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The Coexistence Problem for the Discrete Mathieu Operator

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Abstract. We solve the coexistence problem for the periodic discrete Mathieu operator in all parametric cases. The main tool in the proof will be Bezout's theorem for projective plane curves. As an additional result we obtain the gap opening and gap growth powers for this operator.

0. Introduction and Main Results

Define for $b:\mathbf{Z}\to\mathbf{R}$ the linear recursion operator $H_b:\mathbf{C}^{\mathbf{Z}}\to\mathbf{C}^{\mathbf{Z}}$ by

$$(H_b g)(n) = g(n+1) + b_n g(n) + g(n-1).$$

There exists an actual interest in the cases where the potential b is almost periodic, in particular in the discrete Mathieu operator (with real parameters A, α, v) which one obtains by taking $b = Ab^{(\alpha,v)}$ with $b_n^{(\alpha,v)} = 2\cos(2\pi n\alpha - v)$. Thus explicitly

$$(H_{Ab^{(\alpha,v)}} g)(n) = g(n+1) + 2A\cos(2\pi n\alpha - v)g(n) + g(n-1).$$

This operator, a discrete version of Mathieu's periodic differential operator, is interesting from the mathematical as well from the physical point of view [1-7, 9, 12, 16, 18, 19, 21]. Here we shall occupy ourselves only with the periodic potential case, i.e., where α is a rational number p/q (always written in its canonical form). The following theorem completely solves the coexistence problem for the periodic discrete Mathieu operator.

Theorem 1. *For all*[1] $A\in\mathbf{R}^*$, $p/q\in\mathbf{Q}$, $v\in\mathbf{R}$ *all the* $q-1$ *gaps of the periodic discrete Mathieu operator* $H_{Ab^{(p/q,v)}}$ *are non-degenerate, with the exception of the middle gaps (which are* $\{0\}$*) in the cases* $v\in(2\pi/q)\mathbf{Z}$, $q\equiv 0 \pmod 4$ *and the cases* $v\in(\pi/q)+(2\pi/q)\mathbf{Z}$, $q\equiv 2 \pmod 4$. ◇

The coexistence problem for Mathieu's differential operator has been solved by Ince [14]. Partial results for Theorem 1 had already been obtained in [3, 6]. The

[1] For a subset X of C we denote $X\backslash\{0\}$ by X^*

punch to complete these results will be Bezout's theorem for projective plane curves. The needed intersection numbers are provided by the gap opening and gap growth powers of Theorem 2.

In order to define these powers in general we now review quickly some spectral theory for periodic recursion operators. For this theory we recommend [13, 15, 16, 20].[2]

Let $b:\mathbf{Z}\to\mathbf{R}$ be periodic with positive period q.[3] Consider the operator family $(H_{Ab})_{A\in\mathbf{R}}$. For all $A\in\mathbf{R}$ the spectrum $\sigma(H_{Ab|l^2(\mathbf{Z})})$ consists of q bands $E_1(A)$, $E_2(A),\ldots,E_q(A)$ which we number from the left to the right on the real axis. The $q-1$ closed intervals between[4] them $G_1(A)$, $G_2(A),\ldots,G_{q-1}(A)$ are the gaps. The two boundary points of each gap G_N are real-analytic functions of the real parameter A. Denote the 2-tuple of these functions by λ_N^+, λ_N^-. For $A=0$ all gaps are degenerated, thus $\lambda_N^+(0)=\lambda_N^-(0)$. Denote the power series expansion about $A=0$ of the signed gap length $\lambda_N^- - \lambda_N^+$ by $\sum_{j\geqq 0}\alpha_{j,N}A^j$. One has a Laurent series expansion for $A\to+\infty$ of $\lambda_N^- - \lambda_N^+$ of the form $\sum_{j\geqq -1}\beta_{j,N}(1/A)^j$. We call the leading powers of these expansions respectively gap opening powers $O(b)_N$ and gap growth powers $I(b)_N$. Thus

$$O(b)_N=\inf\{j\geqq 1\,|\,\alpha_{j,N}\neq 0\},$$
$$I(b)_N=\inf\{j\geqq -1\,|\,\beta_{j,N}\neq 0\}.$$

Notice that these powers may be $+\infty$, which means that the corresponding gap is degenerate for all $A\in\mathbf{R}$. The way to calculate these powers is Rayleigh-Schrödinger perturbation theory which can become very complex in general. However it turns out that for the periodic discrete Mathieu operator tricks exist that make the determination of these powers easier.

Defining for $p/q\in\mathbf{Q}$ the permutation $\tau_{p/q}$ of $\{0,1,\ldots,q-1\}$ by

$$\tau_{p/q}(n)=pn(\mathrm{mod}\,q)$$

we can state our Theorem 2.[5]

Theorem 2

A) *For the gap opening powers of the family* $(H_{Ab^{(p/q,\nu)}})_{A\in\mathbf{R}}$ *one has*

$$O(b^{(p/q,\nu)})_N=\min(\tau_{p/q}^{-1}(N),q-\tau_{p/q}^{-1}(N)) \quad \textit{if } N\neq\frac{q}{2}.$$

$$O(b^{(p/q,\nu)})_{q/2}=\frac{q}{2} \quad \textit{if } \nu\in\mathbf{R}\setminus\frac{\pi}{q}\mathbf{Z}.$$

$$=+\infty \quad \textit{if } q\equiv 0\ (\mathrm{mod}\,4),\ \nu\in\frac{2\pi}{q}\mathbf{Z}.$$

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[2] It is possible to translate almost everything in the classical book [14] into the discrete case.

[3] In the following q always denotes a positive integer.

[4] Also numbered from the left to the right.

[5] See also Remark 1

$$= \frac{q}{2} \qquad \textit{if } q \equiv 2 \ (\mathrm{mod}\, 4),\ \nu \in \frac{2\pi}{q}\mathbf{Z}.$$

$$= +\infty \qquad \textit{if } q \equiv 2 \ (\mathrm{mod}\, 4),\ \nu \in \frac{\pi}{q} + \frac{2\pi}{q}\mathbf{Z}.$$

$$= \frac{q}{2} \qquad \textit{if } q \equiv 0 \ (\mathrm{mod}\, 4),\ \nu \in \frac{\pi}{q} + \frac{2\pi}{q}\mathbf{Z}.$$

B) *For the gap growth powers of the family* $(H_{Ab^{(p/q,\nu)}})_{A \in \mathbf{R}}$ *one has*

$$I(b^{(p/q,\nu)})_N = O(b^{(p/q,\nu)})_N - 1$$

if $\nu \in (2\pi/q)\mathbf{Z}$, $N + q$ *even or* $\nu \in (\pi/q) + (2\pi/q)\mathbf{Z}$, $N + q$ *odd.* ◇

1. Some Basic Objects

In addition to the previous paragraph we shall state here some notations, conventions and properties of basic objects.

1. We define $T_t (t \in \mathbf{Z})$, $S: \mathbf{C}^{\mathbf{Z}} \to \mathbf{C}^{\mathbf{Z}}$ by $(T_t g)(n) = g(n + t)$, $(Sg)(n) = g(-n)$ and define

$$\begin{aligned} \mathscr{F}_{t,z} &= \ker (T_t - z\,\mathrm{Id}) && (t \in \mathbf{Z}^*, z \in \mathbf{C}^*), \\ E_{s,w} &= \ker (S T_{2s} - w\,\mathrm{Id}) && (s \in \tfrac{1}{2}\mathbf{Z}, w \in \{-1, 1\}), \\ \mathscr{S}_\varepsilon(H_b) &= \ker (H_b - \varepsilon\,\mathrm{Id}) && (\varepsilon \in \mathbf{C}). \end{aligned}$$

2.
$$\begin{aligned} \dim (\mathscr{F}_{q,z}) &= q && (z \in \mathbf{C}^*). \\ \dim (\mathscr{F}_{q,1} \cap E_{s,1}) &= \left[\frac{q}{2} + 1\right] && \text{if } s \in \mathbf{Z}, \\ &= \left[\frac{q+1}{2}\right] && \text{if } s \in \tfrac{1}{2} + \mathbf{Z}. \\ \dim (\mathscr{F}_{q,1} \cap E_{s,-1}) &= \left[\frac{q-1}{2}\right] && \text{if } s \in \mathbf{Z}, \\ &= \left[\frac{q}{2}\right] && \text{if } s \in \tfrac{1}{2} + \mathbf{Z}. \\ \dim (\mathscr{F}_{q,-1} \cap E_{s,1}) &= \left[\frac{q+1}{2}\right] && \text{if } s \in \mathbf{Z}, \\ &= \left[\frac{q}{2}\right] && \text{if } s \in \tfrac{1}{2} + \mathbf{Z}. \\ \dim (\mathscr{F}_{q,-1} \cap E_{s,-1}) &= \left[\frac{q}{2}\right] && \text{if } s \in \mathbf{Z}, \\ &= \left[\frac{q+1}{2}\right] && \text{if } s \in \tfrac{1}{2} + \mathbf{Z}. \end{aligned}$$

3. We denote the eigenvalues counted with multiplicities of a linear transformation H in a finite dimensional vector space by $\bar{\sigma}_p(H)$.
4. Let $b:\mathbf{Z}\to\mathbf{R}$ periodic with period q. Define the discriminant $\Delta_q:\mathbf{C}\to\mathbf{C}$ by $\Delta_q(\varepsilon)=\mathrm{Tr}\,(T_{q|S_\varepsilon(H_b)})$. Δ_q is a monic polynomial of degree q with real coefficients, thus $\Delta_q\in\mathbf{R}[\varepsilon]$. The main properties of Δ_q are described by the so-called oscillation theorem. We explicitly call attention to the fact that Δ_q has $q-1$ real critical points $\kappa_1<\kappa_2<\cdots<\kappa_{q-1}$; these satisfy $\kappa_N\in G_N$. Gap G_N is called a periodic gap if $N+q$ even and an anti-periodic gap if $N+q$ odd. We call $\bar{\sigma}_p(H_{b|\mathscr{F}_{q,1}})$ the q-periodic spectrum and $\bar{\sigma}_p(H_{b|\mathscr{F}_{q,-1}})$ the q-anti-periodic spectrum. They determine the bands and gaps completely.
5. Sometimes we denote explicitly the dependence of the discriminant on b; for example $\Delta_q(A,p/q,v,\varepsilon)$ for $H_{Ab^{(p/q,v)}}$.

2. Some Useful Spectral Topics

Here we shall present four propositions that we need to get around the proofs of our two theorems. We start with the Aubry-Wannier duality [1, 21] which we would like to present in the following form.

Proposition 1. (Aubrey-Wannier Duality). *Fix* $p/q\in\mathbf{Q}$, k, $v\in\mathbf{R}$. *Then* $\mathscr{A}:\mathscr{F}_{q,e^{-ivq}}\to\mathscr{F}_{q,e^{ikq}}$ *defined by*

$$(\mathscr{A}g)(n)=\frac{e^{ikn}}{\sqrt{q}}\sum_{j=1}^{q}g(j)e^{-i(2\pi(p/q)n-v)j}$$

is a linear isomorphism such that the diagram

$$\begin{array}{ccc}\mathscr{F}_{q,e^{-ivq}} & \xrightarrow{\quad AH_{(1/A)b^{(p/q,k)}}\quad} & \mathscr{F}_{q,e^{-ivq}}\\ \mathscr{A}\downarrow & & \downarrow\mathscr{A}\\ \mathscr{F}_{q,e^{ikq}} & \xrightarrow[\quad H_{Ab^{(p/q,v)}}\quad]{} & \mathscr{F}_{q,e^{ikq}}\end{array}$$

commutes for all $A\in\mathbf{R}^*$. $\diamond$

The following relation has been proved [4, 5] in the context of a generalisation of the discrete Mathieu operator.

Proposition 2. (Butler-Brown-Chambers Phase Relation). *For all* $p/q\in\mathbf{Q}$, A, $v\in\mathbf{R}$, $\varepsilon\in\mathbf{C}$ *one has the relation*

$$\Delta_q(A,p/q,v,\varepsilon)=\Delta_q(A,p/q,\pi/2q,\varepsilon)-2A^q\cos(qv).\quad\diamond$$

Our third proposition gives an useful alternative way to look at gap opening powers.

Proposition 3. *Let* $b:\mathbf{Z}\to\mathbf{R}$ *periodic with period* q. *Consider the family* $(H_{Ab})_{A\in\mathbf{R}}$. *For* $A\in\mathbf{R}$ *denote by* $\kappa_1(A)<\cdots<\kappa_{q-1}(A)$ *the* $q-1$ *critical points of the polynomial* $\Delta_q(A,\cdot)\in\mathbf{R}[\varepsilon]$. *Fix* $N\in\{1,2,\ldots,q-1\}$. *Then*

1. $\kappa_N:\mathbf{R}\to\mathbf{R}$ *is real-analytic.*

2. $M:\mathbf{R}\to\mathbf{R}$ *defined by* $M(A)=(-1)^{N+q}\Delta_q(A,\kappa_N(A))$-2 *is real-analytic,* $M(0)=0$, $M\geqq 0$.
3. *The leading power of the power series expansion about* $A=0$ *of* M *equals* $2O(b)_N$. ◇

Proof.
1. The critical points of $\Delta_q(A,.)$ are all different. Apply an analytic version of the implicit function theorem.
2. Because $\Delta_q\in\mathbf{R}[A,\varepsilon]$ it follows that M is also real-analytic. The oscillation theorem implies that $M\geqq 0$. An analysis for $A=0$ gives $M(0)=0$ (see also for example [10]).
3. Notice that $(\partial\Delta_q/\partial\varepsilon)(A,\kappa_N(A))=0, (\partial^2\Delta_q/\partial\varepsilon^2)(A,\kappa_N(A))\neq 0$. Applying the Morse lemma with parameters [8, page 19] to the function $(-1)^{N+q}\Delta_q(A,\varepsilon)-2$ gives the desired result. □

The following proposition is a generalisation of an idea used in [3].

Proposition 4. *Let* b, $b':\mathbf{Z}\to\mathbf{R}$ *be periodic functions with period* q. *Consider the recursion operators* H_b, $H_{b'}$. *Suppose there exists* $c\in\mathbf{R}$ *such that*

$$\Delta_q(b',\cdot)=\Delta_q(b,\cdot)+c.$$

Then for $c>0$ *all periodic gaps of* $H_{b'}$ *are non-degenerate and for* $c<0$ *all anti-periodic gaps of* $H_{b'}$ *are non-degenerate.* ◇

Proof. Let κ_N be the critical point that lies in gap G_N of H_b. Then if $c>0$ ($c<0$) we have for any periodic (anti-periodic) gap G_N the inequality $\Delta_q(b',\kappa_N)=\Delta_q(b,\kappa_N)+c\geqq 2+c>2(\leqq -2+c<-2)$. $\Delta_q(b',\cdot)$ and $\Delta_q(b,\cdot)$ have the same critical points, so gap N of $H_{b'}$ is non-degenerate. □

3. Proof of Theorem 1; Extreme Cases

Because $Ab^{(p/q,\nu)}=-Ab^{(p/q,\nu+\pi)}$ the validity of Theorem 1 for $A>0$ implies its validity for $A<0$, we may suppose $A>0$. Because bands and gaps of a given H_b remain unchanged if one replaces the potential b by a translated $T_t(b)$ and because for our $b^{(p/q,\nu)}$ we have $\{T_t(b^{(p/q,\nu)})|t\in\mathbf{Z}\}=\{b^{(p/q,\nu+m(2\pi/q))}|m\in\mathbf{Z}\}$, we may also suppose that $\nu\in[0,2\pi/q)$. In order to clarify our proof of Theorem 1 it is convenient to introduce the following terminology for the gaps $G_N(1\leqq N\leqq q-1)$ of $H_{Ab^{(p/q,\nu)}}(p/q\in\mathbf{Q}, A>0, \nu\in[0,2\pi/q))$: We call extremal gaps the gaps G_N in the following parametric cases

I $\nu=0$, $q\equiv 0 \pmod 4$, $N=\dfrac{q}{2}$.

II $\nu=\dfrac{\pi}{q}$, $q\equiv 2 \pmod 4$, $N=\dfrac{q}{2}$.

III $\nu\in\left(0,\dfrac{2\pi}{q}\right)\Big\backslash\left\{\dfrac{\pi}{q}\right\}$.

IV $\nu=0$, $N+q$ odd.

V $v = \frac{\pi}{q}$, $N + q$ even.

We call intermediate gaps the gaps in all other allowed parametric cases[6]. The validity of Theorem 1 for the extreme cases I, II, III was proved by [3, 6]. Continuing the analysis in [3] one can even prove IV, V: they are straightforward consequences of Proposition 4 in combination with Proposition 2. □

4. Proof of Theorem 2

Starting from the validity of Lemma 1 below and using the spectral topics presented in 2, we shall be able to prove Theorem 2 in a fairly quick way.[7] The proof of this lemma can be found essentially in the proof of Proposition 4.2 and the remark following it in [3]; one has to use our numbering of gaps to obtain our formulation[8] of Lemma 1.

Lemma 1. *In the cases $v = 0$, $N + q$ even, $N \neq q/2$ and in the cases $v = \pi/q, q$ even, $N + q$ odd, $N \neq q/2$ one has for the gap growth powers of the family $(H_{Ab^{(p/q,v)}})_{A \in \mathbf{R}}$*

$$I(b^{(p/q,v)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) - 1. \quad \diamond$$

Here is the proof of Theorem 2:
B) The Aubry-Wannier duality implies for $A \in \mathbf{R}^*$ the spectral identities

$$\bar{\sigma}_p(H_{Ab^{(p/q,0)}|\mathscr{F}_{q,1}}) = A\bar{\sigma}_p(H_{(1/A)b^{(p/q,0)}|\mathscr{F}_{q,1}}),$$
$$\bar{\sigma}_p(H_{Ab^{(p/q,\pi/q)}|\mathscr{F}_{q,-1}}) = A\bar{\sigma}_p(H_{(1/A)b^{(p/q,\pi/q)}|\mathscr{F}_{q,-1}}).$$

These in turn imply the desired results.
A) In view of Lemma 1 and **B**) we already have the results

$$(*):\quad O(b^{(p/q,0)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad N \neq \frac{q}{2},\ N + q \text{ even}.$$

$$(**):\quad O(b^{(p/q,\pi/q)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad N \neq \frac{q}{2},\ q \text{ even},\ N + q \textit{ odd}.$$

We shall now prove the remaining cases in the steps below.

Step 1.
We have seen already that the middle gap $q/2$ is degenerate for all $A \in \mathbf{R}$ in the cases $v = 0$, $q \equiv 0 \pmod 4$ and in the cases $v = \pi/q$, $q \equiv 2 \pmod 4$. Thus

$$O(b^{(p/q,0)})_{q/2} = +\infty \quad \text{if} \quad q \equiv 0 \pmod 4,$$
$$O(b^{(p/q,\pi/q)})_{q/2} = +\infty \quad \text{if} \quad q \equiv 2 \pmod 4.$$

Moreover because these middle gaps are $\{0\}$ (see [3, 16]) one has for all $A \in \mathbf{R}$ $\Delta_q(A, p/q, 0, 0) = 2$ if $q \equiv 0 \pmod 4$ and $\Delta_q(A, p/q, \pi/q, 0) = -2$ if $q \equiv 2 \pmod 4$.

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6 See also Remark 2.
7 I would like to thank J. Bellissard for pointing out to me the usefulness of Lemma 1.
8 We shall use a minimal version here

Proposition 2 gives $\Delta_q(A, p/q, \pi/q, 0) = \Delta_q(A, p/q, 0, 0) + 4A^q$ ($A \in \mathbf{R}$). Using these three identities Proposition 3 leads to

$$O(b^{(p/q,\pi/q)})_{q/2} = \frac{q}{2} \quad \text{if} \quad q \equiv 0 \pmod 4,$$

$$O(b^{(p/q,0)})_{q/2} = \frac{q}{2} \quad \text{if} \quad q \equiv 2 \pmod 4.$$

Step 2.
Let q be even. We have $\Delta_q(A, p/q, v, \varepsilon) = \Delta_q(A, p/q, 0, \varepsilon) - 2A^q(\cos qv - 1)$. Because of Step 1 we have $(-1)^{q/2+q}\Delta_q(A, p/q, v, 0) - 2 = -2A^q(\cos qv - 1)$. Proposition 3 implies now

$$O(b^{(p/q,v)})_{q/2} = \frac{q}{2} \quad \text{if} \quad v \in \left(0, \frac{2\pi}{q}\right) \Big\backslash \left\{\frac{\pi}{q}\right\}, \quad q \equiv 0 \pmod 4.$$

In the same way we prove

$$O(b^{(p/q,v)})_{q/2} = \frac{q}{2} \quad \text{if} \quad v \in \left(0, \frac{2\pi}{q}\right) \Big\backslash \left\{\frac{\pi}{q}\right\}, \quad q \equiv 2 \pmod 4.$$

Step 3.
Because $KH_{Ab} = -H_{-Ab}K$, where $(Kg)(n) = (-1)^n g(n)$, one has $O(b)_N = O(b)_{q-N}$ ($1 \leqq N \leqq q-1$). Now let q be odd, $N+q$ odd. Then $q-N$ is odd and $(*)$ gives $O(b^{(p/q,0)})_N = O(b^{(p/q,0)})_{q-N} = \min(\tau_{p/q}^{-1}(q-N),\ q - \tau_{p/q}^{-1}(q-N)) = \min(q - \tau_{p/q}^{-1}(N), q - (q - \tau_{p/q}^{-1}(N)))$. Thus

$$O(b^{(p/q,0)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad q \text{ odd, } N+q \text{ odd.}$$

Step 4.
Let q be odd, $N+q$ odd. We proved in Step 3 that $O(b^{(p/q,0)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$. Denote for $A \in \mathbf{R}$ with $\kappa_N(A)$ the N^{th} critical point of the polynomial $\Delta_q(A, p/q, 0, .) \in \mathbf{R}[\varepsilon]$. Proposition 3 gives that the leading power of the power series expansion about $A = 0$ of $(-1)^{N+q}\Delta_q(A, p/q, 0, \kappa_N(A)) - 2$ equals $2O(b^{(p/q,0)})_N$. Because of Proposition 2 we have the identity $(-1)^{N+q}\Delta_q(A, p/q, \pi/q, \kappa_N(A)) - 2 = (-1)^{N+q}\Delta_q(A, p/q, 0, \kappa_N(A)) - 2 + 4(-1)^{N+q}A^q$. Thus because $2O(b^{(p/q,0)})_N < q$, the leading power of $(-1)^{N+q}\Delta(A, p/q, \pi/q, \kappa_N(A)) - 2$ equals also $2O(b^{(p/q,0)})_N$. Because $\kappa_N(A)$ is also the N^{th} critical point of the polynomials $\Delta_q(A, p/q, \pi/q, .) \in \mathbf{R}[\varepsilon]$, we obtain using again Proposition 3 that $2O(b^{(p/q,\pi/q)})_N = 2O(b^{(p/q,0)})_N$. Thus

$$O(b^{(p/q,\pi/q)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad q \text{ odd, } N+q \text{ odd.}$$

Step 5.
Using as in Step 4. Propositions 2, 3 we deduce from $(*)$

$$O(b^{(p/q,\pi/q)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad N \neq \frac{q}{2}, N+q \text{ even}$$

and from $(**)$

$$O(b^{(p/q,0)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad N \neq \frac{q}{2}, q \text{ even, } N+q \text{ odd.}$$

Step 6.
We have already $O(b^{(p/q,0)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$ if $N \neq q/2$. Using again Propositions 2, 3 we obtain

$$O(b^{(p/q,v)})_N = \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) \quad \text{if} \quad v \in \left(0, \frac{2\pi}{q}\right) \setminus \left\{\frac{\pi}{q}\right\}, \quad N \neq \frac{q}{2}. \qquad \square$$

5. Proof of Theorem 1; Intermediate Cases

Here we shall prove, using Bezout's theorem, Theorem 1 for the gaps G_N in the parametric cases

VI. $v = 0, N + q$ even, $N \neq \frac{q}{2}$, $A \in \mathbf{R}^*$.

VII. $v = \frac{\pi}{q}, N + q$ odd, $N \neq \frac{q}{2}$, $A \in \mathbf{R}^*$.

These cases include all intermediate gap cases and so a proof of VI, VII will prove Theorem 1 completely. Bezout's theorem is the following statement [11]: Let F, G be projective plane curves over $\mathbf{C}$ without common components, then (degree F)(degree G) $= \sum$(intersection-numbers of F and G).

Proof of VI. Consider the q-tuple of real-analytic functions that describe the q-periodic spectrum $\bar{\sigma}_p(H_{Ab^{(p/q,0)}|\mathscr{F}_{q,1}})$ as function of the real parameter A. Because $S(Ab^{(p/q,0)}) = Ab^{(p/q,0)}$ one has $[H_{Ab^{(p/q,0)}}, S] = 0$, which implies that these real-analytic functions can be split in a $[q/2 + 1]$-tuple that describes the q-periodic 0-symmetric spectrum

$$\bar{\sigma}_p(H_{Ab^{(p/q,0)}|\mathscr{F}_{q,1} \cap E_{0,1}})$$

and in a $[(q-1)/2]$-tuple that describes the q-periodic 0-anti-symmetric spectrum

$$\bar{\sigma}_p(H_{Ab^{(p/q,0)}|\mathscr{F}_{q,1} \cap E_{0,-1}})$$

as a function of the real parameter A. Define the polynomials[9] $r^{\pm} \in \mathbf{R}[A, \varepsilon]$ by

$$r^{\pm} = \det(\varepsilon Id - H_{Ab^{(p/q,0)}})_{|\mathscr{F}_{q,1} \cap E_{0,\pm 1}}.$$

We apply Bezout's theorem to the curves defined by the roots of $r^+ = 0$, $r^- = 0$, after homogenising $r^{\pm}$. One knows that the two boundary points of a gap consist of a point of the symmetric as well as of a point of the anti-symmetric spectrum. This makes that we can interpret the gap opening powers and the gap growth powers $+1$ of the periodic gaps of the family $(H_{Ab^{(p/q,0)}})_{A \in \mathbf{R}}$ as intersection numbers (there may be others). However, this is only correct if there are no common components, that is, if there are no permanently degenerate (i.e. for all $A \in \mathbf{R}$) gaps. Theorem 2 shows that such a common component occurs only for the middle gap

[9] [17] shows that indeed $r^{\pm} \in \mathbf{R}[A, \varepsilon]$ and that degree $(r^+) = [(q/2) + 1]$, degree $(r^-) = [(q-1)/2]$

in case $q \equiv 0 \pmod 4$. Because this gap is $\{0\}$ we can handle this case by factoring out the component $\varepsilon = 0$.

We shall now show that our gap opening and gap growth powers provide us with all intersection numbers. The conclusion is then that there are no other intersections. In particular we have that all periodic gaps are non-degenerate for $A \in \mathbf{R}^*$. We shall distinguish three cases. In the following computations one has to use Theorem 2 and some simple properties of the permutation $\tau_{p/q}^{-1}$.

Case q Odd.

$$\sum_{N=1,N\,\mathrm{odd}}^{q-1} O(b^{(p/q,0)})_N + \sum_{N=1,N\,\mathrm{odd}}^{q-1} (I(b^{(p/q,0)})_N + 1)$$

$$= 2 \sum_{N=1,N\,\mathrm{odd}}^{q-1} O(b^{(p/q,0)})_N = 2\left(\sum_{N=1,N\,\mathrm{odd}}^{(q-1)/2} + \sum_{N=(q+1)/2,N\,\mathrm{odd}}^{q-1}\right)\min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$$

$$= 2 \sum_{N=1,N\,\mathrm{odd}}^{(q-1)/2} \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$$

$$+ 2 \sum_{N=1,N\,\mathrm{even}}^{(q-1)/2} \min(\tau_{p/q}^{-1}(q-N), q - \tau_{p/q}^{-1}(q-N))$$

$$= 2\left(\sum_{N=1,N\,\mathrm{odd}}^{(q-1)/2} + \sum_{N=1,N\,\mathrm{even}}^{(q-1)/2}\right)\min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$$

$$= 2 \sum_{N=1}^{(q-1)/2} \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) = 2 \sum_{N=1}^{(q-1)/2} N = \left(\frac{q+1}{2}\right)\left(\frac{q-1}{2}\right)$$

$$= (\text{degree } r^+)(\text{degree } r^-).$$

Case q Even, $q \equiv 2 \pmod 4$.

$$\sum_{N=1,N\,\mathrm{even}}^{q-1} O(b^{(p/q,0)})_N + \sum_{N=1,N\,\mathrm{even}}^{q-1} (I(b^{(p/q,0)})_N + 1)$$

$$= 2 \sum_{N=1,N\,\mathrm{even}}^{q-1} O(b^{(p/q,0)})_N$$

$$= 2\left(\sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} + \sum_{N=q/2+1,N\,\mathrm{even}}^{q-1}\right)\min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$$

$$= 2 \sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$$

$$+ 2 \sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} \min(\tau_{p/q}^{-1}(q-N), q - \tau_{p/q}^{-1}(q-N))$$

$$= 4 \sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} \min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N)) = 4 \sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} N = \left(\frac{q}{2}+1\right)\left(\frac{q}{2}-1\right)$$

$$= (\text{degree } r^+)(\text{degree } r^-).$$

Case q Even, $q \equiv 0 \pmod 4$.

$$\sum_{N=1,N\,\mathrm{even},N\neq q/2}^{q-1} O(b^{(p/q,0)})_N + \sum_{N=1,N\,\mathrm{even},N\neq q/2}^{q-1} (I(b^{(p/q,0)})_N + 1)$$

$$= 2 \sum_{N=1,N\,\mathrm{even},N\neq q/2}^{q-1} O(b^{(p/q,0)})_N$$

$$= 2\left(\sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} + \sum_{N=q/2+1,N\,\mathrm{even}}^{q-1}\right)\min(\tau_{p/q}^{-1}(N), q - \tau_{p/q}^{-1}(N))$$

$$= 4 \sum_{N=1,N\,\mathrm{even}}^{(q/2)-1} N = \frac{q}{2}\left(\frac{q}{2} - 2\right) = (\mathrm{degree}\, r^+ - 1)(\mathrm{degree}\, r^- - 1). \quad \square$$

Proof of VII. When we proceed in a manner analogous to VI, this gives the desired result in this case as well. Notice $ST_{\tau_{p/q}^{-1}(1)}(Ab^{(p/q,\pi/q)}) = Ab^{(p/q,\pi/q)}$; Now

$$r^{\pm} = \det(\varepsilon \mathrm{Id} - H_{Ab^{(p/q,\pi/q)}})_{|\mathscr{F}_{q,-1} \cap E_{s,\pm 1}}.$$

Here $s = \frac{1}{2}\tau_{p/q}^{-1}(1)$. One has to factor out the common component $\varepsilon = 0$ in the case q even $q \equiv 2 \pmod 4$. $\square$ $\square$

6. Remarks

1. Theorem 2 gives the gap opening powers for all parametric cases. It is not hard to show that the missing gap growth powers in Theorem 2 are all -1 [16].
2. The cases which we called extreme are just those for which the associated gap growth powers are -1 or $+\infty$.
3. Our results for rational α may imply some results for irrational α[3].
4. Another route without using Lemma 1 that proves Theorem 1 is given in [16].
5. We would like to conjecture that for any q-periodic b each gap opening power which is not $+\infty$ is less or equal $q/2$.
6. We understood that Elliott and collaborators [9] too has recently found a proof of Theorem 1 using a C^*-algebra approach.

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