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Cyclic Cocycles from Graded KMS Functionals

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Abstract. Each "graded KMS functional" of a $Z/2$-graded C^*-algebra with respect to a "supersymmetric" one-parameter automorphism group gives rise to a cyclic cocycle.

In order to match algebras of primary mathematical interest for which there are no p-summable Fredholm modules, A. Connes introduced the wider notion of *θ-summable Fredholm module* [1], which also encompasses the Dirac operator on loop space rigorously constructed by A. Jaffe and collaborators [2] – and subsequently developed the corresponding generalizations of cyclic cohomology and of the Chern character [3]. For constructing the latter, Connes had to resort to a "formal square root" (Ref. [3], p. 20), so to speak enforcing supersymmetry, and thus leading to conjecture a deep relationship between cyclic cohomology, supersymmetry, and the modular theory of Von Neumann algebras [4]. On the other hand A. Jaffe, A. Lesniewski and K. Osterwalder were led by the investigation of supersymmetric field theoretical models [2] to propose (under a different name) an interesting alternative construction of the Chern character of a θ-summable Fredholm module [5] (cf. [9]).

The purpose of the present note is two-fold: first, using a $Z/2$-graded version of cyclic cohomology [6, 7], we enrich the (slightly adapted) Jaffe et al. (overall even) cocycle by a second component (odd both for the degree-of-form and the intrinsic grading)[1]. Second, we point out, as a first step towards the program [4], that the Jaffe et al. construction may be reinterpreted to pertain to "graded-KMS functionals" with respect to one-parameter automorphism groups "supersymmetric" in that they possess infinitesimal generators "with a square root." Under this aspect, [5] appears as describing the cocycle attached to the "superextension" of KMS-states of a type-I flavour. We defer to a later publication the discussion of more general cases.

[1] We in fact also treat the overall odd case (cf. 9 below)

1. Definition. Let $A = A^0 + A^1$ be a $Z/2$-graded C^*-algebra (i.e. A^0 and A^1 are closed linear spaces with $A^i A^j \subset A^{i+j} \bmod 2)$[2] possessing a unit $\mathbb{1}$. A continuous one-parameter automorphism group of A is called *supersymmetric* whenever

(i) α preserves the $Z/2$ grading:

$$\alpha_t(A^i) \subset A^i, \quad i = 1, 2, \quad t \in \mathbb{R}, \tag{1}$$

(ii) the infinitesimal generator of α:

$$D = \frac{d}{dt}\bigg|_{t=0} \alpha_t \tag{2}$$

is the square of an odd derivation δ of A, i.e. one has on the domain $\mathscr{D}_\delta$ of δ (contained in the domain $\mathscr{D}_D$ of D):

$$D = \delta^2, \tag{3}$$

$$\delta(ab) = (\delta a)b + (-1)^{\partial a} a \delta b, \quad a, b \in \mathscr{D}_\delta \cap A^0 \cap A^1, \tag{4}$$

[note that (1, 2), (1, 3), and (1, 4) hold on the *-subalgebra A_∞ of infinitely differentiable (= smooth) elements of A].

2. Definition. With (α, δ) a supersymmetric one-parameter automorphism group of the $Z/2$-graded C^*-algebra $A = A^0 + A^1$, and with $t \in R$, a (bounded) linear form φ of A is called *graded* t-KMS whenever one has[3]

$$\varphi(ba) = (-1)^{\partial a \partial b} \varphi(a \alpha_{it}(b)), \quad a, b \in A_\infty \cap A^0 \cap A^1, \tag{5}$$

and

$$\varphi \circ \alpha_t = \varphi, \quad t \in R \quad (\text{hence } \varphi \circ \delta = 0). \tag{6}$$

With these definitions one has

3. Theorem. *Given a $Z/2$-graded C^*-algebra $A = A^0 + A^1$, a supersymmetric one-parameter automorphism group (α, δ) of A in the sense* [1], *and an (even*[4]*) graded* t-KMS *form φ of A in the sense* [2], *setting, for $a_0, a_1, \ldots, a_n \in A$,*

$$\varphi^t(a_0 da_1 \ldots da_n) = t^{-\frac{n}{2}} i^n \varphi\left(a_0 \int_{I_t^n} \alpha_{it_1}(\delta a_1) \ldots \alpha_{it_n}(\delta a_n) dt\right), \tag{7}$$

where

$$I_t^n = \{t \in (t_1, \ldots, t_n);\ 0 \leqq t_1 \leqq \ldots \leqq t_n \leqq t\} \tag{8}$$

yields a cyclic cocycle of A in the sense that one has

$$\varphi^t(\beta\varepsilon + \mathbb{B}) = 0, \tag{9}$$

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[2] We shall denote by ∂a the grade of $a \in A^0 \cup A^1$, and by θ the grading automorphism of A (for $a \in A^0$, $\partial a = 0$ and $\theta a = a$; for $a \in A^1$, $\partial a = 1$ and $\theta a = -a$)

[3] Condition (6) is not independent of (5). Note that in restriction to A^0, φ is t-KMS in the usual sense

[4] Even in the sense that φ vanishes on A^1 (could be left out, cf. 9)

where[5] $\beta\varepsilon=\beta'\varepsilon-\alpha\varepsilon$ *with, for* $a_0, a_1, \dots, a_{n+1}\in A^0\cup A^1$,

$$\beta'\varepsilon(a_0da_1\dots da_{n+1})=(-1)^{\partial a_0}a_0a_1da_1\dots da_{n+1} + \sum_{j=1}^{n}(-1)^{j+\sum_{k=0}^{j}\partial a_k}a_0da_1\dots d(a_ja_{j+1})\dots da_{n+1}, \quad (10)$$

$$\alpha\varepsilon(a_0da_1\dots da_{n+1})=(-1)^{(1+\partial a_{n+1})\left(n+\sum_{k=0}^{n}\partial a_k\right)}a_{n+1}a_0da_1\dots da_n, \quad (11)$$

and $\mathbb{B}=\mathbb{B}_0A$ with

$$\mathbb{B}_0(a_0da_1\dots da_n)=\mathbb{1}da_0da_1\dots da_n+(1)^{n+\sum_{k=0}^{n}\partial a_k}a_0da_1\dots da_nd\mathbb{1}, \quad (12)$$

and $A=\sum_{k=0}^{n}\lambda^n$ *on* Ω^n, *where*

$$\lambda(a_0da_1\dots da_n)=(-1)^{(1+\partial a_n)\left(n+\sum_{k=0}^{n-1}\partial a_k\right)}a_nda_0da_1\dots da_{n-1}. \quad (13)$$

In fact one has

$$\varphi^t\circ\beta\varepsilon(a_0da_1\dots da_n)=t^{\frac{n-1}{2}}i^{n-1}\varphi\left(\delta a_0\int_{I^n}\alpha_{it_1}(\delta a_1)\dots\alpha_{it_n}(\delta a_n)dt\right) = -\varphi^t\circ\mathbb{B}(a_0da_1\dots da_n), \quad (14)$$

The proof follows from a sequence of lemmas.

4. Lemma. *With* u_i, $i=1,\dots,n$ *differentiable functions:* $\mathbb{R}\to A$, *setting* $f^t_{(1)}=\mathbb{1}$ *and*

$$f^t_{(n)}(u_1,\dots,u_n)=\int_{I^n_t}u_1(t_1)\dots u_n(t_n)dt, \qquad t\in\mathbb{R}, \quad (15)$$

we have that, with $\dot{u}_i=\frac{d}{dt}u_i$, $i=1,\dots,n$, *for* $1<k<n$, $n=1,2,\dots$:

$$\begin{aligned} f^t_{(n)}(\dot{u}_1,u_2,\dots,u_n)&=f^t_{(n-1)}(u_1u_2,u_3,\dots,u_n)-u_1(0)f^t_{(n-1)}(u_2,\dots,u_n)\\ f^t_{(n)}(u_1,\dots,\dot{u}_k,\dots,u_n)&=f^t_{(n-1)}(u_1,\dots,u_ku_{k+1},\dots,u_n)-f^t_{(n-1)}(u_1,\dots,u_{k-1}u_k,\dots,u_n)\\ f^t_{(n)}(u_1,\dots,u_{n-1},\dot{u}_n)&=f^t_{(n-1)}(u_1,\dots,u_{n-1})u_n(t)-f^t_{(n-1)}(u_1,\dots,u_{n-2},u_{n-1}u_n) \end{aligned} \quad (16)$$

and, with $\mathbb{1}$ *the constant unit function,*

$$\sum_{k=1}^{n-1}f^t_{(n+1)}(u_1,\dots,u_k,\mathbb{1},u_{k+1},\dots,u_n)=tf^t_{(n)}(u_1,\dots,u_n). \quad (17)$$

Proof. Equation (16) follows straightforwardly from (15); and (17) by termwise adding the relations obtained by making $\dot{u}_k=\mathbb{1}(u_k(t)=t\mathbb{1})$ in (16) for $k=1,\dots,n$.

5. Lemma. *Setting, for* $a_0,a_1,\dots,a_n\in A^0\cup A^1$,

$$\Psi^t(a_0da_1\dots da_n)=a_0f^t_{(n)}(\delta\underline{a}_1,\dots,\delta\underline{a}_n), \quad (18)$$

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[5] We have used the definition of the Hochschild boundary $\beta\varepsilon$ and the operator λ of $Z/2$-graded cyclic cohomology as formulated within the differential envelope $\Omega=\bigoplus_{n\in\mathbb{N}}\Omega^n$ [6]. For the formulation in terms of multilinear forms, see 6 below

where $\underline{a}_n$ *denotes the function* $t \to \alpha_{it}(a_n)$, $k=1,\ldots,n$, *(so that* $\varphi^t = t^{-\frac{n}{2}} i^n \varphi \circ \Psi^t$, *cf.* (7)*)* *we have, for*[6] $\omega \in \Omega^0 \cup \Omega^1$, $a \in A^0 \cup A^1$, $b \in A$:

$$\Psi^t(\beta'\varepsilon(ad\omega db)) - (-1)^{\partial(ad\omega)} \Psi^t(ad\omega)\Psi^t(\alpha_{it}(b)) = \delta \Psi^t(ad\omega db) - \delta a \Psi^t(\mathbb{1}\omega db), \quad (19)$$

where $\beta'\varepsilon$ *is the operator* (10).

Proof. For $a_0, a_1, \ldots, a_n \in A^0 \cup A^1$ we have, using the derivation rule (4), and relations (3) and (16),

$$\begin{aligned} &-(1)^{\partial a_0} a_0 \delta\{f^t_{(n)}(\delta \underline{a}_1, \ldots, \delta \underline{a}_n)\} \\ &= (-1)^{\partial a_0} a_0 a_1 f^t_{(n-1)}(\delta \underline{a}_2, \ldots, \delta \underline{a}_n) \\ &\quad + \sum_{j=1}^{n-1} (-1)^{j + \sum_{k=0}^{j} \partial a_k} a_0 f^t_{(n-1)}(\delta \underline{a}_1, \ldots, \delta(\underline{a}_j \underline{a}_{j+1}), \ldots, \delta \underline{a}_n) \\ &\quad - (-1)^{n-1+\sum_{k=0}^{n-1} \partial a_k} a_0 f^t_{(n-1)}(\delta \underline{a}_1, \ldots, \delta \underline{a}_{n-1}) \underline{\alpha}_{it}(a_n) \\ &= -\delta\{a_0 f^t_{(n)}(\delta \underline{a}_1, \ldots, \delta \underline{a}_n)\} + \delta a_0 f^t_{(n)}(\delta \underline{a}_1, \ldots, \delta \underline{a}_n), \end{aligned} \quad (20)$$

yielding (19) for $a_0 = a$, $a_n = b$, $\omega = da_1, \ldots, da_{n-1}$.

Equating the values for both sides of (19) of a graded t-KMS linear form φ of A then yields the first equations (14), since[7]

$$\begin{aligned} (-1)^{\partial(ad\omega)} \varphi\{\Psi^t(ad\omega)\Psi^t(\alpha_{it}(b))\} &= (-1)^{\partial(ad\omega)(\partial b + 1)} \varphi\{\Psi^t(\kappa)\Psi^t(ad\omega)\} \\ &= \varphi\{\Psi^t(\alpha(ad\omega d\kappa))\}. \end{aligned} \quad (21)$$

For the proof of the second equation (14) we need

6. Lemma. *Let* φ *be an even graded* t-KMS *linear form of* A, *and set, for* $a_0, a_1, \ldots, a_n \in A$,

$$F^t_{(n)}(a_0, a_1, \ldots, a_n) = \varphi(a_0 f^t_{(n)}(\underline{a}_1, \ldots, \underline{a}_n)). \quad (22)$$

We have the properties

$$F^t_{(n)}(a_n a_0, a_1, \ldots, a_{n-1}) = (-1)^{\partial a_n} F^t_{(n)}(a_0, a_1, \ldots, a_n), \qquad a_n \in A^0 \cup A^1, \quad (23)$$

and

$$\sum_{k=0}^{n} F^t_{(n+1)}(a_0, \ldots, a_k, \mathbb{1}, \ldots, a_n) = t F^t_{(n)}(a_0, a_1, \ldots, a_n). \quad (24)$$

Proof. Using (5) and (6) we have

$$\begin{aligned} &F^t_{(n)}(a_0, a_1, \ldots, a_n) \\ &= \int_{t \in I^n_t} \varphi\{a_0 \alpha_{it_1}(a_1) \ldots \alpha_{it_n}(a_n)\} dt \\ &= (-1)^{\partial a_n \sum_{k=0}^{n-1} \partial a_k} \int_{t \in I^n_t} \varphi\{a_n \alpha_{i(t-t_n)}(a_0) \alpha_{i(t+t_n-t_1)}(a_1) \ldots \alpha_{i(t+t_{n-1}-t_n)}(a_{n-1})\} dt, \end{aligned} \quad (25)$$

[6] Ω^0 and Ω^1 are the even, respectively odd parts of the differential envelope Ω for its total grading (sum of the n-grading and the intrinsic grading). The total grade of $\omega \in \Omega^0 \cup \Omega^1$ is denoted $\partial\omega$

[7] Note that the first equation (14) holds for all graded t-KMS linear forms of A, irrespective of parity

however, with $s=(s_1,\dots,s_n)$, $s_1=t-t_n$, $s_2=t-t_n+t_1,\dots,s_n=t-t_n+t_{n-1}$, one has $t\in I_t^n$ iff $s\in I_t^n$; and φ is even, i.e. vanishes unless $\sum_{k=0}^{n}\partial a_n=0$: this proves (23). As for (24), it immediately follows from (22) and (17).

We now check the second equation (14): rewriting definition (7) as

$$\varphi^t(a_0da_1\dots da_n)=t^{-\frac{n}{2}}i^nF_{(n)}^t(a_0,\delta a_1,\dots,\delta a_n),\tag{7.a}$$

we have from (12), since $\delta\mathbb{1}=0$, and using (23),

$$\varphi^t\circ\mathbb{B}_0(a_0da_1\dots da_n)=t^{-\frac{n+1}{2}}i^{n+1}F_{(n+1)}^t(\delta a_0,\delta a_1,\dots,\delta a_n,\mathbb{1}),\tag{26}$$

hence, since φ, and thus $F_{(n+1)}^t$, is even

$$\varphi^t\circ\mathbb{B}_0\lambda^k(a_0da_1\dots da_n)=t^{-\frac{n+1}{2}}i^{n+1}F_{(n+1)}^t(\delta a_0,\dots,\delta a_{n-k},\mathbb{1},\dots,\delta a_n),\tag{27}$$

whence our result, by termwise addition.

7. *Remark.* As explained in [6] Remark [3, 5], the following regauging of φ^t:

$$\tau^t(a_0,a_1,\dots,a_n)=(-1)^{\sum_{k\text{ odd}}\partial a_k+n\sum_{k=0}^{n}\partial a_k}\varphi^t(a_0da_1\dots da_n)\tag{28}$$

will produce the cocycle condition $(b+B)\tau^t=0$, where

$$\begin{aligned}(b\tau^t)(a_0,a_1,\dots,a_n)=&\sum_{j=0}^{n-1}(-1)^j\tau^t(a_0,\dots,a_ja_{j+1},\dots,a_n)\\&-(-1)^{n-1+\partial a_n\sum_{k=0}^{n-1}\partial a_k}\tau^t(a_na_0,a_1,\dots,a_{n-1}),\end{aligned}\tag{29}$$

and $B=AB_0$ with

$$(B_0\tau^t)(a_0,a_1,\dots,a_n)=\tau^t(\mathbb{1},a_0,\dots,a_n)\tag{30}$$

and $A=\sum_{k=0}^{n}\lambda^k$, where

$$(\lambda\tau^t)(a_0,\dots,a_n)=(-1)^{n+\partial a_n\sum_{k=0}^{n-1}\partial a_k}\tau^t(a_n,a_0,a_1,\dots,a_{n-1}).\tag{31}$$

8. *Remark.* In a quantum field theory situation we know from [8] that any extremal invariant β-KMS (temperature) state of the bosonic part A^0 extends uniquely to a state φ of A invariant for $\alpha(\mathbb{R})$ and θ and such that

$$\varphi(ba)=\varphi\{a(\alpha_{i\beta}\circ\gamma)(b)\},\qquad a,b\in A\tag{32}$$

with $\gamma=\text{id}$ but, for φ odd, (32) is a reformulation of (5).

9. *Remark.* Theorem 3 holds as well for odd (graded = ordinary) t-KMS forms. Indeed, as one checks easily, for φ odd relation (23) holds without the sign factor right hand side, whilst (26) and (27) hold as they stand.

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Note added in proof. Theorem 3 suggests the following questions:

(i) In which situations is the entire cohomology class independant of temperature (as found in [5])? If this prevails in physics, to which extent is the construction of relativistic supersymmetric field theories tantamount to computing the entire cyclic cohomology of a universal algebra (array of local type IIIs with intermediate type Is)?

(ii) Are the KMS-states the adequate generalization of elliptic operators to the non-commutative (possibly type III) frame?