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Polynomial Integrals of Evolution Equations

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Abstract. In this paper a complete description is achieved, for the first time, of polynomial integrals (conservation laws) for a broad range of evolution equations: the Sivashinsky-Kuramoto equation, the Burgers equation and others.

1. Introduction

We shall consider an evolution differential equation of the following type:

$$v_t = \mathscr{P}(v, v^{(1)}, v^{(2)}, \dots); \qquad v^{(i)} = \frac{\partial^i v}{\partial x^i}, \qquad v = v(t,x); \qquad t, x \in R^1. \tag{1.1}$$

We put

$$I(R(v)) = \int_a^b R(v, v^{(1)}, v^{(2)}, \dots)dx (= I(R) = I), \tag{1.2}$$

and consider the complete derivative d/dt of the integral $I(R(v))$ with respect to the time t, where v is a solution of Eq. (1.1):

$$\frac{dI}{dt}(R(v)) = \int_a^b \left(\sum_{k \geqq 0} \frac{\partial R(v)}{\partial v^{(k)}} \cdot \frac{\partial v^{(k)}}{\partial t} \right) dx = \int_a^b \left(\sum_{k \geqq 0} \frac{\partial R(v)}{\partial v^{(k)}} \cdot \frac{\partial^k \mathscr{P}(v)}{\partial x^k} \right) dx. \tag{1.3}$$

In this paper we shall study $dI(R(v))/dt$. Since this integral does not depend explicitly on t, we may treat $v(t,x)$ for any fixed t as an element of some set E of functions of x, $x \in [a,b]$. An element of E will be denoted by $u = u(x)$, $u^i = (d/dx)^i u$. The integral (1.3) will henceforth be interpreted as an integral in which $v = v(t,x)$ is replaced by $u \in E$.

We shall assume throughout that the elements of E are infinitely differentiable functions on $[a,b]$, satisfying the periodicity condition $u^{(k)}(a) = u^{(k)}(b)$, $k = 0, 1, 2, \dots$.

It should be noted that

$$\frac{dI}{dt}(R(u)) = \int_a^b \left(\sum_{k \geqq 0} \frac{\partial R(u)}{\partial u^{(k)}} \cdot \frac{\partial^k \mathscr{P}(u)}{\partial x^k} \right) dx \tag{1.4}$$

can also be studied with reference to Eq. (1.1), when it is not known whether the latter has solutions in the space E.

Definition 1. If $dI(R(u))/dt=0$ for all $u\in E$, where $I(R(u))$ is defined by (1.4), then I is called an integral of the motion described by Eq. (1.1) (relative to E); briefly, we write IE(1.1).

Let A_N denote the set of all homogeneous polynomials of degree N in an arbitrary finite sequence of abstract symbols $\{u_0, u_1, \dots\}$ over the real field; A_0 will denote the set of constants. We put $A=\bigcup_{N=1}^{\infty} A_N$, $\hat{A}=A_0\cup A$.

The algebra A is spanned by the set of all products $u_0^{\alpha_0}u_1^{\alpha_1}u_2^{\alpha_2}\dots$, where α_i are nonnegative integers, only finitely many of which do not vanish. The standard partial differentiation operations $\partial/\partial u_i$ are defined in A; we shall also define an operation

$$\frac{d}{dx}\overset{\text{def}}{=}\sum_{i\geqq 0} u_{i+1}\frac{\partial}{\partial u_i}, \tag{1.5}$$

relative to which A becomes a differential algebra:

$$\frac{du_j}{dx}=u_{j+1}; \tag{1.6}$$

$$\begin{gathered}\frac{d}{dx}(\lambda_1 a_1+\lambda_2 a_2)=\lambda_1\frac{da_1}{dx}+\lambda_2\frac{da_2}{dx};\\ \frac{d(a_1a_2)}{dx}=a_1\frac{da_2}{dx}+a_2\frac{da_1}{dx}, \qquad \forall a_i\in A, \qquad \forall \lambda_i\in R^1, \qquad i=1,2.\end{gathered} \tag{1.7}$$

It is easy to see that if $H\in\hat{A}$, there exists a unique expansion

$$H=\sum_{N\geqq 0} H_N, \qquad H_N\in A_N. \tag{1.8}$$

Remark 1. If $F\in\hat{A}$, $u\in E$, then $F(u)$ will always denote the element of E obtained from F by the substitution $\{u_i\to u^{(i)},\ i=0,1,2,\dots\}$.

Definition 2. $I(R)$ will be called a trivial IE if there exists $F=F(u,u^{(1)},u^{(2)},\dots)$ such that $R=R(u,u^{(1)},\dots)=dF/dx$, where d/dx is defined by (1.5). If no such function F exists, $I(R)$ will be called a nontrivial *IE*.

In the trivial case, $I(R(u))\equiv 0$ for all $u\in E$, because of the boundary conditions in E. We shall confine attention in this paper to the case in which $\mathscr{P}\in A$, and $R\in\hat{A}$. We shall call such IEs polynomial integrals, or briefly PIEs.

There exists a well-developed theory for the Korteweg-de Vries equations, in view of which it is legitimate to investigate the state of affairs regarding PIEs in larger classes of evolution equations. It is clear that in the general case one cannot expect the set of nontrivial PIEs to be sufficiently rich. The present paper, confirming this point of view, will present an exact description of all nontrivial PIEs for a very large class of evolution equations and indicate the reasons that the class of nontrivial PIEs is not rich. Of the previous work in this direction we mention [2].

Write the fundamental equation (1.1), $\mathscr{P} \in A$, as

$$v_t = \sum_{i \geqq 0} \mu_i v^{(i)} + \sum_{i=2}^{m} \mathscr{P}_i(v, v^{(1)}, v^{(2)} \ldots), \tag{1.9}$$

where $\left(\sum_{i \geqq 0} \mu_i v^{(i)}\right) \in A_1$, $\mathscr{P}_i \in A_i$. Two of the main results of this paper can now be stated:

Theorem 1. *Let $\mu_0 \neq 0$ in Eq. (1.9). Then all* PIEs *of (1.9) are trivial.*

Theorem 2. *Let $\mu_0 = 0$ in Eq. (1.9). Assume that there exists $p \geqq 1$ such that $\mu_{2p} \neq 0$ and there exists $F \in A$ such that $\dfrac{dF}{dx} = \sum_{i=2}^{m} \mathscr{P}_i$.*

Then there exists exactly one nontrivial PIE *(1.9) of the form $C \int_a^b u dx$, where C is a nonzero constant.*

Thanks to this theorem, we can offer a qualitative interpretation of our results: the presence of at least one even-order derivative in the linear part of Eq. (1.9) implies that, in the informal sense, the equation has no nontrivial PIE, since a PIE of the form $C \int_a^b u dx$ corresponds to the fact that

$$\frac{dI}{dt}(cu) = c \int_a^b \mathscr{P}(u) dx = 0 \quad \forall u \in E, \tag{1.10}$$

and this equality, for all $\mathscr{P}$ actually occurring in equations of type (1.1), seems more or less obvious a priori.

As to the restrictions on the nonlinear part, note that, for example, such a common nonlinearity as $uu^{(1)}$ satisfies the condition of Theorem 2: $uu^{(1)} = d(u^2/2)/dx$.

However, we shall prove a much more general result – Theorem 3 – which makes it possible to investigate equations with nonlinear parts not of the form dF/dx, such as $(u^{(1)})^2$.

Before formulating Theorem 3, we must first introduce a transformation Γ of elements of A – a special case of a transformation defined in [1, Chap. 1, p. 82].

Definition 3. For primitive polynomials $u_{i_1} u_{i_2} u_{i_3} \ldots u_{i_N} \in A$, $i_k \in [0, 1, \ldots \infty)$

$$\Gamma(u_{i_1} u_{i_2} u_{i_3} \ldots u_{i_N}) \stackrel{\text{def}}{=} \frac{1}{N!} \sum_{\{j_1, j_2 \ldots j_N\} \in S_N} \xi_{j_1}^{i_1} \xi_{j_2}^{i_2} \ldots \xi_{j_N}^{i_N}, \tag{1.11}$$

where the sum extends over all permutations $\{j_1, j_2, \ldots, j_N\}$ in the symmetric group S_N. If a_j are primitive polynomials in A, and $\lambda_j \in R^1$, then we define

$$\Gamma\left(\sum_j \lambda_j a_j\right) \stackrel{\text{def}}{=} \sum_j \lambda_j \Gamma(a_j). \tag{1.12}$$

Thus, Γ maps A_N into the algebra $SP(N)$ of all symmetric polynomials in the abstract symbols $\xi_1, \xi_2, \ldots, \xi_N$ over the real field R^1.

Lemma 1. *For all $N \geqq 1$, Γ is an isomorphism of the linear spaces A_N and $SP(N)$. (The proof will be presented in Sect. 4).*

The following three properties (1.13), (1.14), and (1.16) justify our definition of Γ:

Let $F \in A_N$. Then

$$\Gamma\left(\frac{\partial F}{\partial u_p}\right) = z_N \cdot \frac{N}{P!}\left(\frac{\partial}{\partial \xi_N}\right)^P \cdot \Gamma(F), \tag{1.13}$$

where z_N transforms the set of symbols $(\xi_1, \ldots, \xi_N)$ in accordance with the rule: $z_N(\xi_i) = \xi_i$ for $i < N$ and $z_N(\xi_N) = 0$. Note moreover that since $\partial/\partial u_p$ maps A_N into A_{N-1}, it follows that $\Gamma(\partial F/\partial u_p) \in SP(N-1)$,

$$\Gamma\left(\frac{dF}{dx}\right) = (\xi_1 + \xi_2 + \ldots + \xi_N)\Gamma(F). \tag{1.14}$$

Note that d/dx [see (1.5)] maps A_N into A_N. Put

$$\frac{\delta}{\delta u} \stackrel{\text{def}}{=} \sum_{k \geqq 0}\left(-\frac{d}{dx}\right)^k \frac{\partial}{\partial u_k}. \tag{1.15}$$

Clearly $\delta/\delta u$ maps A_N into A_{N-1}. We have

$$\Gamma\left(\frac{\delta F}{\delta u}\right) = N \cdot (\Gamma F)\left(\xi_1, \xi_2, \ldots, \xi_{N-1}, -\sum_{i=1}^{N-1} \xi_i\right), \tag{1.16}$$

where $\Gamma F \in SP(N)$, but the substitution $\xi_N \to \left(-\sum_{i=1}^{N-1} \xi_i\right)$ takes this polynomial into $SP(N-1)$, so that

$$(\Gamma F)\left(\xi_1, \xi_2, \ldots, \xi_{N-1}, -\sum_{i=1}^{N-1} \xi_i\right) \in SP(N-1).$$

Brief proofs of (1.13), (1.14), and (1.16) may be found in [1, Chap. 1]. Since the transformation Γ seems to be a quite interesting and promising analogue of the Fourier transform for the differential algebra A, the proofs will be presented in detail below (Sect. 4) for the reader's convenience.

It is convenient to introduce the following notation:

$$\frac{\tilde{\partial}}{\partial u_p} \stackrel{\text{def}}{=} z_N \cdot \frac{N}{P!}\left(\frac{\partial}{\partial \xi_N}\right)^P, \tag{1.17}$$

$$\frac{\tilde{\delta}}{\delta u} : \frac{\tilde{\delta}}{\delta u}(\xi_i) = \xi_i \quad \text{if} \quad i = 1, 2, \ldots, N-1; \qquad \frac{\tilde{\delta}}{\delta u}(\xi_N) = -\sum_{i=1}^{N-1} \xi_i. \tag{1.18}$$

Note that

$$\Gamma \frac{\delta}{\delta u} = N \cdot \frac{\tilde{\delta}}{\delta u} \Gamma. \tag{1.19}$$

This follows from property (1.16). Put

$$\Phi_N \stackrel{\text{def}}{=} \sum_{j \geqq 0} \mu_j \left\{(-1)^j \left(\sum_{k=1}^{N} \xi_k\right)^j + \sum_{i=1}^{N} \xi_i^j\right\}. \tag{1.20}$$

In the sequel we shall need the expansion (1.8) of R:

$$R = \sum_{i \geqq 0} R_i, \qquad R_i \in A_i, \tag{1.21}$$

then

$$\frac{\delta R}{\delta u} = \sum_{i \geqq 1} \frac{\delta R_i}{\delta u}, \qquad \frac{\delta R_0}{\delta u} = 0, \qquad \frac{\delta R_1}{\delta u} = \text{const}. \tag{1.22}$$

We have to consider the system

$$\left\{ \Phi_{i-1} \cdot \Gamma\left(\frac{\delta R_i}{\delta u}\right) + C \cdot \frac{\tilde{\delta}}{\delta u} \cdot \Gamma(\mathscr{P}_i) = 0,\ i = 2, 3, 4, \ldots \right\}, \tag{1.23}$$

where the $\mathscr{P}_i$ are as in (1.9), $\mathscr{P}_i = 0$ for $i > m$, C is an arbitrary fixed constant. Denote the set of solutions $\{R_i\}$, $R_i \in A_i$, $i = 2, 3, 4, \ldots$, of this system by $\zeta(C)$. Of course, it may occur that $\zeta(C) = \emptyset$, but $\zeta(0) \neq \emptyset$, since $\{R_i = 0,\ i = 2, 3, \ldots\} \in \zeta(0)$. Note that $\zeta(C) = \emptyset$ if there exists $i_0 \geqq 2$ such that the i_0-th equation in (1.23) is not satisfied for any $R_i \in A_i$.

Theorem 3. *Let $\mu_0 = 0$ in Eq. (1.9). Assume that for all C such that $\zeta(C) \neq \emptyset$ and all $\{R_i',\ i = 2, 3, \ldots\} \in \zeta(C)$,*

$$\frac{\delta R_i'}{\delta u} = 0, \qquad i = 2, 3, 4, \ldots.$$

Then:

1) *If $\zeta(C) = \emptyset$ for all $C \neq 0$, then all* PIE *(1.9) are trivial.*

2) *If there exists $C_1 \neq 0$ such that $\zeta(C_1) \neq \emptyset$, then there exists a unique nontrivial* PIE *(1.9) of the form $C \int_a^b u\,dx$.*

2. Proofs (in Outline)

Our main task is to investigate the existence and structure of the solution of the equation

$$\int_a^b \left(\sum_{k \geqq 0} \frac{\partial R}{\partial u_k}(u) \cdot \frac{\partial^k \mathscr{P}(u)}{\partial x^k} \right) dx = 0 \quad \forall u \in E \tag{2.1}$$

for $R \in \hat{A}$ [see (1.3)]. Using the following nontrivial lemma, we shall reduce Eq. (2.1) to a purely algebraic equation:

Lemma 2. *The relationship $\int_a^b Q(u)dx = 0$ $\forall u \in E$, $Q \in \hat{A}$ is true if and only if there exists $G \in A$ such that $Q = dG/dx$.*

The proof will be presented in Sect. 4.

Note that integration by parts, using the boundary conditions in E, yields

$$\frac{dI(R(u))}{dt} = \int_a^b \left(\sum_{k \geqq 0} \frac{\partial R}{\partial u_k}(u) \cdot \frac{\partial^k \mathscr{P}(u)}{\partial x^k} \right) dx = \int_a^b \frac{\delta R}{\delta u}(u) \cdot \mathscr{P}(u) dx, \tag{2.2}$$

where $\delta/\delta u$ is as in (1.15). $\left(\mathscr{P}_0 = 0, \text{ then } \mathscr{P} \in A \text{ and } \dfrac{\delta R}{\delta u}(u)\mathscr{P}(u) \in A\right)$.

It now follows from Lemma 2 that Eq. (2.1) is equivalent to the following relationship

$$\frac{\delta R}{\delta u}\cdot\mathscr{P}=\frac{dG}{dx}\tag{2.3}$$

for some $G\in A$. Using (1.8) we see that (2.3) is equivalent to the following system of equations:

$$\sum_{k=0}^{i}\frac{\delta R_{k+1}}{\delta u}\cdot\mathscr{P}_{i-k}=\frac{dG_i}{dx},\qquad i=0,1,2,\ldots,\tag{2.4}$$

where $\mathscr{P}_{i-k}\in A_{i-k}$; $\mathscr{P}_0=0$ and $\mathscr{P}_1=\sum_{j\geqq 0}\mu_j u_j$ by (1.9); $\mathscr{P}_2,\ldots,\mathscr{P}_m$ are as in the nonlinear part of (1.9); $\mathscr{P}_i=0$ for $i>m$.

Note that we are using the same notation $\mathscr{P}_i$ for the polynomial $\mathscr{P}_i(u,u^{(1)},u^{(2)},\ldots)$ in the right-hand side of (1.9) and for the polynomial $\mathscr{P}_i(u,u_1,u_2,\ldots)\in\hat{A}$ obtained by the substitution $u^{(i)}\to u_i$; this should not cause any confusion.

The following lemma is technically important:

Lemma 3.

$$\frac{\tilde{\delta}}{\delta u}\cdot\Gamma\left(\frac{\delta R_N}{\delta u}\cdot\Big(\sum_{j\geqq 0}\mu_j u_j\Big)\right)=\Phi_{N-1}\cdot\frac{\tilde{\delta}}{\delta u}\Gamma(R_N)$$

for $N=2,3,\ldots$, where Φ_{N-1} is defined in (1.20).

For the proof, see Sect. 4.

Proof of Theorem 1. We shall prove that system (2.4), as a system of equations in the unknowns $\partial R_k/\partial u$, has only the trivial solution.

If $i=0$: $\frac{\delta R_1}{\delta u}\cdot\mathscr{P}_0=\frac{dG_0}{dx}$, since $\mathscr{P}_0=0$, $G_0=\text{const}$, and this equality holds identically for any $\delta R_1/\delta u=\text{const}$.

If $i=1$: $\frac{\delta R_2}{\delta u}\cdot\mathscr{P}_0+\frac{\delta R_1}{\delta u}\cdot\mathscr{P}_1=\frac{\delta R_1}{\delta u}\cdot\mathscr{P}_1=\frac{dG_1}{dx}$.

Since $\delta R_1/\delta u=\text{const}$, it follows that G_1 exists if and only if $\mu_0=0$. But this contradicts the assumptions of the theorem, and so $\delta R_1/\delta u=0$.

Both here and later we shall need the following proposition:

$$\operatorname{Ker}\frac{\delta}{\delta u}=\operatorname{Im}\frac{d}{dx}\quad\text{in}\quad A.\tag{2.5}$$

(For the proof see [1, Chap. 1, p. 81].) Note that $\delta\mathscr{P}_1/\delta u=\mu_0$.

If $i=2$;

$$\mathscr{P}_0\frac{\delta R_3}{\delta u}+\mathscr{P}_1\frac{\delta R_2}{\delta u}+\mathscr{P}_2\frac{\delta R_1}{\delta u}=\mathscr{P}_1\frac{\delta R_2}{\delta u}=\frac{dG_2}{dx}.\tag{2.6}$$

Apply the operator $(\delta/\delta u)\Gamma$ to both sides of (2.6), using (2.5) and the fact that (as follows from Lemma 1) $\operatorname{Ker}\Gamma=0$. The result is that (2.6) is equivalent to the following equation:

$$\frac{\tilde{\delta}}{\delta u}\Gamma\left(\frac{\delta R_2}{\delta u}\Big(\sum_{j\geqq 0}\mu_j u_j\Big)\right)=0;\tag{2.7}$$

and from Lemma 3 and (1.19) it follows that (2.6) is equivalent to

$$\Phi_1 \cdot \Gamma\left(\frac{\delta R_2}{\delta u}\right) = 0. \tag{2.8}$$

It now follows from (1.20) that

$$\Phi_1 = 2 \sum_{p \geqq 0} \mu_{2p} \xi_1^{2p} \neq 0 \quad \text{since} \quad \mu_0 \neq 0.$$

Hence $\Gamma(\delta R_2/\delta u) = 0$ and $(\delta R_2/\delta u) = 0$. It remains to carry out the induction step: Assume that

$$\frac{\delta R_1}{\delta u} = \frac{\delta R_2}{\delta u} = \ldots = \frac{\delta R_n}{\delta u} = 0.$$

Considering the equation of system (2.4) for $i = n+1$, we obtain

$$\frac{\delta R_{n+1}}{\delta u} \cdot \mathscr{P}_1 = \frac{dG_{n+1}}{dx}. \tag{2.9}$$

Repeating the arguments used in the case $i = 2$ word for word, noting that $\Phi_n \neq 0$, we obtain $\delta R_{n+1}/\delta u = 0$.

Consequently,

$$\frac{\delta R}{\delta u} = \frac{\delta}{\delta u}\left(\sum_{i \geqq 0} R_i\right) = 0,$$

and this completes the proof of Theorem 1.

Remark 2. By (2.5) $\delta R/\delta u = 0$ if and only if $R = dF/dx$ for $F \in A$, i.e., R is trivial. Similarly, if $\delta R/\delta u$ is given, then R is defined up to a term dF/dx.

Proof of Theorem 2. Proceeding just as in the proof of Theorem 1, we consider the equation of system (2.4) for $i = 0$; it holds identically.

If $i = 1$ the equation is satisfied by any $\delta R_1/\delta u = C$, since $\mu_0 = 0$. In that case $R_1 = Cu + dF_1/dx$, where $F_1 \in A_1$.

If $i = 2$: $\mathscr{P}_1 \dfrac{\delta R_2}{\delta u} + \mathscr{P}_2 \dfrac{\delta R_1}{\delta u} = \dfrac{dG_2}{dx}$. Proceeding just as in the case (2.6), we obtain the equivalent equation

$$\Phi_1 \cdot \Gamma\left(\frac{\delta R_2}{\delta u}\right) + \frac{\tilde{\delta}}{\delta u} \cdot \Gamma\left\{\mathscr{P}_2 \cdot \frac{\delta R_1}{\delta u}\right\} = 0. \tag{2.10}$$

Since $\mathscr{P}_2 = dF_2/dx$ for $F = \sum_{i \geqq 2} F_i$, $F_i \in A_i$, and $dF/dx = \sum_{i \geqq 2} \mathscr{P}_i$, it follows in view of (2.5) that

$$\frac{\tilde{\delta}}{\delta u} \Gamma\{\mathscr{P}_2 \cdot C\} = \Gamma\left\{\frac{C}{2} \cdot \frac{\delta \mathscr{P}_2}{\delta u}\right\} = 0; \quad \frac{\delta R_1}{\delta u} = C.$$

Since $\Phi_1 \neq 0$ (because $\mu_2 \neq 0$), it follows that $\delta R_2/\delta u = 0$.

If $i = 3$: $\mathscr{P}_1 \dfrac{\delta R_3}{\delta u} + \mathscr{P}_3 \dfrac{\delta R_1}{\delta u} = \dfrac{dG_3}{dx}$.

Repeating the arguments used in the case $i=2$ word for word and noting that $\Phi_i \neq 0$ for $i=1,2,\ldots$, we obtain $\delta R_3/\delta u=0$.

The proof now continues by induction; analogous arguments yield the equality $\delta R/\delta u = C = \text{const}$, completing the proof.

Proof of Theorem 3. Proceeding just as in the proofs of Theorems 1 and 2, we see that the equation of system (2.4) for $i=0$ reduces to an identity; and for $i=1$, $\delta R_1/\delta u = C$, where C is an arbitrary constant. Let $C=C_1$ be such that $\zeta(C_1) \neq \emptyset$.

If $i=2$, we obtain Eq. (2.10), but by assumption $\delta R_2/\delta u=0$. Repeating the procedure for $i=3,4,\ldots$, using induction and the assumption concerning $\zeta(C)$, we see that $\delta R_i/\delta u=0$, $i=2,3,4,\ldots$.

Thus

$$\frac{\delta R}{\delta u} = \sum_{i \geqq 1} \frac{\delta R_i}{\delta u} = C_1 .$$

If C_1 can be assumed distinct from 0 (i.e., Case 2 holds), then $R=C_1 u$ and, multiplying by an arbitrary constant, we see that $R=Cu$ up to a trivial term. Otherwise, we obtain the assertion of Case 1, and this completes the proof of Theorem 3.

3. Examples

1) Burgers equation $u_t = v u_{xx} - u u_x$, $v=\text{const}$. Since we can use Theorem 2, whence it follows that $I = C\int_a^b u dx$ is the unique nontrivial PIE of the Burgers equation in the space E.

2)
$$u_t = u_{xxxx} + \frac{\partial}{\partial x}\{[1-(u_x)^2]u_x\} + \alpha u .$$

This equation describes Bénard convection in a nearly isolated liquid layer [3–5].

Theorem 1 states that in this case, if $\alpha \neq 0$, there are no nontrivial PIEs in E.

3) The Kuramoto-Sivashinsky equation [6, 7]:

$$u_t = -u_{xxxx} - u_{xx} - u_x^2 .$$

The first equation of system (1.23) for $i=2$, gives

$$-2(\xi_1^2+\xi_1^4)\frac{\tilde{\delta}}{\delta u}\Gamma(R_2) + C\xi_1^2 = 0 , \qquad \Phi_1 = -2(\xi_1^2+\xi_1^4), \qquad C = \frac{\delta R_1}{\delta u},$$

$$\frac{\tilde{\delta}}{\delta u}\Gamma\left\{-[u^{(1)}]^2\frac{\delta R_1}{\delta u}\right\} = -\frac{C}{2}\frac{\tilde{\delta}}{\delta u}\cdot\{\xi_1\xi_2+\xi_2\xi_1\} = C\xi_1^2 .$$

Next, $\Gamma(R_2)$ is a symmetric polynomial in (ξ_1, ξ_2), but $\tilde{\delta}\Gamma(R_2)/\delta u$ is simply a polynomial in ξ_1:

$$\frac{\tilde{\delta}}{\delta u}\Gamma(R_2) = \sum_{i=0}^{k} a_i \xi_1^i ,$$

where a_i are constants. Thus, we can rewrite the equation of (1.23) for $i=2$ in the following form:

$$(\xi_1^2+\xi_1^4)\left(\sum_{i=0}^{k} a_i \xi_1^i\right) - \frac{c}{2}\xi_1^2 = 0 . \tag{3.1}$$

We now prove by induction on k that

$$a_i = c = 0 \quad \text{for} \quad i = 0, 1, \ldots, k, \tag{3.2}$$

i.e., $\delta R_2/\delta u = C = 0$.

a) $k=0$: $(\xi_1^2 + \xi_1^4)a_0 - \frac{C}{2}\xi_1^2 = 0$, whence it follows that $a_0 = 0$, and so $C = 0$.

b) $k+1$: $(\xi_1^2 + \xi_2^4)\left[\sum_{i=0}^{k} a_i \xi_1^i + a_{k+1}\xi_1^{k+1}\right] - \frac{C}{2}\xi_1^2 = 0$, or

$$\left\{(\xi_1^2 + \xi_1^4)\left(\sum_{i=0}^{k} a_i \xi_1^i\right) - \frac{C}{2}\xi_1^2\right\} + a_{k+1}\xi_1^{k+1}(\xi_1^2 + \xi_1^4) = 0.$$

$a_{k+1}\xi^{k+5}$ is the highest-degree term in this polynomial, and therefore $a_{k+1} = 0$. Thus by the induction hypothesis, $a_i = C = 0$, $i = 0, 1, \ldots, k$.

This proves (3.2). We now consider the equations of (1.23) with $i \geqq 3$:

$$\left\{\left(\sum_{k=1}^{i-1} \xi_k\right)^2 + \sum_{k=1}^{i-1} \xi_k^2 + \left(\sum_{k=1}^{i-1} \xi_k\right)^4 + \sum_{k=1}^{i-1} \xi_k^4\right\} \cdot \Gamma\left(\frac{\delta R_i}{\delta u}\right) = 0, \tag{3.3}$$

where

$$\Phi_{i-1} = -\left\{\left(\sum_{k=1}^{i-1} \xi_k\right)^2 + \sum_{k=1}^{i-1} \xi_k^2 + \left(\sum_{k=1}^{i-1} \xi_k\right)^4 + \sum_{k=1}^{i-1} \xi_k^4\right\}; \quad \frac{\delta R_1}{\delta u} = C = 0;$$

$\mathscr{P}_i = 0$ for $i \geqq 3$.

It follows from (3.3) that $\Gamma(\delta R_i/\delta u) = 0$ and, by Lemma 1, $\delta R_i/\delta u = 0$. Thus the assumptions of Theorem 3 are satisfied, and $C = 0$ if $\zeta(C) \neq \emptyset$. Thus all PIEs of the Kuramoto-Sivashinsky equation are trivial in E.

4) $$h_t = -h_{xxxx} - h_{xx} - h_x h.$$

This equation describes the evolution of the surface of a disturbed film of viscous liquid flowing down a vertical plane [9].

Theorem 2 enables us to state that this equation has a unique nontrivial PIE in E, namely, $I = C\int_a^b u\,dx$.

5) $$u_t = 6uu_x - u_{xxxx}.$$

It is interesting to observe how the technique we have developed here breaks down in the case of the KdV equation. The obstruction here is the equality $\Phi_1 = 2\sum_{p \geqq 0} \mu_{2p}\xi_1^{2p} = 0$ for the KdV equation, i.e., the fact that the linear part does not contain even-ordered derivatives.

The nonlinear part of the KdV equation is a total differential: $6uu_x = 3\frac{d}{dx}u^2$. One can therefore use the argument of Theorem 2, but since $\Phi_1 = 0$, it does not follow from (2.10) that $\delta R_2/\delta u = 0$. Hence we cannot arrive at the conclusion of Theorem 2, and we cannot state that the KdV equation has PIE's of the form $c\int_a^b u\,dx$ only.

As to the equality $\Phi_1=0$, in view of the fact that $\frac{\tilde{\delta}}{\delta u}\Gamma(6uu_x)=0$ and $\mathscr{P}_i=0$ for $i=3,4,\ldots$, we can infer that system (1.23) assumes the form $\left\{\Phi_{i-1}\Gamma\left(\frac{\delta R_i}{\delta u}\right)=0,\ i=2,3,4,\ldots\right\}$ and the derivative $\delta R_2'/\delta u$ in $\zeta(c)$ does not necessarily vanish. Thus Theorem 3 cannot be applied, as its assumptions are violated.

In sum, we see that the methods developed in this paper are only illustrated by the above five examples. It is quite clear that they may be applied over a considerably broader range of problems. At the same time, the exact descriptions obtained here of classes of PIEs provide radically new information about classical objects.

4. Proofs of Lemmas etc.

Proof of Lemma 1. Let

$$\theta(\xi_1,\xi_2,\ldots,\xi_N)=\sum_{\bar{i}\in Z^+(N)} a_{\bar{i}}\cdot\xi_1^{i_1}\cdot\xi_2^{i_2}\ldots\xi_N^{i_N}, \qquad \bar{i}=(i_1,i_2\ldots i_N),$$

where $Z^+(N)$ is the set of all vectors in R^N with nonnegative integer coordinates. It is easy to show that $\theta\in SP(N)$ if and only if $a_{\pi\bar{i}}=a_{\bar{i}}$ for all $\bar{i}\in Z^+(N)$ and all $\pi\in S_N$, where $\pi\bar{i}=(i_{\pi(1)},\ i_{\pi(2)},\ldots,i_{\pi(N)})$.

Assume that $\theta\in SP(N)$. Then

$$\theta(\xi_1,\xi_2,\ldots,\xi_N)=\sum_{\bar{i}} a_{\bar{i}}\xi_1^{i_1}\xi_2^{i_2}\ldots\xi_N^{i_N}=\sum_{\bar{i}} a_{\pi\bar{i}}\xi_1^{i_1}\xi_2^{i_2}\ldots\xi_N^{i_N}=\sum_{\bar{i}} a_{\bar{i}}\xi_{\pi(2)}^{i_1}\xi_{\pi(2)}^{i_2}\ldots\xi_{\pi(N)}^{i_N}.$$

Summing both sides of the equality $\theta=\sum_{\bar{i}} a_{\bar{i}}\xi_{\pi(1)}^{i_1}\xi_{\pi(2)}^{i_2}\ldots\xi_{\pi(N)}^{i_N}$ over all $\pi\in S_N$, we obtain

$$\theta=\frac{1}{N!}\sum_{\bar{i}} a_{\bar{i}}\left\{\sum_{\pi\in S_N}\xi_{\pi(1)}^{i_1}\cdot\xi_{\pi(2)}^{i_2}\ldots\xi_{\pi(N)}^{i_N}\right\}. \tag{4.1}$$

This equality shows that for all $\theta\in SP(N)$ there exists $r\in A_N$, $r=\sum_{\bar{i}} a_{\bar{i}}u_{i_1}u_{i_2}\ldots u_{i_N}$, such that $\Gamma(r)=\theta$, i.e., $\operatorname{Im}\Gamma=SP(N)$.

We now show that $\operatorname{Ker}\Gamma=0$. Let $\bar{a}=\sum_{\bar{i}} a_{\bar{i}}u_{i_1}u_{i_2}u_{i_N}$, $\bar{i}=\{i_1,i_2,\ldots,i_N\}\in Z^+(N)$ denote an arbitrary element of A_N. Since A_N is commutative, we may assume without loss of generality that

$$a_{\pi\bar{i}}=a_{\bar{i}} \quad \text{for any} \quad \pi\in S_N \quad \text{and any} \quad \bar{i}\in z^+(N). \tag{4.2}$$

Define

$$W:\xi_k\to 0 \quad \text{for} \quad k=1,2,\ldots,N;$$

$$D^{\bar{j}}\overset{\text{def}}{=}\frac{1}{j_1!}\left(\frac{\partial}{\partial\xi_1}\right)^{j_1}\frac{1}{j_2!}\left(\frac{\partial}{\partial\xi_2}\right)^{j_2}\frac{1}{j_N!}\left(\frac{\partial}{\partial\xi_N}\right)^{j_N}.$$

We have

$$WD^{\bar{j}}\Gamma(\bar{a})=\frac{1}{N!}\sum_{\pi\in S_N} a_{\pi\bar{j}}. \tag{4.3}$$

Suppose that $\Gamma(\bar{a})=0$. Noting that $WD^{\bar{j}}0=0$, we obtain

$$WD^{\bar{j}}\Gamma(\bar{a})=0=\frac{1}{N!}\sum_{\pi\in S_N} a_{\pi\bar{j}}. \tag{4.4}$$

It follows from (4.4) and (4.2) that $a_j=0$, and so $a=0$. Thus $\operatorname{Ker}\Gamma=0$, completing the proof of Lemma 1.

For the reader's convenience, we present the proofs of properties (1.13), (1.14), and (1.16), which were only outlined in [1]. We begin with the proof of (1.13):

$$\begin{aligned}\Gamma(u_{i_1}u_{i_2}\dots u_{i_N})&=\frac{1}{N!}\sum_{\pi\in S_N}\xi^{i_1}_{\pi(1)}\xi^{i_2}_{\pi(2)}\dots\xi^{i_N}_{\pi(N)}\\&=\frac{1}{N!}\sum_{\pi\in S_N}\xi_1^{i_{\pi(1)}}\xi_2^{i_{\pi(2)}}\dots\xi_N^{i_{\pi(N)}}.\end{aligned}\tag{4.5}$$

Since A is commutative, we may assume without loss of generality that $i_1\leqq i_2\leqq\dots\leqq i_N$. Now,

$$\begin{aligned}\frac{Z_N}{P!}\left(\frac{\partial}{\partial\xi_N}\right)^P\Gamma(u_{i_1}u_{i_2}\dots u_{i_N})&=\frac{1}{N!}\sum_{\pi\in S_N}\xi_1^{i_{\pi(1)}}\xi_2^{i_{\pi(2)}}\dots\xi_{N-1}^{i_{\pi(N-1)}}\left\{\frac{Z_N}{P!}\left(\frac{\partial}{\partial\xi_N}\right)^P\xi_N^{i_{\pi(N)}}\right\}\\&=\frac{1}{N!}\sum_{\pi\in S_N}\xi_1^{i_{\pi(1)}}\xi_2^{i_{\pi(2)}}\dots\xi_N^{i_{\pi(N-1)}}\delta_{p,i_{\pi(N)}}.\end{aligned}\tag{4.6}$$

Continuing, we can establish a one-to-one correspondence between $S_N^q=\{\pi:\pi\in S_N,\ \pi(N)=q\}$ and S_{N-1}, where q is a fixed number, $q\in[1,\dots,N]$. The correspondence of $\sigma\leftrightarrow\pi$, $\pi\in S_N^q$, is defined by means of the function

$$\chi_q(s)=\begin{cases}S, & S<q\\ S+1, & S\geqq q\end{cases}$$

putting $\chi_q(\sigma(t))=\pi(t)$, $t\in[1,\dots,N-1]$, $\sigma\in S_{N-1}$.

Assume that p has the property: there exists $i_q\in\{i_1,\dots,i_N\}$ such that $i_q=p$. To be precise, q is such that

$$i_{q-1}<i_q=i_{q+1}=\dots=i_{q+p}=p<i_{q+p+1}.$$

Then

$$\begin{aligned}\frac{1}{N!}\sum_{\pi\in S_N}\xi_1^{i_{\pi(1)}}\xi_2^{i_{\pi(2)}}\dots\xi_{N-1}^{i_{\pi(N-1)}}\cdot\delta_{p,i_{\pi(N)}}&=\frac{1}{N!}\sum_{j=0}^{p}\sum_{p\in S_N^{q+j}}\xi_1^{i_{\pi(1)}}\xi_2^{i_{\pi(2)}}\dots\xi_{N-1}^{i_{\pi(N-1)}}\\&=\frac{1}{N!}\sum_{j=0}^{p}\sum_{\sigma\in S_{N-1}}\xi_1^{i_{\chi_{q+j}(\sigma(1))}}\dots\xi_{N-1}^{i_{\chi_{q+j}(\sigma(N-1))}}\end{aligned}\tag{4.7}$$

But

$$\frac{1}{(N-1)!}\sum_{\sigma\in S_{N-1}}\xi_1^{i_{\chi_{q+j}(\sigma(1))}}\dots\xi_{N-1}^{i_{\chi_{q+j}(\sigma(N-1))}}=\Gamma(u_{i_1}u_{i_2}\dots u_{i_{q+j-1}}u_{i_{q+j+1}}\dots u_{i_N}). \tag{4.8}$$

Combining (4.8), (4.7), and (4.6), we obtain (1.13). If $i_s\neq p$ for all $s\in[1,\dots,N]$, then the sum (4.6) vanishes, and (1.13) is identically true.

Proof of (1.14).

$$\Gamma\left\{\frac{d}{dx}(u_{i_1}u_{i_2}\dots u_{i_N})\right\}=\Gamma\left\{\sum_{s=1}^{N} u_{i_1}u_{i_2}\dots u_{i_{s-1}}u_{i_s+1}u_{i_{s+1}}\dots u_{i_N}\right\}$$
$$=\sum_{s=1}^{N}\frac{1}{N!}\left\{\sum_{\pi\in S_N}\xi_{\pi(1)}^{i_1}\xi_{\pi(2)}^{i_2}\dots\xi_{\pi(s)}^{i_s+1}\dots\xi_{\pi(N)}^{i_N}\right\}$$
$$=(\xi_1+\xi_2+\dots+\xi_N)\Gamma(u_{i_1}u_{i_2}\dots u_{i_N}).$$

Proof of (1.16).

$$\Gamma\left(\frac{\delta F}{\delta u}\right)=\Gamma\left(\sum_{p\geqq 0}\left(-\frac{d}{dx}\right)^p\frac{\partial F}{\partial u_p}\right)=\sum_{p\geqq 0}\left(-\sum_{i=1}^{N-1}\xi_i\right)^p\Gamma\left(\frac{\partial F}{\partial u_p}\right)$$
$$=\sum_{p\geqq 0}\left(-\sum_{i=1}^{N-1}\xi_i\right)^p Z_N\frac{N}{P!}\left(\frac{\partial}{\partial\xi_N}\right)^p\Gamma F$$
$$=Z_N\cdot N\sum_{p\geqq 0}\left(-\sum_{i=1}^{N-1}\xi_i\right)^p\frac{1}{P!}\left(\frac{\partial}{\partial\xi_N}\right)^p(\Gamma F)(\xi_1,\dots,\xi_N)$$
$$=Z_N\cdot N(\Gamma F)\left(\xi_1,\dots,\xi_{N-1},\xi_{N-1}-\sum_{i=1}^{N-1}\xi_i\right)$$
$$=N\cdot(\Gamma F)\left(\xi_1,\dots,\xi_{N-1},-\sum_{i=1}^{N-1}\xi_i\right).$$

Proof of Lemma 2. If $Q=dG/dx$, then $\int_a^b Q(u)(x)dx=0$ because of the boundary conditions in E.

Conversely, let $\int_a^b Q(u)(x)dx=0$ for all $u\in E$, $Q\in\hat{A}$. Put $u=0\in E$, then $\int_a^b Q(u)dx=\int_a^b Q_0dx=0$ and $Q_0=0$, where $Q(u)=Q_0+\sum_{i\geqq 1}Q_i$; $Q_0\in A_0$, $\sum_{i\geqq 1}Q_i\in A$. Thus $Q\in A$. Define

$$W_\varepsilon(u;x)\overset{\text{def}}{=}u(x)+\varepsilon\frac{\delta Q(u)}{\delta u}(x),\tag{4.9}$$

where u is arbitrary but fixed, $\varepsilon\in[0,1]$. Note that

$$\left(\frac{d}{dx}\right)^k W_\varepsilon(u;x)=u^{(k)}(x)+\varepsilon\left(\frac{d}{dx}\right)^k\frac{\delta Q(u)}{\delta u}(x),\qquad k=1,2,\dots.\tag{4.10}$$

Hence we obtain the expansion

$$Q(W_\varepsilon(u;x))=Q(u)(x)+\varepsilon\sum_{k\geqq 0}\frac{\partial Q(u)}{\partial u^{(k)}}(x)\cdot\left(\frac{d}{dx}\right)^k\frac{\delta Q(u)}{\delta u}(x)+o(\varepsilon).\tag{4.11}$$

Since $W_\varepsilon(u;x)\in E$, it follows that $\int_a^b Q(W_\varepsilon(u;x))dx=0$ for all $u\in E$. Hence

$$\int_a^b Q(u)(x)dx+\varepsilon\int_a^b\sum_{k\geqq 0}\frac{\partial Q(u)}{\partial u^{(k)}}(x)\left(\frac{d}{dx}\right)^k\frac{\delta Q(u)}{\delta u}(x)dx+o(\varepsilon)=0$$

identically in $\varepsilon \in [0,1]$, whence it follows that

$$\int_a^b \sum_{k \geqq 0} \frac{\partial Q(u)}{\partial u^{(k)}}(x)\left(\frac{d}{dx}\right)^k \frac{\delta Q(u)}{\delta u}(x)dx = 0 \quad \text{for all} \quad u \in E. \tag{4.12}$$

Using the boundary conditions in E, we obtain

$$\int_a^b \left(\frac{\delta Q(u)}{\delta u}\right)^2 (\mathrm{x})\mathrm{dx} = 0, \quad \text{for all} \quad u \in E, \tag{4.13}$$

whence, since the functions involved are continuous, it follows that $\delta Q(u)/\delta u(x) = 0$ for all $u \in E$. This implies that $\delta Q/\delta u = 0$ as an element of $\hat{A}$. In order to prove this almost obvious fact, we observe that the sequence $u(x_0), u^{(1)}(x_0), \ldots, u^{(N)}(x_0)$ may assume arbitrary given real values $\alpha_0, \alpha_1, \ldots, \alpha_N$; x_0 is arbitrary but fixed. In order to complete the proof, we reason by *reductio ad absurdum*, using Theorem 14 of [8, Chap. 1, Sect. 18]. From $\delta Q/\delta u = 0$ we now deduce, using (2.5), that $Q = dG/dx$.

Proof of Lemma 3. For convenience, we denote

$$\alpha = \frac{\tilde{\delta}}{\delta u}\Gamma\left\{\frac{\delta R_N}{\delta u}\left(\sum_{j \geqq 0} \mu_j u_j\right)\right\} = \frac{\tilde{\delta}}{\delta u}\Gamma\left\{\frac{\delta R_N}{\delta u}\left(\sum_{j \geqq 0} \mu_j \left(\frac{d}{dx}\right)^j u\right)\right\}.$$

Using (2.5), we can state that

$$\frac{\tilde{\delta}}{\delta u}\Gamma\left\{W\frac{dV}{dx}\right\} = \frac{\tilde{\delta}}{\delta u}\Gamma\left\{V\left(-\frac{dW}{dx}\right)\right\} \quad \text{for any} \quad V, W \in \hat{A}. \tag{4.14}$$

$\left(\text{Note that } W\dfrac{dV}{dx},\ V\dfrac{dW}{dx},\ \dfrac{dVW}{dx} \in A.\right)$

Hence

$$\alpha = \frac{\tilde{\delta}}{\delta u}\Gamma\left\{u_0 \sum_{j \geqq 0} \mu_j \left(-\frac{d}{dx}\right)^j \frac{\delta R_N}{\delta u}\right\} = \frac{\tilde{\delta}}{\delta u}\left\{\sum_{i=1}^N \left[\sum_{j \geqq 0} \mu_j (-1)^j \left(\sum_{k=1, k \neq i}^N \xi_k\right)^j\right]\right.$$
$$\left. \times \Gamma(R_N)\left(\xi_1, \ldots, \check{\xi}_i, \ldots, \xi_N, -\sum_{s=1, s \neq i}^N \xi_s\right)\right\},$$

where $\check{\xi}_i$ means that the element ξ_i is omitted. Noting that $\dfrac{\tilde{\delta}}{\delta u}\left(\sum_{k=1, k \neq i} \xi_k\right)^j$ equals $\left(\sum_{k=1}^{N-1} \xi_k\right)^j$ if $i = N$, and equals $(-\xi_i)^j$ if $i \neq N$, we obtain

$$\alpha = \left\{\sum_{j \geqq 0} \mu_j (-1)^j \sum_{i=1}^{N-1} (-\xi_i)^j\right\} \Gamma(R_N)\left(\xi_1, \ldots, \check{\xi}_i, \ldots, \xi_N - 1, -\sum_{i=1}^{N-1} \xi_i, \xi_i\right)$$
$$+ \left\{\sum_{j \geqq 0} \mu_j (-1)^j \left(\sum_{i=1}^{N-1} \xi_i\right)^j\right\} \Gamma(R_N)\left(\xi_i, \ldots, \xi_{N-1}, -\sum_{i=1}^{N-1} \xi_i\right)$$
$$= \Phi_{N-1} \cdot \frac{\tilde{\delta}}{\delta u}\Gamma(R_N).$$

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