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Analytic Extrapolation in L^{∞}-Norm: An Alternative Approach to "QCD Sum Rules"

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Abstract. In view of physical applications (especially to "QCD Sum Rules"), the following problem, pertaining to analytic extrapolation techniques, is studied. We are considering "amplitudes," which are (real) analytic functions in the complex plane cut along $\Gamma=[s_0,\infty)$. A model $F_0(s)$ of the amplitude is given through the values of $F_0(s)$ on some interval $\gamma=[s_2,s_1]$ (with $s_1<s_0$) and the values of its discontinuity on Γ. These values are approximate, and are supplemented by prescribed error channels, measured in L^{∞}-norm (both on Γ and γ). Investigating the compatibility between these data leads to an extremum problem which is solved up to a point where numerical methods can be implemented.

I. Introduction

Analytic extrapolation has been widely used in elementary particle theory to convey information between space-like and time-like domains of various amplitudes. In quantum chromodynamics (QCD) for instance, perturbation expansions, together with the inclusion of some non-perturbative effects, allow us to approximate the two-point functions of hadronic currents in the distant, space-like region in terms of a few parameters (the values of the so-called "condensates"). On the other hand, the discontinuity of these amplitudes in the time-like region is related to more directly measurable quantities. Although analyticity strongly correlates the values of the amplitudes in these two regions, the errors affecting both types of "data" make the correlation much looser. Any procedure aimed to build up acceptable amplitudes must take account of these errors in a reasonable way.

Several methods, generically called "QCD sum rules" have been devised to deal with this problem [1]. Most of them include the "theoretical errors" in the space-like region only at a qualitative level, and/or need (explicit or implicit) assumptions on the derivatives of the amplitudes. The application of fully controlled analytic extrapolation techniques should remedy these defects. As a matter of fact, a method of this sort has been already proposed, in which the error channels in

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the space-like region are defined through L^2-norms [2]. Although this approach is quite satisfactory from the mathematical point of view, it involves weight functions devoided of a clear physical meaning (this is especially manifest in the fact that these weight functions are not invariant against conformal transformations of the energy variable).

We have argued [3] that a fully unbiased treatment should rely on i) a well defined model of the errors, both in the space-like and the time-like ranges, ii) an intrinsic (conformally invariant) definition of the error channels, implying the use of L^∞-norms. The aim of the present paper is to study the mathematical aspects of this problem. Clearly, our results may be useful in various circumstances. Applications to "QCD sum rules" will be presented elsewhere [4].

Let us now describe our purpose more explicitly. We are considering a set of admissible amplitudes $F(s)$ expressed in terms of some (squared energy) variable s. By "admissible", it is meant that

i) $F(s)$ has the usual analyticity properties of a two-point function: it is a real analytic function in the complex s-plane cut along the time-like interval $\Gamma = [s_0, \infty)$,

ii) the asymptotic behaviour of $F(s)$ is restricted, fixing the number of subtractions in the dispersion relation between $F(s)$ and $f(s) \equiv \operatorname{Im} F(s+io)|_{s\in\Gamma}$.

A "theoretical" model $F_0(s)$ for the amplitude is given in some space-like interval $\gamma = [s_2, s_1]$, together with a positive function $\sigma_L(s)$ defining the error channel $\pm\sigma_L(s)$. We do not exclude "a priori" the case $s_2 = -\infty$. Of course, the functions $F_0(s)$ and $\sigma_L(s)$ have no need to conform to the analyticity properties i) (if $s_2 = -\infty$, the behaviour of $F_0(s) \pm \sigma_L(s)$ for real $s \to -\infty$ must however be in agreement with the asymptotics prescribed in ii)).

Similarly, an "experimental" model $f_0(s)$ is given for the imaginary part of the amplitude on Γ, together with an allowed error channel $\pm\sigma_R(s)$.

We shall say that the above data are compatible if there is at least one admissible amplitude $F(s)$ such that:

$$|F(s) - F_0(s)| \leqq \sigma_L(s) \quad \text{on} \quad \gamma \tag{I.1}$$

and

$$|f(s) - f_0(s)| \leqq \sigma_R(s) \quad \text{on} \quad \Gamma. \tag{I.2}$$

Clearly, to decide about compatibility amounts to find the infimum of the functional:

$$\sup_{s\in\gamma} \frac{|F(s) - F_0(s)|}{\sigma_L(s)} \tag{I.3}$$

in the set of admissible functions $F(s)$ subjected to the constraint (I.2).

It is worth mentioning that a similar problem was considered in ref. [5]. There however, a constraint of the type (I.2) was imposed on the modulus of $F(s)$, not on its imaginary part. This made the problem a more difficult one, and a solution was provided in particular cases only. It turns out that by proceeding here essentially in the same spirit, one can "solve" our problem up to a point where numerical methods may be implemented. We have first to make sure that the infimum of the functional (I.3) is attained. This is the main result of Sect. II below (to avoid inessential complications, we first restrict ourselves to the unsubtracted case). In

Sect. III, we derive the structural properties of the extremal functions in the case of a bounded interval γ. The imaginary parts of these functions on Γ are found to have a rather simple characterization. The extensions to subtracted dispersion relations and unbounded intervals γ (especially relevant to QCD applications) are discussed in Sect. IV. Although the former extension is very easy, it is not so for the latter: one then faces the usual difficulties encountered when the data region in the holomorphy domain and the boundary of this domain are not disconnected sets. In fact, we shall content ourselves with a strategy to reach the infimum from below. Some concluding comments are made in Sect. V.

II. Preliminary Results

We have first to formulate our problem in precise mathematical terms. There are given:

i) on $\Gamma=[s_0,\infty)$ (with $s_0>0$): a real (not necessarily continuous) function $f_0(x)$ and a continuous function $\sigma_R(x)$, strictly positive for $x>s_0(\sigma_R(s_0)\neq 0$ is not required). It is assumed that:

$$f_0(x)/x\in L^1(\Gamma), \tag{II.1}$$

$$\sigma_R(x)/x\in L^1(\Gamma). \tag{II.2}$$

ii) On $\gamma=[s_2,s_1]$ (with $s_1<s_0$ and possibly $s_2=-\infty$): a continuous, real function $F_0(y)$ and a continuous strictly positive function $\sigma_L(y)$.

Let $\mathscr{F}$ be the class of real analytic functions $F(s)$ of the form:

$$F(s)=\frac{1}{\pi}\int_{s_0}^{\infty}dx\,\frac{f(x)}{x-s}\quad(s\in\mathbb{C}\setminus\Gamma),\quad\text{where}\quad\frac{f(x)-f_0(x)}{\sigma_R(x)}\in L^\infty(\Gamma). \tag{II.3}$$

One notices that our assumptions (II, 1–2) are sufficient to make full sense of this definition of $\mathscr{F}$ through unsubtracted dispersion relations. One also remarks that $\mathscr{F}$ is a convex subset of the space of holomorphic functions on $\mathbb{C}\setminus\Gamma$.

We now define the two functionals over $\mathscr{F}$:

$$\chi_R[F]=\|\Delta(x)\|_\infty\quad\text{(the norm is the ess sup-norm on } L^\infty(\Gamma)), \tag{II.4}$$

$$\chi_L[F]=\|\psi(y)\|_\infty\quad\text{(the norm is the sup-norm on } C(\gamma)), \tag{II.5}$$

where

$$\Delta(x)=\frac{f(x)-f_0(x)}{\sigma_R(x)},\quad \psi(y)=\frac{F(y)-F_0(y)}{\sigma_L(y)}. \tag{II.6}$$

Let $\mathscr{D}$ be the set of points (χ_R,χ_L) in $\mathbb{R}_+^2$ such that $\{F\in\mathscr{F}\,|\,\chi_R[F]\leqq\chi_R,\ \chi_L[F]\leqq\chi_L\}$ is not empty. The physical problem is to determine if $\mathscr{D}$ contains points $(\chi_R\leqq 1,\chi_L\leqq 1)$. Clearly, it is enough to construct the "lower boundary" $\partial\mathscr{D}_-$ of $\mathscr{D}$, i.e. to solve the extremum problem:

$$\chi_-(\chi)=\inf_{\substack{F\in\mathscr{F}\\ \chi_R[F]\leqq\chi}}\chi_L[F] \tag{II.7}$$

for all $\chi \geqq 0$. $\partial\mathscr{D}_-$ is the graph of the function $\chi_-(\chi)$. It will be shown below that the infimum in Eq. (II.7) is attained.

Our aim is then i) to describe the general properties of the function $\chi_-(\chi)$ ii) to describe the structural properties of the extremal functions (i.e. of those F which saturate the bound $\chi_-(\chi)$). To this end, it is convenient to introduce the "discrepancy function":

$$D_0(y) = F_0(y) - \frac{1}{\pi} \int_{s_0}^{\infty} dx \frac{f_0(x)}{x-y}. \tag{II.8}$$

Then

$$\psi(y) = \frac{1}{\sigma_L(y)} \left[\frac{1}{\pi} \int_{s_0}^{\infty} dx \frac{\sigma_R(x)}{x-y} \Delta(x) - D_0(y) \right], \tag{II.9}$$

which allows us to consider the functional (II.5) over $\mathscr{F}$ as a functional $\chi_L[\Delta]$ over $L^\infty(\Gamma)$. The extremum problem (II.7) is now restated as:

$$\chi_-(\chi) = \inf_{\substack{\Delta \in L^\infty(\Gamma) \\ \|\Delta\|_\infty \leqq \chi}} \chi_L[\Delta], \quad \text{where} \quad \chi_L[\Delta] = \|\psi(y)\|_\infty. \tag{II.10}$$

A. *The Infimum is Attained*

When $s_2 > -\infty$, $\|\psi(y)\|_\infty < \infty$ for all Δ in $L^\infty(\Gamma)$, and $\chi_-(\chi)$ is a well defined (finite) function on the whole interval $[0, \infty)$. This is not necessarily so in the case $s_2 = -\infty : \chi_-(\chi) = \infty$ cannot be excluded "a priori." We are of course assuming that the behaviours of $F_0(y)$ and $\sigma_L(y)$ when $y \to -\infty$ are such that there exist $F \in \mathscr{F}$ for which $\chi_L[F] < \infty$. This clearly implies that $\chi_-(\chi)$ is finite on some (maximal) interval (χ_0, ∞) with $0 \leqq \chi_0 < \infty$.

Proposition 1. $\forall \chi > \chi_0$, *the infimum in Eq. (II.10) is attained for a (non-necessarily unique) function* $\Delta_\chi \in L^\infty(\Gamma) : \chi_-(\chi) = \chi_L[\Delta_\chi]$.

Proof. From the definition of $\chi_-(\chi)$, there is a sequence $\{\Delta_n\}$ in $L^\infty(\Gamma)$ with $\|\Delta_n\|_\infty \leqq \chi \forall n$ and such that:

$$\lim_{n \to \infty} \chi_L[\Delta_n] = \chi_-(\chi). \tag{II.11}$$

This sequence is contained in B_χ, the ball of radius χ in $L^\infty(\Gamma)$. Now, according to the Banach–Alaoglu theorem [6], B_χ is compact in the weak-* topology of $L^\infty(\Gamma)$. Since $L^\infty(\Gamma)$ is the dual of $L^1(\Gamma)$, this means that there exists a subsequence $\{\Delta_{n_r}\} \subset \{\Delta_n\}$ and some $\Delta_\chi \in B_\chi$ such that:

$$\lim_{r \to \infty} \int_{s_0}^{\infty} dx\, g(x) \Delta_{n_r}(x) = \int_{s_0}^{\infty} dx\, g(x) \Delta_\chi(x) \quad \forall g \in L^1(\Gamma). \tag{II.12}$$

In particular, the assumption (II.2) allows us to take $g(x) = \sigma_R(x)/(x-y)$ in Eq. (II.12), as long as $y \in \gamma$:

$$\lim_{r \to \infty} \int_{s_0}^{\infty} dx \frac{\sigma_R(x)}{x-y} \Delta_{n_r}(x) = \int_{s_0}^{\infty} dx \frac{\sigma_R(x)}{x-y} \Delta_\chi(x) \quad \forall y \in \gamma. \tag{II.13}$$

Let $\psi_r(y)$ and $\psi_\chi(y)$ be the functions respectively associated to Δ_{n_r} and Δ_χ through

Eq. (II.9). Then, from Eq. (II.11):

$$\overline{\lim_{r\to\infty}} |\psi_r(y)| \leqq \chi_-(\chi) \quad \forall y \in \gamma \tag{II.14}$$

and from Eq. (II.13):

$$\lim_{r\to\infty} |\psi_r(y) - \psi_\chi(y)| = 0 \quad \forall y \in \gamma \tag{II.15}$$

(notice that the latter limit is not necessarily uniform on γ if this interval is unbounded).

Hence, taking the $\overline{\lim}_{r\to\infty}$ in $|\psi_\chi(y)| \leqq |\psi_r(y)| + |\psi_r(y) - \psi_\chi(y)|$, we deduce that:

$$|\psi_\chi(y)| \leqq \chi_-(\chi) \quad \forall y \in \gamma, \tag{II.16}$$

which yields:

$$\chi_L[\Delta_\chi] = \sup_{y\in\gamma} |\psi_\chi(y)| \leqq \chi_-(\chi). \tag{II.17}$$

Since $\chi_L[\Delta_\chi] < \chi_-(\chi)$ would contradict the very definition of $\chi_-(\chi)$, the proof is completed.

To $\Delta_\chi(x)$ corresponds the extremal function in $\mathscr{F}$:

$$F_\chi(x) = \frac{1}{\pi}\int_{s_0}^{\infty} dx \frac{f_0(x)}{x-s} + \frac{1}{\pi}\int_{s_0}^{\infty} dx \frac{\sigma_R(x)}{x-s} \Delta_\chi(x). \tag{II.18}$$

B. *Properties of the Function* $\chi_-(\chi)$

They essentially reflect the convexity of the set $\mathscr{F}$ and of the norms:

Theorem 1. $\chi_-(\chi)$ *is a non-increasing, convex (and thus continuous) function on* (χ_0, ∞).

Proof. That the function $\chi_-(\chi)$ is non-increasing is a trivial consequence of the definition (II.10), through set inclusion. In order to prove that it is convex, let us take any χ_1 and χ_2 with $\chi_0 < \chi_1 < \chi_2$, together with corresponding saturating functions $\Delta_1(x)$ and $\Delta_2(x)$:

$$\begin{cases} \chi_-(\chi_i) = \chi_L[\Delta_i], \\ \|\Delta_i\|_\infty \leqq \chi_i, \qquad (i = 1, 2). \end{cases} \tag{II.19}$$

Consider

$$\bar{\chi} = \lambda\chi_1 + (1-\lambda)\chi_2 \quad \text{with} \quad 0 \leqq \lambda \leqq 1, \tag{II.20}$$

and define

$$\bar{\Delta}(x) = \lambda\Delta_1(x) + (1-\lambda)\Delta_2(x). \tag{II.21}$$

If $\psi_1(y)$, $\psi_2(y)$ and $\bar{\psi}(y)$ are respectively constructed from $\Delta_1(x)$, $\Delta_2(x)$ and $\bar{\Delta}(x)$ through Eq. (II.9), then:

$$\bar{\psi}(y) = \lambda\psi_1(y) + (1-\lambda)\psi_2(y). \tag{II.22}$$

Hence:

$$\begin{aligned} \chi_L[\bar{\Delta}] &= \|\bar{\psi}\|_\infty \leqq \lambda\|\psi_1\|_\infty + (1-\lambda)\|\psi_2\|_\infty = \lambda\chi_L[\Delta_1] + (1-\lambda)\chi_L[\Delta_2] \\ &\leqq \lambda\chi_-(\chi_1) + (1-\lambda)\chi_-(\chi_2). \end{aligned} \tag{II.23}$$

On the other hand, Eq. (II.21) implies:

$$\|\bar{\Delta}\|_\infty \leqq \lambda \|\Delta_1\|_\infty + (1-\lambda)\|\Delta_2\|_\infty \leqq \lambda\chi_1 + (1-\lambda)\chi_2 = \bar{\chi}. \tag{II.24}$$

Therefore $\chi_-(\bar{\chi}) \leqq \chi_L[\bar{\Delta}]$, which, together with Eq. (II.23), entails the announced convexity property:

$$\chi_-(\lambda\chi_1 + (1-\lambda)\chi_2) \leqq \lambda\chi_-(\chi_1) + (1-\lambda)\chi_-(\chi_2). \tag{II.25}$$

Let us add that $F_\chi(y)$ may accidentally coincide with the data function $F_0(y)$ if this function has itself an analytic continuation in the whole of $\mathbb{C}\backslash\Gamma$ and if χ is large enough. In such a case, the function $\chi_-(\chi)$ will vanish identically above some critical value of χ.

III. Properties of the Extremal Functions for a Bounded Interval γ

In this section, we need to assume that γ is bounded ($s_2 > -\infty$). Only then we shall be able to gain detailed information on the extremal functions. What can be done in the unbounded case will be presented in sub-sect. IV.A. Of course, we are also assuming that the extremal functions $F_\chi(s)$ under consideration do not identify with the data function $F_0(y)$ on γ, i.e. that $\chi_-(\chi) > 0$.

In order to describe conveniently certain properties of $F_\chi(s)$, it is useful to introduce the number of its "effective extrema." This essentially amounts to count the number of times the continuous function $\psi(y) \equiv [F_\chi(y) - F_0(y)]/\sigma_L(y)$ reaches its extrema $\chi_-(\equiv \chi_-(\chi))$ or $-\chi_-$ on γ, but counting only for 1 any set of *successive* extrema of the same sign (and not excluding the possibility that $\psi(y) = \pm\chi_-$ on a whole subinterval of γ). To have a precise counting procedure, let us define two decreasing sequences $\{u_j\}$ and $\{v_j\}$ in $\gamma = [s_2, s_1]$ as follows (Fig. 1):

$$\begin{cases} u_1 = \text{largest value of } y \leqq s_1 \text{ for which } |\psi(y)| = \chi_-; \\ \qquad \text{we set } \varepsilon = \psi(u_1)/\chi_- \ (= \pm 1) \\ u_j = \text{largest value of } y \leqq u_{j-1} \text{ for which } \psi(y) = (-)^{j+1}\varepsilon\chi_- \\ \qquad (j = 2, 3, \ldots). \end{cases} \tag{III.1}$$

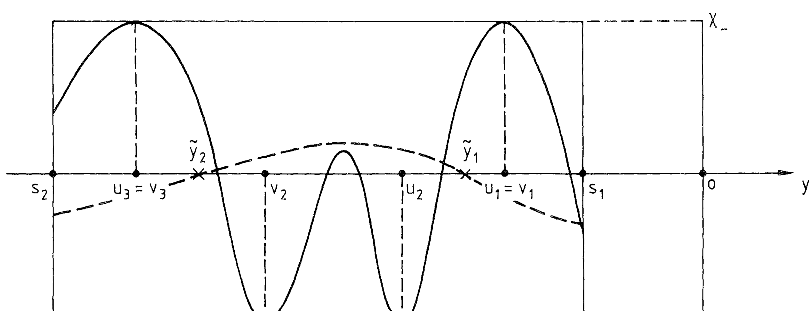

Fig. 1. Construction of the sequences $\{u_j\}$, $\{v_j\}$ and typical graphs of the functions $\psi(y)$ (full line) and $\delta\psi(y)$ (broken line). Here: number of effective extrema $n = 3$ and $\varepsilon = 1$

$$v_j = \text{smallest value of } y \geqq u_{j+1} \text{ for which } \psi(y) = (-)^{j+1}\varepsilon\chi_- \quad (j = 1, 2, \ldots). \tag{III.2}$$

Evidently:

$$u_{j+1} < v_j \leqq u_j \quad \forall j = 1, 2, \ldots \tag{III.3}$$

(the strict $<$ sign in Eq. (III.3) follows from the continuity of $\psi(y)$). Moreover, the two sequences must be finite. Otherwise, the limit of $\{u_j\}$ would belong to the *closed* interval γ and would be a point of discontinuity of $\psi(y)$ (since $\psi(u_{j+1}) = -\psi(u_j) \forall j$).

Then, let n be the largest index j for which u_j exists (and v_n be defined by setting $u_{n+1} = s_2$ in Eq. (III.2)). One observes here that our "compactness" assumptions $s_2 > -\infty$ and $\sigma_L(s_2) \neq 0$ are essential to guarantee that n is finite. In general, this property would be lost by relaxing any one of those assumptions.

Definition 1. The number of *effective extrema* of $F_\chi(s)$ is the integer $n \geqslant 1$ just constructed.

We are now in a position to establish the

Lemma 1. *Let χ be such that $\chi_-(\chi) \neq 0$. Then any function Δ_χ solving the problem (II.10) obeys the condition*:

$$|\Delta_\chi(x)| = \chi \quad \text{a.e. on } \Gamma \quad (\text{and thus } \|\Delta_\chi\|_\infty = \chi). \tag{III.4}$$

In other words, the discontinuity on Γ of any extremal function $F_\chi(s)$ not identical with the data function saturates its bound almost everywhere.

Proof. Assume Eq. (III.4) to be wrong. Then $\exists d > 0$ and a set $E \subset \Gamma$ with positive Lebesgue measure $\mu(E)$ such that:

$$|\Delta_\chi(x)| \leqq \chi - d \quad \text{a.e. on} \quad E. \tag{III.5}$$

Let n be the number of effective extrema of $F_\chi(s)$. Consider the covering of Γ by the intervals of length $\alpha = \mu(E)/2n$:

$$J_l = [s_0 + l\alpha, s_0 + (l+1)\alpha], \quad l = 0, 1, 2, \ldots. \tag{III.6}$$

Since $\mu(E \cap J_l) \leqq \alpha$, there are at least $2n$ such intervals, J_{l_k}, for which $\mu(E \cap J_{l_k}) > 0$. Pick any n of them with the proviso that there remains no pair of adjacent ones, and excluding J_0. Denote by I_k these closed, non-overlapping intervals. Using $\mu(E \cap I_k) > 0$, and the compactness of the I_k's, it is readily shown that in each I_k, there is a point x_k such that

$$\mu(E \cap [x_k - \eta, x_k + \eta]) > 0 \quad \forall \eta > 0. \tag{III.7}$$

For an appropriate choice of index assignment:

$$s_0 < x_1 < x_2 < \cdots < x_n \text{ (our construction implies strict} < \text{signs).} \tag{III.8}$$

We also set:

$$\begin{cases} I_k^\eta = [x_k - \eta, x_k + \eta], \\ \mu_k^\eta = \mu(E \cap I_k^\eta), \qquad (k = 1, \ldots, n) \\ X_k^\eta(x) = \text{characteristic function of } E \cap I_k^\eta, \end{cases} \tag{III.9}$$

and observe that $0 < \mu_k^\eta \leqq 2\eta$.

Let us now define, for $\lambda > 0$ and any real set $\{r_1, r_2, \ldots, r_n\}$ with $r_1 = 1$:

$$\tilde{\Delta}(x) = \Delta_\chi(x) - \lambda\varepsilon\pi \sum_{k=1}^{n} \frac{r_k}{\mu_k^\eta} \cdot \frac{X_k^\eta(x)}{\sigma_R(x)} \tag{III.10}$$

(here: $\varepsilon = \pm 1$ as specified in Eq. (III.1)). First of all, Eq. (III.5) implies:

$$\|\tilde{\Delta}\|_\infty \leqq \chi \text{ as soon as } \lambda \text{ is small enough.} \tag{III.11}$$

Next, using $\tilde{\Delta}(x)$ in place of $\Delta_\chi(x)$ in Eq. (II.9) gives:

$$\tilde{\psi}(y) = \psi(y) - \frac{\lambda\varepsilon}{\sigma_L(y)} \sum_{k=1}^{n} \frac{r_k}{\mu_k^\eta} \int_{I_k^\eta} dx \frac{X_k^\eta(x)}{x - y}. \tag{III.12}$$

Since for $y < s_0$:

$$\frac{1}{x_k + \eta - y} < \frac{1}{\mu_k^\eta} \int_{I_k^\eta} dx \frac{X_k^\eta(x)}{x - y} < \frac{1}{x_k - \eta - y}, \tag{III.13}$$

we can rewrite Eq. (III.12) as:

$$\delta\psi(y) \equiv \tilde{\psi}(y) - \psi(y) = -\lambda\varepsilon\left[\frac{R(y)}{\sigma_L(y)} + O(\eta)\right], \tag{III.14}$$

where

$$R(s) = \sum_{k=1}^{n} \frac{r_k}{x_k - s}, \tag{III.15}$$

and $O(\eta)$ is uniform in y over γ. In a final step, choose $\{y_1, y_2, \ldots, y_{n-1}\}$ in γ so that $u_{j+1} < y_j < v_j$ (see Eq. (III.3)). According to Appendix A, the r_k's in Eq. (III.15) can be adjusted in such a way that $R(s)$ has exactly $(n-1)$ simple zeros at $s = y_j$. Moreover, since $r_1 = 1$:

$$\begin{gathered}(-1)^j R(y) > 0 \quad \text{for} \quad y_{j+1} < y < y_j \\ (j = 0, 1, \ldots, n; \text{ we set } y_0 = s_1, y_{n+1} = s_2).\end{gathered} \tag{III.16}$$

For η small enough the function $\delta\psi(y)$ has, like $R(y)$, exactly $(n-1)$ simple zeros on γ. They are now located at $\tilde{y}_j = y_j + O(\eta)$ with:

$$u_{j+1} < \tilde{y}_j < v_j \quad (j = 1, 2, \ldots, n-1). \tag{III.17}$$

Using the continuity of $\psi(y)$ and $\delta\psi(y)$ on γ, one readily infers from Eqs. (III.1–3), (III.14) and (III.16–17) that there are positive v and ρ such that (see Fig. 1):

$$\left\{\begin{array}{l} -\chi_- < (-1)^{j+1}\varepsilon\psi(y) \leqq \chi_- - v \\ 0 \leqq (-)^{j+1}\varepsilon\delta\psi(y) \leqq \lambda\rho \end{array}\right. \quad \text{for} \quad y \in]u_{j+1}, \tilde{y}_j], \tag{III.18}$$

$$\left\{\begin{array}{l} -\chi_- + v \leqq (-)^{j+1}\varepsilon\psi(y) \leqq \chi_- \\ -\lambda\rho \leqq (-)^{j+1}\varepsilon\delta\psi(y) < 0 \end{array}\right. \quad \text{for} \quad y \in]\tilde{y}_j, u_j]. \tag{III.19}$$

Choosing $\lambda < \nu/\rho$, we deduce:

$$-\chi_- < (-)^{j+1}\varepsilon\tilde{\psi}(y) < \chi_- \quad \text{for} \quad y \in]y_{j+1}, y_j] \quad (j=1,\ldots,n-1). \quad \text{(III.20)}$$

Hence $|\tilde{\psi}(y)| < \chi_-$ on the whole interval γ, and $\chi_L[\tilde{\Delta}] < \chi$. This, together with Eq. (III.11), contradicts the fact that $F_\chi(s)$ is an extremal function. Therefore the hypothesis (III.5) cannot be true, and the proof is completed.

In fact, any function $\Delta_\chi(x)$ solving the problem (II.10) must belong to a much more restricted class than the one defined by Eq. (III.4). To show this, let us introduce a partition of Γ in two sets Γ_+ and Γ_- defined by:

$$\Gamma_\pm = \{x \in \Gamma \,|\, \Delta_\chi(x) = \pm\chi\}. \quad \text{(III.21)}$$

Ignoring the trivial (but possible!) cases where $\mu(\Gamma_+) = 0$ or $\mu(\Gamma_-) = 0$, we need the following

Lemma 2. *Suppose that there are p intervals $K_i = [a_i, b_i] \subset \Gamma$ subjected to the conditions*:

$$\begin{cases} b_i \leqq a_{i+1} & (i = 1, 2, \ldots, p-1;\ b_p \text{ may be infinite}), \\ \text{for} \quad \bar{\varepsilon} = + \text{ or } -, & \mu(K_i \cap \Gamma_{\bar{\varepsilon}(-)^{i+1}}) > 0 \quad \forall i = 1, \ldots, p. \end{cases} \quad \text{(III.22)}$$

Then

α) $p \leqq n$
β) if $p = n$: $\bar{\varepsilon}\varepsilon = -1$

(again, n is the number of effective extrema of $F_\chi(s)$ and ε is defined in Eq. (III.1)).

Proof. As in the proof of Lemma 1, one proceeds "per absurdo" and uses a "perturbation" argument based on Appendix A. Therefore, we shall content ourselves with a very brief sketch. Assume first $p \geqq n+1$.

i) If $\bar{\varepsilon}\varepsilon = 1$, one selects the intervals $K_1, \ldots, K_n$ and shows that by a suitable decrease of $|\Delta_\chi(x)|$ on each set $K_i \cap \Gamma_{\bar{\varepsilon}(-)^{i+1}}$, $|\psi(y)|$ can be decreased on the whole interval γ (Eq. (A.3) is used in this step).
ii) If $\bar{\varepsilon}\varepsilon = -1$, the same procedure works with the intervals $K_2, \ldots, K_{n+1}$.
In both cases, we run into contradiction. Hence $p \leqq n$.

When $p = n$, the argument i) is still valid and leads again to a contradiction. Hence $\bar{\varepsilon}\varepsilon = -1$.

Lemmas 1 and 2 allow us to deduce in a fairly simple way the general structure of the function $\Delta_\chi(x)$.

Definition 2. A function defined on Γ will be called a *p-step function* if it is right continuous, takes only the values $\pm\chi$ and has $(p-1)$ jumps.

Theorem 2. *Let χ be such that $\chi_-(\chi) \neq 0$ and $F_\chi(s)$ an extremal function with n effective extrema. Then its imaginary part $f_\chi(x) = \operatorname{Im} F_\chi(x+io)$ on Γ has the form*:

$$f_\chi(x) = f_0(x) + \sigma_R(x)\Delta_\chi(x), \quad \text{(III.23)}$$

where $\Delta_\chi(x)$ is a m-step function. Moreover:

$$m \leqq n, \tag{III.24}$$

$$\text{if} \quad m = n: \varepsilon \Delta_\chi(s_0) < 0. \tag{III.25}$$

Proof. Let t_+ (respectively t_-) be the smallest value of $x \geqq s_0$ for which $\Delta_\chi(x) = +\chi$ (respectively $-\chi$) a.e. on $[s_0, t_+]$ (respectively $[s_0, t_-]$). It is easily deduced from Lemmas 1 and 2α) that either $t_+ > 0$, $t_- = 0$, or $t_+ = 0$, $t_- > 0$. We then define a non-decreasing sequence $\{t_i\}$ in Γ by:

$$\begin{aligned} &t_i = t_{\bar\varepsilon} > 0 \ (\bar\varepsilon = \pm); \text{ we can set } \Delta_\chi(s_0) = \bar\varepsilon\chi, \\ &t_i = \text{smallest value of } x \geqq t_{i-1} \text{ such that } \Delta_\chi(x) = \bar\varepsilon(-)^{i+1}\chi \\ &\qquad \text{a.e. on } [t_{i-1}, t_i] \ (i = 2, 3, \ldots). \end{aligned} \tag{III.26}$$

Applying again Lemmas 1 and 2α), we find that:

$$t_i > t_{i-1} \quad \forall i = 2, 3, \ldots \tag{III.27}$$

and that the sequence must stop at some index $i = m - 1$ with $m \leqq n$ (i.e. $t_m = \infty$). This just means that $\Delta_\chi(x)$ is a m-step function subjected to the condition (III.24). As for the result (III.25), it immediately follows from Lemma 2β).

IV. Extensions

A. Unbounded Interval γ

As already mentioned in Sect. III, nothing prevents the number of effective extrema of $F_\chi(s)$ to become infinite when $\gamma = (-\infty, s_1]$. More than that, for the applications we have in mind [3, 4], the data functions $F_0(y)$, $f_0(x)$ and the error functions $\sigma_L(y)$, $\sigma_R(x)$ are such that this phenomenon is likely to occur when χ is large enough. In this case, very little can be said about the corresponding function $\Delta_\chi(x)$. In particular, there is no guarantee that it is still a step function (with possibly an infinite number of jumps). We were even unable to prove that $|\Delta_\chi(x)|$ saturates its bound χ a.e. on Γ (and actually we see no compelling reason for this to be true in general).

Although we know that the infimum χ_- in Eq. (II.10) is still attained, we are lacking of a practical procedure to construct it, since we cannot restrict at once the class of Δ's to step functions. Fortunately, a partial way out is provided by the following fact: χ_- can be approached arbitrarily close *from below* by solving as previously the extremum problem (II.10) for bounded sub-intervals $[s^{(i)}, s_1]$ (which involves only step functions Δ) and letting $s^{(i)}$ go to $-\infty$. This results from:

Theorem 3. *Consider an arbitrary decreasing sequence $\{s^{(i)}\} \to -\infty$ and define:*

$$\chi_-^{(i)} = \inf_{\substack{\Delta \in L^\infty(\Gamma) \\ |\Delta|_\infty \leqq \chi}} \chi_L^{(i)}[\Delta], \tag{IV.1}$$

where

$$\chi_L^{(i)}[\Delta] = \sup_{y \in \gamma^{(i)} = [s^{(i)}, s_1]} |\psi(y)| \tag{IV.2}$$

and $\psi(y)$ is still given by Eq. (II.9). Then $\{\chi_-^{(i)}\}$ is a non-decreasing sequence and $\lim_{i \to \infty} \chi_-^{(i)} = \chi_-$.

Proof. For all $\Delta \in L^\infty(\Gamma)$:

$$\chi_L^{(i)}[\Delta] \leqq \chi_L^{(i+1)}[\Delta] \leqq \chi_L[\Delta] \equiv \sup_{y \in \gamma = (-\infty, s_1]} |\psi(y)| \tag{IV.3}$$

by set inclusion. This implies that the sequence $\{\chi_-^{(i)}\}$ is non-decreasing and bounded from above by χ_-. Thus it has a limit $\chi_-^{(\infty)} \leqq \chi_-$. One has to show that $\chi_-^{(\infty)} = \chi_-$. Suppose that this is wrong:

$$\chi_-^{(\infty)} < \chi_- . \tag{IV.4}$$

From Sect. II.A, we know that for each i, $\chi_-^{(i)}$ is attained for some (step) function $\Delta^{(i)}$. Using again the Banach–Alaoglu theorem, we can assert that a subsequence $\{\Delta^{(i_r)}\} \subset \{\Delta^{(i)}\}$ exists, which converges to some $\Delta^{(\infty)}$ (with $\|\Delta^{(\infty)}\|_\infty \leqq \chi$) in the weak-* topology of $L^\infty(\Gamma)$. Define $\psi^{(r)}(y)$ and $\psi^{(\infty)}(y)$ from $\Delta^{(i_r)}$ and $\Delta^{(\infty)}$ as in Eq. (II.9). Then, given *any* $y \in \gamma$:

$$\lim_{r \to \infty} \psi^{(r)}(y) = \psi^{(\infty)}(y) \tag{IV.5}$$

by the weak-* convergence (and noticing that $y \in \gamma^{(i_r)}$ for i_r large enough). Now, for $y \in \gamma^{(i_r)}$:

$$|\psi^{(r)}(y)| \leqq \chi_L^{(i_r)}[\Delta^{(i_r)}] = \chi_-^{(i_r)} \leqq \chi_-^{(\infty)}. \tag{IV.6}$$

Using Eq. (IV.5), we deduce that for any $y \in \gamma$:

$$|\psi^{(\infty)}(y)| \leqq \chi_-^{(\infty)}. \tag{IV.7}$$

Hence $\chi_L[\Delta^{(\infty)}] \leqq \chi_-^{(\infty)}$ and, according to our assumption (IV.4):

$$\chi_L[\Delta^{(\infty)}] < \chi_- , \tag{IV.8}$$

which is a contradiction.

B. *The Subtracted Case*

The extension of our analysis to subtracted amplitudes requires only minor changes in definitions and proofs. The results remain essentially the same.

Consider the N-subtracted case ($N \geqq 1$). This means that Eqs. (II.1–2) must be replaced by:

$$\begin{cases} f_0(x)/x^{N+1} \in L^1(\Gamma), \\ \sigma_R(x)/x^{N+1} \in L^1(\Gamma), \end{cases} \tag{IV.9}$$

whereas the equations defining the class $\mathscr{F}$ are now:

$$\begin{aligned} F(s) &= \sum_{n=0}^{N-1} A_n s^n + \frac{s^N}{\pi} \int_{s_0}^{\infty} dx \frac{f(x)}{x^N(x-s)}, \\ \Delta(x) &\equiv \frac{f(x) - f_0(x)}{\sigma_R(x)} \in L^\infty(\Gamma) \end{aligned} \tag{IV.10}$$

(we have fixed the subtraction point at $s=0$ for pure notational convenience). Equation (II.9) for $\psi(y) \equiv [F(y) - F_0(y)]/\sigma_L(y)$ becomes:

$$\psi(y) = \frac{1}{\sigma_L(y)} \left[\sum_{n=0}^{N-1} A_n y^n + \frac{y^N}{\pi} \int_{s_0}^{\infty} dx \frac{\sigma_R(x)}{x^N(x-y)} \Delta(y) - D_0(y) \right] \tag{IV.11}$$

with

$$D_0(y) = F_0(y) - \frac{y^N}{\pi} \int_{s_0}^{\infty} dx \frac{f_0(x)}{x^N(x-y)}, \tag{IV.12}$$

so that the functional $\chi_L[F] \equiv \|\psi(y)\|_\infty$ now appears as a functional $\chi_L[\Delta, \vec{A}]$ over $L^\infty(\Gamma) \times \mathbb{R}^N$ (with $\vec{A} = (A_0, A_1, \ldots, A_{N-1}) \in \mathbb{R}^N$). The extremum problem (II.10) is changed into:

$$\chi_-(\chi) = \inf_{\substack{\|\Delta\|_\infty \leqq \chi \\ \vec{A} \in \mathbb{R}^N}} \chi_L[\Delta, \vec{A}]. \tag{IV.13}$$

We are assuming consistency of the data, that is: $\chi_-(\chi) < \infty$ for χ larger than some $\chi_0 \geqq 0$. Again, given $\chi > \chi_0$, the infimum (IV.13) is attained for some function $\Delta_\chi(\chi) \in L^\infty(\Gamma)$ and some subtraction constants $\vec{A}_\chi$. This is established by modifying the proof of Proposition 1 as follows. Since $\chi > \chi_0$, there is a $F^{(0)} \in \mathscr{F}$ such that $\|\psi^{(0)}(y)\|_\infty = K < \infty$. Then $\chi_-(\chi) \leqq K$ and one is allowed to include the supplementary constraint $\|\psi(y)\|_\infty \leqq K$ in the right-hand side of Eq. (IV.13). From Eq. (IV.11), this constraint implies:

$$\left| \sum_{n=0}^{N-1} A_n y^n \right| \leqq K \sigma_L(y) + \frac{|y|^N}{\pi} \chi \int_{s_0}^{\infty} dx \frac{\sigma_R(x)}{x^N(x-y)} + |D_0(y)| \quad \forall y \in \gamma. \tag{IV.14}$$

Hence, by assigning to y N arbitrary (but distinct) values $z_i \in \gamma$:

$$\left| \sum_{n=0}^{N-1} A_n z_i^n \right| \leqq C_i, \quad i = 1, \ldots, N, \tag{IV.15}$$

where the C_i's depend only on χ. Eq. (IV.15) reads as well:

$$|\vec{A}\ \vec{z}_i| \leqq C_i, \quad i = 1, \ldots, N, \tag{IV.16}$$

where the N vectors $\vec{z}_i = (1, z_i, \ldots, z_i^{N-1})$ are linearly independent. From this, one readily deduces that $\vec{A}$ is contained in some ball of finite radius R_χ. Therefore, Eq. (IV.13) can be rewritten as:

$$\chi_-(\chi) = \inf_{\substack{\|\Delta\|_\infty \leqq \chi \\ \|\vec{A}\| \leqq R_\chi}} \chi_L[\Delta, \vec{A}]. \tag{IV.17}$$

The proof then proceeds exactly as in Sect. II.A, by noticing that the set $\{\Delta \in L^\infty(\Gamma) \mid \|\Delta\|_\infty \leqq \chi\} \times \{\vec{A} \in \mathbb{R}^N \mid \|\vec{R}\| \leqq R_\chi\}$ is compact in the product topology of $L^\infty(\Gamma) \times \mathbb{R}^N$ (the topology of $L^\infty(\Gamma)$ being still taken in the weak-$*$ sense).

The existence of saturating function $\Delta_\chi(x)$ and subtraction constants $\vec{A}_\chi$ being now established, it is easily checked that Theorem 1 and Theorem 2 (up to Eq. (III.24)) keep unchanged. As for Theorem 3, it still holds with obvious alterations in the definition (IV.1–2) of $\chi_-^{(i)}$.

Of course, our contention that allowing for subtractions is rather trivial at the purely mathematical level does not mean that it is harmless in numerical applications. The occurrence of the extra parameters A_i in Eqs. (IV.11–13) may lead to noticeable difficulties in actual minimization procedures.

V. Conclusions

We have defined the compatibility of "space-like and time-like data" by introducing L^∞ norms. The answer to the question of compatibility then amounts to finding the infimum of some functional, which defines a curve in $\mathbb{R}^2_+$ (the lower boundary $\partial\mathscr{D}_-$ of the allowed region $\mathscr{D}$ for the two error parameters χ_R, χ_L). We have shown that the infimum was attained indeed, and we have derived several structural properties of the extremal functions.

The infimum problem then is reduced to a minimization over a finite number of variables (the abscissas of the jumps of the step function Δ_χ defined in Theorem 2). In the generic case, only a small number of jumps are involved in the low χ_L region of the boundary $\partial\mathscr{D}_-$, which is then easily computed.

Finally, let us notice that, although our proofs rely heavily on the properties of rational functions, we suspect that our main results hold not only for the Cauchy kernel, but also for more general ones.

Appendix A

Let be given two set of distinct, real numbers $\{y_i\}_{i=1}^{n-1}$ and $\{x_k\}_{k=1}^{n}$ ordered as:

$$y_{n-1} < \cdots < y_2 < y_1 < x_1 < x_2 < \cdots < x_n. \tag{A.1}$$

We assert that there are n (uniquely defined) numbers $r_1 = 1$, $r_k \neq 0$ $(k = 2, \ldots, n)$ such that the function

$$R(s) = \sum_{k=1}^{n} \frac{r_k}{x_k - s} \tag{A.2}$$

has exactly $(n-1)$ simple zeros, located at $s = y_i$ $(i = 1, \ldots, n-1)$. Moreover:

$$(-1)^{k+1} r_k > 0 \quad (k = 1, \ldots, n). \tag{A.3}$$

Indeed, given the y_i's, the r_k's for $2 \leqq k \leqq n$ are obtained by solving the linear system:

$$\sum_{k=2}^{n} \frac{1}{x_k - y_i} r_k = -\frac{1}{x_1 - y_i}, \quad i = 1, \ldots, n-1, \tag{A.4}$$

The determinant D of this system is:

$$D = (-1)^{(n-1)(n-2)/2} \prod_{\substack{2 \leqq k < l \leqq n \\ 1 \leqq i < j \leqq n-1}} (x_k - x_l)(y_i - y_j) \Big/ \prod_{\substack{2 \leqq k \leqq n \\ 1 \leqq i \leqq n-1}} (x_k - y_i) \tag{A.5}$$

(an easy consequence of the fact that D must vanish for $x_k = x_l$ and for $y_i = y_j$). As $D \neq 0$, the existence and uniqueness of the r_k's is insured. Furthermore

$$r_q = \frac{D_q}{D} \quad (q = 2, \ldots, n), \tag{A.6}$$

where D_q is obtained from D (up to a sign) by the substitution $x_q \to x_1$ in the right-hand side of Eq. (A.5). Again $D_q \neq 0$, so that $r_q \neq 0$.

Since $R(s) = N(s) \Big/ \prod_{k=1}^{n} (x_k - s)$, where $N(s)$ is a polynomial of degree $(n-1)$, the $(n-1)$ y_i's necessarily exhaust the set of zeros of $R(s)$. This statement also implies that the signs of the r_k's must alternate. Hence Eq. (A.3).

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