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P-Adic Feynman and String Amplitudes

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Abstract. We derive an explicit representation for p-adic Feynman and Koba–Nielsen amplitudes and we briefly outline the connection between the scalar models of p-adic quantum field theory and Dyson's hierarchical models.

1. Introduction

As we have shown previously (see [1], submitted to "Theoretical and Mathematical Physics" in May 1987), the scalar models of the field theory over the p-adic field Q_p are the natural continuous analogs of Dyson's hierarchical models (see [2–5]). More precisely, the discretization of the field theory over Q_p on the hierarchical lattice of p-adic numbers with zero integer part is a model of Dyson's type. The traditional methods of quantum field theory such as Feynman diagrams, renormalization theory and Wilson's renormalization group have analogs in the p-adic case. The main results of [1] are briefly outlined in Sect. 2.

On the other hand, there has been recently some interest on the possibility of a p-adic formulation of string theory (see [5–12]).

All this explains our interest in the Feynman amplitudes over the p-adic field. The remarkable feature of p-adic models is the exact representation of Feynman and string scattering amplitudes as a sum of elementary functions. Namely, let us consider a general Feynman amplitude over Q_p in coordinate representation,

$$F(x_v; v\in V_{\text{ext}}) = \int \prod_{v,v'\in V_{\text{ext}}\cup V_{\text{int}}} \|x_v - x_{v'}\|_p^{a(v,v')} \prod_{v\in V_{\text{int}}} dx_v, \tag{1.1}$$

where the integral taken over $Q_p^{|V_{\text{int}}|}$, p is a fixed prime number, Q_p is a p-adic field, $\|\cdot\|_p$ is a p-adic norm (further on, the sign p will be omitted), V_{ext} (V_{int}) is a set of external (internal) vertices, $a(v,v')\in\mathbb{C}$ for each pair $v\in V_{\text{ext}}\cup V_{\text{int}}$, $v'\in V_{\text{ext}}\cup V_{\text{int}}$ (we identify a pair (v,v') with (v',v)), dx is a Haar measure on Q_p, normalized such

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that

$$\int_{\|x\| \leqq 1} dx = 1. \tag{1.2}$$

Every given vector of external variables $x = (x_v; v \in V_{\text{ext}}) \in Q_p^{|V_{\text{ext}}|}$ generates on V_{ext} a hierarchy A_x. We recall (see, for example [4, 12]) that a hierarchy A on a finite set V is a family of subsets of V, such that $V \in A$, $\{v\} \in A$ for every $v \in V$, and for each pair $V' \in A'$, $V'' \in A$ either $V' \cap V'' = \emptyset$ or $V' \subset V''$, or $V'' \subset V'$. For every $V' \in A$ we denote by $\tau(V')$ the minimal set in A, which contains V', but does not coincide with V' (we assume $V' \neq V$). In the following we shall consider only hierarchies such as

$$1 < |K(V')| \leqq p, \quad V' \in A', \tag{1.3}$$

where

$$K(V') = \{V'' \in A \mid \tau(V'') = V'\}, \tag{1.4}$$

$$A' = \{V' \in A : |V'| > 1\}. \tag{1.5}$$

In our case, the hierarchy A_x on V_{ext} is defined as

$$A_x = \left\{ V \subset V_{\text{ext}} : \max_{\substack{v \in V \\ v' \in V}} \|x_v - x_{v'}\| < \min_{\substack{v \in V \\ v' \in V_{\text{ext}} \setminus V}} \|x_v - x_{v'}\| \right\}. \tag{1.6}$$

Note, that for every A on V_{ext} there exists $x \in Q_p^{|V_{\text{ext}}|}$ such that $A_x = A$.

Let

$$F_A(x) = \begin{cases} F(x), & \text{if } A_x = A \\ 0, & \text{otherwise} \end{cases}. \tag{1.7}$$

Then

$$F(x) = \sum_A F_A(x), \tag{1.8}$$

where the sum goes over all hierarchies on V_{ext}.

Let $A_x = A$. The main result of Sect. 3 is the following:

$$F_A(x) = \sum_{I(A)} \prod_{V \in A'} C(V, I) \max_{\substack{v \in V \\ v' \in V}} \|x_v - x_{v'}\|^{\lambda(V, I)}, \tag{1.9}$$

where the sum taken over all partitions of V_{int}, indexed by the elements of A':

$$I(A) = \{I(V), V \in A'\}, \quad I(V') \cap I(V'') = \emptyset, \quad \text{if} \quad V' \neq V'', \quad \left(\bigcup_{V \in A'} I(V') \right) = V_{\text{int}}, \tag{1.10}$$

$$\lambda(V, I) = a(V(I)) - \sum_{V' \in K(V)} a(V'(I)) + |I(V)|, \tag{1.11}$$

$$V(I) = \left(\bigcup_{V' \subseteq V} I(V') \right) \cup V, \tag{1.12}$$

$$C(V, I) = \int \prod_{v, v' \in I(V)} \|y_v - y_{v'}\|^{a(v, v')} \prod_{v \in I(V)} \left(\prod_{V' \in K(V)} \|y_v - \alpha_{V'}\|^{a(v, V'(I))} \right) dy. \tag{1.13}$$

Integral (1.13) taken over $Q_p^{|I(V)|}$, $\{\alpha_{V'}, V' \in K(V)\}$ is an arbitrary set of p-adic numbers, such that $\|\alpha_{V'} - \alpha_{V''}\| = 1$, if $V' \neq V''$ (coefficient $C(V, I)$ does not depend on the choice of $\alpha = \{\alpha_V, V' \in K(V)\}$. Here we use the notations

$$a(W, W') = \sum_{v \in W, v' \in W'} a(v, v'), \quad W, W' \subset V_{\text{ext}} \cup V_{\text{int}}, \tag{1.14}$$

$$a(W) = a(W, W). \tag{1.15}$$

We see that the integral representation for $C(V, I)$ is a generalization of the integral representation of string scattering amplitude in the Koba–Nielsen form.

Let

$$F_1 = \int \prod_{v, v' \in V} \|y_v - y_{v'}\|^{a(v,v')} \prod_{v \in V} \prod_{i=0}^{k} \|y_v - \alpha_i\|^{a(v,i)} \, dy_v, \tag{1.16}$$

where $0 \leq k < p$, $\alpha = \{\alpha_i, i = 0, \ldots, k\}$ is an arbitrary set of p-adic numbers, such that $\|\alpha_i - \alpha_j\| = 1$, if $i \neq j$, V is a finite set, $a(v, v') \in C$ for each pair $v \in V$, $v' \in V$ (we identify (v, v') with (v', v)). In Sect. 4 we prove that the calculation of this integral can be reduced to that of the following one:

$$F_2 = \int_{\substack{|y_v| < 1 \\ v \in V \setminus \{v_0\}}} \prod_{v, v' \in V} \|y_v - y_{v'}\|^{a(v,v')} \prod_{v \in V \setminus \{v_0\}} dy_v, \tag{1.17}$$

where $y_{v_0} = 0$. We shall prove that

$$F_2(a_V) = p^{-|V|+1} \sum_A \prod_{V' \in A'} \frac{1}{p^{a(V') + |V'| - 1} - 1} \cdot \frac{(p-1)!}{(p - |K(V')|)!}, \tag{1.18}$$

the sum taken over all hierarchies A on V.

2. *P*-Adic Scalar Models and Dyson's Hierarchical Models

Let a generalized random field $P(\phi)$ on the Q_p be given, i.e. a system of probability distributions $P\{(\varphi, f_1), \ldots, (\varphi, f_m)\}$ with the usual conditions of accordance. Here $f_i = f_i(x)$, $i = 1, \ldots, m$ are arbitrary test functions in the space $S(Q_p)$. $S(Q_p)$ is a space of locally piece-wise finite complex-valued functions on Q_p (see [15]).

We define the scaling operator

$$R_\lambda^a P(\varphi) = P(\|\lambda\|^{1-(a/2)} \varphi_\lambda), \tag{2.1}$$

where a is a real number, $1 \leq a \leq 2$, $\lambda \in Q_p$,

$$(\varphi_\lambda, f) = (\varphi(\lambda x), f(x)),$$

and $P(\|\lambda\|^{1-(a/2)} \varphi_\lambda)$ is a generalized random field with probability distributions

$$P\{(\|\lambda\|^{1-(a/2)} \varphi_\lambda, f_1), \ldots, (\|\lambda\|^{1-(a/2)} \varphi_\lambda, f_m)\}.$$

A generalized random field is scaling invariant if

$$R_\lambda^a P(\varphi) = P(\varphi) \tag{2.2}$$

for all $\lambda \in Q_p$. A generalized random field is called translation invariant if $P(\varphi(x)) = P(\varphi(x + a))$ for any $a \in Q_p$.

It is easy to see that a generalized gaussian random field with zero mean and binary correlation function

$$\langle \varphi(x_1), \varphi(x_2) \rangle = \mathrm{const}\, \| x_1 - x_2 \|^{a-2} \tag{2.3}$$

is a scaling and translation invariant random field.

By $\mathscr{D}_p$ we denote the set of all p-adic numbers with zero integer part. If

$$x = \sum_{i=n}^{\infty} a_i p^i, \quad 0 \leqq a_i < p-1 \tag{2.4}$$

is a p-adic number, then $\{x\}$ denotes its fraction part

$$\{x\} = \sum_{i=n}^{-1} a_i p^i, \tag{2.5}$$

$$\mathscr{D}_p = \{x \in Q_p : x = \{x\}\}. \tag{2.6}$$

$\mathscr{D}_p$ has a natural hierarchical structure, which consists from all sets of the type

$$V_i^n = \{x \in \mathscr{D}_p : \| p^n x - i \| \leqq 1\}, \quad i \in \mathscr{D}_p, \quad n = 0, 1, 2, \dots . \tag{2.7}$$

The discretization of a generalized random field φ on $\mathscr{D}_p$ is defined as a random field ξ on $\mathscr{D}_p$ such that

$$\xi = \{\xi_j = (\varphi, \chi_j), j \in \mathscr{D}_p\}, \tag{2.8}$$

where $\chi_j(x) = \chi(x-j)$, $\chi(x)$ is the characteristic function of the ball $Z_p = \{x \in Q_p : \| x \| \leqq 1\}$. The following operations on a random field $\xi = \{\xi_j, j \in \mathscr{D}_p\}$ are defined by the formulae

$$r_\lambda^a : \xi_j \to \xi'_j = \| \lambda \|^{-(a/2)} \sum_{i : \{i\lambda^{-1}\} = j, j \in \mathscr{D}_p} \xi_i, \quad \| \lambda \| \geqq 1, \tag{2.9}$$

$$t_i : \xi_j \to \xi_{\{j+i\}}, \quad i \in \mathscr{D}_p. \tag{2.10}$$

If a generalized random field φ is a translation and scaling invariant, then its discretization is invariant relative to actions $t_i, i \in \mathscr{D}_p$ and r_λ^a, $\| \lambda \| > 1$. The gaussian scaling and translation invariant random field on Q_p also may be defined by the hamiltonian

$$H_0 = \tfrac{1}{2} \int \| x - y \|^{-a} \varphi(x) \varphi(y) dx dy. \tag{2.11}$$

One can show that the discretization of the random field with the hamiltonian (2.11) is a gaussian random field on $\mathscr{D}_p$ with the hamiltonian

$$\tilde{H}_0 = \frac{1}{2} \sum_{i,j \in \mathscr{D}_p} d(i,j) \xi_i \xi_j, \tag{2.12}$$

where

$$d(i,j) = \begin{cases} \| i - j \|^{-a}, & \text{if } i \neq j \\ \dfrac{p^{-a} - p^{1-a}}{1 - p^{1-a}}, & \text{otherwise} \end{cases} . \tag{2.13}$$

Note that the hamiltonian $\tilde{H}_0$ is the hamiltonian of the gaussian Dyson hierarchical model.

Further we shall have to deal with the hamiltonians in momentum representation. A hamiltonian in the ball $\Omega = \{k \in Q_p : \|k\| \leqq R\}$ is an expression of the form

$$H(\sigma) = \sum_{m=1}^{n} \int_{\Omega^m} h_m(k_1, \ldots, k_m)\delta(k_1 + \cdots + k_m) \prod_{i=0}^{k} \sigma(k_i) dk_i. \tag{2.14}$$

A formal hamiltonian is a formal series in ε

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots, \tag{2.15}$$

whose coefficients are finite-particle hamiltonians of the type (2.14). In what follows the coefficient H_0 will be fixed:

$$H_0 = \frac{1}{2}\int_{\Omega} \|k\|^{a-1} |\sigma(k)|^2 dk. \tag{2.16}$$

Wilson's renormalization transformation, as in a real case, is defined by the formula

$$R^a_{\Omega,\lambda} H = \ln \langle \exp(R^a_\lambda H(\sigma_0 + \sigma_1)) \rangle_{\mu(d\sigma_1)}, \tag{2.17}$$

where $(R^a_\lambda H)(\sigma) = H(\|\lambda\|^{-a/2}\sigma(\lambda^{-1}))$ is a hamiltonian of the random field in the ball $\lambda\Omega$, $\sigma(k)$ is a configuration in the ball $\lambda\Omega$, $\sigma(k) = \sigma_0(k) + \sigma_1(k)$, $\sigma_0(k) = \chi_\Omega(k)\sigma(k)$, $\sigma_1(k) = (\chi_{\lambda\Omega}(k) - \chi_\Omega(k))\sigma(k)$, $\chi_\Omega(k)$ is a characteristic function of the ball Ω, and the average taken with respect to the gaussian measure $\mu(d\sigma_1)$ with zero mean and binary correlation function

$$\langle \sigma_1(k)\sigma_1(k') \rangle = \delta(k + k')(\chi_{\lambda\Omega}(k) - \chi_\Omega(k)) \|k\|^{1-a}. \tag{2.18}$$

The branch of gaussian fixed points of Wilson's renormalization group is determined by the hamiltonian $H_0 = H_0(a)$. An investigation of the spectrum of the differential of the renormalization group on this branch shows that $a_0 = \frac{3}{2}$ is the bifurcation point. One can try to construct a new branch of non-gaussian fixed points as power series in the deviation of the parameter a from the bifurcation value a_0. As in the real case, (see [16]), we seek a solution in the class of projection hamiltonians, in the form

$$H = H(\sigma, \varepsilon) = \ln \langle \exp\{u(\varepsilon)\varphi^4(\sigma_0 + \sigma_1)\} \rangle_{\mu(d\sigma_1)}, \tag{2.19}$$

where

$$\varphi^4(\sigma) = \int \delta(k_1 + \cdots + k_4) \prod_{i=1}^{4} \sigma(k_i) dk_i, \tag{2.20}$$

σ_0 is a configuration in the ball Ω, and the average taken with respect to the gaussian measure with binary correlation function

$$\langle \sigma_1(k)\sigma(k') \rangle = \delta(k + k')(1 - \chi_\Omega(k)) \|k\|^{1-a} = \delta(k + k')(1 - \chi_\Omega(k)) \|k\|^{-(1/2)-\varepsilon}. \tag{2.21}$$

The only quantity which is not defined here is $u(\varepsilon)$, which we assume to be a formal power series in ε:

$$u(\varepsilon) = \sum_{i=1}^{\infty} u_j \varepsilon^j. \tag{2.22}$$

In the computation of a projection hamiltonian divergences appear. Namely, in

φ^4 theory with the propagator

$$(1-\chi_\Omega(k))\|k\|^{-(1/2)-\varepsilon} \tag{2.23}$$

the diagrams with two and four external lines have poles when $\varepsilon \to 0$. The theory of analytic renormalization analogous to the real case exists in the p-adic case. One can show that

$$\begin{aligned} &R^a_{\Omega,\lambda}\text{A.R.}\ln\langle \exp\{u(\varepsilon)\varphi^4(\sigma_0+\sigma_1)\}\rangle_{\mu(d\sigma_1)} \\ &= \exp\left(\tau\beta(u)\frac{d}{du}\right)\text{A.R.}\ln\langle \exp\{u(\varepsilon)\varphi^4(\sigma_0+\sigma_1)\}\rangle_{\mu(d\sigma_1)}, \end{aligned} \tag{2.24}$$

where A.R. denotes the analytic renormalization with minimal subtractions, $\tau = 2\ln\|\lambda\|$,

$$\beta(u) = \varepsilon u + \sum_{n=2}^{\infty} c_n u^n, \tag{2.25}$$

and the coefficients c_n for $n \geqq 2$ do not depend on ε.

The renormalization projection hamiltonian A.R.$\ln\langle \exp(u(\varepsilon)\varphi_4)\rangle_{\mu(d\sigma_1)}$ is invariant under the action of the renormalization group, if

$$\beta(u) = 0. \tag{2.26}$$

This equation has two solutions in the formal power in ε. The solution $u=0$ corresponds to a gaussian fixed point. A nontrivial non-gaussian solution is obtained from the solution of

$$\varepsilon + \sum_{n=2}^{\infty} c_n u^{n-1} = 0. \tag{2.27}$$

It is easy to check that $c_2 \neq 0$ and therefore Eq. (2.27) is indeed solvable. This solution is a p-adic field-theoretical description of the non-trivial solution in Dyson's hierarchical model, which was investigated in [2–5].

3. Feynman Amplitudes in Coordinate Representation

First of all we shall prove that A_x, defined by (1.6) with the function

$$m_x(V) = \max_{\substack{v\in V \\ v'\in V}} \|x_v - x_{v'}\| \tag{3.1}$$

on it, is an indexed hierarchy. We recall that an indexed hierarchy on V is a pair (A, m), where A denotes a given hierarchy on V and m is a positive function on A, satisfying the following conditions:

1) $m(V') = 0$ if and only if $|V'| = 1$,
2) if $V' \subset V''$, then $m(V') < m(V'')$.

In addition to these conditions we shall consider the functions $m(V)$ for which

$$m(V) = p^{n(V)}, \quad n(V) \in Z \tag{3.2}$$

for every $V \in A$.

It suffices to show that if $V' \in A_x$, $V'' \in A_x$ and $V' \cap V'' \neq \emptyset$, then $V' \subset V''$ or $V'' \subset V'$. In fact, let $x \in V' \cap V''$, $y \in V' \setminus V''$, $z \in V'' \setminus V'$. From the definition of V' $\| x - y \| < \| y - z \|$ follows and hence, from the ultrametricity of the p-adic norm $\| y - z \| = \| x - z \|$, but this contradicts the definition of V''.

It is easy to see that for every indexed hierarchy (A, m) on V_{ext} there exists $x = (x_v, v \in V_{\text{ext}})$ such that $(A_x, m_x) = (A, m)$. To every indexed hierarchy (A, m) corresponds a family of sets $\{\mathscr{D}(V, m), V \in A\}$, where

$$\mathscr{D}(V, m) = \{ y \in Q_p : m(V) \leqq \| y - x_v \| < m(\tau(V)), \forall v \in V \}. \tag{3.3}$$

We put $m(\tau(V_{\text{ext}})) = \infty$.

Let $\varphi: A \to V_{\text{ext}}$ be any function such that $\varphi(V) \in V$ for every $V \in A$.

Lemma 1. 1) *If $V \in A$, $V' \in A$, $V \subset V'$, $V \neq V'$ and $y \in \mathscr{D}(V, m)$, $y \in \mathscr{D}(V', m)$. then*

$$\| y' - y \| = \| y' - x_{\varphi(V)} \|.$$

2) *If $V \cap V' = \emptyset$, $V \in A$, $V' \in A$ and V'' is a minimal set in A, containing $V \cup V'$, $y \in \mathscr{D}(V, m)$, $y' \in \mathscr{D}(V', m)$, then*

$$\| y' - y \| = m(V'').$$

3) *$\{\mathscr{D}(V, m); V \in A\}$ is a partition of Q_p.*

Proof. Let $y \in \mathscr{D}(V, m)$, $y' \in \mathscr{D}(V', m)$, $V \subset V'$, $V \neq V'$.
Then

$$\| y - x_{\varphi(V)} \| < m(\tau(V)) \leqq m(V') \leqq \| y' - x_{\varphi(V)} \|,$$

and therefore $\| y - y' \| = \| y' - x_{\varphi(V)} \|$. If $V \cap V' = \emptyset$, $y \in \mathscr{D}(V, m)$, $y' \in \mathscr{D}(V', m)$, then

$$\| y - x_{\varphi(V)} \| < m(\tau(V)) \leqq m(V'') = \| x_{\varphi(V)} - x_{\varphi(V')} \|,$$

$$\| y' - x_{\varphi(V')} \| < m(\tau(V')) \leqq m(V''),$$

and hence

$$\| y' - x_{\varphi(V)} \| = \| y - y' \| = m(V'').$$

Let $V \neq V'$ and $y \in \mathscr{D}(V, m) \cap \mathscr{D}(V', m)$. If $V \subset V'$, $V \neq V'$, then

$$m(V) \leqq \| y - x_{\varphi(V)} \| < m(\tau(V)) \leqq \| y - x_{\varphi(V)} \|,$$

but this contradicts the first part of the lemma. If $V \cap V'' = \emptyset$, then

$$\| x_{\varphi(V)} - x_{\varphi(V')} \| < \max(\| y - x_{\varphi(V)} \|, \| y - x_{\varphi(V')} \|) < m(V'').$$

At last

$$\bigcup_{V \in A} \mathscr{D}(V, m) = Q_p.$$

Lemma 1 is proved.

Everywhere below we have used the next notations: $a_V = (a(v, v'))_{v \in V}^{v' \in V}$ is a matrix, $b_V = (b(v))_{v \in V}$ is a vector,

$$a(V, V') = \sum_{v \in V, v' \in V'} a(v, v'), \tag{3.4}$$

$a(V)=a(V,V)$, $a_V^{V'}=(a(v,V'))_{v\in V}$ is a vector, $a(v,v')\in\mathbb{C}, b'(v)\in\mathbb{C}$ for every $v\in V$, $v'\in V$.

Let r_1, r_2 be real numbers, $\infty \geqq r_2 > r_1 \geqq 0$. Denote by

$$f(r_1,r_2;a_V,b_V)=\int\limits_{r_1\leqq\|y_v\|<r_2, v\in V}\prod_{v,v'\in V}\|y_v-y_{v'}\|^{a(v,v')}\prod_{v\in V}\|y_v\|^{b(v)}dy_v. \tag{3.5}$$

Lemma 2.

$$f(r_1,r_2;a_V,b_V)=\sum_{V_1\cup V_2=V}f(r_1,\infty;a_{V_1},b_{V_1})f(0,r_2;a_{V_2},b_{V_2}+a_{V_2}^{V_1}) \tag{3.6}$$

where the sum goes over all partitions (V_1, V_2) of V.

Proof. We prove this lemma by induction with respect to the number of elements of V. For $|V|=1$ this formula is checked directly:

$$\begin{aligned}\int\limits_{r_1\leqq\|y\|<r_2}\|y\|^b dy &= \int\limits_{r_1\leqq\|y\|<r_2}\|y\|^b dy+\int\limits_{\|y\|<r_1}\|y\|^b dy+\int\limits_{r_1\leqq\|y\|}\|y\|^b dy\\ &=\int\limits_{\|y\|<r_2}\|y\|^b dy+\int\limits_{r_1\leqq\|y\|}\|y\|^b dy\end{aligned}$$

(we used $\int\limits_{Q_p}\|y\|^b dy=0$).

Note that

$$f(r_1,r_2;a_V,b_V)=f(0,r_2;a_V,b_V)-\sum_{V_1\subseteq V, V_1\neq\varnothing}f(0,r_1;a_{V_1},b_{V_1})f(r_1,r_2;a_{V_2},b_{V_2}+a_{V_2}^{V_1}). \tag{3.7}$$

Applying the assumption of induction to $f(r_1,r_2;a_{V_2},b_{V_2}+a_{V_2}^{V_1})$, we have

$$\begin{aligned}f(r_1,r_2;a_V,b_V)=f(0,r_2;a_V,b_V)&-\sum_{V_1\subseteq V, V_1\neq\varnothing}(f(0,r_1;a_{V_1},b_{V_1})\\ &\cdot\sum_{V_2\subseteq V\setminus V_1}f(r_1,\infty;a_{V_2},b_{V_2}+a_{V_2}^{V_1})f(0,r_2;a_{V_3},b_{V_3}+a_{V_3}^{V_1}+a_{V_3}^{V_2})),\end{aligned} \tag{3.8}$$

where $V_3=V\backslash(V_1\cup V_2)$. By denoting $V_4=V_1\cup V_2$ and changing the order of the summation, we get

$$\begin{aligned}f(r_1,r_2;a_V,b_V)=f(0,r_2;a_V,b_V)-\sum_{V_4\subseteq V, V_4\neq\varnothing}\Bigg\{\Bigg(&\sum_{V_1\subseteq V_4, V_1\neq\varnothing}f(0,r_1;a_{V_1},b_{V_1})\\ &\cdot f(r_1,\infty;a_{V_2},b_{V_2}+a_{V_2}^{V_1})f(0,r_2;a_{V_3},b_{V_3}+a_{V_3}^{V_4})\Bigg)\Bigg\}.\end{aligned} \tag{3.9}$$

But

$$\sum_{V_1\subseteq V_4}f(0,r_1;a_{V_1},b_{V_1})f(r_1,\infty;a_{V_2},b_{V_2}+a_{V_2}^{V_1})=f(0,\infty;a_{V_4},b_{V_4})=0, \tag{3.10}$$

and hence

$$\sum_{V_1\subseteq V_4, V_1\neq\varnothing}f(0,r_1;a_{V_1},b_{V_1})f(r_1,\infty;a_{V_2},b_{V_2}+a_{V_2}^{V_1})=-f(r_1,\infty;a_{V_4},b_{V_4}), \tag{3.11}$$

where the summation is over $V_1\neq\varnothing$, $V_1\subset V_4$. Therefore,

$$f(r_1,r_2;a_V,b_V)=f(r_1,r_2;a_V,b_V)+\sum_{V_2\subseteq V, V_4\neq\varnothing}f(r_1,\infty;a_{V_4},b_{V_4})f(0,r_2;a_{V_3},b_{V_3}+a_{V_3}^{V_4}), \tag{3.12}$$

$V_3=V\backslash V_4$. Lemma 2 is proved.

Theorem 1. *Let $A_x = A$. Then*

$$F(x)=\sum_{I(A)}\prod_{V\in A'} C(V,I)\max_{\substack{v\in V\\ v'\in V}}\|x_v-x_{v'}\|^{\lambda(V,I)}, \tag{3.13}$$

where

$$C(V,I)=\int\prod_{v',v\in I(V)}\|y_v-y_{v'}\|^{a(v,v')}\prod_{v\in I(V)}\left(\prod_{V'\in K(V)}\|y_v-\alpha_{V'}\|^{a(v,V'(I))}dy_v\right), \tag{3.14}$$

$\{\alpha_{V'}, V'\in K(V)\}$ is an arbitrary set of p-adic numbers, such that $\|\alpha_{V'}-\alpha_{V''}\|=1$ if $V'\neq V''$ (for example, $\alpha=\{\alpha_{V'}; V'\in K(V)\}=\{0,1,\dots,|K(V)|-1\}$). $C(V,I)$ do not depend on the choice of α. For the other notations see Sect. 1.

Proof. Let $A_x=A$ and $\varphi: A\to V_{\text{ext}}$ is any function such that $\varphi(V)\in V, V\in A$. Let $T(A)=\{T(V), V\in A\}$ is a partition of V_{int}, indexed by elements of A. Denote by

$$g(V,T,x)=\int_{\substack{y_v\in\mathcal{S}(V,m_x)\\ v\in T(V)}}\prod_{v,v'\in T(V)}\|y_v-y_{v'}\|^{a(v,v')}\prod_{v\in T(V)}\|y_v-x_{\varphi(V)}\|^{a(v,V(T))}dy_v, \tag{3.15}$$

where $V(T)=\left(\bigcup_{V'\subset V}T(V')\right)UV$.

From Lemma 1 it is easy to derive that

$$F(x)=\sum_{T(A)}\prod_{V\in A'} g(V,T,x)m_x(V)^{s(V,T)}, \tag{3.16}$$

where the sum goes over all partitions of V_{int}, indexed by elements of A,

$$s(V,T)=\sum_{\substack{V'V''\in K(V)\\ V'\neq V''}} a(V'(T)\cup T(V'), V''(T)\cup T(V'')). \tag{3.17}$$

According to Lemma 2

$$g(V,T,x)=\sum_{T_1(V)\cup T_2(V)=T(V)} f(m(V),\infty;a_{T_1(V)},a^{V(T)}_{T_1(V)})f(0,m(\tau(V);a_{T_2(V)},a^{V(T)+T_1(V)}_{T_2(V)}), \tag{3.18}$$

the sum goes over all partitions $(T_1(V), T_2(V))$ of $T(V)$.

As $f(0,\infty;a_V,b_V)=0$ for any a_V, b_V, we may write

$$F(x)=\sum_{I(A)}\prod_{V\in A'}\left(\sum_{\substack{(\bigcup T_2(V'))UT_1(V)=I(V)\\ V'\in K(V)}} f(m(V),\infty;a_{T_1(V)},a^{V(I)T_1(V)}_{T_1(V)})\right.$$
$$\left.\cdot\prod_{V'\in K(V)} f(0,m(V);a_{T_2(V)},a^{V'(I)}_{T_2(V)})\right)m_x(V)^{s(V,I)}, \tag{3.19}$$

where the sum goes over all partitions of V_{int}, indexed by elements of A', the internal sum taken over all partitions of

$$I(V)=\left(\bigcup_{V'\in K(V)}T_2(V')\right)\cup T_1(V),\quad V(I)=\left(\bigcup_{V'\subseteq V}I(V')\right)\cup V,$$

$$s(V,I)=\sum_{\substack{V',V''\in K(V)\\ V'\neq V''}} a(V'(I),V''(V)). \tag{3.20}$$

As

$$\left(\bigcup_{V'\in K(V)}\{y\in Q_p: \|y-x_{\varphi(V')}\|<m(V)\}\right)\cup\{y\in Q_p: \|y-x_{\varphi(V)}\|\geqq m(V)\}=Q_p, \quad (3.21)$$

we get

$$F(x)=\sum_{I(A)}\prod_{V\in A'} m_x(V)^{s(V,I)}$$
$$\cdot\left(\int\prod_{v,v'\in I(V)}\|y_v-y_{v'}\|^{a(v,v')}\prod_{v\in I(V)}\left(\prod_{V'\in K(V)}\|y_v-x_{\varphi(V')}\|^{a(v,V'(I))}\right)dy_v\right). \quad (3.22)$$

Performing a change of variables,

$$y_v=y_v p^{-n}, \quad v\in I(V), \quad (3.23)$$

$$\alpha_V=x_{\varphi(V')}p^{-n}, \quad V'\in K(V), \quad (3.24)$$

where $\|p^n\|=m_x(V)$, we obtain

$$F(x)=\sum_{I(A)}\prod_{V\in A'} C(V,I,\alpha)m_x(V)^{\lambda(V,I)}, \quad (3.25)$$

$$C(V,I,\alpha)=\int\prod_{v,v'\in I(V)}\|y_v-y_{v'}\|^{a(v,v')}\prod_{v\in I(V)}\left(\prod_{V'\in K(V)}\|y_v-\alpha_{V'}\|^{a(v,V'(I))}\right)dy, \quad (3.26)$$

$$\lambda(V,I)=a(V(I))-\sum_{V'\in K(V)}a(V'(I))+|I(V)|. \quad (3.27)$$

Note that $\|\alpha_{V'}-\alpha_{V''}\|=1$ for each pair $V',V''\in K(V)$ $V'\neq V''$. In the next section we shall see that $C(V,I,\alpha)$ does not depend on the choice of α.

Note also that $C(V,I)$ is a Feynman amplitude of the contracted graph

$$G_V|_{\{G_{V'},V'\in K(V)\}}.$$

Here G_V is a graph with the set of vertices $V(I)$. To each external vertice $\{V'\}$ of this contracted graph we must assign an external variable $\alpha_{V'}$.

$\lambda(V,I)$ also has a simple geometrical description.

4. The Calculation of Coefficients $C(V,I)$ and String Amplitudes in the Koba–Nielsen Form

Let us consider an integral

$$F_1=\int\prod_{v,v'\in V}\|y_v-y_{v'}\|^{a(v,v)}\prod_{v\in V}\left(\prod_{i=0}^{k}\|y_v-\alpha_i\|^{b_i(v)}\right)dy_v, \quad (4.1)$$

where $1\leqq k\leqq p-1$, $\|\alpha_i-\alpha_j\|=1$ if $i\neq j$. This integral is a generalization of Koba–Nielsen amplitude (some examples of this amplitude were calculated in [10, 11]).

We introduce the next partition of Q_p:

$$\mathscr{D}_i=\{y\in Q_p: \|y-\alpha_i\|<1\}, \quad i=0,1,\dots,k, \quad (4.2)$$

$$\mathscr{D}_j=\{y\in Q_p: \|y-\alpha_0\|=1, (y-\alpha_0)\bmod p=j\}, \quad j\in J. \quad (4.3)$$

where $J=\{0,1,\ldots,p-1\}\backslash\{(\alpha_i-\alpha_0) \bmod p, i=0,\ldots,k\}$. As $|J|=p-k-1$, it is convenient to renumerate the family $\mathscr{D}_j, j\in J$ by $i=k+1,\ldots,p-1$. Finally,

$$\mathscr{D}_p=\{y\in Q_p: \| y-\alpha_0\| > 1\}. \tag{4.4}$$

As above, it is easy to show that $\{\mathscr{D}_i, i=0,\ldots,p\}$ is a partition of Q_p. Therefore

$$\begin{aligned}F_1 = &\sum_{\bigcup_{i=0}^p V_i=V} \prod_{i=0}^k \int_{\substack{y_v\in\mathscr{D}_i\\ v\in V_i}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_i} \| y_v-\alpha_i\|^{b_i(v)} dy_v\\ &\cdot \prod_{j=k+1}^{p-1} \int_{\substack{y_v\in\mathscr{D}_j\\ v\in V_j}} \prod_{v,v'\in V_j} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_j} dy_v\\ &\cdot \int_{\substack{y_v\in\mathscr{D}_p\\ v\in V_p}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_i} \| y_v-\alpha_0\|^{\sum_{i=0}^k b_i(v)+a(v,V\backslash V_p)} dy_v.\end{aligned} \tag{4.5}$$

But for $i=0,\ldots,k$,

$$\begin{aligned}&\int_{\substack{y_v\in\mathscr{D}_i\\ v\in V_i}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_i} \| y_v-\alpha_i\|^{b_i(v)} dy_v\\ &= \int_{\substack{\|y_v\|<1\\ v\in V_i}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_i} \| y_v-\alpha_i\|^{b_i(v)} dy_v,\end{aligned} \tag{4.6}$$

for $i=k+1,\ldots,p-1$

$$\int_{\substack{y_v\in\mathscr{D}_i\\ v\in V_i}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} dy_v = \int_{\substack{\|y_v\|<1\\ v\in V_i}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} dy_v, \tag{4.7}$$

and

$$\begin{aligned}&\int_{\substack{y_v\in\mathscr{D}_p\\ v\in V_p}} \prod_{v,v'\in V_p} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_p} \| y_v-\alpha_0\|^{\sum_{i=0}^k b_i(v)+a(v,V\backslash V_p)} dy_v\\ &= \int_{\substack{y_v<1\\ v\in V_p}} \sum_{v,v'\in V_p} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_p} \| y_v\|^{(\sum_{i=0}^k b_i(v)+a(v,V)+2)} dy_v.\end{aligned} \tag{4.8}$$

In the last reduction we used the change of variables $y_v\to(1/y_v-\alpha_0)$, $v\in V_p$. Therefore, we may rewrite

$$F_1=\sum_{\bigcup_{i=0}^p V_i=V} \prod_{i=0}^p g(a_{V_i}, b_i), \tag{4.9}$$

where

$$g(a_{V_i}, b_i)= \int_{\substack{\|y_v\|<1\\ v\in V_i}} \prod_{v,v'\in V_i} \| y_v-y_{v'}\|^{a(v,v')} \prod_{v\in V_i} \| y_v\|^{b_i(v)} dy_v, \tag{4.10}$$

$$b_i(v)=0, \quad v\in V_i, \quad i=k+1,\ldots,p-1, \tag{4.11}$$

$$b_p(v)=-\left(\sum_{i=0}^k b_i(v)+a(v,V)+2)\right), \quad v\in V_p. \tag{4.12}$$

So, we must calculate the integral of the type

$$F_2(a_V) = \int_{\substack{|y_v|<1\\ v\in V\setminus\{v_0\}}} \prod_{v,v'\in V} \|y_v - y_{v'}\|^{a(v,v')} \prod_{v\in V\setminus\{v_0\}} dy_v, \tag{4.13}$$

where v_0 is a fixed element of V, $y_{v_0}=0, a(v,v')\in\mathbb{C}$ for every pair (v,v') (we identify each pair (v,v') with (v',v)), $a(v,v)=0$ for every $v\in V$ (in our case $V\setminus\{v_0\}=V_i$, $a(v,v_0)=b_i(v)$, $i=0,1,\dots,p$).

Lemma 3. *Let* (A,m) *be an indexed hierarchy on* V, *and*

$$\chi_{A,m}(y_v; v\in V\setminus\{v_0\}) = \begin{cases} 1, & \text{if } (A_y, m_y) = (A,m) \\ 0, & \text{otherwise} \end{cases}, \tag{4.14}$$

where $y=\{y_v, v\in V\}$, $y_{v_0}=0$. *Then*

$$\int \chi_{A,m}(y_v; v\in V\setminus\{v_0\}) \prod_{v\in V\setminus\{v_0\}} dy_v = \prod_{V'\in A'} \left(\frac{m(V')}{p}\right)^{|K(V')|-1} \frac{(p-1)!}{(p-|K(V)|)!}. \tag{4.15}$$

Proof of Lemma 3. Let T be a tree with the set of vertices V and the set of lines $L=\{l=(\varphi(V'),\varphi(V'')), \varphi(V')\neq\varphi(V'')\,|\,V'\in A,\ \ V''\in K(V')\}$. Here $\varphi: A\to V$ is any function such that $\varphi(V')\in V'$ for every $V'\in A$, $\varphi(V)=v_0$. To each line $l=(\varphi(V'),\varphi(V''))$ we assign a variable $S_l = y_{\varphi(V')} - y_{\varphi(V'')}$. Then for every $y_v = S_{l_1} + S_{l_2} + \cdots + S_{l_j}$, where $\{l_1,\dots,l_j\}$ is a path joining v with v_0. It is clear that jacobian of this change of variables is equal to 1. And moreover, $(A_y, m_y)=(A,m)$ if and only if

1) $\|S_l\| = m(V') = p^{n(V')}$ for every $l=(\varphi(V'), \varphi(V''))$, $V'\in A'$, $V''\in K(V')$.
2) $S_{l_1}p^{n(V')} \bmod p \neq S_{l_2}p^{n(V')} \bmod p$ for each pair $l_1 = (\varphi(V'), \varphi(V_1))$,

$l_2 = (\varphi(V'), \varphi(V_2))$, $V'\in A'$, $V_1\in K(V')$, $V_2\in K(V')$, $V_1\neq V_2$.

Using

$$\int_{\substack{|s|=m(V')\\ sp^{n(V')}\bmod p = r}} ds = \frac{m(V)}{p}, \tag{4.16}$$

$r=1,2,\dots,p-1$, we obtain the formula (4.15).

Theorem 2. *Let* $a(V')+|V'|-1>0$ *for every* $V'\subseteq V$, $|V'|>1$. *Then*

$$F_2(a_V) = p^{-|V|+1} \sum_A \prod_{V'\in A'} \frac{1}{p^{a(V')+|V'|-1}-1} \cdot \frac{(p-1)!}{(p-|K(V')|)!}, \tag{4.17}$$

the sum goes over all hierarchies A *on* V.

Proof.

$$g(a_V) = \sum_{(A,m):m(V)<1} \int \prod_{v,v'\in V} \|y_v - y_{v'}\|^{a(v,v')} \chi_{A,m}(y_v; v\in V\setminus\{v_0\}) \prod_{v\in V\setminus\{v_0\}} dy_v, \tag{4.18}$$

where the sum taken over all indexed hierarchies on V, such that $m(V)<1$. Note that this sum is equal to the double sum

$$\sum_A \sum_{m:n(V)<0} \int \prod_{v,v'\in V} \|y_v - y_{v'}\|^{a(v,v')} \chi_{A,m}(y_v; v\in V\setminus\{v_0\}) \prod_{v\in V\setminus\{v_0\}} dy_v, \tag{4.19}$$

where the external sum goes over all hierarchies on V and the internal sum goes over all set of integer numbers $\{n(V'), V' \in A'\}$, $n(V') \in Z$, $m(V') = p^{n(V')}$, $n(V') < n(\tau(V'))$ for every $V' \in A'$ and $n(V) < 0$.

Let

$$a'(V') = a(V') - \sum_{V'' \in K(V')} a(V''), \quad V' \in A'. \tag{4.20}$$

Obviously, that

$$\prod_{v,v' \in V} \|y_v - y_{v'}\|^{a(v,v')} \chi_{A,m}(y_v; v \in V \setminus \{v_0\}) = \prod_{V' \in A'} m(V')^{a'(V')} \chi_{A,m}(y_v; v \in V \setminus \{v_0\}).$$

Note that if $\beta > 0$, then

$$\sum_{n<m} p^{n\beta} = \frac{p^{m\beta}}{p^{\beta}-1}. \tag{4.21}$$

Using (4.21) and Lemma 3, we obtain

$$\sum_{A} \sum_{n:n(V)<0} \sum_{V' \in A'} p^{n(V')(a'(V') + |K(V')| + 1)} p^{-|K(V')|+1} \cdot \frac{(p-1)!}{(p-|K(V')|)!}$$

$$= \sum_{A} \prod_{V' \in A'} p^{-|K(V')|+1} \cdot \frac{(p-1)!}{(p-|K(V')|)!} \frac{1}{p^{\beta(V')}-1}. \tag{4.22}$$

Here we used that

$$\beta(V') = \sum_{V'' \subseteq V', V'' \in A'} (a'(V'') + |K(V'')| - 1) > 0. \tag{4.23}$$

As

$$\sum_{V'' \subseteq V', V'' \in A'} a'(V'') = a(V'), \tag{4.24}$$

$$\sum_{V'' \subseteq V', V'' \in A'} (|K(V''| - 1) = |V'| - 1, \tag{4.25}$$

we have

$$\beta(V') = a(V') + |V'| - 1 > 0, \quad V' \subseteq V \tag{4.26}$$

by the assumption of the theorem. Finally we get

$$F_2(a_V) = p^{-|V|+1} \sum_{A} \prod_{V' \in A'} \frac{1}{p^{a(V')+|V'|-1}-1} \cdot \frac{(p-1)!}{(p-|K(V')|)!}. \tag{4.27}$$

The theorem is proved.

This last formula defines also an analytic continuation of $F_2(a_V)$ from the domian (4.29) to the whole complex plane as a meromorphic function of $a(v, v')$, $v, v' \in V$.

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