b k|^2 - M^2 + 2Mv, \tag{6.11}$$

$$|\pi_b k - My|^2 - \left|\pi_b k - M\frac{x}{|x|}\right|^2 \leqq 2M|\pi_b k|\left|\frac{x}{|x|}-y\right| \leqq 2M\varepsilon|\pi_b k|, \tag{6.12}$$

$$\beta^2 \leqq |\pi_a k - My|^2 \leqq |\pi_b k - My|^2. \tag{6.13}$$

We add up (6.11), (6.12) and (6.13) and get

$$\beta^2 \leqq |\pi_b k|^2 - M^2 + 2Mv + 2M\varepsilon|\pi_b k|.$$

Thus

$$|\pi_b k| \geqq -M\varepsilon + \sqrt{M^2+\beta^2-2Mv+M^2\varepsilon^2}. \tag{6.14}$$

If $\varepsilon\to 0$ and $v\to 0$, then the right-hand side of (6.14) goes to $\sqrt{M^2+\beta^2}$. Thus we will find $\varepsilon, v>0$ such that the right-hand side of (6.14) is greater than $M+v$. QED.

Let us fix β_0 such that $0<\beta_0<\beta$ (recall that β is the number that appears in Proposition 6.11). Assume also that v_0 and ε_0 are determined by Lemma 6.12.

As in the proof of Proposition 6.7, to show Proposition 6.11 we will analyze separately two regions of the phase space. Our next proposition describes the two regions that we have in mind. It implies immediately Proposition 6.11.

Proposition 6.13. *Let M, a and y be as in Proposition 6.11. Let ε_0 be as described above.*

a) *Let $1 > \sigma^- > 1 - (\varepsilon_0^2/2)$ and $\lambda^- < M\sigma^-$. Then*

$$\mathrm{Cone}(y, \sqrt{2(1-\sigma^-)}) \times \{k \in K\colon k\cdot y < \lambda^{-1}, |\pi_a k - My| \geqq \beta\} \in \mathcal{NP}_{E,M}.$$

b) *Let $1 > \sigma^+ > 1 - (\varepsilon_0^2/2)$ and $\lambda^+ > M\sigma^+$. Then*

$$\mathrm{Cone}(y, \sqrt{2(1-\sigma^+)}) \times \{k \in K\colon k\cdot y > \lambda^+, |\pi_a k - My| \geqq \beta\} \in \mathcal{NP}_{E,M}.$$

Proof. Choose $w_1, w_2, \zeta_1, \zeta_2, \lambda_1^\pm$ and $\sigma_1^\pm$ such that:

$$M - v_0 < w_1 < w_2 < M < \zeta_2 < \zeta_1 < M + v_0,$$
$$1 > \sigma^\pm > \sigma_1^\pm > 1 - (\varepsilon_0^2/2),$$
$$\lambda^- < \lambda_1^- < w_1\sigma_1^-,$$

and $\lambda^+ > \lambda_1^+ > \zeta_1\sigma^+$.

Choose $f_0, f_\pm, F_0, F_\pm, \tilde{q}_\pm, \tilde{Q}_\pm, q_\pm, Q_\pm, \tilde{g}_\pm$ and $g_\pm$ in the same way as in the proof of Proposition 6.8. Also fix a function $\tilde{g}_1 \in C^\infty(\mathbb{R})$ such that $1 \geqq \tilde{g}_1 \geqq 0, \tilde{g}_1(t) = 1$ for $t > \beta$ and $\operatorname{supp} \tilde{g}_1 \subset [\beta_0, \infty)$. Set $G_\pm(k_a) = g_\pm(k_a)\tilde{g}_1(|k_a - My|)$. Define

$$\Phi_\pm = \tfrac{1}{2}F_0(\gamma)Q_\pm(x)G_\pm^2(D_a) + \tfrac{1}{2}F_\pm(\gamma)(1 - Q_\pm(x)G_\pm^2(D_a)) + hc$$

and

$$\Psi_\pm = G_\pm(D_a)\sqrt{Q_\pm(x)}\frac{1}{\sqrt{\langle x\rangle}}f_0(\gamma).$$

Now our proposition will follow from Lemma 6.14 that we present below by an argument similar to the one used in the proof of Proposition 6.8. QED.

Lemma 6.14. *There exists an open interval Δ containing $E, c > 0$ and operators B_i, B_i' that are H-smooth on Δ such that*

$$E_\Delta i[H, \Phi_\pm]E_\Delta \geqq cE_\Delta \Psi_\pm^* \Psi_\pm E_\Delta + \sum_{i=1}^{k} B_i^* B_i'.$$

Proof. The proof is most of the time similar to that of Lemma 6.9. We start with an analog of equality (6.6) with $G_\pm$ replacing $g_\pm$. We deal with the second, third and fourth terms of this equality exactly as in the proof of Lemma 6.9. To handle the first term we use Proposition 6.4 in the following way.

We choose positive numbers v, κ and δ such that $(M - v_0)^2 < (w_2 - v)^2$ and $0 < 2(M + v_0)^2 - \delta - 2\zeta_2^2$. Now:

$$\Theta \geqq \inf\left\{\theta^\kappa(E, x, k)\colon (x, k) \in \mathrm{Cone}(y, \sqrt{2(1-\sigma^\pm)}) \times K,\right.$$
$$\left.|\pi_a k - My| > \beta_0,\ w_2 - v \leqq \frac{x}{|x|}\cdot k \leqq \zeta_2 + v\right\}$$

$$\geqq \inf\left\{\theta^\kappa(E,x,k)\colon (x,k)\in\mathrm{Cone}(y,\varepsilon_0)\times K, |\pi_a k - My| > \beta_0,\ M - v_0 \leqq \frac{x}{|x|}\cdot k\right\}$$

$$\geqq \inf\{\theta_b^\kappa(E,\pi_b k)\colon b\subset a,\ k\in K, |\pi_b k|^2 \geqq (M+v_0)^2\} \geqq 2(M+v_0)^2.$$

Thus by Proposition 6.4 there exists an open interval Δ containing E and a number c_1 such that

$$E_\Delta(H-\lambda)i[F_0(\gamma),(H-\lambda)^{-1}]Q_\pm(x)G_\pm^2(D_a)(H-\lambda)E_\Delta$$
$$\geqq (2(M+v_0)^2 - \delta - 2\zeta_2^2)E_\Delta \Psi_\pm^* \Psi_\pm E_\Delta - c_1 E_\Delta \langle x\rangle^{-1-\mu}E_\Delta.$$

This implies the statement of our lemma which $c = 2(M+v_0)^2 - \delta - 2\zeta_2^2 > 0$.

QED.

Propositions 6.5, 6.7 and 6.11 provide us with a lot of information on $\mathcal{NP}_{E,M}$. We can put this information together and formulate the following proposition.

Proposition 6.15. *Let $M\in\mathbb{R}$. Define*

$$PS_{E,M} = \bigcup_{\substack{a\in\mathscr{A}\\ \text{such that } M\in\Sigma_a(E)}} \bigcup_{\substack{x\in X_a\\ x\neq 0}} \left\{(x,k)\colon k\in K, \pi_a k = M\frac{x}{|x|}\right\}.$$

Let $\varepsilon,\beta>0$. Then

$$X\times K\backslash (PS_{E,M})^{\varepsilon,\beta} \in \mathcal{NP}_{E,M}.$$

Proof. Let $g\in L^\infty(K)$ and $Q\in S^{-\frac{1}{2}}(X)$ such that $\mathrm{supp}\, Q(\cdot)g(\cdot)\subset X\times K\backslash(PS_{E,M})^{\varepsilon,\beta}$. For any $a\in\mathscr{A}$ and $y\in Z_a\cap S$ let ε_y, $\tilde{\Delta}_y$ and Δ_y be determined in the obvious way by Proposition 6.7 in the case $M\notin\Sigma_a(E)$ and by Proposition 6.11 in the case $M\in\Sigma_a(E)$. We may assume that $0<\varepsilon_y<\varepsilon$. The sets $\mathrm{Cone}(y,\varepsilon_y)\cap S$ for $y\in S$ form an open cover of S. We can choose a finite subcover labelled by $y_1,\ldots,y_N$. We can also choose a partition of unity $j_1,\ldots,j_N$ such that $j_i\in S^0(x)$, $0\leqq j_i\leqq 1$, $\sum_{i=1}^N j_i = 1$ and $\mathrm{supp}\, j_i\subset B(1)\cup\mathrm{Cone}(y_i,\varepsilon_{y_i})$. (See a similar construction in Lemma 8.2). We set $\tilde{\Delta} = \bigcap_{i=1}^N \tilde{\Delta}_{y_i}$ and $\Delta = \bigcap_{i=1}^N \Delta_{y_i}$. Now let $f\in C_0^\infty(\mathbb{R})$, $f\geqq 0$ and $\mathrm{supp} f\subset\Delta$. The proposition will be proved if we show that $g(D)Q(x)f(\gamma)$ is H-smooth on Δ.

To this end we write

$$g(D)Q(x)f(\gamma) = \sum_{i=1}^N g(D)Q(x)j_{y_i}(x)f(\gamma). \tag{6.15}$$

Let us fix our attention on a certain y_i. Let $y_i\in Z_a$ for some $a\in\mathscr{A}$. If $M\notin\Sigma_a(E)$ then

$$\mathrm{supp}\, Q(\cdot)j_{y_i}(\cdot)g(\cdot)\subset\mathrm{Cone}(y_i,\varepsilon_{y_i})\times K,$$

and if $M\in\Sigma_a(E)$ then

$$\mathrm{supp}\, Q(\cdot)j_{y_i}(\cdot)g(\cdot)\subset\mathrm{Cone}(y_i,\varepsilon_{y_i})\times\{k\in K\colon |\pi_a k - My|>\beta\}.$$

Thus all the terms of the sum on the right-hand side of (6.15) are H-smooth either by Proposition 6.7 or by Proposition 6.11. QED.

Now we are ready for the proof of the propagation theorem

Proof of Theorem 4.1. Let $g \in L^\infty(K)$ and $Q \in S^{-\frac{1}{2}}(X)$ such that $\operatorname{supp} Q(\cdot)g(\cdot) \subset X \times K \backslash (PS_E)^{\varepsilon,\beta}$. For any $M \in \mathbb{R}$ let $\tilde{\Delta}_M$ and Δ_E be open intervals containing M and E respectively determined by Proposition 6.15. Let $w_0 = \sup \Sigma_a(E)$ and $w > w_0$.

The sets $\tilde{\Delta}_M$ for $M \in \mathbb{R}$ form an open cover of $[-w, w]$. We can choose a finite subcover labelled by $M_1, \ldots, M_N$. Let $f_0, f_1, \ldots, f_N, f_{N+1}$ be a partition of unity such that $f_i \in C^\infty(\mathbb{R})$, $f_i \geqq 0$, $\sum_{i=1}^{N} f_i = 1$, $\operatorname{supp} f_0 \subset (-\infty, w]$, $\operatorname{supp} f_i \subset \tilde{\Delta}_{M_i}$ for $i = 1, \ldots, N$ and $\operatorname{supp} f_{N+1} \subset [w, \infty)$.

By Proposition 6.2 if $i = 0, N+1$ then $q(D)Q(x)f_i(\gamma)$ are H-smooth on a certain open interval Δ containing E.

Clearly $PS_{E,M} \subset PS_E$. Consequently $X \times K \backslash (PS_{E,M})^{\varepsilon,\beta} \supset X \times K \backslash (PS_E)^{\varepsilon,\beta}$. Thus the operators $g(D)Q(x)f_i(\gamma)$ are H-smooth on Δ_{M_i} for $i = 1, \ldots, N$ by Proposition 6.15. Hence

$$g(D)Q(x) = \sum_{i=0}^{N+1} g(D)Q(x)f_i(\gamma)$$

is H-smooth on $\Delta \cap \bigcap_{i=1}^{N} \Delta_{M_i}$. QED.

7. Proof of Lemma 6.1

In this setion we are going to show Lemma 6.1 that describes the commutator of a function of γ with H. A very similar fact is also proved in [SigSof 1]. Our proof follows that of [SigSof 1] and we have included it for the convenience of the reader.

Let us start with the following lemma.

Lemma 7.1

a) $i[\gamma, (H-\lambda)^{-1}] = \frac{1}{\sqrt{\langle x \rangle}}(H-\lambda)^{-1}(i[H, A] - 2\gamma^2)(H-\lambda)^{-1}\frac{1}{\sqrt{\langle x \rangle}} + 0(\langle x \rangle^{-2})$,

b) $[\gamma, (H-\lambda)^{-1}] = 0(\langle x \rangle^{-1})$,

c) $[\gamma, [\gamma, (H-\lambda)^{-1}]] = 0(\langle x \rangle^{-2})$.

Proof. Note that $\gamma = A(1/\langle x \rangle) + Q(x)$, where $Q \in S^{-1}(X)$. Thus

$$i[\gamma, (H-\lambda)^{-1}] = (H-\lambda)^{-1} i[H, A](H-\lambda)^{-1}\frac{1}{\langle x \rangle} + A(H-\lambda)^{-1} i\left[H_0, \frac{1}{\langle x \rangle}\right](H-\lambda)^{-1} + 0(\langle x \rangle^{-2}). \qquad (7.1)$$

Clearly the first term of the right-hand side of (7.1) equals

$$\frac{1}{\sqrt{\langle x \rangle}}(H-\lambda)^{-1} i[H, A](H-\lambda)^{-1}\frac{1}{\sqrt{\langle x \rangle}}$$

plus $0(\langle x \rangle^{-2})$. Next note that $A = \langle x \rangle \gamma + 0(\langle x \rangle^0)$ and $[H_0, (1/\langle x \rangle)] =$

$2\gamma\langle x\rangle^{-2}+0(\langle x\rangle^{-3})$. Thus the second term of (7.1) equals

$$2\langle x\rangle\gamma(H-\lambda)^{-1}\gamma\langle x\rangle^{-2}(H-\lambda)^{-1} \tag{7.2}$$

plus $0\langle x\rangle^{-2}$. This implies immediately b). Next we note that (7.2) equals

$$2\langle x\rangle\gamma(H-\lambda)^{-1}\gamma(H-\lambda)^{-1}\langle x\rangle^{-2} \tag{7.3}$$

plus $0(\langle x\rangle^{-2})$. Using b) to commute γ and $(H-\lambda)^{-1}$ we obtain that (7.3) equals

$$2\langle x\rangle^{-1/2}(H-\lambda)^{-1}\gamma^2(H-\lambda)^{-1}\langle x\rangle^{-1/2}+0(\langle x\rangle^{-2}).$$

This ends the proof of a).

The proof of c) is left to the reader. Let us remark only that in the proof of c) we use the boundedness of $[[H,A],A](H-\lambda)^{-1}$. QED.

Next we will need the following lemma (which is also taken from [SigSof 1].)

Lemma 7.2. *Let B and C be self adjoint operators. Suppose that B, $[C,B]$ and $[C,[C,B]]$ are bounded. Suppose also that $F\in C^\infty(\mathbb{R})$ and $F'\in B_1C^\infty(\mathbb{R})$. Then $[B,F(C)]$ is bounded and*

$$[F(C),B]=F'(C)[C,B]+R, \tag{7.4}$$

where $R=(2\pi)^{-1}\int\hat{F}(t)tdt\int_0^t ds e^{iC(t-s)}[C,[C,B]]e^{iCs}$.

Proof. If $F\in\mathscr{S}(\mathbb{R})$ then we easily compute (7.4) in the sense of quadratic forms on $\mathscr{D}(C^2)$. Next we apply the density argument. QED.

Now we proceed directly to the proof of Lemma 6.1. We apply formula (7.4) to $B=(H-\lambda)^{-1}$ and $C=\gamma$. By Lemma 7.1 c) $R=0(\langle x\rangle^{-2})$. Thus

$$(H-\lambda)^{-1}i[H,F(\gamma)](H-\lambda)^{-1}=i[F(\gamma),(H-\lambda)^{-1}]=f^2(\gamma)i[\gamma,(H-\lambda)^{-1}]+0(\langle x\rangle^{-2}). \tag{7.5}$$

We insert the formula of Lemma 7.1 a) into (7.5). Now clearly

$$f^2(\gamma)\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1}i[H,A](H-\lambda)^{-1}\frac{1}{\sqrt{\langle x\rangle}}$$
$$=\frac{1}{\sqrt{\langle x\rangle}}f(\gamma)(H-\lambda)^{-1}i[H,A](H-\lambda)^{-1}f(\gamma)\frac{1}{\sqrt{\langle x\rangle}}+0(\langle x\rangle^{-2})$$

and

$$f^2(\gamma)\frac{1}{\sqrt{\langle x\rangle}}[(H-\lambda)^{-1}\gamma][\gamma(H-\lambda)^{-1}]\frac{1}{\sqrt{\langle x\rangle}}$$
$$=\frac{1}{\sqrt{\langle x\rangle}}[(H-\lambda)^{-1}\gamma]f^2(\gamma)[\gamma(H-\lambda)^{-1}]\frac{1}{\sqrt{\langle x\rangle}}+0(\langle x\rangle^{-2}).$$

(The square parentheses suggest the way we commute the factors). QED.

8. Proof of Proposition 6.4

The proof of Proposition 6.4 is based on a careful analysis of the commutator $[H, A]$. It is closely related to the proof of Theorem 7.1 of [SigSof 1], nevertheless we think that our approach is simpler and more transparent.

By Lemma 6.1

$$\tfrac{1}{2} i[F(\gamma), (H-\lambda)^{-1}] Q(x) g^2(D_a) + hc$$

equals

$$\frac{1}{2}\frac{1}{\sqrt{\langle x\rangle}} f(\gamma)(H-\lambda)^{-1} i[H, A](H-\lambda)^{-1} f(\gamma)\frac{1}{\sqrt{\langle x\rangle}} Q(x) g^2(D_a) + hc$$

$$-\frac{1}{2}\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1} f^2(\gamma) 2\gamma^2 (H-\lambda)^{-1}\frac{1}{\sqrt{\langle x\rangle}} Q(x) g^2(D_a) + hc \tag{8.1}$$

plus $0(\langle x\rangle^{-2})$.

It is easy to handle the second term. First note that the second term of (8.1) plus its hermitian conjugate equals

$$-(H-\lambda)^{-1}\frac{1}{\sqrt{\langle x\rangle}}\sqrt{Q(x)} g(D_a) f^2(\gamma) 2\gamma^2 g(D_a)\sqrt{Q(x)}\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1}$$

plus $0(\langle x\rangle^{-2})$. This is greater than or equal to

$$-\sup\{2t^2 : t\in \operatorname{supp} f\}(H-\lambda)^{-1}\frac{1}{\sqrt{\langle x\rangle}}\sqrt{Q(x)} g(D_a) f^2(\gamma) g(D_a)$$

$$\cdot\sqrt{Q(x)}\frac{1}{\sqrt{\langle x\rangle}}(H-\lambda)^{-1}. \tag{8.2}$$

Finally, (8.2) equals

$$-\sup\{2t^2 : t\in \operatorname{supp} f\}(H-\lambda)^{-1}\Psi^*\Psi(H-\lambda)^{-1}$$

plus $0(\langle x\rangle^{-2})$.

Now it remains to deal with the first term of (8.1), which is much more difficult. To this end we need a lemma saying that functions of γ can be approximated by differential operators if we localize them in a sufficiently small cone in the configuration space.

Lemma 8.1. *Let $y\in S$ and $\varepsilon>0$. Let $j\in L^\infty(X)$, $\operatorname{supp} j\subset \operatorname{Cone}(y,\varepsilon)$ and $f\in B_1C^\infty(\mathbb{R})$. Then there exists a number c that depends only on f and there also exists a bounded operator B such that*

$$j(x)(f(\gamma)-f(y\cdot D))\langle D\rangle^{-1} = j(x)B + 0(\langle x\rangle^{-1}) \tag{8.3}$$

and $\|B\|\leqq \varepsilon c$.

Proof. Let $\tilde{j}$ be the characteristic function of $\operatorname{Cone}(y,\varepsilon)$. Clearly $\tilde{j}$ commutes with

γ. Thus the left-hand side of (8.3) equals

$$j(x)(2\pi)^{-1}\int dt\hat{f}(t)(e^{i\gamma t}-e^{iy\cdot Dt})\langle D\rangle^{-1}=j(x)(2\pi)^{-1}\int dt\hat{f}(t)\int_0^t dse^{i\gamma s}(\gamma-y\cdot D)e^{iy\cdot D(t-s)}$$
$$=j(x)\left[(2\pi)^{-1}\int dt\hat{f}(t)\int_0^t dse^{i\gamma s}\tilde{j}(x)\left(\frac{x}{|x|}-y\right)\right.$$
$$\left.\cdot D\langle D\rangle^{-1}e^{iy\cdot D(t-s)}\right]+0(\langle x\rangle^{-1}).$$

Now the integral in the square bracket defines a bounded operator with norm less than

$$\varepsilon\int dt|\hat{f}(t)t|\|D\langle D\rangle^{-1}\|.\quad \text{QED.}$$

The next lemma contains a construction of a partition of unity that we will need in our proof.

Lemma 8.2 *Let $\varepsilon>0$. Then there exist $\varepsilon_1>0$, a finite collection $\{y_i\colon i=1,\dots,N\}$ of points of $S\cap\operatorname{supp}Q$ and a family of functions $\{j_i\colon i=1,\dots,N\}$ such that $j_i\in S^0(X)$, $0\leqq j_i\leqq 1$, $\sum_{i=1}^N j_i^2=1$ on $\operatorname{supp}Q$, $\operatorname{supp}j_i\subset\operatorname{Cone}(y_i,\varepsilon)\cup B(1)$ and if $y_i\in Z_b$ then $\operatorname{supp}j_i\subset Y_b^{\varepsilon_1}$.*

Proof. For every $b\in\mathscr{A}$ and every $y\in S\cap\operatorname{supp}Q\cap Z_b$ we can fix $\varepsilon_y>0$ such that $\varepsilon_y<\varepsilon$ and $\operatorname{Cone}(y,\varepsilon_y)\subset Y_b^{\varepsilon_y}$. The family of sets $S\cap\operatorname{Cone}(y,(\varepsilon_y/2))$, where $y\in S\cap\operatorname{supp}Q$ is an open cover of the compact set $S\cap\operatorname{supp}Q$. We can choose a finite subcover labelled by $y_1,\dots,y_N$. Let $\{\chi_i\colon i=1,\dots,N\}$ be a family of functions on S such that $\chi_i\in C^\infty(S)$, $0\leqq\chi_i\leqq 1$, $\operatorname{supp}\chi_i\subset\operatorname{Cone}(y_i,\varepsilon_{y_i})\cap S$ and $\chi_i=1$ on $\operatorname{Cone}(y_i,(\varepsilon_{y_i}/2))\cap S$. Let $\eta\in C^\infty(\mathbb{R})$ such that $\eta(t)=t$ for $t>1$ and $\eta(t)>\max(t,\frac{1}{2})$ for $t<1$. Set $\tilde{j}_i(y)=\chi_i(y)/\eta\left(\sum_{i=1}^N\chi_i^2(y)\right)$. Now choose $j_i\in C^\infty(X)$ such that $j_i(x)=\tilde{j}_i(x/|x|)$ for $|x|>1$. QED.

Now we proceed directly to estimating the first term of (8.1). First we fix an open interval Δ_1 containing E such that for any $b\subset a$ we have

$$E_{\Delta_1}(H_b)i[H_b,A]E_{\Delta_1}(H_b)\geqq\left(\theta_b^\kappa(E,D_b)-\frac{\delta}{2}\right)E_{\Delta_1}(H_b).\tag{8.4}$$

Let Δ be an open interval containing E such that $\bar{\Delta}\subset\Delta_1$. Choose $h\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}h\subset\Delta_1$, $0\leqq h\leqq 1$ and $h=1$ on Δ. Fix also $\tilde{f}\in B_1C^\infty(\mathbb{R})$ such that $\operatorname{supp}\tilde{f}\subset\{t\colon\operatorname{dist}(t,\operatorname{supp}f)\leqq v\}$ and $\tilde{f}=1$ on $\operatorname{supp}f$. Let $\{j_i\colon i=1,\dots,N\}$ be a partition of unity described in Lemma 8.2 where ε will be fixed later on. For shortness we will denote $\sqrt{Q(x)}\,j_i(x)$ by $J_i(x)$.

We begin our calculations with multiplying the first term of (8.1) from both sides with $h(H)(H-\lambda)$. The expression that we get is the sum of the following terms:

$$h(H)(H-\lambda)\frac{1}{\sqrt{\langle x\rangle}}f(\gamma)(H-\lambda)^{-1}i[H,A](H-\lambda)^{-1}f(\gamma)$$

$$\cdot \frac{1}{\sqrt{\langle x \rangle}} J_i^2(x) g^2(D_a)(H-\lambda)h(H), \tag{8.5}$$

where $i = 1, \ldots, N$.

Let us fix our attention on a certain i such that $y_i \in Z_b$. Now we can apply Proposition 5.9 a) to the above expression with $J_i(x)$ playing the role of $Q_0(x)$. Thus (8.5) equals

$$\frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i(x) \tilde{f}(\gamma) h(H_b) i[H_b, A] h(H_b) \tilde{f}(\gamma) J_i(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}}$$

plus $0(\langle x \rangle^{-1-\mu})$. By Lemma 8.1 this is equal to

$$\frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i(x) \tilde{f}(y_i \cdot D) h(H_b) i[H_b, A] h(H_b) \tilde{f}(y_i \cdot D) J_i(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}}$$

$$+ \frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i(x) B_i J_i(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}} \tag{8.6}$$

plus $0(\langle x \rangle^{-2})$, where $\| B_i \| \leqq c_1 \varepsilon$ and c_1 is independent of ε. The second term of (8.6) is greater than or equal to

$$-c_1 \varepsilon \frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i^2(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}}.$$

In the first term of (8.6) we can commute $g(D_a)$ to the middle and obtain

$$\frac{1}{\sqrt{\langle x \rangle}} f(\gamma) J_i(x) g(D_a) \tilde{f}(y_i \cdot D) h(H_b) i[H_b, A] h(H_b) \tilde{f}(y_i \cdot D) g(D_a) J_i(x) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}} \tag{8.7}$$

plus $0(\langle x \rangle^{-2})$. Now there comes the most crucial step of our estimate. Due to the inequality (8.4), the definition of Θ and the location of the support of $g(D_a) \tilde{f}(y_i \cdot D)$ the expression (8.7) is greater than or equal to

$$\left(\Theta - \frac{\delta}{2} \right) \frac{1}{\sqrt{\langle x \rangle}} f(\gamma) J_i(x) g(D_a) \tilde{f}(y_i \cdot D) h^2(H_b) \tilde{f}(y_i \cdot D) g(D_a) J_i(x) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}}. \tag{8.8}$$

Arguments similar to the ones applied above show that (8.8) is greater than or equal to

$$\left(\Theta - \frac{\delta}{2} \right) \frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i(x) \tilde{f}(\gamma) h^2(H_b) \tilde{f}(\gamma) J_i(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}}$$

$$- c_2 \varepsilon \frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i^2(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}} \tag{8.9}$$

plus $0(\langle x \rangle^{-2})$ where c_2 does not depend on ε. The first term of (8.9) equals

$$\left(\Theta - \frac{\delta}{2} \right) h(H) \frac{1}{\sqrt{\langle x \rangle}} f(\gamma) g(D_a) J_i^2(x) g(D_a) f(\gamma) \frac{1}{\sqrt{\langle x \rangle}} h(H)$$

plus $0(\langle x\rangle^{-1-\mu})$. Eventually, we multiply our estimations from both sides by E_Δ, use the fact that $E_\Delta h(H)=E_\Delta$, add up all the terms with $i=1,\ldots,N$ and do some commuting. Having done this we can conclude that there exist numbers c_1, c_2 and c_3 such that

$$\tfrac{1}{2}E_\Delta(H-\lambda)\frac{1}{\sqrt{\langle x\rangle}}f(\gamma)(H-\lambda)^{-1}i[H,A](H-\lambda)^{-1}f(\gamma)$$

$$\cdot\frac{1}{\sqrt{\langle x\rangle}}Q(x)g^2(D_a)(H-\lambda)E_\Delta+hc$$

$$\geqq\left(\Theta-\frac{\delta}{2}-c_1\varepsilon-c_2\varepsilon\right)E_\Delta\Psi^*\Psi E_\Delta-c_3E_\Delta\langle x\rangle^{-1-\mu}E_\Delta.$$

Since ε was an arbitrary positive number and c_1 and c_2 did not depend on ε we may assume that $c_1\varepsilon+c_2\varepsilon<\delta/2$. This ends the proof of Proposition 6.4.

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